Prof Mark Thomson

Particle Physics Major Option

EXAMPLES SHEET 2

11. a) The elastic form factors for the proton are well described by the form

$$G(q^2) = \frac{G(0)}{(1 + |q^2|/0.71)^2}$$

with q^2 in GeV^2. Show that an exponential charge distribution in the proton

$$\rho(\boldsymbol{r}) = \rho_0 e^{-\lambda r}$$

leads to this form for $G(q^2)$ (insofar as $|q^2| = |\boldsymbol{q}^2|$), and calculate λ.

b) Show that, for any spherically symmetric charge distribution, the mean square radius is given by

$$\langle r^2 \rangle = -6 \left[\frac{\mathrm{d}G(q^2)}{\mathrm{d}|q^2|} \right]_{q^2=0}$$

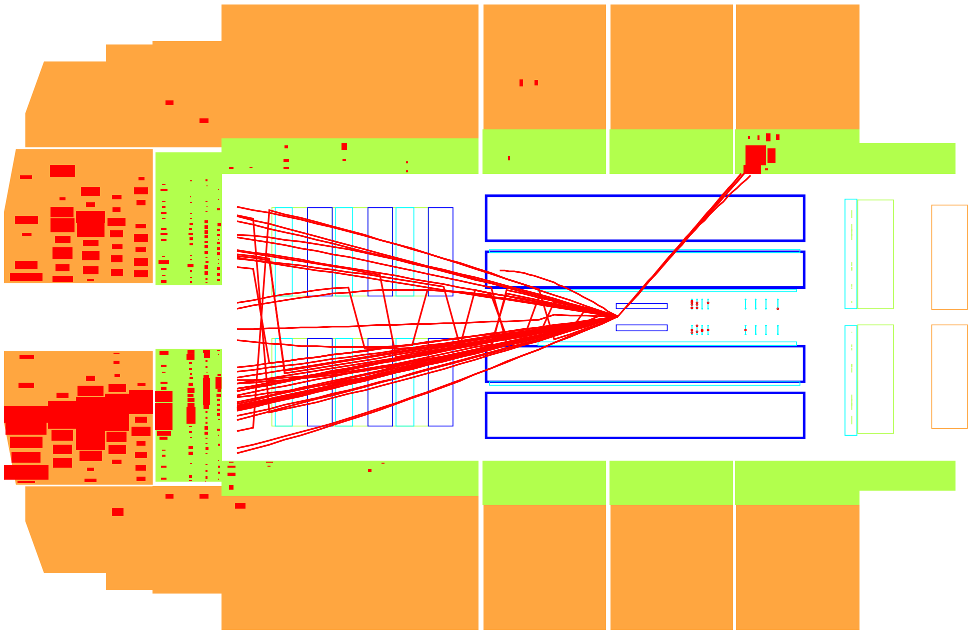
and estimate the r.m.s. charge radius of the proton.

c) The pion form factor may be determined in $\pi \mathrm{e}^-$ scattering. Use the following data to estimate the r.m.s. charge radius of the pion.

$\lvert q^2 \rvert$ (GeV^2)	$G_E^2(q^2)$
0.015	0.944 $\pm$ 0.007
0.042	0.849 $\pm$ 0.009
0.074	0.777 $\pm$ 0.016
0.101	0.680 $\pm$ 0.017
0.137	0.646 $\pm$ 0.027
0.173	0.534 $\pm$ 0.030
0.203	0.529 $\pm$ 0.040
0.223	0.487 $\pm$ 0.049

DEEP-INELASTIC SCATTERING

12. The figure below shows a deep-inelastic scattering event $\mathrm{e^+p \to e^+X}$ recorded by the H1 experiment at the HERA collider. The positron beam, of energy $E_1 = 27.5\,\mathrm{GeV}$, enters from the left and the proton beam, of energy $E_2 = 820\,\mathrm{GeV}$, enters from the right. The energy of the outgoing positron is measured to be $E_3 = 31\,\mathrm{GeV}$.



a) Show that the Bjorken scaling variable x is given by

$$x = \frac{E_3}{E_2}\left[\frac{1-\cos\theta}{2-(E_3/E_1)(1+\cos\theta)}\right]$$

where θ is the angle through which the positron has scattered.

