Particle Physics

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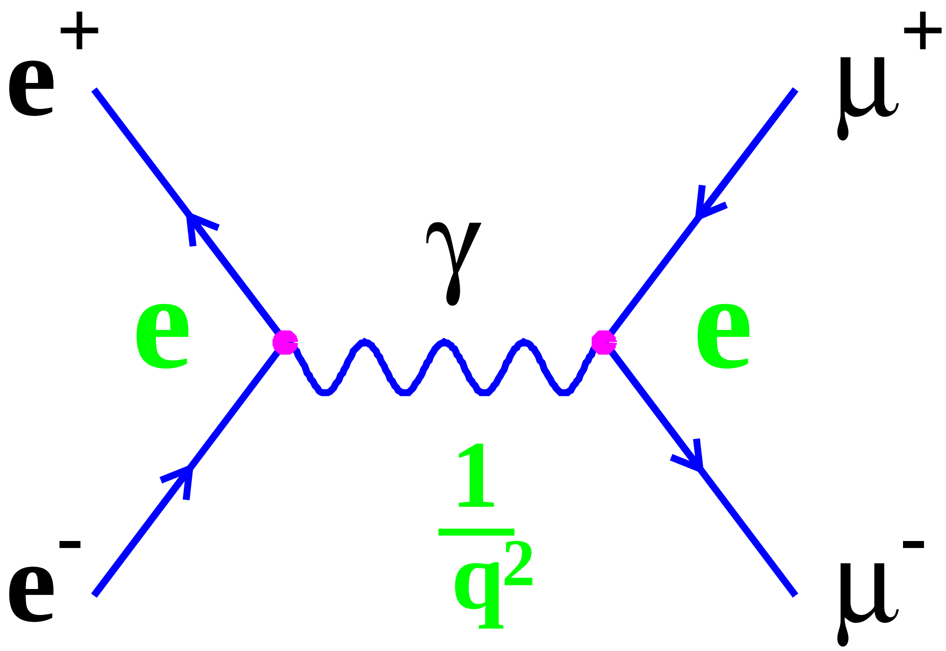


Part II, Lent Term 2004

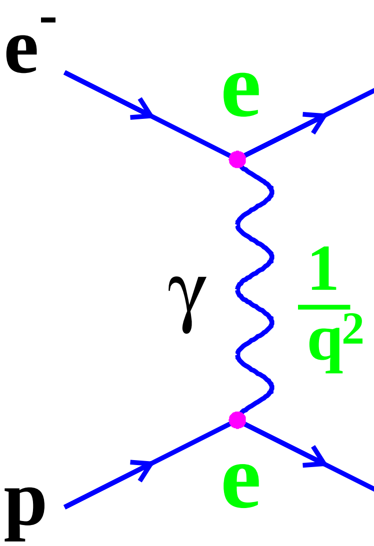
HANDOUT II

Quantum Electrodynamics

QUANTUM ELECTRODYNAMICS: is the quantum theory of the electromagnetic interaction.

CLASSICAL PICTURE: Action at a distance : forces arise from $\tilde{\mathrm{E}}$ and $\tilde{\mathrm{B}}$ fields. Particles act as sources of the fields $\rightarrow V(\tilde{\mathrm{r}})$.

Q.E.D. PICTURE: Forces arise from the exchange of virtual field quanta.



Although a complete derivation of the theory of Q.E.D. and Feynman diagrams is beyond the scope of this course, the main features will be derived.

Interaction via Particle Exchange

NON-EXAMINABLE

FERMI'S GOLDEN RULE for Transition rate, Γ_{fi}:

$$\Gamma_{fi} = \frac{2\pi}{\hbar}|M_{fi}|^2\rho(E_f)$$

$\rho(E_f)$ = density of final states.

★ From 1^{st} order perturbation theory, matrix element M_{fi}:

$$\mathrm{M}_{fi} = \langle\psi_f|\hat{\mathrm{H}}'|\psi_i\rangle$$

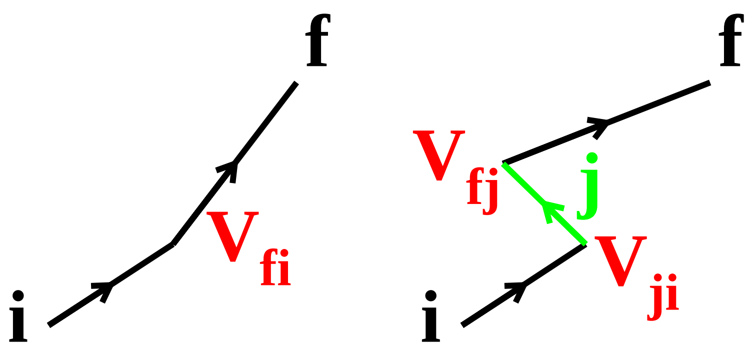
where $\hat{\mathrm{H}}'$ is the operator corresponding to the perturbation to the Hamiltonian.

★ This is only the 1^{st} order term in the perturbation expansion. In 2^{nd} order perturbation theory:

$$\mathrm{M}_{fi} \rightarrow \mathrm{M}_{fi} + \sum_{j\neq i}|\mathrm{M}_{fj}|\frac{1}{E_i - E_j}|\mathrm{M}_{ji}|$$

where the sum is over all intermediate states j, and E_i and E_j are the energies of the initial and intermediate state

★ For scattering, the 1^{st} and 2^{nd} order terms can be viewed as:

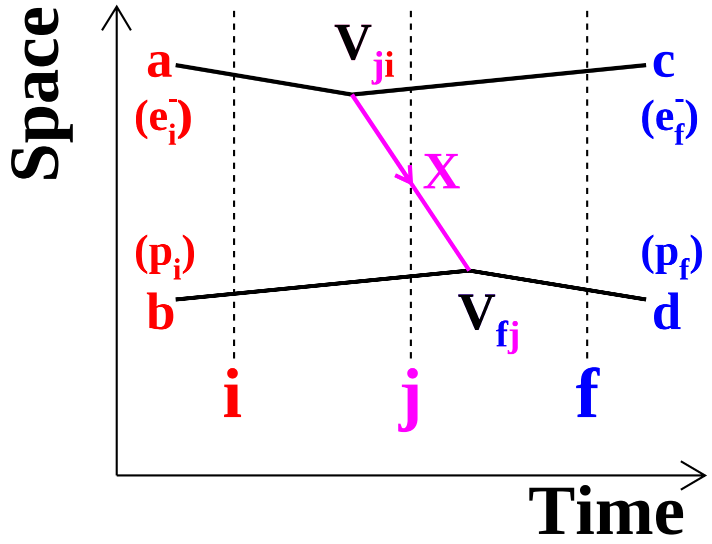


Consider the particle interaction

$$a + b \rightarrow c + d$$

which involves the exchange a particle X. This could be the elastic scattering of electrons and protons, *e.g.* $e^- p \rightarrow e^- p$ where X is an exchanged photon.

