Prof Mark Thomson

Particle Physics Major Option

EXAMPLES SHEET 3

NEUTRINO OSCILLATIONS

21. In the CHOOZ experiment, a neutrino detector was positioned a distance $L \approx 1\,\mathrm{km}$ from a nuclear reactor emitting neutrinos (actually antineutrinos) of mean energy $E \approx 3\,\mathrm{MeV}$. The number of neutrino interactions observed was consistent with the number expected assuming no neutrino oscillations, giving the result $P(\nu_\mathrm{e} \to \nu_\mathrm{e}) = 1.01 \pm 0.04$.

a) Show that neutrino oscillations associated with the (solar) mass-squared difference $|\Delta m^2_{12}| \approx 7 \times 10^{-5}\,\mathrm{eV}^2$ can be neglected for the CHOOZ experiment, and that

$$P(\nu_\mathrm{e} \to \nu_\mathrm{e}) \approx 1 - \sin^2 2\theta_{13} \sin^2 \Delta_{23}$$

where

$$\Delta_{23} \equiv \frac{\Delta m^2_{23} L}{4E}\ .$$

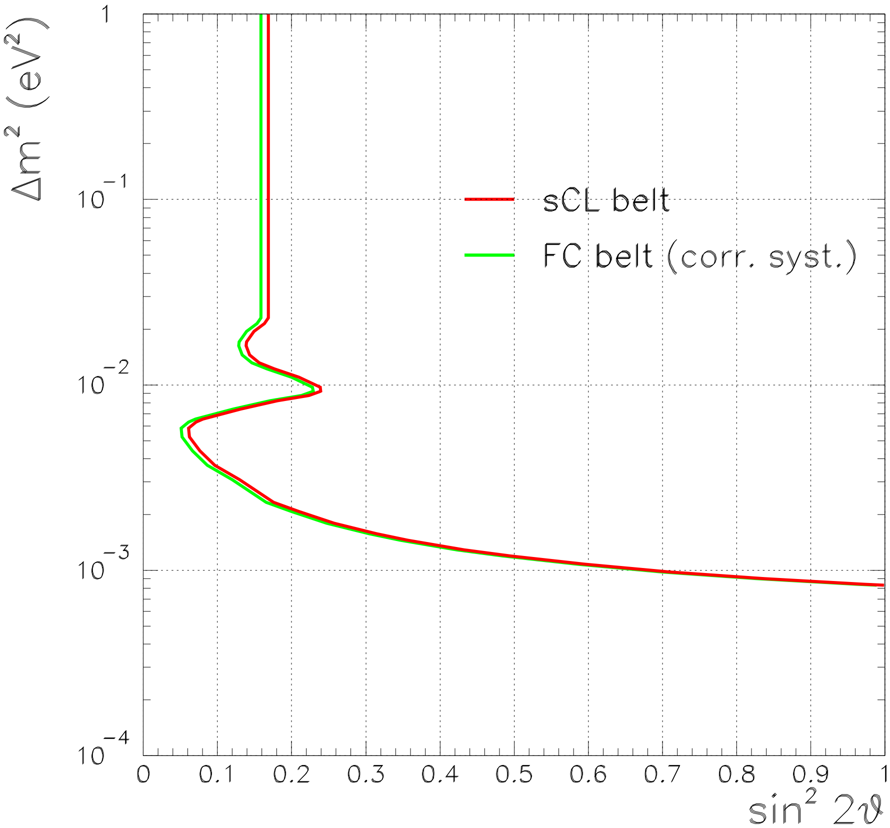
b) In the limit $|\Delta m^2_{23}| \gg (E/L)$, explain why a given measurement, P, of the survival probability $P(\nu_\mathrm{e} \to \nu_\mathrm{e})$ determines the neutrino mixing to be $\sin^2 2\theta_{13} = 2(1 - P)$.

c) In the limit $|\Delta m^2_{23}| \ll (E/L)$, show that a given measurement, P, of the survival probability $P(\nu_\mathrm{e} \to \nu_\mathrm{e})$ determines the neutrino mixing to be $\sin^2 2\theta_{13} \propto 1/(\Delta m^2_{23})^2$, with constant of proportionality $(1 - P)(4E/L)^2$.

d) The null result from the CHOOZ experiment, $P(\overline{\nu}_\mathrm{e} \to \overline{\nu}_\mathrm{e}) = 1.01 \pm 0.04$, can be used to exclude a region of the $(\sin^2 2\theta_{13}, \Delta m^2_{23})$ parameter space. This is conventionally presented as the region which can be excluded at "90% Confidence Level", which for the CHOOZ measurement encompasses all values of $(\sin^2 2\theta_{13}, \Delta m^2_{23})$ which would give a survival probability $P(\nu_\mathrm{e} \to \nu_\mathrm{e}) < 0.92$ (which is about twice the rms precision of the measurement below unity: $P < 1 - 2 \times 0.04$). In the plot overleaf, published by CHOOZ, the curves correspond approximately to the contour $P = 0.92$ and the excluded region lies above and to the right of the curves. [The two similar curves correspond to slightly different statistical approaches to the analysis of the data.]