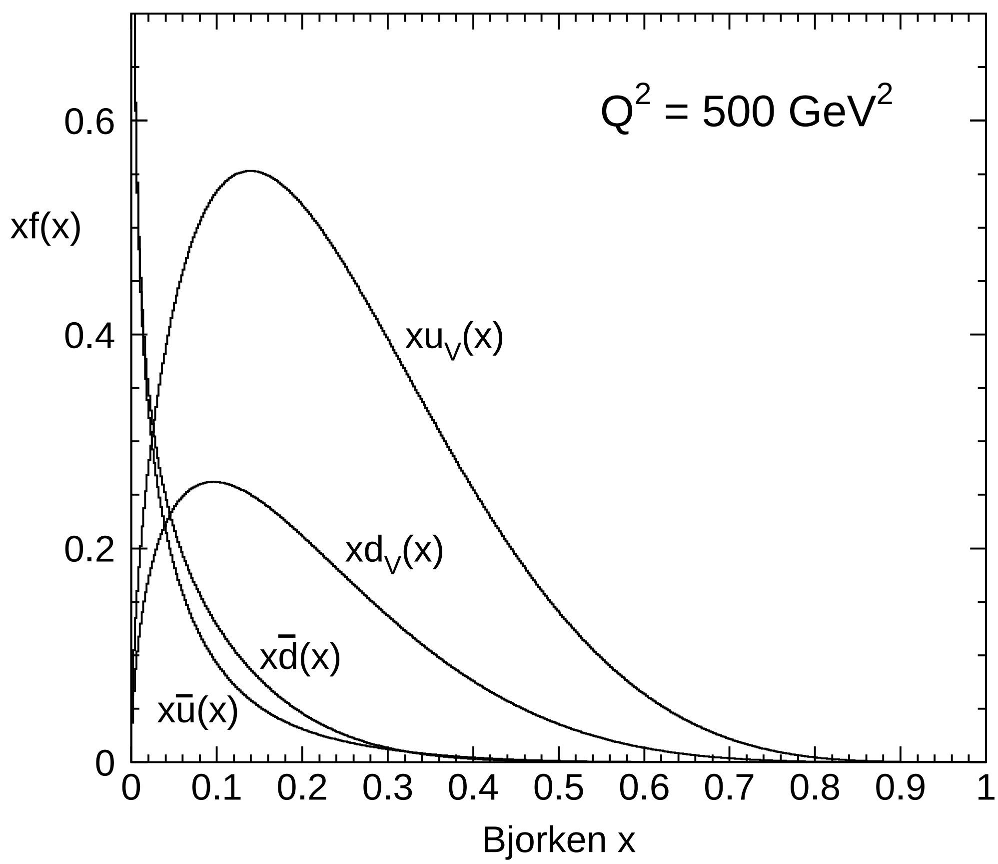
b) Estimate the values of Q^2, x and y for this event.

c) Estimate the invariant mass M_{X} of the final state hadronic system.

d) Draw quark level diagrams to illustrate the possible origins of this event. Using the plot overleaf of the parton distribution functions $xu_{\mathrm{V}}(x)$, $xd_{\mathrm{V}}(x)$, $x\overline{u}(x)$ and $x\overline{d}(x)$, estimate the relative probabilities of the various possible quark-level processes for the event.

[Neglect contributions from the heavier quarks s, c, b, t.]

e) Estimate the relative contributions of the F_1 and F_2 terms to the deep-inelastic cross section for the x and Q^2 values corresponding to this event.



Q^2 = 500 GeV^2
xf(x)
0.6
0.4
0.2
0
xu_V(x)
xd_V(x)
xd̄(x)
xū(x)
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Bjorken x

13. a) Show that the lab frame differential cross section $\mathrm{d}^2\sigma/\mathrm{d}E_3\mathrm{d}\Omega$ for deep-inelastic scattering is related to the Lorentz invariant differential cross section $\mathrm{d}^2\sigma/\mathrm{d}\nu\mathrm{d}Q^2$ via

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}Q^2} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2}$$

where E_1 and E_3 are the energies of the incoming and outgoing lepton, $\nu = E_1 - E_3$, and $Q^2 = -q^2 = -(p_1 - p_3)^2$. Show further that

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2} = \frac{2Mx^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}$$

where M is the mass of the target nucleon and $x = Q^2/2M\nu$.