★ One possible space-time picture for this process is



Initial State, i: a+b
Final State, f: c+d
Intermediate State, j:
b+c+X

★ The Time Ordered interaction consists of $a \rightarrow c + X$ followed by $b + X \rightarrow d$. For example $e^-_i p_i \rightarrow e^-_f p_f$ has the electron emitting a photon $(e^-_i \rightarrow e^-_f \gamma)$ followed by the photon being absorbed by the proton $(p_i \gamma \rightarrow p_f)$.

★ The corresponding term in 2^{nd} order PT:

$$\begin{aligned} \mathrm{M}^{ab}_{fi} &= \frac{\langle\psi_f|\hat{\mathrm{H}}'|\psi_j\rangle\langle\psi_j|\hat{\mathrm{H}}'|\psi_i\rangle}{E_i - E_j} \\ &= \frac{\langle\psi_d|\hat{\mathrm{H}}'|\psi_X\psi_b\rangle\langle\psi_c\psi_X|\hat{\mathrm{H}}'|\psi_a\rangle}{(E_a + E_b) - (E_c + E_X + E_b)} \\ &= \frac{\langle\psi_d|\hat{\mathrm{H}}'|\psi_X\psi_b\rangle\langle\psi_c\psi_X|\hat{\mathrm{H}}'|\psi_a\rangle}{(E_a - E_c - E_X)} \end{aligned}$$

Before we go any further some comments:

★ The superscript ab on M_{fi}^{ab} indicates the time ordering where a interacts with X before b

consequently the results are not Lorentz Invariant

i.e. depend on rest frame.

★ Momentum is conserved in $a \rightarrow c + X$ and $b + X \rightarrow d$.

★ The exchanged particle X is ON MASS SHELL:

$$E_X^2 - p_X^2 = m_X^2$$

★ The matrix elements $\langle\psi_d|\hat{\mathbf{H}}'|\psi_X\psi_b\rangle$ and $\langle\psi_c\psi_X|\hat{\mathbf{H}}'|\psi_a\rangle$ depend on the "strength" of the interaction. *e.g.* the strength of the γe^- and γp interaction which determines the probability that an electron(proton) will emit(absorb) a photon.

★ For the electromagnetic interaction:

$$\langle\psi_k|\hat{\mathbf{H}}'|\psi_j\rangle = e\epsilon_0\langle\psi_k|z|\psi_j\rangle$$

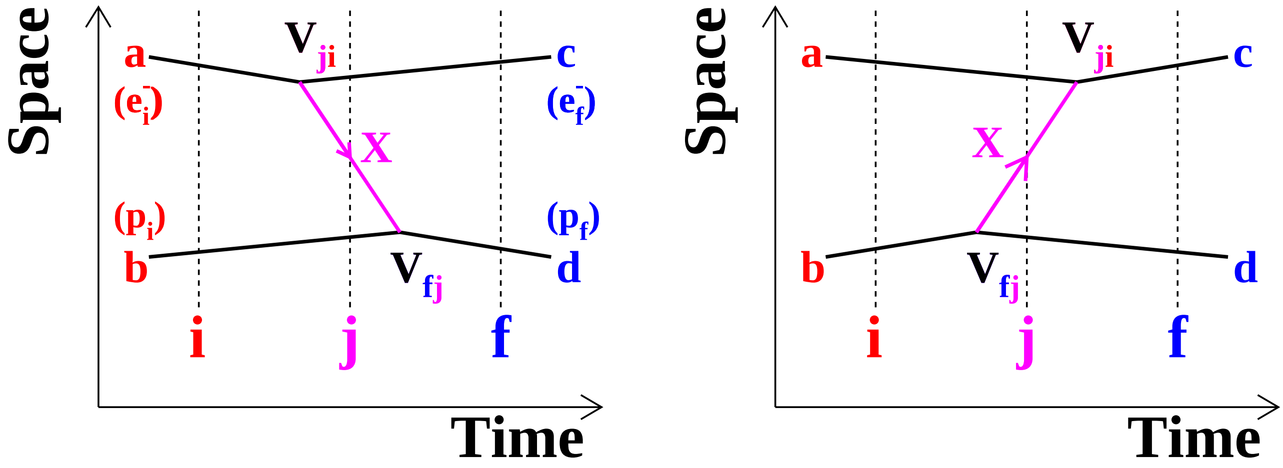
for a photon with polarization in the z-direction. (see Dr Ritchie's QM II lecture 10)

★ Neglecting spin (*i.e.* for assuming all particles are spin-0 *i.e.* scalars) the ME becomes:

$$\langle\psi_d|\hat{\mathbf{H}}'|\psi_X\psi_b\rangle = e$$

★ More generally, $\langle\psi_d|\hat{\mathbf{H}}'|\psi_X\psi_b\rangle = g$, where g is the interaction strength.

Now consider the <u>other</u> time ordering $b \to d + X$ followed by $a + X \to b$



The corresponding term in 2^{nd} order PT:

$$\mathrm{M}_{fi}^{ba} = \frac{\langle\psi_c|\hat{\mathrm{H}}'|\psi_X\psi_a\rangle\langle\psi_d\psi_X|\hat{\mathrm{H}}'|\psi_b\rangle}{(E_a + E_b) - (E_d + E_X + E_a)}$$

$$= \frac{\langle\psi_c|\hat{\mathrm{H}}'|\psi_X\psi_a\rangle\langle\psi_d\psi_X|\hat{\mathrm{H}}'|\psi_b\rangle}{(E_b - E_d - E_X)}$$

$$= \frac{\langle\psi_c|\hat{\mathrm{H}}'|\psi_X\psi_a\rangle\langle\psi_d\psi_X|\hat{\mathrm{H}}'|\psi_b\rangle}{(E_b - E_d - E_X)}$$

Assume a common interaction strength, g, at both vertices,

i.e. $\langle\psi_c|\hat{\mathrm{H}}'|\psi_X\psi_a\rangle = \langle\psi_d\psi_X|\hat{\mathrm{H}}'|\psi_b\rangle = g$

$$\Rightarrow \quad \mathrm{M}_{fi}^{ba} = \frac{g^2}{(E_b - E_d - E_X)} \times \frac{1}{2E_X}$$

WARNING : I have introduced an (unjustified) factor of $\frac{1}{2E_X}$. This arises from the relativistic normalization of the wave-function for particle X (see appendix). For initial/final state particles the normalisation is cancelled by corresponding terms in the flux/phase-space. For the "intermediate" particle X no such cancelation occurs.

