Appendix I: Logitudinal invariance of 'Lorentz Invariant Flux'

The argument in this appendix aims to show that the so-called 'Lorentz Invariant Flux', F, defined only for collinear collisions $a \xrightarrow[v_a,\vec{p}_a]{} \xleftarrow[v_b,\vec{p}_b]{} b$ by

$$F = 2E_a 2E_b \left(v_a + v_b\right)$$

may be written in a Lorentz Invariant way, constant across all frames for which the collision is colliner.

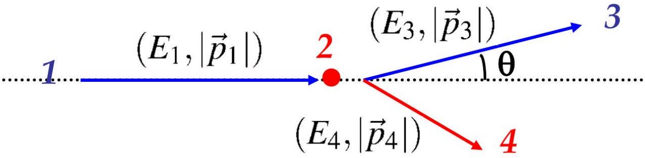
For all such frames: $p_a \cdot p_b = p_a^\mu p_{b\mu} = E_a E_b - \vec{p}_a \cdot \vec{p}_b = E_a E_b + |\vec{p}_a|\,|\vec{p}_b|$. (It is the last step therein which assumes collinearity!) Thus, for all such frames:

$$\begin{aligned}
F^2/16 - (p_a^\mu p_{b\mu})^2 &= \frac{1}{16}\left(2E_a 2E_b\left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b}\right)\right)^2 - (p_a \cdot p_b)^2 \\
&= \left(|\vec{p}_a|\,E_b + |\vec{p}_b|\,E_a\right)^2 - \left(E_a E_b + |\vec{p}_a|\,|\vec{p}_b|\right)^2 \\
&= |\vec{p}_a|^2\left(E_b^2 - |\vec{p}_b|^2\right) + E_a^2\left(|\vec{p}_b|^2 - E_b^2\right) \\
&= |\vec{p}_a|^2\, m_b^2 - E_a^2 m_b^2 \\
&= -m_a^2 m_b^2
\end{aligned}$$

and so

$$F = 4\left[(p_a^\mu p_{b\mu})^2 - m_a^2 m_b^2\right]^{1/2} \qquad \square.$$

Appendix II: General 2 → 2 Body Scattering in lab frame I



$p_1 = (E_1, 0, 0, |\vec{p}_1|)$, $p_2 = (M_2, 0, 0, 0)$, $p_3 = (E_3, E_3 \sin\theta, 0, E_3 \cos\theta)$, $p_4 = (E_4, \vec{p}_4)$ again

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

But now the invariant quantity t :

$$\begin{aligned} t &= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2 \cdot p_4 = m_2^2 + m_4^2 - 2m_2 E_4 \\ &= m_2^2 + m_4^2 - 2m_2 (E_1 + m_2 - E_3) \\ &\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2m_2 \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \end{aligned}$$

Appendix II: General 2 → 2 Body Scattering in lab frame II

Which gives $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_2}{\pi}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$

To determine $\mathrm{d}E_3/\mathrm{d}\left(\cos\theta\right.$, first differentiate $E_3^2 - |\vec{p}_3|^2 = m_3^2$

$$2E_3\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = 2\,|\vec{p}_3|\,\frac{\mathrm{d}\,|\vec{p}_3|}{\mathrm{d}(\cos\theta)} \tag{172}$$

Then equate

$$t = (p_1 - p_3)^2 = (p_4 - p_2)^2 \qquad \text{to give}$$

$$m_1^2 + m_3^2 - 2\left(E_1E_3 - |\vec{p}_1|\,|\vec{p}_3|\cos\theta\right) = m_4^2 + m_2^2 - 2m_2\left(E_1 + m_2 - E_3\right)$$

Differentiate wrt. $\cos\theta$

$$\left(E_1 + m_2\right)\frac{\mathrm{d}E_3}{\mathrm{d}\cos\theta} - |\vec{p}_1|\cos\theta\frac{\mathrm{d}\,|\vec{p}_3|}{\mathrm{d}\cos\theta} = |\vec{p}_1|\,|\vec{p}_3|$$

Using (172)

$$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{|\vec{p}_1|\,|\vec{p}_3|^2}{|\vec{p}_3|\left(E_1 + m_2\right) - E_3\,|\vec{p}_1|\cos\theta} \tag{173}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_2}{\pi}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{m_2}{\pi}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\frac{1}{64\pi s\,|\vec{p}_i^{\,*}|^2}\,|M_{fi}|^2$$

Appendix II: General 2 → 2 Body Scattering in lab frame III

It is easy to show $|\vec{p}_i^{\,*}|\sqrt{s} = m_2\,|\vec{p}_1|$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\,\frac{m_2}{64\pi^2 m_2^2\,|\vec{p}_1|^2}\,|M_{fi}|^2$$

and using (173) obtain

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{m_2\,|\vec{p}_1|}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|\,(E_1+m_2) - E_3\,|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}.$$

Appendix III: Dimensions of the Dirac Matrices I

In a d-dimensional spacetime there will always be d gamma matrices, as one is associated with each spacetime derivative in the Hamiltonian. That is why in 4-dimensional spacetime we have four gamma matrices: γ_0, γ_1, γ_2 and γ_3.

But why does $d = 4$ force those matrices to be (4×4)-matrices ?

Rather than answer the above question, we instead state (and later prove) the more general result (174) linking the $(n \times n)$ size of gamma matrices to the number d of spacetime dimension with which they are associated:

$$n = 2^{\lfloor \frac{d}{2} \rfloor}. \tag{174}$$

The result (174) is a direct consequence of the gamma matrices having to satisfy (as we already saw in (30)) the defining property of a (so called) 'Clifford Algebra', namely that:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} 1_{n\times n}. \tag{175}$$

Warning: the proof we provide for the above statement relies on Schur's Lemma. This may be a source of dissatisfaction for some persons taking the course because **Schur's Lemma**, although stated in the Groups and Representations section of the Part IB Mathematics course within Natural Sciences Tripos, was stated in that coure without proof. If you find that annoying, you will have to find an alternative proof.

Appendix III: Dimensions of the Dirac Matrices II

Aside on size of Pauli matrices:

Although we are mainly interested in proving (174) to substantiate the claim that each γ^μ is a (4×4)-matrix, we note that the same result can be used to explain why the Pauli matrices are (2×2)-matrices. The reason is that the three ($d = 3$) Pauli matrices satisfy their own equivalent of (175), namely: $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$. Hence $n = 2^{\lfloor 3/2 \rfloor} = 2^1 = 2$.

We wish to prove the result stated in (174) is the relationship between the dimension d of spacetime and the dimension n of the (irreducible) $(n \times n)$ irreducible matrices γ_μ satisfying (175) with $\mu, \nu = 0, 1, \cdots, d-1$. Conveniently, the relationship (174) between n and d which we seek to prove does not depend on the signature of the metric since it is possible to convert a representation designed for one signature (say $g_{\mu\nu} = \text{diag}(+,-,-,-)$) to another (say $g_{\mu\nu} = \text{diag}(+,+,+,+)$) without changing n by multiplying appropriate γ-matrices by $i = \sqrt{-1}$.

Therefore, without loss of generality, we actually take as our start point the simplest possibility, namely:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu} \cdot 1_{n\times n}. \tag{176}$$

We nonetheless demand that the γ-matrices are irreducible – i.e. that there is not a similarity transformation that would reduce them all to a (non-trivial) block diagonal form. We start by noting that with those assumptions:

Appendix III: Dimensions of the Dirac Matrices III

- **Every γ^μ is invertible**. [To prove this simply set $\mu = \nu$ in (176) and take the determinant of both sides.]
- **For the matrix $\gamma^* \equiv \gamma^0\gamma^1\cdots\gamma^{d-1}$ we have**

$$\gamma^*\gamma^\mu = (-1)^{d-1}\gamma^\mu\gamma^*. \tag{177}$$

[Proof: When γ^μ commutes with γ^* it must pass $d-1$ dissimilar γ-matrices and a single 'identical' γ-matrix. Given (176) there are therefore $d-1$ anti-commutations and a single commutation. $\square$]

- **The matrix $\gamma^* \equiv \gamma^0\gamma^1\cdots\gamma^{d-1}$ squares to either +1 or -1 depending on d.** [Proof: it takes $\frac{1}{2}(d-1)d$ flips of adjacent pairs to reverse the order of d objects, and since all the γ-matrices in γ^* are dissimilar and thus anti-commute we can deduce that

$$\gamma^* \equiv \gamma^0\gamma^1\cdots\gamma^{d-1} = (-1)^{\frac{1}{2}(d-1)d}\cdot\gamma^{d-1}\cdots\gamma^1\gamma^0$$

Appendix III: Dimensions of the Dirac Matrices IV

and so

$$\begin{aligned}(\gamma^*)^2 &= (-1)^{\frac{1}{2}(d-1)d} \cdot (\gamma^{d-1} \cdots \gamma^1 \gamma^0) \cdot (\gamma^0 \gamma^1 \cdots \gamma^{d-1}) \\ &= (-1)^{\frac{1}{2}(d-1)d} \prod_{\mu=0}^{d-1} \delta^{\mu\mu} \\ &= (-1)^{\frac{1}{2}(d-1)d} \\ &= s(d) \end{aligned} \tag{178}$$

in which $s(d) \equiv (-1)^{\frac{1}{2}(d-1)d}$ is a d-dependent sign in $\{+1, -1\}$.]