b) Show that

$$\frac{2Mx^2}{Q^2}\cdot\frac{y^2}{2} = \frac{1}{M}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}$$

and that

$$1 - y - \frac{M^2x^2y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2}\,.$$

c) Show that the Lorentz invariant cross section for deep-inelastic electromagnetic scattering,

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)\frac{F_2}{x} + \frac{y^2}{2}\frac{2xF_1}{x}\right]$$

becomes

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{F_2}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1}{M}\sin^2\frac{\theta}{2}\right]$$

in the lab frame.

d) An experiment consists of an electron beam of maximum energy $20\,\mathrm{GeV}$ and a variable angle spectrometer which can detect scattered electrons with energies greater than $2\,\mathrm{GeV}$. Find the range of values of θ over which deep-inelastic scattering events can be studied for $x = 0.2$ and $Q^2 = 2\,\mathrm{GeV}^2$.

[You may find it helpful to determine $E_1 - E_3$ (fixed), and E_1E_3 in terms of θ, and then sketch the various constraints on E_1 and E_3 on a 2D plot of E_3 against E_1.]

e) Outline a possible experimental strategy for measuring $F_1(x, Q^2)$ and $F_2(x, Q^2)$ for the above values of x and Q^2.

HADRONS AND QCD

14. Imagine that the u,d and s quarks exist with their observed quantum numbers, except that they have spin zero. Discuss in as much detail as you can the resulting spectrum of hadrons and their properties. You should specifically consider the possible J^{PC} values of the meson multiplets, and the J^P value and multiplicity of the lightest baryon multiplet. Are these results compatible with the data ?

    [Remember that bosons have the same parity as antibosons].

15. a) Show that the short range interaction between the quark and antiquark in a meson is *attractive* if the meson is in the colour singlet state

$$\psi = \frac{1}{\sqrt{3}}(r\overline{r} + g\overline{g} + b\overline{b})$$

but *repulsive* if the meson is in any of the colour octet states

$$\psi = \frac{1}{\sqrt{6}}(r\overline{r} + g\overline{g} - 2b\overline{b}) \qquad \frac{1}{\sqrt{2}}(r\overline{r} - g\overline{g}) \qquad r\overline{g} \quad r\overline{b} \quad g\overline{r} \quad g\overline{b} \quad b\overline{r} \quad b\overline{g}\ .$$

b) Sketch the possible colour quantum numbers of a two-quark system on a plot of colour hypercharge Y^{c} against colour isospin I_3^{c}. Using ladder operators, or otherwise, show that the two-quark colour states consist of a sextet plus a triplet, and determine the colour wavefunctions of each state. Show that the strong interaction potential arising from single-gluon exchange between the two quarks is repulsive for the colour sextet but attractive for the colour triplet. Why, if the potential is attractive, are hadrons consisting of two quarks ("diquarks") not observed ?

c) Use the baryon colour singlet wavefunction

$$\psi = \frac{1}{\sqrt{6}}(rgb - grb + gbr - bgr + brg - rbg)$$

to show that the short range interaction between any pair of quarks in a baryon is attractive. (*i.e.* show that the overall colour factor C is negative for, say, quarks 1 and 2 in the baryon.)

WEAK INTERACTIONS

16. Following on from Question 9, show that, for a free particle spinor ψ:

$$\overline{\psi_{\rm L}}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi_{\rm R} = \overline{\psi_{\rm R}}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi_{\rm L} = \overline{\psi_{\rm R}}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi_{\rm R} = 0$$

$$\overline{\psi_{\rm L}}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi_{\rm L} = \overline{\psi}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi$$

where $\psi_{\rm L} \equiv \frac{1}{2}(1-\gamma^5)\psi$ and $\psi_{\rm R} \equiv \frac{1}{2}(1+\gamma^5)\psi$. Explain the relevance of these results to the weak interactions. What are the equivalent results for currents of the form $\overline{\psi}\gamma^\mu\frac{1}{2}(1+\gamma^5)\psi$?