Now sum over two time ordered transition rates

$$\mathrm{M}_{fi} = \mathrm{M}_{fi}^{ab} + \mathrm{M}_{fi}^{ba}$$
$$= g^2 \left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_b - E_d - E_X} \right) \times \frac{1}{2E_X}$$

$$\text{since} \quad E_a + E_b = E_c + E_d$$
$$\Rightarrow \quad E_b - E_d = E_c - E_a$$

giving:

$$\mathrm{M}_{fi} = g^2 \left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_c - E_a - E_X} \right) \times \frac{1}{2E_X}$$
$$= g^2 \left(\frac{1}{E_a - E_c - E_X} - \frac{1}{E_a - E_c + E_X} \right) \times \frac{1}{2E_X}$$
$$= g^2 \frac{2E_X}{(E_a - E_c)^2 - E_X^2} \times \frac{1}{2E_X}$$

From the first time ordering:

$$E_X^2 = (\tilde{\mathrm{p}}_a - \tilde{\mathrm{p}}_c)^2 + m_X^2$$

therefore

$$\mathrm{M}_{fi} = \frac{g^2}{(E_a - E_c)^2 - (\tilde{\mathrm{p}}_a - \tilde{\mathrm{p}}_c)^2 - m_X^2}$$

$$\mathrm{M}_{fi} = \frac{g^2}{q^2 - m_X^2}$$

$$\text{with} \quad q^2 = q^\mu q_\mu = E^2 - |\tilde{\mathrm{p}}|^2$$

where $(E, |\tilde{\mathrm{p}}|)$ are energy/momentum carried by the virtual particle. The SUM of time-ordered processes depends on q^2 and is therefore Lorentz invariant ! The 'invariant mass' of the exchanged particle, X, $m_{inv}^2 = E^2 - |\tilde{\mathrm{p}}|^2$, is NOT the REST MASS, m_X.

The term

$$\boxed{\frac{1}{q^2-m^2}}$$

is called the PROPAGATOR

It corresponds to the term in the matrix element arising from the exchange of a massive particle which mediates the force. For massless particles e.g. photons :

$$\boxed{\frac{1}{q^2}}$$

NOTE: q^2 is the 4-momentum of the exchanged particle ($q^2 = q^\mu q_\mu = E^2 - |\tilde{\mathrm{p}}|^2$)

Previously (page 35 of HANDOUT 1) we obtained the matrix element for elastic scattering in the YUKAWA potential:

$$\boxed{M_{fi}^{YUK} = -\frac{g^2}{(m^2+|\vec{\mathrm{p}}|^2)}}$$

For elastic scattering $E_X = 0$, and $q^2 = -|\tilde{\mathrm{p}}|^2$

$$\boxed{M_{fi}^{YUK} \rightarrow \frac{g^2}{q^2-m^2}}$$

Which is exactly the expression obtained on the previous page. Hence, elastic scattering via particle exchange in 2nd order P.T. is equivalent to scattering in a Yukawa potential using 1st order P.T.

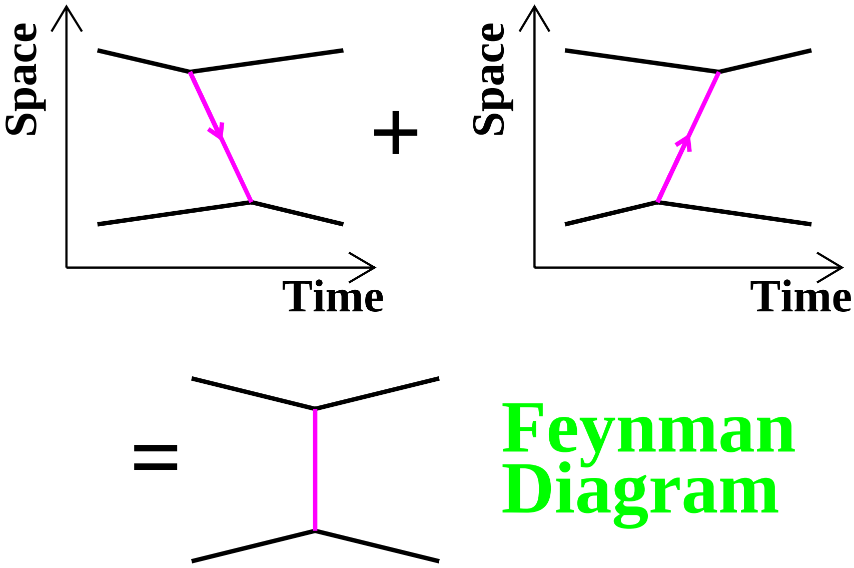
Action at a Distance

NEWTON : "...that one body can act upon another at a distance, through a vacuum, without the mediation of anything else,..., is to me a great absurdity"

- ★ In Classical Mechanics and non-relativistic Quantum Mechanics forces arise from potentials $V(\tilde{r})$ which act instantaneously over all space.
- ★ In Quantum Field theory, forces are mediated by the exchange of virtual field quanta - and there is no mysterious action at a distance.
- ★ Matter and Force described by 'particles'

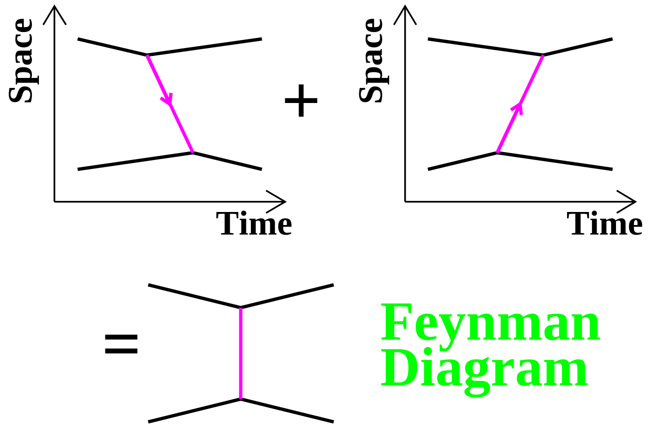
Feynman Diagrams

- ★ **The results of calculations based on a single process in Time-Ordered Perturbation Theory (sometimes called old-fashioned, OFPT) depend on the reference frame.**
- ★ **However, the sum of all time orderings is not frame dependent and provides the basis for our relativistic theory of Quantum Mechanics.**
- ★ **The sum of time orderings are represented by FEYNMAN DIAGRAMS**



- ★ **Energy and Momentum are conserved at the interaction vertices**
- ★ **But the exchanged particle no longer has $m_X^2 = E_X^2 - p_X^2$, it is VIRTUAL**

Virtual Particles



Virtual Particles:

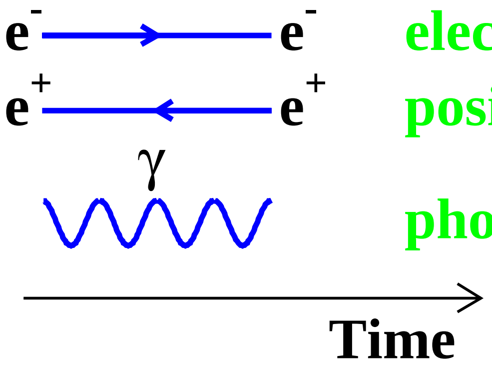
- ★ Forces due to exchanged particle X which is termed VIRTUAL.
- ★ The exchanged particle is off mass-shell, *i.e.* for the unobservable exchanged VIRTUAL particle $E^2 \neq p^2 + m_X^2$.
- ★ *i.e.* $m^2 = E_X^2 - p_X^2$ does not give the physical mass, m_X. The mass of the virtual particle $m^2 = E_X^2 - p_X^2$ can be +ve or -ve.