- **If $d > 1$ then n must be even.** [To prove this, consider $\mu \neq \nu$ (which requires $d > 1$) in (176). In this case (176) becomes $\gamma^\mu \gamma^\nu = -\gamma^\mu \gamma^\mu$ which implies that $\det\{\gamma^\mu\} \det\{\gamma^\nu\} = (-1)^n \det\{\gamma^\nu\} \det\{\gamma^\mu\}$ which (since every γ^μ is invertible) implies that $1 = (-1)^n$ and thus that n is even.]

Appendix III: Dimensions of the Dirac Matrices V

- **Theorem A: Any product of any number of γ-matrices may (up to a sign) be written as a product of at most d gamma matrices in strictly ascending order of their indices.** [This is because (176) states that dissimilar γ-matrices anti-commute, and that individual γ-matrices square to ± 1'. Therefore, an arbitrary product of γ-matrices can always have its γ-matrices permuted into numerical order (with a sign change if an odd number of permutations is required) leaving at most one copy of each γ-matrix as repeats will disappear (up to a sign) on account of the squaring property.]

The last result above motivates the following definition.

Definition

If A is any integer whose binary representation modulo 2^d is $\vec{A}$, i.e. if (A mod 2^d) $= \sum_{i=0}^{d-1} A_i \cdot 2^i$ with each $A_i \in \{0, 1\}$, then define Γ_A by

$$\Gamma_A = \prod_{i=0}^{d-1} \left\{ \begin{matrix} \gamma_i & \text{if } A_i = 1 \\ 1 & \text{otherwise} \end{matrix} \right\} . \tag{179}$$

For example, this definition would make $\Gamma_{13} = \gamma_0\gamma_2\gamma_3$ since $13 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3$.

Appendix III: Dimensions of the Dirac Matrices VI

On account of the modulo 2^d part of the definition, any continuous range of indices of length 2^d would suffice to include every such Γ-matrix. Without loss of generality will always take indices A to be in the set

$$\mathcal{A} = \{1, 2, \cdots, 2^d\},$$

and mapped into that range, if necessary, by an implicit modulo 2^d operation. We therefore define a complete list, L, of Γ-matrices as follows:

$$L = (\Gamma_1, \Gamma_2, \dots, \Gamma_{2^d}) = (\Gamma_A \mid A \in \mathcal{A}). \tag{180}$$

Note that although we have defined 2^d quantities Γ_A in the list L we have not shown that they are all unique. In other words, we cannot assume '$(A \neq B) \implies (\Gamma_A \neq \Gamma_B)$' or '$(\Gamma_A = \Gamma_B) \implies (A = B)$' unless later proved.

We now state and prove two important properties of the Γ-matrices:

Lemma 1:

Appendix III: Dimensions of the Dirac Matrices VII

The most general form of this Lemma is

$$\mathrm{Tr}\left[\Gamma_A\right] = \begin{cases} n & \text{if } A = 0 \mod 2^n \\ 0 & \text{if } (A \neq 0 \mod 2^n) \text{ and } (d \text{ is even or } \sum_{i=1}^{d} A_i \text{ is even}) \\ Tr[\Gamma_A] & \text{otherwise.} \end{cases} \qquad (181)$$

Alternatively, a narrower form could be stated as follows

$$\text{When } d \text{ is even:} \qquad \mathrm{Tr}\left[\Gamma_A\right] = \begin{cases} n & \text{if } A = 0 \mod 2^n \\ 0 & \text{otherwise.} \end{cases} \qquad (182)$$

Proof of Lemma 1:

The trace of Γ_0 is always trivially n as $\Gamma_0 = 1_{n\times n}$. Every other Γ_A is the product of one or more dissimilar γ-matrices. We split the remainder of the proof into two parts: part (i) shows that traces of products are zero where the remaining products contain an **even** number of γ-matrices, while part (ii) shows the same for products containing any **odd** number of γ-matrices. Note the subtle differences between these two parts of of the proof: the first needs to assume that the multiplied gammas are **distinct** but does not need to worry about whether d is even or odd. In contrast the second does not care about distinctness in the gammas but **needs to assume that d is even**.

Appendix III: Dimensions of the Dirac Matrices VIII

Part (i): even products

If k is an integer greater than zero, and if $a_1, a_2, \ldots, a_k$ are k **distinct** integers in $[0, d-1]$ and if $T = \mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right]$ then

$$\begin{aligned}
T &= \mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] \\
&= (-1)^{k-1}\cdot\mathrm{Tr}\left[\gamma_{a_k}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\right] \\
&\qquad\qquad\text{(after } k-1 \text{ anti-commutations using (176) and } k>0\text{)} \\
&= (-1)^{k-1}\cdot\mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] \qquad \text{(trace cyclicity)} \\
&= (-1)^{k-1}\cdot T
\end{aligned}$$

therefore:

"The trace of the product of an **even** number of **distinct** γ-matrices ...

... is zero provided the even number is greater than or equal to two". (183)

Part (ii): odd products

Appendix III: Dimensions of the Dirac Matrices IX

If k is an integer greater than zero, and if $a_1, a_2, \ldots, a_k$ are k integers in $[0, d-1]$ and if $T = \operatorname{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right]$ then

$$\begin{aligned}
& & T &= \operatorname{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] & & \\
\Longrightarrow & & s(d)\cdot T &= \operatorname{Tr}\left[(\gamma^*\gamma^*)\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] & & \text{(by (178))} \\
\Longrightarrow & & s(d)\cdot T &= \operatorname{Tr}\left[\gamma^*\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\gamma^*\right] & & \text{(trace cyclicity)} \\
\Longrightarrow & & s(d)\cdot T &= ((-1)^{d-1})^k\cdot\operatorname{Tr}\left[\gamma^*\gamma^*\gamma_{a_k}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\right] & & \text{(after } k \text{ uses of (177))} \\
\Longrightarrow & & T &= (-1)^{k(d-1)}\cdot\operatorname{Tr}\left[\gamma_{a_k}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\right] & & \text{(by (178) again)} \\
\Longrightarrow & & T &= (-1)^{k(d-1)}\cdot T & &
\end{aligned}$$

therefore:

$$\text{“when } d \text{ is even, the trace of the product of an odd number of } \gamma\text{-matrices is zero”.} \tag{184}$$

This conclues our proof of Lemma 1. ☐

Lemma 2:

$$\Gamma_A\Gamma_B = s(A,B)\cdot\Gamma_{A\oplus B} \tag{185}$$

Appendix III: Dimensions of the Dirac Matrices X

in which '$\oplus$' represents 'BITWISE EXCLUSIVE OR' and $s(A, B)$ is a function mapping pairs of indices to the set $\{+1, -1\}$.

Proof of Lemma 2:

$$\begin{aligned}\Gamma_A\Gamma_B &= \prod_{i=0}^{d-1}\begin{Bmatrix}\gamma_i & \text{if } A_i = 1\\ 1 & \text{otherwise}\end{Bmatrix}\prod_{i=0}^{d-1}\begin{Bmatrix}\gamma_i & \text{if } B_i = 1\\ 1 & \text{otherwise}\end{Bmatrix}\\ &= s_1(A, B)\prod_{i=0}^{d-1}\left(\begin{Bmatrix}\gamma_i & \text{if } A_i = 1\\ 1 & \text{otherwise}\end{Bmatrix}\begin{Bmatrix}\gamma_i & \text{if } B_i = 1\\ 1 & \text{otherwise}\end{Bmatrix}\right)\end{aligned}$$

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where $s_1(A,B) \in \{+1,-1\}$ is a sign which will depend on how many anti-commutations deriving from (176) were needed to re-order the matrices, and so

$$\begin{aligned}
\Gamma_A\Gamma_B &= s_1(A,B)\prod_{i=0}^{d-1}\left\{\begin{matrix}(\gamma_i)^2 & \text{if } A_i = B_i = 1 \\ \gamma_i & \text{if } A_i \oplus B_i = 1 \\ 1 & \text{otherwise}\end{matrix}\right\} \\
&= s_1(A,B)\prod_{i=0}^{d-1}\left\{\begin{matrix}g_{ii} & (\text{no sum } i) & \text{if } A_i = B_i = 1 \\ \gamma_i & & \text{if } A_i \oplus B_i = 1 \\ 1 & & \text{otherwise}\end{matrix}\right\} \qquad (\text{by } (176)) \\
&= s(A,B)\prod_{i=0}^{d-1}\left\{\begin{matrix}1 & \text{if } A_i = B_i = 1 \\ \gamma_i & \text{if } A_i \oplus B_i = 1 \\ 1 & \text{otherwise}\end{matrix}\right\}
\end{aligned}$$