Use the results derived above to justify approximately the shape and position of the exclusion contour in the regions close to the intercepts with the upper horizontal and right-hand vertical axes. Give a qualitative explanation of the shape of the remainder of the contour.

Evaluate the survival probability $P(\nu_\mathrm{e} \to \nu_\mathrm{e})$ for some representative points lying on either side of the contour, specifically for $(\sin^2 2\theta_{13}, \Delta m^2_{23}) = (0.5, 5 \times 10^{-4}\,\mathrm{eV}^2)$ and $(0.5, 5 \times 10^{-3}\,\mathrm{eV}^2)$.



e) Experiments studying atmospheric neutrino oscillations indicate a mass-squared splitting in the range $|\Delta m^2_{23}| \approx 2 - 3 \times 10^{-3}\,\mathrm{eV}^2$. What constraint can now be placed on the angle θ_{13} ?

22. a) It was shown in the lectures (see Equation (14) of Handout 12) that a general expression for the probability that an initial ν_e oscillates into a ν_μ is

$$P(\nu_\mathrm{e} \to \nu_\mu) = 2\sum_{i<j} \mathrm{Re}\left(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j}\left[e^{-i(E_i-E_j)t} - 1\right]\right) \ .$$

Show that

$$P(\nu_\mathrm{e} \to \nu_\mu) = -4\sum_{i<j} \mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin^2\Delta_{ij} + 2\sum_{i<j} \mathrm{Im}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin 2\Delta_{ij}$$

where

$$\Delta_{ij} \equiv \frac{(m_i^2 - m_j^2)L}{4E} \equiv \frac{\Delta m^2_{ij}L}{4E} \ .$$

b) Use the unitarity of the PMNS matrix to show that

$$\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3}) = -\mathrm{Im}(U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3}) = -\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2}) \equiv -J, \text{ say} \ .$$

c) Hence show that

$$P(\nu_\mathrm{e} \to \nu_\mu) = -4\sum_{i<j} \mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin^2\Delta_{ij} + 8J\sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23}$$

[You may wish to use the trigonometric identity

$$\sin A + \sin B - \sin(A+B) = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{A+B}{2} \ .]$$

d) The standard parameterisation of the PMNS matrix is

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. Show that, in this parameterisation,

$$J = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\sin\delta$$

and find the maximum possible value of $|J|$ given the present experimental knowledge of the mixing angles θ_{12}, θ_{23} and θ_{13}.

e) The conversion probabilities for antineutrinos are obtained by replacing U by U^*. Show that

$$P(\nu_\mathrm{e} \to \nu_\mu) - P(\overline{\nu}_\mathrm{e} \to \overline{\nu}_\mu) = 16J\sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23} \ .$$

f) It is proposed to build a "neutrino factory" to search for evidence of CP violation in neutrino oscillations; $P(\nu_\mathrm{e} \to \nu_\mu) \neq P(\overline{\nu}_\mathrm{e} \to \overline{\nu}_\mu)$. A neutrino factory would produce an intense beam of neutrinos with typical energy $10\,\mathrm{GeV}$. Roughly how far away should a neutrino detector be positioned to optimise the chances of observing CP violation, and how large an effect might be expected ?

CP VIOLATION AND THE CKM MATRIX

23. a) Draw Feynman diagrams for the decays $\mathrm{K}^0 \to \pi^+\pi^-$ and $\overline{\mathrm{K}}^0 \to \pi^+\pi^-$, and for the decays $\mathrm{K}^0 \to \pi^0\pi^0$ and $\overline{\mathrm{K}}^0 \to \pi^0\pi^0$.

    b) Draw Feynman diagrams for the decays $\mathrm{K}^0 \to \pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\overline{\mathrm{K}}^0 \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$, and explain why the decays $\overline{\mathrm{K}}^0 \to \pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\mathrm{K}^0 \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$ cannot occur.

    c) How does the decay rate for each of the above decays depend on the Cabibbo angle θ_C ?