Qualitatively: the propagator is inversely proportional to how far the particle is off-shell. The further off-shell, the smaller the probability of producing such a virtual state.

Understanding Feynman Diagrams

★ Feynman diagrams are the language of modern particle physics. They will be used extensively throughout this course.

The Basic Building Blocks



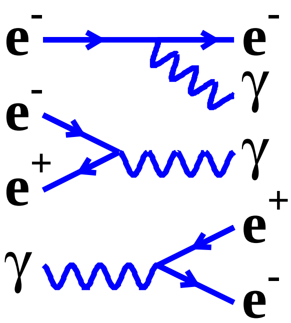
electron

positron

photon

Note : the positron (e^+) line is drawn as a negative energy electron traveling backwards in time

The $e^{\pm}$ − photon interactions



Radiation

Annihilation

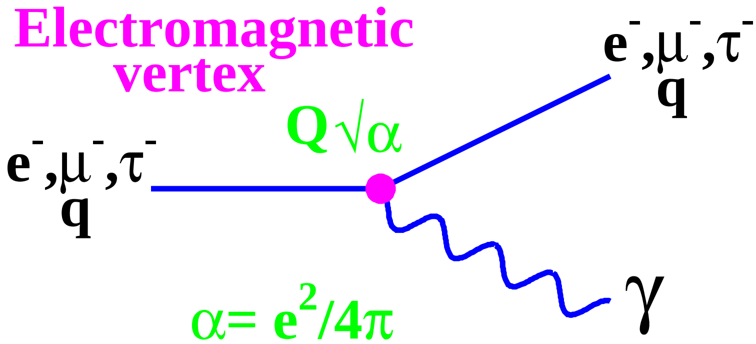
Pair Prod.

Note: none of these processes are allowed in isolation : Forbidden by $(E, \tilde{\mathrm{p}})$ conservation.

★ The strength of the interaction between the virtual photon and fermions is called the coupling strength. For the electromagnetic interaction this is proportional to electric charge e.

The Electromagnetic Vertex

★ The electromagnetic interaction is described by the photon propagator and the vertex:



COUPLING strength proportional to the fermion charge.

★ All electromagnetic interactions can be described in terms of the above diagram

★ Always conserve energy and momentum + (angular momentum, charge)

★ QED Vertex NEVER changes flavour i.e. $e^- \rightarrow e^-\gamma$ but not $e^- \rightarrow \mu^-\gamma$

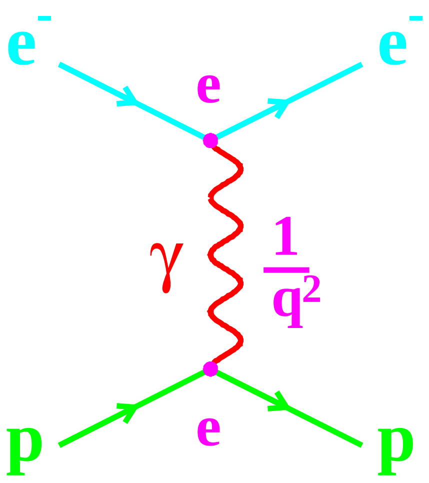
★ QED Vertex also conserves PARITY

★ Qualitatively : $Q\sqrt{\alpha}$ can be thought of the probability of a charged particle emitting a photon, the probability is proportional to $1/q^2$ of the photon.

Physics with Feynman Diagrams

Scattering cross sections calculated from:

- ★ Fermion wave functions
- ★ Vertex Factors : coupling strength
- ★ Propagator
- ★ Phase Space



Electron Current

Propagator

Proton Current

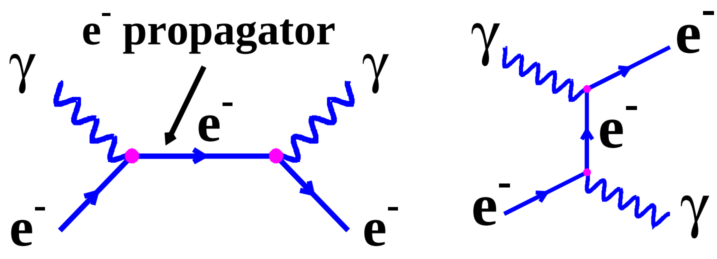
Matrix element M factorises into 3 terms :

$$-iM = \langle \overline{u}_e | ie\gamma^\mu | u_e \rangle \quad \text{Electron Current}$$
$$\times \frac{-ig^{\mu\nu}}{q^2} \quad \text{Photon Propagator}$$
$$\times \langle \overline{u}_p | ie\gamma^\nu | u_p \rangle \quad \text{Proton Current}$$

The factors γ^μ and $g^{\mu\nu}$ are 4×4 matrices which account for the spin-structure of the interaction (described in the lecture on the Dirac Equation).

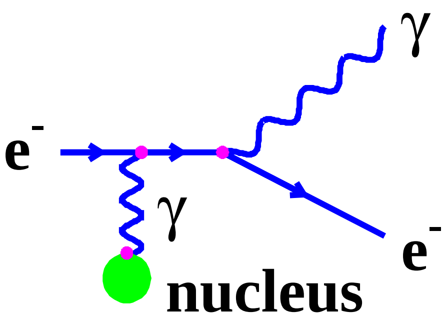
Pure QED Processes

Compton Scattering



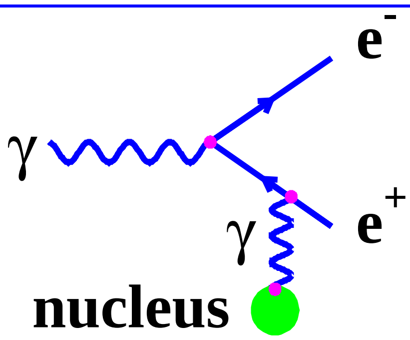
$$M \sim e^2$$
$$\sigma \sim |M|^2 \sim e^4$$
$$\sigma \sim (4\pi)^2\alpha^2$$

Bremsstrahlung



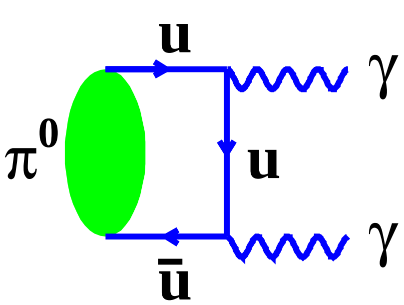
$$M \sim Ze.e.e$$
$$|M|^2 \sim Z^2e^6$$
$$\sigma \sim (4\pi)^3 Z^2\alpha^3$$

e^+e^- Pair Production



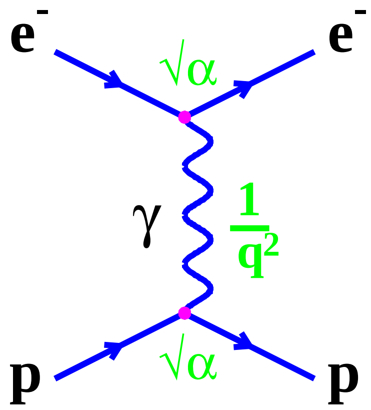
$$M \sim e.e.Ze$$
$$|M|^2 \sim Z^2e^6$$
$$\sigma \sim (4\pi)^3 Z^2\alpha^3$$