Appendix III: Dimensions of the Dirac Matrices XII

where $s(A, B)$ is a new sign function that accounts for our having replaced g_{ii} with 1, and so

$$\begin{aligned}\Gamma_A\Gamma_B &= s(A, B)\prod_{i=0}^{d-1}\left\{\begin{matrix}\gamma_i & \text{if } A_i \oplus B_i = 1\\ 1 & \text{otherwise}\end{matrix}\right\}\\ &= s(A, B)\Gamma_{A\oplus B} \qquad \square.\end{aligned}$$

A corollary of (185) is that every Γ-matrix is invertible. [Proof: setting B equal to A in (185) tells us that $(\Gamma_A)^2 = s(A, A) \cdot \Gamma_0 = s(A, A) \cdot 1_{n\times n} = \pm 1_{n\times n}$ and so

$$(\Gamma_A)^{-1} \text{ is either } \Gamma_A \text{ or } -\Gamma_A. \tag{186}$$

]

Perhaps we can do better. Suppose A has a ones in its binary representation (i.e. $a = \sum_{i=0}^{d-1} A_i$ so that Γ_A is a product of a gamma matrices in ascending order of index). If we then square Γ_A we could attempt to permute adjacent gamma matrices within the product so as to annihilate every identical pairing, leaving behind only a sign. This process would require $a - 1$ anticommutations to annihilate the first pair, $a - 2$ the

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second, *etc*, and none for the last. This is a total of $\frac{1}{2}(a-1)a$ anticommutations, and so we can make the very specific claim that

$$(\Gamma_A)^2 = (-1)^{\frac{1}{2}(a-1)a} \tag{187}$$

or equivalently

$$(\Gamma_A)^{-1} = (-1)^{\frac{1}{2}(a-1)a} \cdot \Gamma_A. \tag{188}$$

Indeed, we see that the already derived result (178) could be viewed with hindsight as a simple corollary of (187).
Knowing that the Γ-matrices are all invertible we may define a matrix S as follows:

$$S = \sum_{X \in \mathcal{A}} (\Gamma_X)^{-1} \cdot Y \cdot \Gamma_X \tag{189}$$

Appendix III: Dimensions of the Dirac Matrices XIV

where Y is an arbitrary $(n \times n)$-matrix whose value we will fix later. It is follows that for any integer A (not summed) in the usual range $\mathcal{A}$:

$$\begin{aligned}
(\Gamma_A)^{-1} \cdot S \cdot \Gamma_A &= \sum_{X \in \mathcal{A}} (\Gamma_X \Gamma_A)^{-1} \cdot Y \cdot (\Gamma_X \Gamma_A) \\
&= \sum_{X \in \mathcal{A}} (s_X \Gamma_{A \oplus X})^{-1} \cdot Y \cdot (s_X \Gamma_{A \oplus X}) \qquad \text{(using (185))} \\
&= \sum_{X \in \mathcal{A}} (\Gamma_{A \oplus X})^{-1} \cdot Y \cdot (\Gamma_{A \oplus X}) \\
&= \sum_{X \in A \oplus \mathcal{A}} (\Gamma_X)^{-1} \cdot Y \cdot (\Gamma_X) \\
&= \sum_{X \in \mathcal{A}} (\Gamma_X)^{-1} \cdot Y \cdot (\Gamma_X) \qquad (\text{since } A \oplus \mathcal{A} \equiv \{A \oplus B, B \in \mathcal{A}\} = \mathcal{A}) \\
&= S
\end{aligned}$$

and thus $S \cdot \Gamma_A = \Gamma_A \cdot S$.

Having found a matrix S which commutes with every element Γ_A of a list L of matrices, one might hope to use Schur's Lemma to claim that S is some multiple of $1_{n \times n}$. However, a precondition of the only version of Schur's Lemma which I understand and which also

Appendix III: Dimensions of the Dirac Matrices XV

allows that conclusion to be drawn requires the elements of L to form an irreducible representation of some group G. Not only have we not yet shown that this precondition is satisfied, it actually looks likely to be false! For example, for the usual γ-matrices in $d = 4$ dimensions we would have $\Gamma_1\Gamma_2 = \gamma_1\gamma_2 = -\ \gamma_2\gamma_1 = -\Gamma_2\Gamma_1$ and so for L to be closed under multiplication it would need to contain both $+\Gamma_2\Gamma_1$ and $-\Gamma_2\Gamma_1$. This seems unlikely as we did not set up L to contain negated copies of every element. It therefore seems unlikely that L is closed under multiplication and so it seems unlikely that L represents a group.
It could be argued that the source of the problem is the annoying sign $s(A, B)$ in (185). If that pesky sign were not there and the constant '+1' were always in its place, products of Γ-matrices would be closed. We cannot arbitrarily dispose of that pesky sign, but it does suggest a resolution: we could double the length of our list L by adding to it another copy of itself but with the sign of every matrix reversed in the second half. The elements of this list will then be closed under multiplication, which is would be a requirement for them to be any kind of representation. We shall call the set containing all those elements G:

$$G = \{+\Gamma_A \mid A \in \mathcal{A}\} \cup \{-\Gamma_A \mid A \in \mathcal{A}\}. \tag{190}$$

This set of matrices is: (i) closed under multiplication, (ii) contains the identity $\Gamma_{2^d} = 1_{n\times n}$, (iii) contains an inverse for every element (see proof in (186)). Finally (iv) matrix multiplication is associative. Therefore G together with the operation of matrix multiplication forms a group. As it is a finite matrix group it is also representation of

Appendix III: Dimensions of the Dirac Matrices XVI

itself. This representation must be irreducible since the representation contains elements which are copies of the original γ-matrices (e.g. $\Gamma_1 = \gamma_0$, $\Gamma_2 = \gamma_1$, ... $\Gamma_{2^d} = \gamma_d$), and those original γ-matrices were taken to be be irreducible at the outset by assumption (see paragraph containing (176)). Although we have increased the number of elements in G relative to L, we can be sure that our old S will commute with every element of the new G because

$$([S, +\Gamma_A] = 0) \iff ([S, -\Gamma_A] = 0).$$

We have thus established all the preconditions necessary to allow us to use Schur's Lemma to state that S is a multiple of the identity, or more specifically:

$$\lambda \cdot 1_{n\times n} = \sum_{X\in\mathcal{A}} (\Gamma_A)^{-1} \cdot Y \cdot \Gamma_A \tag{191}$$

Appendix III: Dimensions of the Dirac Matrices XVII

for some scalar λ that will depend on Y. Taking the trace of both sides of (191) and using the cyclicity of the trace gives us:

$$\begin{aligned} n\lambda &= \sum_{X\in\mathcal{A}} \mathrm{Tr}\left[(\Gamma_A)^{-1}\cdot Y\cdot\Gamma_A\right] \\ &= \sum_{X\in\mathcal{A}} \mathrm{Tr}\left[Y\cdot\Gamma_A\cdot(\Gamma_A)^{-1}\right] \\ &= \sum_{X\in\mathcal{A}} \mathrm{Tr}\, Y \\ &= 2^d\cdot\mathrm{Tr}\, Y \end{aligned}$$

and thus

$$\lambda = \frac{2^d}{n}\cdot\mathrm{Tr}\, Y. \tag{192}$$

Putting this value for λ back into (191) yields

$$\frac{2^d}{n}\cdot\mathrm{Tr}\, Y\cdot 1_{n\times n} = \sum_{X\in\mathcal{A}}(\Gamma_A)^{-1}\cdot Y\cdot\Gamma_A. \tag{193}$$

Appendix III: Dimensions of the Dirac Matrices XVIII

We now exercise our remaining freedom to choose Y to be any $(n \times n)$-matrix we wish, deciding to let

$$[Y]_{ij} = \delta_{is}\delta_{jt}$$

where s and t are integers in $[1, n]$ which we may choose to fix later. With that choice in mind, and with i and j being other arbitrary integers also in $[1, n]$, (193) can be expanded as:

$$\left[\frac{2^d}{n} \cdot \mathrm{Tr}\, Y \cdot 1_{n\times n}\right]_{ij} = \left[\sum_{X\in\mathcal{A}} (\Gamma_A)^{-1} \cdot Y \cdot \Gamma_A\right]_{ij}$$

or equivalently

$$\frac{2^d}{n} \cdot (\delta_{ms}\delta_{mt}) \cdot \delta_{ij} = \sum_{X\in\mathcal{A}} ((\Gamma_A)^{-1})_{im} \cdot (\delta_{ms}\delta_{nt}) \cdot (\Gamma_A)_{nj}$$

which simplifies to

$$\frac{2^d}{n} \cdot \delta_{st} \cdot \delta_{ij} = \sum_{X\in\mathcal{A}} ((\Gamma_A)^{-1})_{is} \cdot (\Gamma_A)_{tj}. \qquad (194)$$