17. a) In Question 5, the decay rate for $\pi^-\to{\rm e}^-\overline{\nu}_{\rm e}$ was found to be 1.28×10^{-4} times that for $\pi^-\to\mu^-\overline{\nu}_\mu$, whereas, on the basis of phase space alone, one would expect a higher decay rate to electrons. Explain why the weak interaction gives such a small decay rate to electrons.

b) The Lorentz invariant matrix element for $\pi^-\to\mu^-\overline{\nu}_\mu$ decay is

$$M_{\rm fi} = \frac{g_{\rm W}^2}{4m_{\rm W}^2}g_{\mu\nu}f_\pi p_1^\mu\overline{u}(p_3)\gamma^\nu\tfrac{1}{2}(1-\gamma^5)v(p_4)$$

where p_1, p_3 and p_4 are the 4-momenta of the π^-, μ^- and $\overline{\nu}_\mu$, respectively, and f_π is a constant which must be determined experimentally. Verify that this matrix element follows from the Feynman rules, with the quark current $\overline{u}\gamma^\mu(1-\gamma^5)v$ taken to be of the form $-f_\pi p_1^\mu$.

[The free particle spinors u, v cannot be used for quarks and antiquarks in a hadronic bound state; a quark current of the form given can be shown to be the most general possibility.]

c) Show that (as in Question 9) the Lorentz-invariant matrix element squared is

$$|M_{\rm fi}|^2 = 2G_{\rm F}^2f_\pi^2m_\mu^2(m_\pi^2-m_\mu^2)\ .$$

[Use the spinors u_1, u_2, v_1, v_2 for this calculation rather than the spinors $u_\uparrow$, $u_\downarrow$, $v_\uparrow$, $v_\downarrow$. Work in the π^- rest frame, and choose the 4-momenta of the μ^- and $\overline{\nu}_\mu$ to be $p_3=(E,0,0,p)$ and $p_4=(p,0,0,-p)$, with $E=\sqrt{p^2+m_\mu^2}$.]

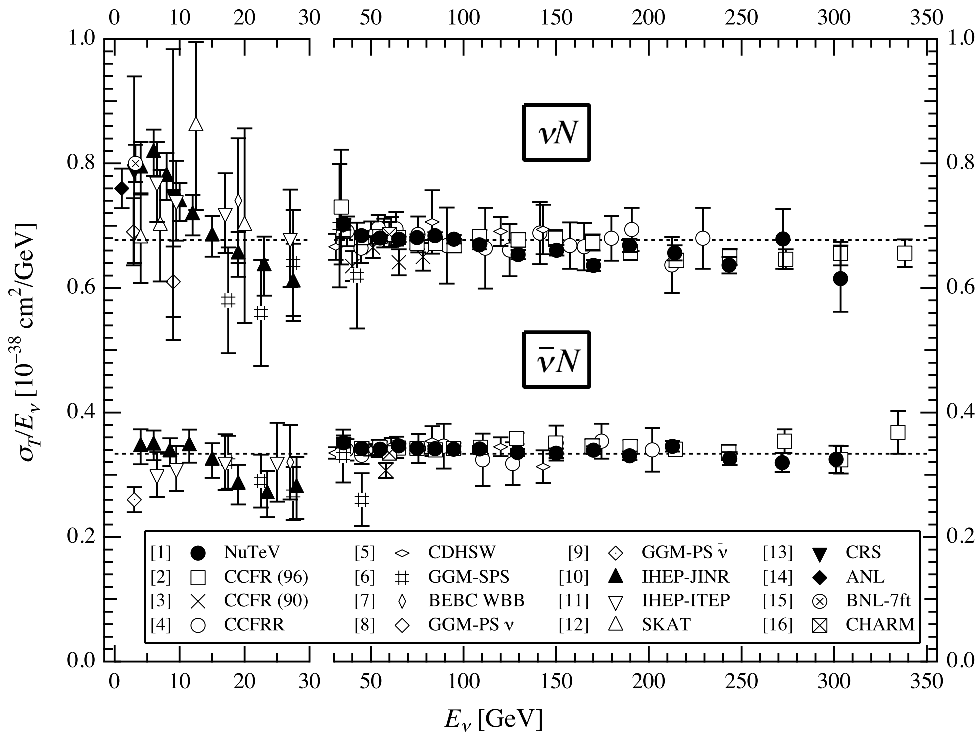
d) Show that the square of the *non-invariant* matrix element $T_{\rm fi}$ is proportional to $1-\beta$:

$$|T_{\rm fi}|^2 = \frac{G_{\rm F}^2}{2}f_\pi^2m_\pi\,(1-\beta)$$

where β is the velocity of the μ^-.