π^0 Decay



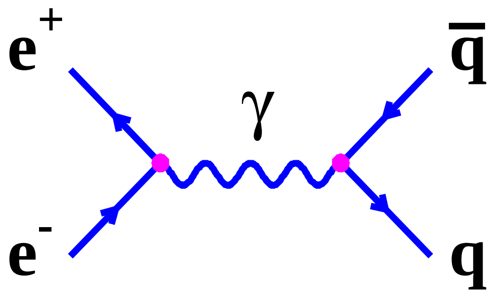
$$M \sim Q_u e.Q_u e$$
$$|M|^2 \sim Q_u^4 e^4$$
$$\sigma \sim (4\pi)^2 Q_u^4\alpha^2$$

Electron-Proton Scattering



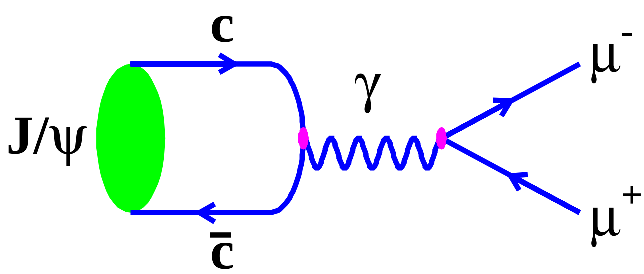
$$M \sim e.e$$
$$|M|^2 \sim e^4$$
$$\sigma \sim (4\pi)^2\alpha^2$$

e^+e^- Annihilation



$$M \sim e.Q_u e$$
$$|M|^2 \sim Q_u^2 e^4$$
$$\sigma \sim (4\pi)^2 Q_u^2 \alpha^2$$

$J/\psi \rightarrow \mu^+\mu^-$



$$M \sim Q_c e.e$$
$$|M|^2 \sim Q_c^2 e^4$$
$$\sigma \sim (4\pi)^2 Q_c^2 \alpha^2$$

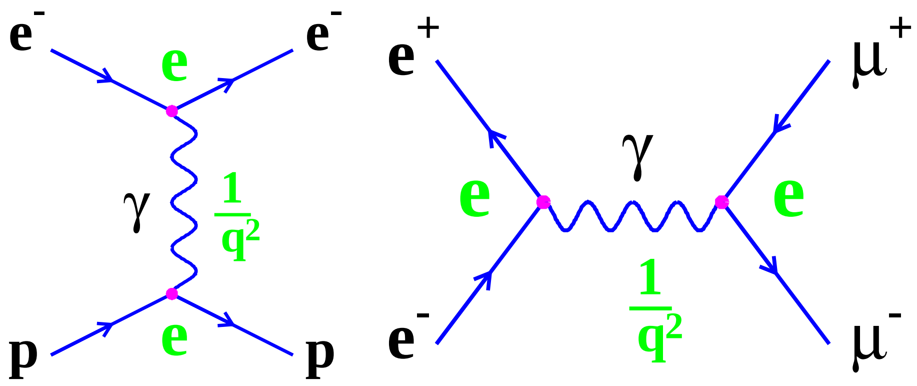
Coupling strength determines 'order of magnitude' of matrix element. For particles interacting/decaying via electromagnetic interaction: typical values for cross sections/lifetimes

$$\sigma_{em} \sim 10^{-2}\text{ mb}$$
$$\tau_{em} \sim 10^{-20}\text{ s}$$

Scattering in QED

EXAMPLE Calculate the "spin-less" cross sections for the two processes:

- ★ electron-proton scattering
- ★ electron-positron annihilation



Here we will consider the case where all particles are spin-0, (see lecture on Dirac Equation for complete treatment)

Fermi's Golden rule and Born Approximation:

$$\frac{d\sigma}{d\Omega} = 2\pi|M|^2 d\rho(E_f)/d\Omega$$

For both processes write the SAME matrix element

$$M = \frac{e^2}{q^2} = \frac{4\pi\alpha}{q^2}$$

However, the four-momentum transfer ($q^2 = E^2 - \tilde{q}^2$) is very different ($\tilde{q}$ is the 3-momentum of the virtual photon)

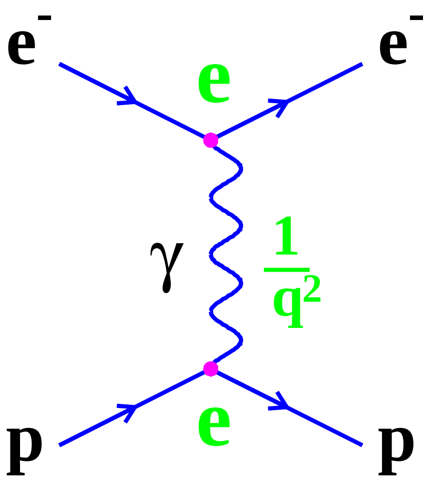
- ★ Elastic e^--proton scattering : $q = (0, \tilde{q})$

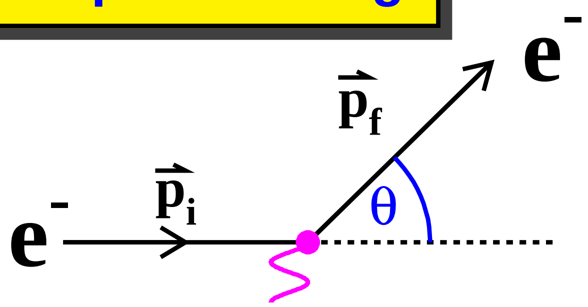
  $$q^2 = -|\tilde{q}|^2$$

- ★ e^+e^- annihilation : $q = (2E, 0)$

  $$q^2 = +4E^2$$

"Spin-less" e-p Scattering





$$M = \frac{e^2}{q^2} = \frac{4\pi\alpha}{q^2}$$

From Handout 1, pages 31-34:

$$\frac{d\sigma}{d\Omega} = 2\pi|M|^2\frac{E^2}{(2\pi)^3}$$
$$= 2\pi\frac{(4\pi\alpha)^2}{q^4}\frac{E^2}{(2\pi)^3} = \frac{4\alpha^2E^2}{q^4}$$

q^2 is the four-momentum transfer:

$$q^2 = q^\mu q_\mu = (E_f - E_i)^2 - (\tilde{\mathrm{p}}_\mathrm{f} - \tilde{\mathrm{p}}_\mathrm{i})^2$$
$$= E_f^2 + E_i^2 - 2E_fE_i - \tilde{\mathrm{p}}_\mathrm{f}^2 - \tilde{\mathrm{p}}_\mathrm{i}^2 + 2.\tilde{\mathrm{p}}_\mathrm{f}.\tilde{\mathrm{p}}_\mathrm{i}$$
$$= 2m_e^2 - 2E_fE_i + 2|\tilde{\mathrm{p}}_\mathrm{f}||\tilde{\mathrm{p}}_\mathrm{i}|\cos\theta$$

neglecting electron mass: *i.e.* $m_e^2 = 0$ and $|\tilde{\mathrm{p}}_\mathrm{f}| = E_f$