Appendix III: Dimensions of the Dirac Matrices XIX

Since (194) is true for any i,j,s,t in $[1, n]$, let us set $s \to i$ and $t \to j$ and then sum over i and j. Making use of the summation convention over i and j we find that:

$$\frac{2^d}{n} \cdot \delta_{ij} \cdot \delta_{ij} = \sum_{A \in \mathcal{A}} ((\Gamma_A)^{-1})_{ii} \cdot (\Gamma_A)_{jj}$$

which simplifies to

$$\frac{2^d}{n} \cdot n = \sum_{A \in \mathcal{A}} \mathrm{Tr}\left[(\Gamma_A)^{-1}\right] \cdot \mathrm{Tr}[\Gamma_A]$$

or

$$2^d = \sum_{A \in \mathcal{A}} \mathrm{Tr}\left[(\Gamma_A)^{-1}\right] \cdot \mathrm{Tr}[\Gamma_A]. \tag{195}$$

d-**even**

Appendix III: Dimensions of the Dirac Matrices XX

For the case that d is even we may now use (182) to simplify (195) to

$$\begin{aligned} 2^d &= \mathrm{Tr}\left[(\Gamma_0)^{-1}\right] \cdot \mathrm{Tr}[\Gamma_0] \\ &= \mathrm{Tr}\left[(1_{n\times n})^{-1}\right] \cdot \mathrm{Tr}[1_{n\times n}] \\ &= \mathrm{Tr}[1_{n\times n}] \cdot \mathrm{Tr}[1_{n\times n}] \\ &= n \cdot n = n^2 \\ \Longrightarrow \qquad n &= 2^{d/2} \qquad \text{(but only for } d \text{ even!).} \qquad (196) \end{aligned}$$

d-**odd**

This is a bit of a trick. One may always generate an irreducible representation of the gamma matrices for an **odd** spacetime dimension $d+1$ from an irreducible representation valid for an **even** number of spacetime dimensions d. The way to do this is surprisingly simple: if

$$\left\{\gamma^0, \gamma^1, \ldots, \gamma^{d-1}\right\}$$

is an irrep of (176) for an **even** number of spacetime dimensions d, and if we define

$$\gamma^* \equiv \gamma^0\gamma^1 \cdots \gamma^{d-1}$$

Appendix III: Dimensions of the Dirac Matrices XXI

and if we recall the definition of $s(d)$ from (178), then

$$\left\{\gamma^0, \gamma^1, \ldots, \gamma^{d-1}\right\} \cup \{\sqrt{s(d)} \cdot \gamma^*\} \tag{197}$$

will be an irrep of (176) valid for dimension $d + 1$ spacetime dimensions. That (197) is the irrep it is claimed to be is a consequence of three things: (i) γ^* was proved in (177) to anticommute with all the other gamma matrices when d is **even** and this anti-commutation is the property enforced/required by (176) whenever $\mu \neq \nu$, (ii) that $\sqrt{s(d)}\gamma^*$ squares to 1 was proved in (178), and this is the property enforced/required by (176) whenever $\mu = \nu$, and (iii) the representation (197) is an irrep as the first d gammas formed an irrep by themselves (i.e. as there was no transformation which could 'reduce' them, there cannot be an irrep that could 'reduce' both then and γ^*). It may be observed that this argument cannot be used to grow irreps without limit, since once an irrep for even d is grown to an irrep for odd d, the 'next' γ^* would fail to anticommute as desired. Nonetheless, the clear message is that the dimension of the gamma matrices for odd spacetime dimension d is always the same as the even dimension $d - 1$, and so (196) now informs us that

$$n = 2^{(d-1)/2} \qquad \text{(but only when } d \text{ is odd!).} \tag{198}$$

Appendix III: Dimensions of the Dirac Matrices XXII

General spacetime dimension d (even or odd)

There result (196) for even d can be merged with the result (198) for odd d into a single expression valid for any d:

$$\begin{aligned} n &= \begin{cases} 2^{d/2} & \text{(when } d \text{ is even)} \\ 2^{(d-1)/2} & \text{(when } d \text{ is odd)} \end{cases} \\ \implies \qquad n &= 2^{\lfloor d/2 \rfloor} \qquad \text{(for any } d\text{).} \end{aligned} \tag{199}$$

This concludes the proof of (174) which is also a proof of the lesser claim that Dirac Spinors have four components in the usual 4-dimensional spacetime.

Appendix IV: Magnetic Moment I

- In the part II Relativity and Electrodynamics course it was shown that the motion of a charged particle in an electromagnetic field $A^\mu = (\phi, \vec{A})$ can be obtained by making the minimal substitution $\vec{p} \to \vec{p} - q\vec{A}; \qquad E \to E - q\phi$
- Applying this to (37) and (38)

$$\begin{aligned}(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_B &= (E - m - q\phi)u_A \\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (E + m - q\phi)u_B \qquad (200)\end{aligned}$$

Multiplying (200) by $(E + m - q\phi)$

$$\begin{aligned}(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_B &= (E - m - q\phi)u_A \\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (E + m - q\phi)u_B \qquad (201)\end{aligned}$$

where kinetic energy $T = E - m$

- In the non-relativistic limit $T \ll m$ (201) becomes

$$\begin{aligned}(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &\approx 2m(T - q\phi)u_A \\ \left[(\vec{\sigma}.\vec{p})^2 - q(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p}) - q(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{A}) + q^2(\vec{\sigma}.\vec{A})^2\right] u_A &\approx 2m(T - q\phi)u_A (202)\end{aligned}$$

Appendix IV: Magnetic Moment II

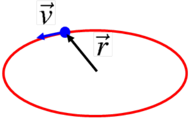
- Now $\vec{\sigma}.\vec{A} = \begin{pmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{pmatrix}$; $\vec{\sigma}.\vec{B} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$; which leads to $(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = \vec{A}.\vec{B} + i\vec{\sigma}.(\vec{A} \wedge \vec{B})$ and $(\vec{\sigma}.\vec{A})^2 = |\vec{A}|^2$
- The operator on the LHS of (202):

$$\begin{aligned}
&= \vec{p}^2 - q\left[\vec{A}.\vec{p} + i\vec{\sigma}.\vec{A} \wedge \vec{p} + \vec{p}.\vec{A} + i\vec{\sigma}.\vec{p} \wedge \vec{A}\right] + q^2\vec{A}^2 \\
&= (\vec{p} - q\vec{A})^2 - iq\vec{\sigma}.\left[\vec{A} \wedge \vec{p} + \vec{p} \wedge \vec{A}\right] \\
&= (\vec{p} - q\vec{A})^2 - q^2\vec{\sigma}.\left[\vec{A}.\vec{\nabla} + \vec{\nabla}.\vec{A}\right] \qquad (\text{since } \vec{p} = -i\vec{\nabla}) \\
&= (\vec{p} - q\vec{A})^2 - q\vec{\sigma}.(\vec{\nabla} \wedge \vec{A}) \qquad (\text{since } (\vec{\nabla} \wedge \vec{A})\psi = \vec{\nabla} \wedge (\vec{A}\psi) + \vec{A} \wedge (\vec{\nabla}\psi)) \\
&= (\vec{p} - q\vec{A})^2 - q\vec{\sigma}.\vec{B} \qquad (\text{since } \vec{B} = \vec{\nabla} \wedge \vec{A})
\end{aligned}$$

Substituting back into (202) gives the Schrödinger-Pauli equation for the motion of a non-relativisitic spin $\frac{1}{2}$ particle in an EM field:

$$\left[\frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{\sigma}.\vec{B} + q\phi\right] u_A = Tu_A.$$

Appendix IV: Magnetic Moment III



- Since the energy of a magnetic moment in a field is we can identify the intrinsic magnetic moment of a spin-half particle to be:

$$\vec{\mu} = \frac{q}{2m}\vec{\sigma}$$

In terms of the spin: $\vec{S} = \frac{1}{2}\vec{\sigma}$

$$\vec{\mu} = \frac{q}{m}\vec{S}$$

- Classically, for a charged particle current loop

$$\mu = \frac{q}{2m}\vec{L}$$

- The intrinsic magnetic moment of a spin half Dirac particle is twice that expected from classical physics. This is often expressed in terms of the gyromagnetic ratio is g=2.

$$\vec{\mu} = g\frac{q}{2m}\vec{S}$$

Appendix V: Generators of Lorentz Transformations I

It will shortly be seen that the quantities

$$(M^{\alpha\beta})^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta} \tag{203}$$

or the equivalent (but less symmetric) quantities

$$(M^{\alpha\beta})^{\mu}{}_{\nu} = g^{\mu\alpha} \delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu} g^{\mu\beta} \tag{204}$$

are generators of Lorentz Transformations. The indices $\alpha\beta$ choose between generators $M^{\alpha\beta}$, while ${}^{\mu}{}_{\nu}$ in $(M^{\alpha\beta})^{\mu}{}_{\nu}$ are there to act on vector indices. Evident antisymmetry in the $\alpha\beta$ of (203) means that there are only six independent non-zero generators. Suppressing