18. Find the maximum possible value of Q^2 in deep-inelastic neutrino scattering for a neutrino beam energy of 400 GeV, and compare with m_{W}^2.

19. The figure below shows the measured total cross sections $\sigma(\nu_\mu + \mathrm{N} \to \mu^- + \text{hadrons})/E_\nu$ and $\sigma(\overline{\nu}_\mu + \mathrm{N} \to \mu^- + \text{hadrons})/E_{\overline{\nu}}$ for charged-current neutrino and antineutrino scattering, averaged over proton and neutron targets.



a) Draw Feynman diagrams for the quark-level processes which contribute to neutrino-nucleon and antineutrino-nucleon scattering. (Neglect the s, c, b and t quark flavours).

b) Show that the parton model predicts total cross sections of the form

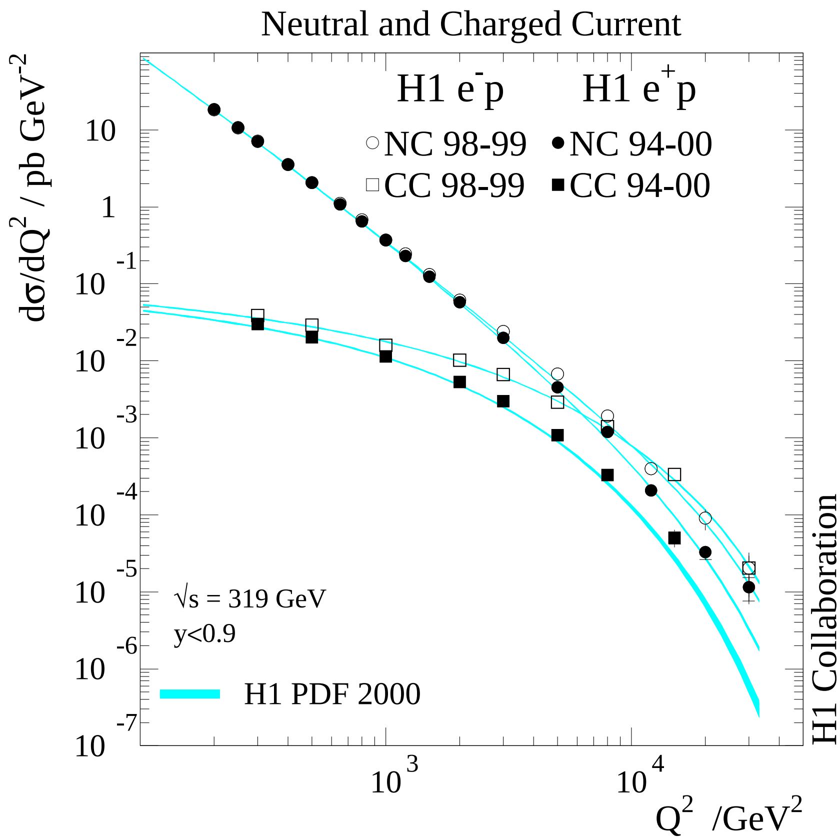
$$\sigma^{\nu N} \equiv \tfrac{1}{2}\left(\sigma^{\nu \mathrm{p}} + \sigma^{\nu \mathrm{n}}\right) = \frac{G_{\mathrm{F}}^2 s}{2\pi}\left[f_{\mathrm{q}} + \tfrac{1}{3} f_{\overline{\mathrm{q}}}\right]$$
$$\sigma^{\overline{\nu} N} \equiv \tfrac{1}{2}\left(\sigma^{\overline{\nu} \mathrm{p}} + \sigma^{\overline{\nu} \mathrm{n}}\right) = \frac{G_{\mathrm{F}}^2 s}{2\pi}\left[\tfrac{1}{3} f_{\mathrm{q}} + f_{\overline{\mathrm{q}}}\right]$$

where s is the neutrino-nucleon centre of mass energy squared, and $f_{\mathrm{q}} = f_{\mathrm{u}} + f_{\mathrm{d}}$ and $f_{\overline{\mathrm{q}}} = f_{\overline{\mathrm{u}}} + f_{\overline{\mathrm{d}}}$ are the average momentum fractions carried by u and d quarks and antiquarks.

c) Estimate the average fractions of the nucleon momentum carried by quarks, antiquarks and gluons.