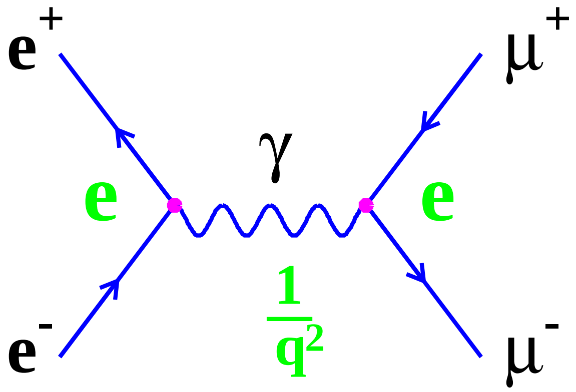
$$q^2 = -2E_iE_f(1-\cos\theta)$$
$$q^2 = -4E_iE_f\sin^2\frac{\theta}{2}$$

Therefore for ELASTIC scattering $E_i = E_f$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}$$

i.e. the Rutherford scattering formula (Handout 1 p.36)

"Spin-less" e^+e^- Annihilation



$$M = \frac{4\pi\alpha}{q^2}$$

$$\frac{d\sigma}{d\Omega} = 2\pi\frac{(4\pi\alpha)^2}{q^4}\frac{E^2}{(2\pi)^3} = \frac{4\alpha^2E^2}{q^4}$$

same formula, but different four-momentum transfer:

$$q^2 = q^\mu q_\mu = (E_{e+} + E_{e-})^2 - (\tilde{p}_{e+} + \tilde{p}_{e-})^2$$

Assuming we are in the centre-of-mass system

$$E_{e+} = E_{e-} = E$$

$$\tilde{p}_{e-} = -\tilde{p}_{e+}$$

$$\rightarrow q^2 = (2E)^2 = s$$

$$\frac{d\sigma}{d\Omega} = \frac{4\alpha^2E^2}{q^4} = \frac{4\alpha^2E^2}{16E^4} = \frac{\alpha^2}{s}$$

Integrating gives total cross section:

$$\sigma = 4\pi\frac{\alpha^2}{s}$$

This is not quite correct - because we have neglected spin. The actual cross section (see lecture on Dirac Equation) is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$$

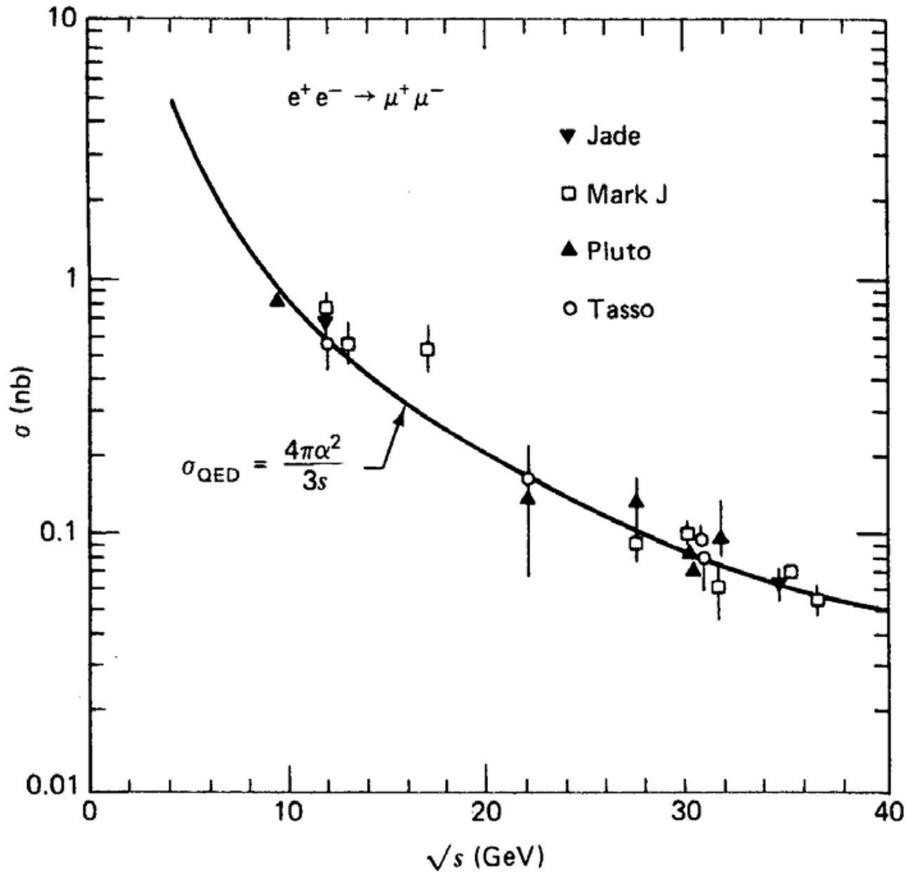
Natural Units Example cross section at $\sqrt{s} = 22\,\mathrm{GeV}$ i.e. 11 GeV electrons colliding with 11 GeV positrons.

$$\sigma_{e^+e^-\to\mu^+\mu^-} = \frac{4\pi\alpha^2}{3s} = \frac{4\pi}{137^2}\frac{1}{3\times 22^2}$$