Appendix V: Generators of Lorentz Transformations II

the vector indices (taken to be ${}^{\mu}{}_{\nu}$) and taking $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$ the six independent generators are:

$$K_1 = M^{01} = -M^{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_2 = M^{02} = -M^{20} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_3 = M^{03} = -M^{30} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Appendix V: Generators of Lorentz Transformations III

and

$$J_1 = M^{23} = -M^{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}$$

$$J_2 = M^{31} = -M^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J_3 = M^{12} = -M^{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

or, for short:

$$J_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$$
$$K_i = M^{0i}.$$

Appendix V: Generators of Lorentz Transformations IV

[Aside: The generators obey commutation relations

$$[J_i, J_j] = \epsilon_{ijk} J_k, \qquad [J_i, K_j] = \epsilon_{ijk} K_k, \qquad [K_i, K_j] = -\epsilon_{ijk} J_k.$$

The first of these says that the J's generate rotations in three-dimensional space and fixes the overall sign of the Js. The second says the Ks transform as a vector under rotations. End of aside]

With above definition[1] one could represent and arbitrary Lorentz transformation (boost, rotation or both) as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

with

$$\Lambda^{\mu}{}_{\nu} = \left(\exp \left[\frac{1}{2} w_{\alpha\beta} (M^{\alpha\beta})^{\bullet}{}_{\bullet} \right] \right)^{\mu}{}_{\nu} \tag{205}$$

$$= \delta^{\mu}_{\nu} + \frac{1}{2} \omega_{\alpha\beta} (M^{\alpha\beta})^{\mu}{}_{\nu} + O(\omega^2) \tag{206}$$

using a set of parameters $w_{\alpha\beta}$ which may as well be antisymmetric in $\alpha\beta$ (since any symmetric part would not participate in (206) on account of the ($\alpha \leftrightarrow \beta$)-antisymmetry of $M^{\alpha\beta}$) and so contain six independent degrees of freedom (controlling three boosts and

Appendix V: Generators of Lorentz Transformations V

three rotations) as required. In most of the proofs which follow we use the infinitesimal transformations to first order in ω since if some properties can be proved for infinitesimal transformations then it is always be possible to generalise that result to the exponential form for a finite transformation.

Appendix V Why do $(M^{\alpha\beta})^{\mu}{}_{\nu}$ generate Lorentz transformations? I

Lorentz transformations should be continuously connected to the identity (which (206) is, when $\omega_{\alpha\beta} = 0$) and should preserve inner products. The transformation in Eq. (206) preserves inner products because:

$$
\begin{aligned}
x' \cdot y' &= g_{\mu\nu} x'^{\mu} y'^{\nu} \\
&= g_{\mu\nu} (\Lambda^{\mu}{}_{\sigma} x^{\sigma})(\Lambda^{\nu}{}_{\tau} y^{\tau}) \\
&= g_{\mu\nu} (\delta^{\mu}_{\sigma} + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}{}_{\sigma})(\delta^{\nu}_{\tau} + \frac{1}{2}\omega_{\bar{\alpha}\bar{\beta}}(M^{\bar{\alpha}\bar{\beta}})^{\nu}{}_{\tau}) x^{\sigma} y^{\tau} + O(\omega)^2 \\
&= \left[g_{\sigma\tau} + \frac{1}{2}\left(\omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} + \omega_{\bar{\alpha}\bar{\beta}}(M^{\bar{\alpha}\bar{\beta}})_{\sigma\tau} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \\
&= \left[g_{\sigma\tau} + \frac{1}{2}\left(\omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} + \omega_{\alpha\beta}(M^{\alpha\beta})_{\sigma\tau} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \qquad \text{relabelling} \\
&= \left[g_{\sigma\tau} + \frac{1}{2}\left(\omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} - \omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \qquad \text{antisymmetry of } M \\
&= g_{\sigma\tau} x^{\sigma} y^{\tau} + O(\omega^2) \\
&= x \cdot y + O(\omega^2).
\end{aligned}
$$

Appendix V Why do $(M^{\alpha\beta})^{\mu}{}_{\nu}$ generate Lorentz transformations? II

If the above argument seems too abstract, a more concrete way of checking that we have generators of Lorentz transformations might instead be to compute

$$\exp\{(\eta K_1)\} = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{207}$$

as this will be recognised by some as a boost in the positive x-direction with rapidity η (that is with $\cosh\eta = \gamma$ and $\sinh\eta = \beta\gamma$) while

$$\exp\{(\theta J_1)\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{208}$$

will be recognised by most as a rotation by an angle θ about the x-axis.

Appendix V: Lorentz covariance of the Dirac equation I

If the Dirac Equation:

$$i\gamma^{\mu}\partial_{\mu}\psi = m\psi \tag{209}$$

is to be Lorentz covariant, there would have to exist a matrix $S(\Lambda)$ such that $\psi' = S(\Lambda)\psi$ is the solution of the Lorentz transformed Dirac Equation

$$i\gamma^{\mu}\partial'_{\mu}\psi' = m\psi'. \tag{210}$$

Equation (210) implies

$$i\gamma_{\mu}\partial'^{\mu}\psi' = m\psi' \tag{211}$$

and so

$$i\gamma_{\mu}\Lambda^{\mu}{}_{\nu}\partial^{\nu}S(\Lambda)\psi = mS(\Lambda)\psi \tag{212}$$

and so since $S(\Lambda)$ is independent of position

$$i\gamma_{\mu}S(\Lambda)\Lambda^{\mu}{}_{\nu}\partial^{\nu}\psi = S(\Lambda)m\psi \tag{213}$$

which using (209) becomes

$$i\gamma_{\mu}S(\Lambda)\Lambda^{\mu}{}_{\nu}\partial^{\nu}\psi = S(\Lambda)i\gamma^{\mu}\partial_{\mu}\psi$$

Appendix V: Lorentz covariance of the Dirac equation II

and hence

$$i\gamma^{\mu} S(\Lambda)\Lambda_{\mu}{}^{\nu}\partial_{\nu}\psi = S(\Lambda) i\gamma^{\nu}\partial_{\nu}\psi$$

or

$$i\left[\gamma^{\mu} S(\Lambda)\Lambda_{\mu}{}^{\nu} - S(\Lambda)\gamma^{\nu}\right]\partial_{\nu}\psi = 0. \tag{214}$$

Therefore, if we can show that there exists a matrix $S(\Lambda)$ satisfying

$$\gamma^{\mu} S(\Lambda)\Lambda_{\mu}{}^{\nu} = S(\Lambda)\gamma^{\nu} \tag{215}$$

we will have found a solution to (214) and thus will have found that the Dirac Equation is Lorentz covariant as desired. Thought it would be entirely possible to work directly with (215) it is perhaps nicer to bring both S matrices to the left hand side

$$S^{-1}(\Lambda)\gamma^{\mu} S(\Lambda)\Lambda_{\mu}{}^{\nu} = \gamma^{\nu}$$

and then use the identity

$$\Lambda_{\mu}{}^{\nu}\Lambda^{\sigma}{}_{\nu} \equiv \delta^{\sigma}_{\mu} \tag{216}$$

Appendix V: Lorentz covariance of the Dirac equation III

so that (215) ends up being written in the more common and (perhaps) more suggestive and useful form:

$$S^{-1}(\Lambda)\gamma^{\sigma}S(\Lambda) = \Lambda^{\sigma}{}_{\nu}\gamma^{\nu}. \tag{217}$$

[Aside: Here is (for infinitesimal Lorentz transformations) a proof of the identity (216):

$$\begin{aligned}
\Lambda_{\mu}{}^{\nu}\Lambda^{\sigma}{}_{\nu} &= \left(g_{\mu}{}^{\nu} + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu}{}^{\nu}\right)\left(g^{\sigma}{}_{\nu} + \frac{1}{2}\omega_{\bar\alpha\bar\beta}(M^{\bar\alpha\bar\beta})^{\sigma}{}_{\nu}\right) + O(\omega^2)\\
&= \delta^{\sigma}_{\mu} + \frac{1}{2}\left[\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu}{}^{\sigma} + \omega_{\bar\alpha\bar\beta}(M^{\bar\alpha\bar\beta})^{\sigma}{}_{\mu}\right] + O(\omega^2)\\
&= \delta^{\sigma}_{\mu} + \frac{1}{2}\left[\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu}{}^{\sigma} + \omega_{\alpha\beta}(M^{\alpha\beta})^{\sigma}{}_{\mu}\right] + O(\omega^2) \qquad \text{(relabelling)}\\
&= \delta^{\sigma}_{\mu} + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})_{\mu}{}^{\sigma} + (M^{\alpha\beta})^{\sigma}{}_{\mu}\right] + O(\omega^2) \qquad \text{(factorising)}\\
&= \delta^{\sigma}_{\mu} + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})^{\tau\sigma} + (M^{\alpha\beta})^{\sigma\tau}\right]g_{\mu\tau} + O(\omega^2) \qquad \text{(tidying)}\\
&= \delta^{\sigma}_{\mu} + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})^{\tau\sigma} - (M^{\alpha\beta})^{\tau\sigma}\right]g_{\mu\tau} + O(\omega^2) \qquad \text{(antisymmetry of } M\text{)}\\
&= \delta^{\sigma}_{\mu} + O(\omega^2).
\end{aligned}$$

Appendix V: Lorentz covariance of the Dirac equation IV

End of aside]