24. In the CPLEAR experiment at CERN, neutral kaons are produced in low energy proton-antiproton collisions via the channels $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\overline{\mathrm{K}}^0$ and $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\mathrm{K}^0$. The strangeness of the initial $\overline{\mathrm{K}}^0$ or K^0 is tagged by the charge of the accompanying K^+ or K^-, and the K^0 or $\overline{\mathrm{K}}^0$ is subsequently detected via decays into the semileptonic final states $\pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$.

    a) Draw Feynman diagrams for the reactions $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\overline{\mathrm{K}}^0$ and $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\mathrm{K}^0$, and explain why the reactions $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\mathrm{K}^0$ and $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\overline{\mathrm{K}}^0$ cannot occur.

    b) Show that, for a system which is initially in a pure K^0 state, the decay rates R_+ and R_- to the semileptonic final states $\pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$ depend on the proper decay time t as

$$R_+ \equiv \Gamma(\mathrm{K}^0_{t=0} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) = N_{\pi e\nu}\tfrac{1}{4}\left[e^{-\Gamma_\mathrm{S}t} + e^{-\Gamma_\mathrm{L}t} + 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}\cos\Delta m t\right]$$
$$R_- \equiv \Gamma(\mathrm{K}^0_{t=0} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}) \approx N_{\pi e\nu}\tfrac{1}{4}\left[1 - 4\mathrm{Re}\epsilon\right]\left[e^{-\Gamma_\mathrm{S}t} + e^{-\Gamma_\mathrm{L}t} - 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}\cos\Delta m t\right]$$

    where $\Gamma_S = 1/\tau_S$, $\Gamma_L = 1/\tau_L$, $\Delta m = m_\mathrm{L} - m_\mathrm{S}$, ϵ is the CP violation parameter, and $N_{\pi e\nu}$ is an overall normalisation constant. Show that the corresponding expressions for a system which is initially in a pure $\overline{\mathrm{K}}^0$ state are

$$\bar{R}_+ \equiv \Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) \approx N_{\pi e\nu}\tfrac{1}{4}\left[1 + 4\mathrm{Re}\epsilon\right]\left[e^{-\Gamma_\mathrm{S}t} + e^{-\Gamma_\mathrm{L}t} - 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}\cos\Delta m t\right]$$
$$\bar{R}_- \equiv \Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}) = N_{\pi e\nu}\tfrac{1}{4}\left[e^{-\Gamma_\mathrm{S}t} + e^{-\Gamma_\mathrm{L}t} + 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}\cos\Delta m t\right] .$$

    c) The figure overleaf shows a measurement from the CPLEAR experiment of the asymmetry

$$A_{\Delta m} \equiv \frac{(R_+ + \overline{R}_-) - (\overline{R}_+ + R_-)}{(R_+ + \overline{R}_-) + (\overline{R}_+ + R_-)}$$

    as a function of the proper decay time $\tau = t$ (plotted in units of the $\mathrm{K_S}$ lifetime $\tau_S = 0.9 \times 10^{-10}$ s). Show that $A_{\Delta m}$ is given by

$$A_{\Delta m} = \frac{2\cos(\Delta m t)\, e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}}{e^{-\Gamma_\mathrm{S}t} + e^{-\Gamma_\mathrm{L}t}}$$

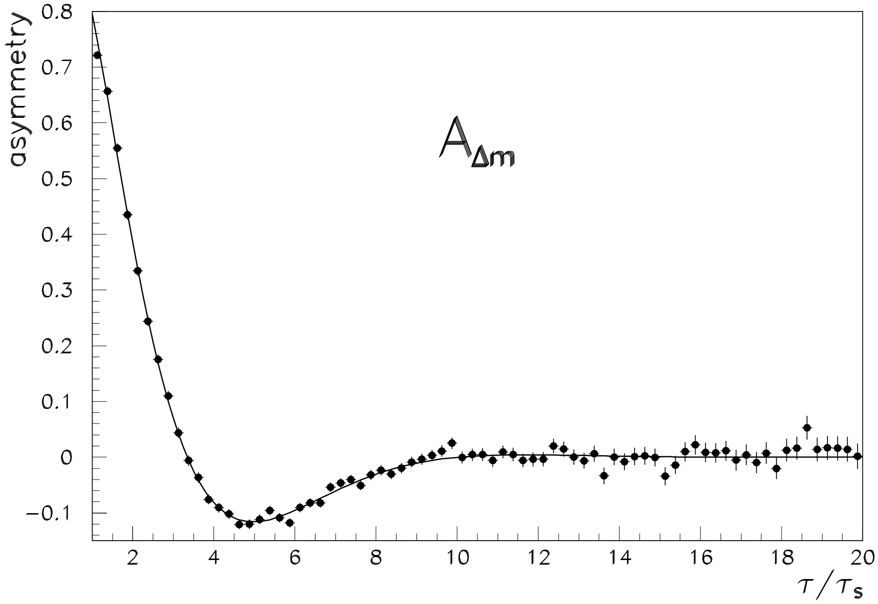
    and obtain an estimate of the mass difference Δm.