[Take $G_{\mathrm{F}} = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$.]

20. The figure below shows measurements of the cross section $\mathrm{d}\sigma/\mathrm{d}Q^2$ from the H1 experiment at HERA for the neutral current (NC) processes $\mathrm{e}^-\mathrm{p} \to \mathrm{e}^-\mathrm{X}$ and $\mathrm{e}^+\mathrm{p} \to \mathrm{e}^+\mathrm{X}$, and the charged current (CC) processes $\mathrm{e}^-\mathrm{p} \to \nu_\mathrm{e}\mathrm{X}$ and $\mathrm{e}^+\mathrm{p} \to \overline{\nu}_\mathrm{e}\mathrm{X}$, with unpolarised incoming e^+ or e^- and proton beams:



a) Draw Feynman diagrams for the quark-level processes which contribute to CC $\mathrm{e}^-\mathrm{p} \to \nu_\mathrm{e}\mathrm{X}$ and $\mathrm{e}^+\mathrm{p} \to \overline{\nu}_\mathrm{e}\mathrm{X}$ scattering. (Neglect the s, c, b and t quark flavours).

b) The HERA data extends to values of $Q^2 > m_\mathrm{W}^2$. Starting from the parton model cross sections $\mathrm{d}^2\sigma/\mathrm{d}x\mathrm{d}Q^2$ for (anti)neutrino-nucleon scattering derived in the lectures for $Q^2 \ll m_\mathrm{W}^2$, explain why the CC cross sections can be written down directly as

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}(\mathrm{e}^+\mathrm{p} \to \overline{\nu}_\mathrm{e}\mathrm{X}) = \frac{G_\mathrm{F}^2 m_\mathrm{W}^4}{2\pi x(Q^2+m_\mathrm{W}^2)^2} x\left[\overline{u}(x) + (1-y)^2 d(x)\right]$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}(\mathrm{e}^-\mathrm{p} \to \nu_\mathrm{e}\mathrm{X}) = \frac{G_\mathrm{F}^2 m_\mathrm{W}^4}{2\pi x(Q^2+m_\mathrm{W}^2)^2} x\left[u(x) + (1-y)^2 \overline{d}(x)\right]$$

c) Explain why the $\mathrm{e}^-\mathrm{p}$ CC cross section is always higher than the $\mathrm{e}^+\mathrm{p}$ CC cross section.

d) Explain why the CC cross sections become approximately constant as Q^2 decreases, while the NC cross sections grow indefinitely large. Account approximately for the observed slope of the NC cross sections at low values of Q^2.

e) Explain why the NC cross sections become similar in magnitude to the CC cross sections at high values of $Q^2 \sim m_\mathrm{Z}^2$.

f) (optional) Explain why the two NC cross sections are equal at low Q^2, but differ at high Q^2.

NUMERICAL ANSWERS

11. a) $\lambda = 0.84\,\mathrm{GeV}$; b) 0.81 fm; c) $\approx 0.68\,\mathrm{fm}$

12. b) $x \approx 0.09$, $Q^2 \approx 610\,\mathrm{GeV}^2$, $y \approx 0.075$; c) $M_{\mathrm{X}} \approx 78\,\mathrm{GeV}$

    d) relative probabilities that scattering is from u, d, $\overline{\mathrm{u}}$, $\overline{\mathrm{d}}$ are

$$\mathrm{u} : \mathrm{d} : \overline{\mathrm{u}} : \overline{\mathrm{d}} \approx 0.73 : 0.12 : 0.12 : 0.04\,.$$

    e) the F_1 term contributes only $\approx 0.3\%$ of events.

13. d) $4.7^\circ < \theta < 21.3^\circ$

15. b) sextet: rr, gg, bb, $(rg + gr)/\sqrt{2}$, $(rb + br)/\sqrt{2}$, $(gb + bg)/\sqrt{2}$;
    triplet: $(rg - gr)/\sqrt{2}$, $(rb - br)/\sqrt{2}$, $(gb - bg)/\sqrt{2}$

18. $(Q^2)_{\mathrm{max}} \approx 750\,\mathrm{GeV}^2$

19. $f_{\mathrm{q}} \approx 0.41$, $f_{\overline{\mathrm{q}}} \approx 0.08$, $f_{\mathrm{g}} \approx 0.51$