Lemma

A valid choice of $S(\Lambda)$ (for an infinitesimal Lorentz transformation) is given by:

$$S(\Lambda) = 1 + \frac{1}{4}\omega_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta} + O(\omega^2). \tag{218}$$

Appendix V: Lorentz covariance of the Dirac equation V

Proof.

$$
\begin{aligned}
S^{-1}(\Lambda)\gamma^\sigma S(\Lambda) &= \left(1-\frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\right)\gamma^\sigma\left(1+\frac{1}{4}\omega_{\bar\alpha\bar\beta}\gamma^{\bar\alpha}\gamma^{\bar\beta}\right)+O(\omega^2)\\
&=\gamma^\sigma+\frac{1}{4}\left(\omega_{\bar\alpha\bar\beta}\gamma^\sigma\gamma^{\bar\alpha}\gamma^{\bar\beta}-\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\gamma^\sigma\right)+O(\omega^2)\\
&=\gamma^\sigma+\frac{1}{4}\omega_{\alpha\beta}\left(\gamma^\sigma\gamma^\alpha\gamma^\beta-\gamma^\alpha\gamma^\beta\gamma^\sigma\right)+O(\omega^2)\\
&=\gamma^\sigma+\frac{1}{4}\omega_{\alpha\beta}\left((\gamma^\sigma\gamma^\alpha+\gamma^\alpha\gamma^\sigma)\gamma^\beta-\gamma^\alpha(\gamma^\sigma\gamma^\beta+\gamma^\beta\gamma^\sigma)\right)+O(\omega^2)\\
&=\gamma^\sigma+\frac{1}{4}\omega_{\alpha\beta}\left(2g^{\sigma\alpha}\gamma^\beta-\gamma^\alpha 2g^{\sigma\beta}\right)+O(\omega^2)\qquad\text{since }\{\gamma^\mu,\gamma^\nu\}\equiv 2g^{\mu\nu}\\
&=\left(\delta^\sigma_\nu+\frac{1}{2}\omega_{\alpha\beta}\left(g^{\sigma\alpha}\delta^\beta_\nu-\delta^\alpha_\nu g^{\sigma\beta}\right)\right)\gamma^\nu+O(\omega^2)\\
&=\left(\delta^\sigma_\nu+\frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^\sigma{}_\nu\right)\gamma^\nu+O(\omega^2)\qquad\text{using (204)}\\
&=\Lambda^\sigma{}_\nu\gamma^\nu+O(\omega^2)\qquad\text{using (206).}
\end{aligned}
$$

$\square$

Appendix V: Lorentz covariance of the Dirac equation VI

[Aside: Since $\gamma^\alpha\gamma^\beta = \frac{1}{2}\{\gamma^\alpha, \gamma^\beta\} + \frac{1}{2}[\gamma^\alpha, \gamma^\beta]$ we can also rewrite (218) in the more frequently seen (conventional) form:

$$S(\Lambda) = 1 + \frac{1}{8}\omega_{\alpha\beta}[\gamma^\alpha, \gamma^\beta] + O(\omega^2). \tag{219}$$

End of aside]

Appendix V: Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. I

Each of the expressions $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$ is of the form $\overline{\phi}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi$. To understand how any of them is affected by a Lorentz transformation it is therefore interesting to consider the following set of manipulations:[2]

$$\begin{aligned}
\overline{\phi'}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi' &= \overline{(S(\Lambda)\phi)}[\gamma^\mu\gamma^\nu\cdots\gamma^\tau](S(\Lambda)\psi) \\
&= \phi^\dagger S^\dagger(\Lambda)\gamma^0[\gamma^\mu S(\Lambda)S^{-1}(\Lambda)\gamma^\nu S(\Lambda)\cdots S^{-1}(\Lambda)\gamma^\tau]S(\Lambda)\psi \\
&= \phi^\dagger S^\dagger(\Lambda)\gamma^0 S(\Lambda)(S^{-1}(\Lambda)\gamma^\mu S(\Lambda))(S^{-1}(\Lambda)\gamma^\nu S(\Lambda))\cdots(S^{-1}(\Lambda)\gamma^\tau S(\Lambda))\psi \\
&= \phi^\dagger S^\dagger(\Lambda)\gamma^0 S(\Lambda)(\Lambda^\mu{}_\alpha\gamma^\alpha)(\Lambda^\nu{}_\beta\gamma^\gamma)\cdots(\Lambda^\tau{}_\lambda\gamma^\lambda)\psi \qquad \text{using (217)}
\end{aligned}$$

which itself suggests that if we can show that

$$S^\dagger(\Lambda)\gamma^0 S(\Lambda) = \gamma^0 \tag{220}$$

then we will have proved that

$$\overline{\phi'}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi' = \overline{\phi}(\Lambda^\mu{}_\alpha\gamma^\alpha)(\Lambda^\nu{}_\beta\gamma^\gamma)\cdots(\Lambda^\tau{}_\lambda\gamma^\lambda)\psi$$

which will itself have showed that each of the expressions under consideration transforms like a tensor of the appropriate rank.

Appendix V: Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. II

We must therefore prove (220). To do so is a two-stage process. First we compute $S^\dagger(\Lambda)$. Then we combine it with $\gamma^0 S(\Lambda)$. Starting with (218):

$$\begin{aligned} S^\dagger(\Lambda) &= \left[1 + \frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\right]^\dagger + O(\omega^2) \\ &= 1 + \frac{1}{4}\omega_{\alpha\beta}(\gamma^\alpha\gamma^\beta)^\dagger + O(\omega^2) \qquad (\omega_{\alpha\beta} \text{ are real}) \\ &= 1 + \frac{1}{4}\omega_{\alpha\beta}(\gamma^\beta)^\dagger(\gamma^\alpha)^\dagger + O(\omega^2) \\ &= 1 + \frac{1}{4}\omega_{\alpha\beta}(\gamma^0\gamma^\beta\gamma^0)(\gamma^0\gamma^\alpha\gamma^0) + O(\omega^2) \\ &= 1 + \frac{1}{4}\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0 + O(\omega^2) \end{aligned} \tag{221}$$

Appendix V: Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. III

from which we can deduce (using (218)) that

$$\begin{aligned}
S^\dagger(\Lambda)\gamma^0 S(\Lambda) &= \left(1+\frac{1}{4}\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0\right)\gamma^0\left(1+\frac{1}{4}\omega_{\bar{\alpha}\bar{\beta}}\gamma^{\bar{\alpha}}\gamma^{\bar{\beta}}\right)+O(\omega^2)\\
&= \gamma^0+\frac{1}{4}\left(\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0\gamma^0+\omega_{\bar{\alpha}\bar{\beta}}\gamma^0\gamma^{\bar{\alpha}}\gamma^{\bar{\beta}}\right)+O(\omega^2)\\
&= \gamma^0\left[1+\frac{1}{4}\left(\omega_{\alpha\beta}\gamma^\beta\gamma^\alpha+\omega_{\beta\alpha}\gamma^\beta\gamma^\alpha\right)\right]+O(\omega^2) \qquad ((\bar{\alpha},\bar{\beta})\to(\beta,\alpha))\\
&= \gamma^0\left[1+0\right]\psi+O(\omega^2) \qquad (\omega_{\alpha\beta}=-\omega_{\beta\alpha})\\
&= \gamma^0+O(\omega^2)
\end{aligned}$$

verifying (220) as required. This completes our proof that:

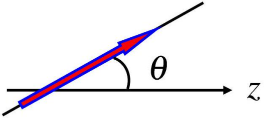
- $\overline{\phi}\psi$ is Lorentz invariant scalar,
- $\overline{\phi}\gamma^\mu\psi$ transforms as a Lorentz vector, and
- $\overline{\phi}\gamma^\mu\gamma^\nu\psi$ transforms as a second-rank tensor, *etc.*

Appendix VI: Spin-1 Rotation Matrices I

- Consider the spin-1 state with spin $+1$ along the axis defined by unit vector

$$\vec{n} = (\sin\theta, 0, \cos\theta)$$

- Spin state is an eigenstate of $\vec{n}\cdot\vec{S}$ with eigenvalue $+1$



$$(\vec{n}.\vec{S})|\psi\rangle = +1|\psi\rangle \tag{222}$$

- Express in terms of linear combination of spin 1 states which are eigenstates of S_z

$$|\psi\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$$

with

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

Appendix VI: Spin-1 Rotation Matrices II

- (222) becomes:

$$(\sin\theta S_x + \cos\theta S_z)\left(\alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle\right) = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle \tag{223}$$

- Write S_x in terms of ladder operators $S_x = \frac{1}{2}(S_+ + S_-)$ where

$$S_+|1,1\rangle = 0 \quad S_+|1,0\rangle = \sqrt{2}|1,1\rangle \quad S_+|1,-1\rangle = \sqrt{2}|1,0\rangle$$

$$S_-|1,1\rangle = \sqrt{2}|1,0\rangle \quad S_-|1,0\rangle = \sqrt{2}|1,-1\rangle \quad S_-|1,-1\rangle = 0$$