    d) Show that the time-reversal asymmetry

$$A_T \equiv \frac{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \mathrm{K}^0) - \Gamma(\mathrm{K}^0_{t=0} \to \overline{\mathrm{K}}^0)}{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \mathrm{K}^0) + \Gamma(\mathrm{K}^0_{t=0} \to \overline{\mathrm{K}}^0)}$$

    is independent of the decay time t and that

$$A_T \approx 4\mathrm{Re}(\epsilon) = 4|\epsilon|\cos\phi .$$



25. a) Draw the Feynman (box) diagrams responsible for $\mathrm{K}^0-\overline{\mathrm{K}}^0$, $\mathrm{D}^0-\overline{\mathrm{D}}^0$, $\mathrm{B}^0_\mathrm{d}-\overline{\mathrm{B}}^0_\mathrm{d}$ and $\mathrm{B}^0_\mathrm{s}-\overline{\mathrm{B}}^0_\mathrm{s}$ mixing.

[The K^0, D^0, B^0_d and B^0_s mesons have quark content $\mathrm{d\bar{s}}$, $\mathrm{c\bar{u}}$, $\mathrm{d\overline{b}}$ and $\mathrm{s\overline{b}}$, respectively.]

b) The mass difference Δm between the mass eigenstates resulting from mixing in neutral meson systems is proportional to the magnitude of the matrix element derived from the box diagrams: $\Delta m \propto |M_\mathrm{fi}|$. For $\mathrm{K}^0-\overline{\mathrm{K}}^0$ mixing, for example, the box diagrams involving virtual quarks of flavour q and q′, with masses m_q and $m_{q'}$, lead to the prediction

$$\Delta\,\mathrm{mK} \approx \frac{G_\mathrm{F}^2}{3\pi^2} f_\mathrm{K}^2\,\mathrm{mK}\left|V_\mathrm{qd}V^*_\mathrm{qs}V_\mathrm{q'd}V^*_\mathrm{q's}\right| m_q m_{q'}$$

where f_K is a constant and the V_{ij} are CKM matrix elements. Show that the dominant contribution to $\Delta\,\mathrm{mK}$ comes from the box diagram containing two virtual charm quarks. Estimate $\Delta\,\mathrm{mK}$ and compare with experiment. [Take $f_\mathrm{K} = 100\,\mathrm{MeV}$.]

c) Show that the dominant contributions to $\mathrm{D}^0-\overline{\mathrm{D}}^0$ and $\mathrm{B}^0-\overline{\mathrm{B}}^0$ mixing come from the box diagrams containing two virtual strange quarks and two virtual top quarks, respectively. Obtain estimates of Δm_D and Δm_B. [Take $f_\mathrm{K} = f_\mathrm{D} = f_\mathrm{B}$]. Explain why $\mathrm{D}^0-\overline{\mathrm{D}}^0$ mixing has not been (and is unlikely to be) observed. [Hint: convert Δm_D to a time and compare with the measured D^0 lifetime of 0.41 ps.]

NUMERICAL ANSWERS

21. d) $P(\nu_\mathrm{e} \to \nu_\mathrm{e}) = 0.98$ and 0.63; e) $\sin^2\theta_{13} < 0.058$

22. d) $|J|_\mathrm{max} = 0.053$; f) about $5000\,\mathrm{km}$, $|\Delta P|_\mathrm{max} \approx 0.04$

25. b) $\Delta\,\mathrm{mK} \sim 2 \times 10^{-12}$ MeV;
c) $\Delta m_\mathrm{D} \sim 10^{-12}$ MeV, $\Delta m_{B_d} \sim 10^{-9}$ MeV, $\Delta m_{B_s} \sim 10^{-8}$ MeV