$$= 4.6\times 10^{-7}\,\mathrm{GeV}^{-2}$$

$$= 4.6\times 10^{-7}(\hbar c)^2/(1.6\times 10^{-10})^2\,\mathrm{m}^2$$

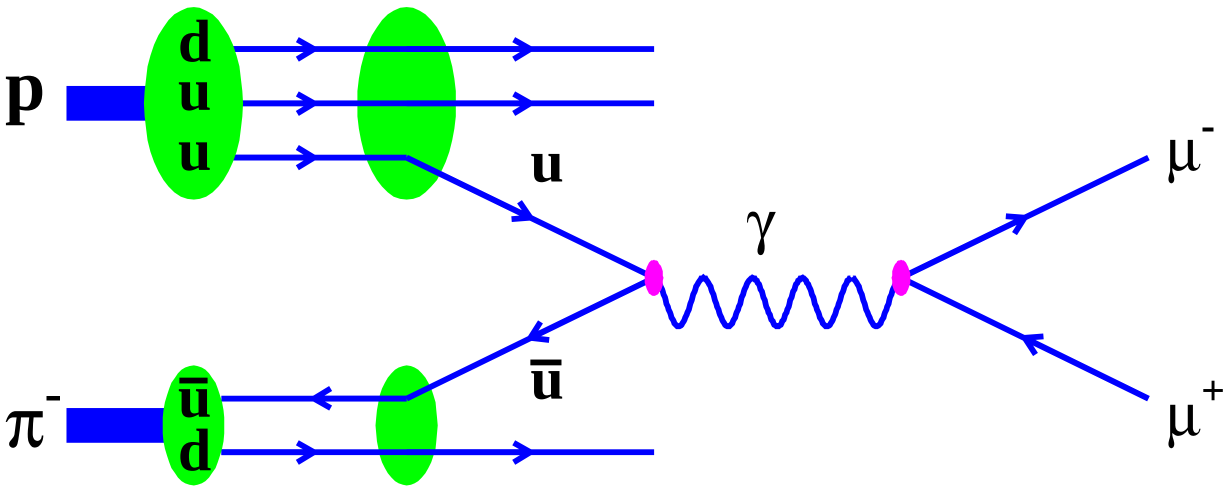
$$= 1.8\times 10^{-38}\,\mathrm{m}^2 = 0.18\,\mathrm{nb}$$



The Drell-Yan Process

★ Can also annihilate $q\overline{q}$ as in the Drell-Yan process

e.g. $\pi^- p \rightarrow \mu^+\mu^- + hadrons$



$$\sigma(\pi^- p \rightarrow \mu^+\mu^- + \text{hadrons}) \propto Q_u^2\alpha^2$$

(see Question 3 on the problem sheet)

Experimental Tests of QED

★ QED is an incredibly successful theory

Example

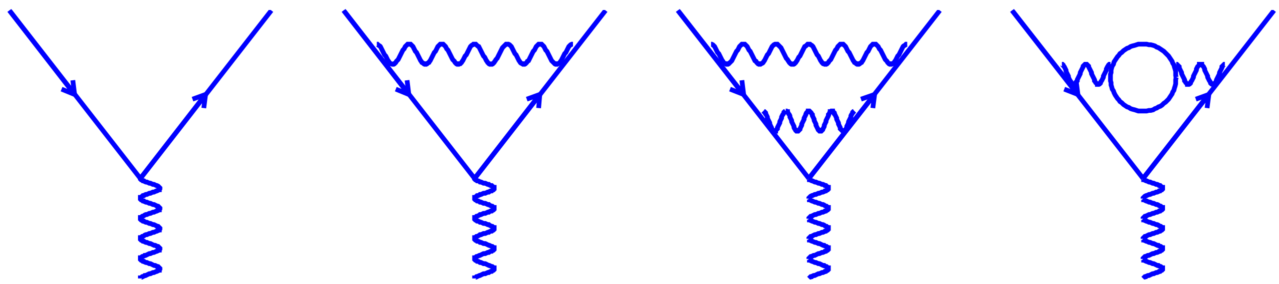
★ Magnetic moments of $e^{\pm}, \mu^{\pm}$

$$\tilde{\mu} = g\frac{e}{2m}\tilde{s}$$

★ For a point-like spin 1/2 particle :

g = 2

However higher order terms induce an anomalous magnetic moment i.e. g not quite 2.



$$\frac{(g_e\text{-}2)}{2} = 11596521.869 \pm 0.041 \times 10^{-10} \text{ EXPT}$$
$$\frac{(g_e\text{-}2)}{2} = 11596521.3 \pm 0.3 \times 10^{-10} \text{ THEORY}$$

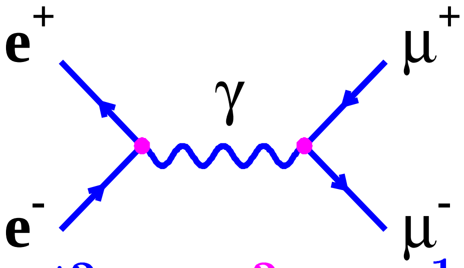
★ Agreement at the level of 1 in 10^8

★ Q.E.D. provides a remarkably precise description of the electromagnetic interaction !

Higher Orders

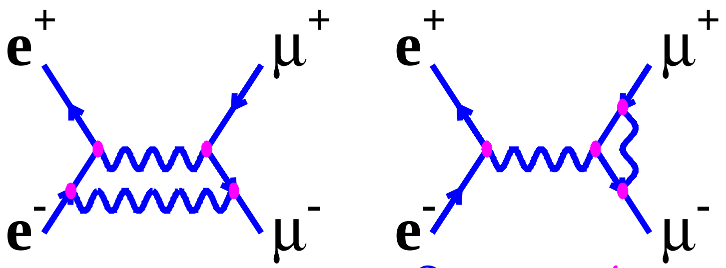
So far only considered lowest order term in the perturbation series. Higher order terms also contribute

Lowest Order:



$$|M|^2 \propto \alpha^2 \sim \frac{1}{137^2}$$

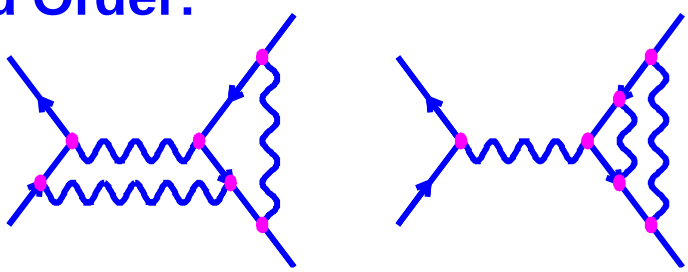
Second Order:



+....

$$|M|^2 \propto \alpha^4 \sim \frac{1}{137^4}$$

Third Order:



+....

$$|M|^2 \propto \alpha^6 \sim \frac{1}{137^6}$$

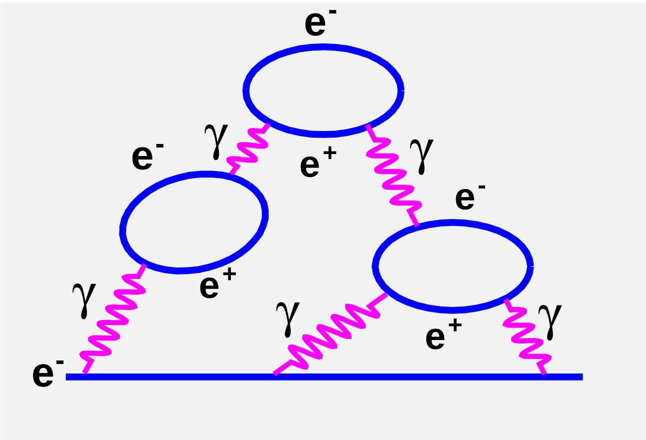
Second order suppressed by α^2 relative to first order. Provided α is small, *i.e.* perturbation is small, lowest order dominates.

Running of α

★ $\alpha = \frac{e^2}{4\pi}$ specifies the strength of the interaction between an electron and photon.