- from which we find $S_x|1,1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle$
- (223) becomes

$$S_x|1,0\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$$

$$S_x|1,-1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle$$

$$\sin\theta\left[\frac{\alpha}{\sqrt{2}}|1,0\rangle + \frac{\beta}{\sqrt{2}}|1,-1\rangle + \frac{\beta}{\sqrt{2}}|1,1\rangle + \frac{\gamma}{\sqrt{2}}|1,0\rangle\right] +$$

$$\alpha\cos\theta|1,1\rangle - \gamma\cos\theta|1,-1\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$$

Appendix VI: Spin-1 Rotation Matrices III

- which gives

$$\left.\begin{array}{r}\beta\frac{\sin\theta}{\sqrt{2}}+\alpha\cos\theta=\alpha\\(\alpha+\gamma)\frac{\sin\theta}{\sqrt{2}}=\beta\\\beta\frac{\sin\theta}{\sqrt{2}}-\gamma\cos\theta=\gamma\end{array}\right\}.$$

- Using $\alpha^2+\beta^2+\gamma^2=1$ the above equations yield

$$\alpha=\frac{1}{\sqrt{2}}(1+\cos\theta)\quad\beta=\frac{1}{\sqrt{2}}\sin\theta\quad\gamma=\frac{1}{\sqrt{2}}(1-\cos\theta)$$

- hence

$$\psi=\frac{1}{2}(1-\cos\theta)|1,-1\rangle+\frac{1}{\sqrt{2}}\sin\theta|1,0\rangle+\frac{1}{2}(1+\cos\theta)|1,+1\rangle.$$

- The coefficients α,β,γ are examples of what are known as quantum mechanical rotation matrices. The express how angular momentum eigenstate in a particular direction is expressed in terms of the eigenstates defined in a different direction $d^j_{m',m}(\theta)$.
- For spin-1 ($j=1$) we have just shown that

$$d^1_{1,1}(\theta)=\frac{1}{2}(1+\cos\theta)\quad d^1_{0,1}(\theta)=\frac{1}{\sqrt{2}}\sin\theta\quad d^1_{-1,1}(\theta)=\frac{1}{2}(1-\cos\theta).$$

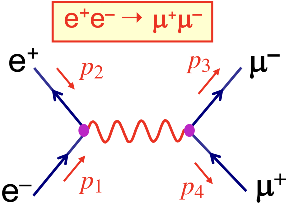
Appendix VI: Spin-1 Rotation Matrices IV

- For spin-1/2 it is straightforward to show

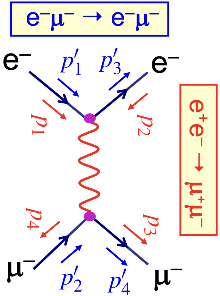
$$d^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta) = \cos\frac{\theta}{2} \quad d^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\theta) = \sin\frac{\theta}{2}.$$

Appendix VII: Crossing Symmetry

- Having derived the Lorentz invariant matrix element for $e^-e^+ \to \mu^-\mu^+$ 'rotate' the diagram to correspond to $e^-\mu^- \to e^-\mu^-$ and apply the principle of crossing symmetry to write down the matrix element !



rotates to



- The transformation: $p_1 \to p'_1$; $p_2 \to -p'_3$; $p_3 \to p'_4$; $p_4 \to -p'_2$ changes the spin averaged matrix element (see page 142) for

$\boxed{e^-e^+ \to \mu^-\mu^+}$ to that for $\boxed{e^-\mu^- \to e^-\mu^-}$:

$$\boxed{\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_3)^2+(p_1.p_4)^2}{(p_1.p_2)^2}} \quad\to\quad \boxed{\langle|M_{fi}|^2\rangle = 2e^4\frac{(p'_1.p'_4)^2+(p'_1.p'_2)^2}{(p'_1.p'_3)^2}}.$$

Appendix VIII: the $SU(2)$ anti-quark representation

Define an anti-quark doublet

$$\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$$

from which it follows that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q} = \begin{pmatrix} u^* \\ d^* \end{pmatrix}. \qquad (224)$$

The quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$ transforms as $\begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix}$ which complex conjugates to

$$\begin{pmatrix} u'^* \\ d'^* \end{pmatrix} = U^* \begin{pmatrix} u^* \\ d^* \end{pmatrix}$$

which using (224) can be re-written as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q}' = U^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q}.$$

Therefore, multiplying both sides of the last equation by the inverse of its left-most matrix, we see that $\bar{q}$ transforms as follows:

$$\bar{q}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q}. \qquad (225)$$

An arbitrary 2×2 unitary matrix with unit determinant can always be written in the form

$$U = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix}$$

provided that one chooses c_{11} and c_{12} such that $|c_{11}|^2 + |c_{12}|^2 = 1$. Therefore, (225) can be re-written to express an arbitrary $SU(2)$ trasformation of $\bar{q}$ as:

$$\begin{aligned} \bar{q}' &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_{11}^* & c_{12}^* \\ -c_{12} & c_{11} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q} \\ &= \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix} \bar{q} \\ &= U\bar{q} \end{aligned}$$

which proves that the anti-quark doublet $\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$ transforms in the same way as the quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$ – thus allowing us to use the same ladder operators on q and $\bar{q}$.

This is a special property of SU(2). For $\mathrm{SU}(3)$ there is no analogous representation of the anti-quarks.

Appendix IX: Electromagnetism

⋆ In Heaviside-Lorentz units $\varepsilon_0 = \mu_0 = c = 1$ Maxwell's equations in the vacuum become

$$\vec{\nabla}\cdot\vec{E} = \rho; \quad \vec{\nabla}\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t}; \quad \vec{\nabla}\cdot\vec{B} = 0; \quad \vec{\nabla}\wedge\vec{B} = \vec{J} + \frac{\partial\vec{E}}{\partial t}$$

⋆ The electric and magnetic fields can be expressed in terms of scalar and vector potentials

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi; \quad \vec{B} = \vec{\nabla}\wedge\vec{A}$$

⋆ In terms of the 4-vector potential $A^\mu = (\phi, \vec{A})$ and the 4-vector current $j^\mu = (\rho, \vec{J})$ Maxwell's equations can be expressed in the covariant form:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{226}$$

where $F^{\mu\nu}$ is the anti-symmetric field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{227}$$

-Combining (226) and (227)

$$\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = j^\nu$$

which can be written

$$\Box^2 A^\mu - \partial^\mu \left(\partial_\nu A^v\right) = j^\mu \tag{228}$$

where the D'Alembertian operator

$$\Box^2 = \partial_\nu \partial^v = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

-Acting on (228) with ∂_V gives

$$\begin{aligned} \partial_v j^v &= \partial_\nu \partial_\mu \partial^\mu A^v - \partial_\mu \partial_\nu \partial^v A^\mu = 0 \\ \Rightarrow \quad & \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{Conservation of Electric Charge} \end{aligned}$$

Appendix X: Gauge Invariance

- Conservation laws are associated with symmetries. Here the symmetry is the GAUGE INVARIANCE of electro-magnetism

⋆ The electric and magnetic fields are unchanged for the gauge transformation:

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi; \quad \phi \to \phi' = \phi - \frac{\partial \chi}{\partial t}$$

where $\chi = \chi(t, \vec{x})$ is any finite differentiable function of position and time
⋆ In 4-vector notation the gauge transformation can be expressed as:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$$

- Using the fact that the physical fields are gauge invariant, choose χ to be a solution of

⋆ In this case we have

$$\partial^\mu A'_\mu = \partial^\mu \left(A_\mu + \partial_\mu \chi\right) = \partial^\mu A_\mu + \Box^2 \chi = 0$$

⋆ Dropping the prime we have a chosen a gauge in which

$$\partial_\mu A^\mu = 0 \qquad \text{The Lorentz Condition}$$

- With the Lorentz condition, equation (228) becomes:

$$\boxed{\Box^2 A^\mu = j^\mu}. \tag{229}$$

- Having imposed the Lorentz condition we still have freedom to make a further gauge transformation, i.e.