★ BUT α isn't a constant

Consider a free electron: Quantum fluctuations lead to a 'cloud' of virtual electron/positron pairs

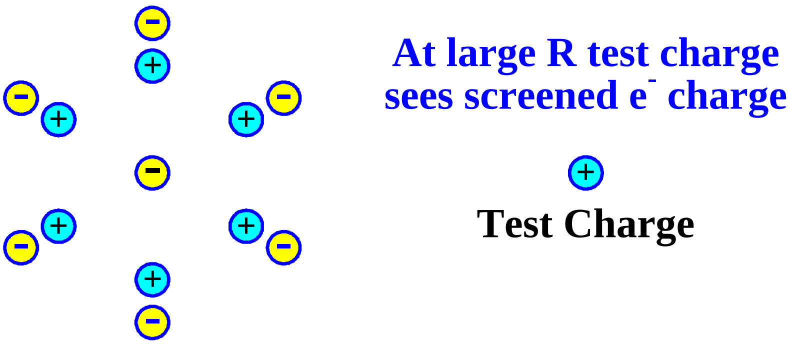


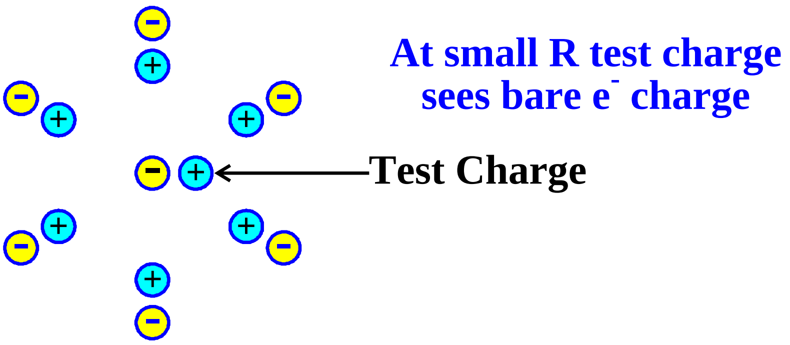
this is just one of many (an infinite set) such diagrams.

★ The vacuum acts like a dielectric medium

★ The virtual e^+e^- pairs are polarized

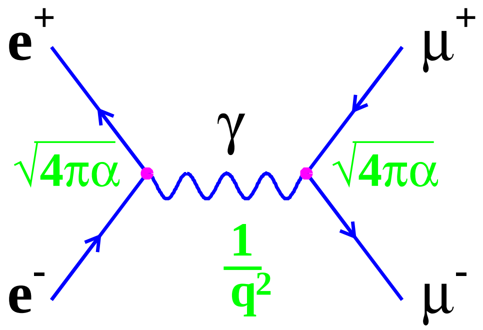
★ At large distances the bare electron charge is screened.

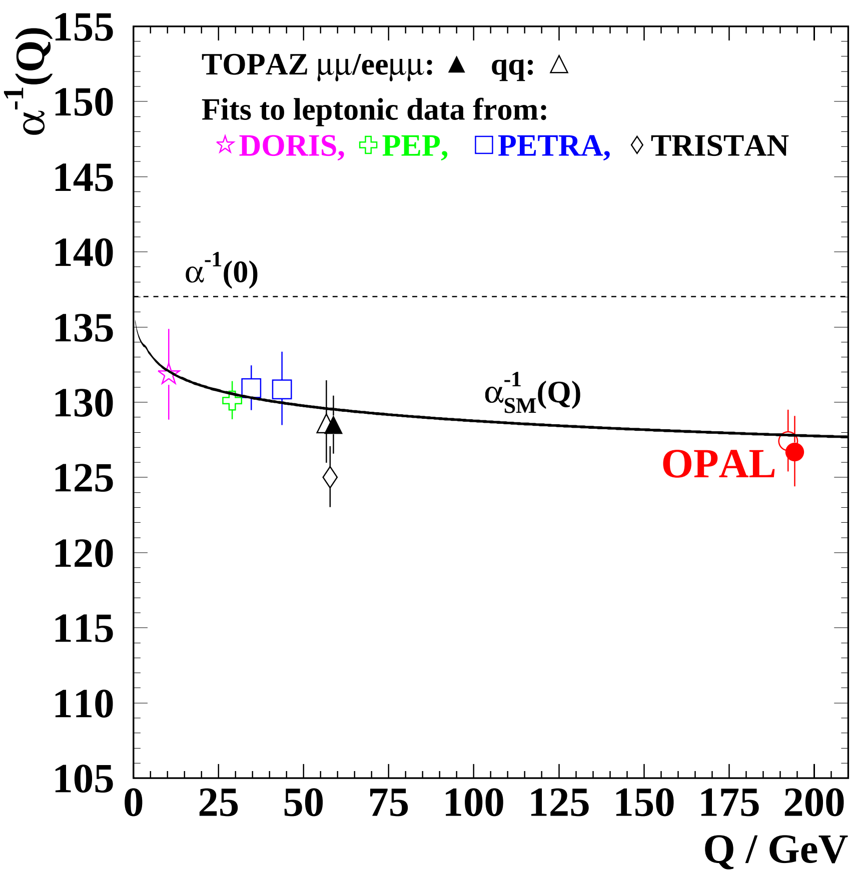




Running of α

Measure $\alpha(q^2)$ from $e^+e^- \rightarrow \mu^+\mu^-$ etc.



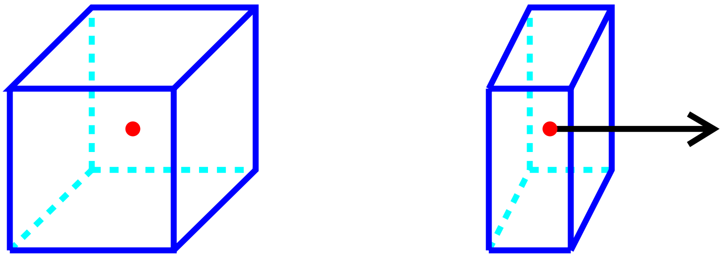


- ★ **α increases with the increasing q^2 (i.e. closer to the bare charge).**
- ★ **At $q^2 = 0$: $\alpha = 1/137$**
- ★ **At $q^2 = (100\ \mathrm{GeV})^2$: $\alpha = 1/128$**

Appendix: Relativistic Phase Space

NON-EXAMINABLE

★ Previously normalized wave-functions to 1 particle in a box of side L (see Handout 1, pages 33-34).



Rest Frame — 1 particle/V

Lab. Frame — 1 particle/(V/γ)

★ In relativity, box will be Lorentz Contracted by a factor of

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{E}{m}$$

$$i.e. \quad V' = V(\frac{m}{E})$$

i.e. E/m particles per volume V

NEED to adjust normalization volume with energy

<u>Conventional choice:</u>

$$N = \frac{1}{\sqrt{2E}}$$

★ In most scattering process the factors of $\sqrt{2E}$ in the wave-function normalization cancel with corresponding factors in the expressions for flux and density of states, just as the factors of L^3 were canceled previously (Handout 1, pages 35)