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$$

where $\Lambda(t, \vec{x})$ is any function that satisfies

$$\Box^2 \Lambda = 0 \tag{230}$$

⋆ Clearly (229) remains unchanged, in addition the Lorentz condition still holds:

$$\partial^\mu A'_\mu = \partial^\mu \left(A_\mu + \partial_\mu \Lambda\right) = \partial^\mu A_\mu + \Box^2 \Lambda = \partial^\mu A_\mu = 0$$

Appendix XI: Photon Polarization

- For a free photon (i.e. $j^\mu = 0$) equation (229) becomes

$$\boxed{\Box^2 A^\mu = 0} \qquad (231)$$

(note have chosen a gauge where the Lorentz condition is satisfied)

- Equation (230) has solutions (i.e. the wave-function for a free photon)

$$A^\mu = \varepsilon^\mu(q)e^{-iq\cdot x}$$

where ε^μ is the four-component polarization vector and q is the photon four-momentum

$$0 = \Box^2 A^\mu = -q^2 \varepsilon^\mu e^{-iq\cdot x}$$

$$\Rightarrow q^2 = 0$$

- Hence equation (231) describes a massless particle.
- But the solution has four components - might ask how it can describe a spin-1 particle which has three polarization states?
- But for (230) to hold we must satisfy the Lorentz condition:

$$0 = \partial_\mu A^\mu = \partial_\mu \left(\varepsilon^\mu e^{-iq\cdot x}\right) = \varepsilon^\mu \partial_\nu \left(e^{-iq\cdot x}\right) = -i\varepsilon^\mu q_\mu e^{-iq\cdot x}$$

Hence the Lorentz condition gives

$$q_\mu \varepsilon^\mu = 0 \qquad (232)$$

⋆ However, in addition to the Lorentz condition still have the addional gauge freedom of
$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$ with (230) $\Box^2 \Lambda = 0$
-Choosing $\Lambda = iae^{-iq\cdot x}$ which has $\Box^2 \Lambda = q^2 \Lambda = 0$

$$\begin{aligned} A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda &= \varepsilon_\mu e^{-iq\cdot x} + ia\partial_\mu e^{-iq\cdot x} \\ &= \varepsilon_\mu e^{-iq\cdot x} + ia\left(-iq_\mu\right) e^{-iq\cdot x} \\ &= \left(\varepsilon_\mu + aq_\mu\right) e^{-iq\cdot x} \end{aligned}$$

⋆ Hence the electromagnetic field is left unchanged by

$$\varepsilon_\mu \to \varepsilon'_\mu = \varepsilon_\mu + aq_\mu$$

⋆ Hence the two polarization vectors which differ by a mulitple of the photon four-momentum describe the same photon. Choose *a* such that the time-like component of ε_μ is zero, i.e. $\varepsilon_0 \equiv 0$
⋆ With this choice of gauge, which is known as the COULOMB GAUGE, the Lorentz condition (232) gives

$$\vec{\varepsilon} \cdot \vec{q} = 0$$

i.e. only 2 independent components, both transverse to the photons momentum

⋆ A massless photon has two transverse polarisation states. For a photon travelling in the z direction these can be expressed as the transversly polarized states:

$$\varepsilon_1^\mu = (0, 1, 0, 0); \quad \varepsilon_2^\mu = (0, 0, 1, 0)$$

⋆ Alternatively take linear combinations to get the circularly polarized states

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0); \quad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0, 1, i, 0)$$

- It can be shown that the ε_+state corresponds the state in which the photon spin is directed in the $+$**z** direction, i.e. $S_z = +1$

Appendix XII: Massive Spin-1 particles

- For a massless photon we had (before imposing the Lorentz condition) we had from equation (228):

$$\Box^2 A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = j^\mu$$

⋆ The Klein-Gordon equation for a spin-0 particle of mass m is

$$\left(\Box^2 + m^2\right)\phi = 0$$

suggestive that the appropriate equations for a massive spin-1 particle can be obtained by replacing $\Box^2 \to \Box^2 + m^2$

- This is indeed the case, and from QFT it can be shown that for a massive spin 1 particle equation (228): becomes

$$\left(\Box^2 + m^2\right) B^\mu - \partial^\mu \left(\partial_\nu B^\nu\right) = j^\mu$$

- Therefore a free particle must satisfy

$$\left(\Box^2 + m^2\right) B^\mu - \partial^\mu \left(\partial_\nu B^\nu\right) = 0 \qquad (233)$$

- Acting on equation (233) with ∂_v gives

$$\begin{aligned}\left(\Box^2 + m^2\right)\partial_\mu B^\mu - \partial_\mu\partial^\mu\left(\partial_v B^v\right) &= 0\\ \left(\Box^2 + m^2\right)\partial_\mu B^\mu - \Box^2\left(\partial_v B^v\right) &= 0\\ m^2\partial_\mu B^\mu &= 0 \qquad (234)\end{aligned}$$

- Hence, for a massive spin-1 particle, unavoidably have $\partial_\mu B^\mu = 0$; note this is not a relation that reflects to choice of gauge.

-Equation (233) becomes

$$\boxed{\left(\Box^2 + m^2\right)B^\mu = 0} : \qquad (235)$$

⋆ For a free spin-1 particle with 4-momentum, p^μ, equation (235): admits solutions

$$B_\mu = \varepsilon_\mu e^{-ip.x}$$

- Substituting into equation (234) gives

$$p_\mu \varepsilon^\mu = 0$$

⋆ The four degrees of freedom in ε^μ are reduced to three, but for a massive particle, equation (235) does not allow a choice of gauge and we can not reduce the number of degrees of freedom any further

⋆ Hence we need to find three orthogonal polarisation states satisfying

$$p_\mu \varepsilon^\mu = 0 \tag{236}$$

⋆ For a particle travelling in the z direction, can still admit the circularly polarized states.

$$\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0, 1, -i, 0); \quad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0, 1, i, 0)$$

⋆ Writing the third state as

$$\varepsilon^\mu_L = \frac{1}{\sqrt{\alpha^2 + \beta^2}}(\alpha, 0, 0, \beta)$$

equation (236) gives $\alpha E - \beta p_z = 0$

$$\Rightarrow \quad \varepsilon^\mu_L = \frac{1}{m}\left(p_z, 0, 0, E\right)$$

- This longitudinal polarisation state is only present for massive spin-1 particles, i.e. there is no analogous state for a free on-shell photon.

Appendix XIII: Local Gauge Invariance

⋆ The Dirac equation for a charged particle in an electro-magnetic field can be obtained from the free particle wave-equation by making the minimal substitution

$$\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi \quad (q = \text{ charge })$$

In QM: $i\partial_\mu \to i\partial_\mu - qA_\mu$ and the Dirac equation becomes

$$\gamma^\mu \left(i\partial_\mu - qA_\mu\right)\psi - m\psi = 0$$

- In Appendix X: saw that the physical EM fields where invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$$

⋆ Under this transformation the Dirac equation becomes

$$\gamma^\mu \left(i\partial_\mu - qA_\mu + q\partial_\mu\chi\right)\psi - m\psi = 0$$

which is not the same as the original equation. If we require that the Dirac equation is invariant under the Gauge transformation then under the gauge transformation we need to modify the wave-functions

$$\psi \to \psi' = \psi e^{iq\chi}$$

⋆ To prove this, applying the gauge transformation :

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi \quad \psi \to \psi' = \psi e^{iq\chi}$$

to the original Dirac equation gives

$$\gamma^\mu \left(i\partial_\mu - qA_\mu + q\partial_\mu \chi\right) \psi e^{iq\chi} - m\psi e^{iq\chi} = 0 \tag{237}$$

⋆ But

$$i\partial_\mu \left(\psi e^{iq\chi}\right) = ie^{iq\chi}\partial_\mu \psi - q\left(\partial_\mu \chi\right) e^{iq\chi}\psi$$

⋆ Equation (237) becomes

$$\begin{aligned} &\gamma^\mu e^{iq\chi}\left(i\partial_\mu - qA_\mu + q\partial_\mu\chi - q\partial_\mu\chi\right)\psi - m\psi e^{iq\chi} = 0 \\ &\Rightarrow \quad \gamma^\mu e^{iq\chi}\left(i\partial_\mu - qA_\mu\right)\psi - m\psi e^{iq\chi} = 0 \\ &\Longrightarrow \\ &\gamma^\mu\left(i\partial_\mu - qA_\mu\right)\psi - m\psi = 0 \end{aligned}$$

which is the original form of the Dirac equation

Appendix XIV : Local Gauge Invariance 2

⋆ Reverse the argument of Appendix XIII. Suppose there is a fundamental symmetry of the universe under local phase transformations

$$\psi(x) \to \psi'(x) = \psi(x)e^{iq\chi(x)}$$

- Note that the local nature of these transformations: the phase transformation depends on the space-time coordinate $x = (t, \vec{x})$

⋆ Under this transformation the free particle Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

becomes $\quad i\gamma^\mu \partial_\mu \left(\psi e^{iq\chi}\right) - m\psi e^{iq\chi} = 0$

$$ie^{iq\chi}\gamma^\mu \left(\partial_\mu \psi + iq\psi\partial_\mu\chi\right) - m\psi e^{iq\chi} = 0$$
$$i\gamma^\mu \left(\partial_\mu + iq\partial_\mu\chi\right)\psi - m\psi = 0$$

Local phase invariance is not possible for a free theory, i.e. one without interactions

- To restore invariance under local phase transformations have to introduce a massless "gauge boson" A^μ which transforms as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi$$

and make the substitution

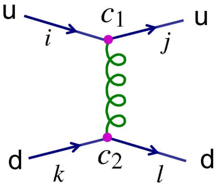
$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$$

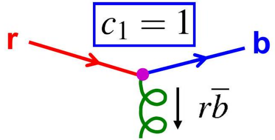
Appendix XV: Alternative evaluation of colour factors

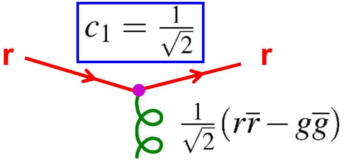
⋆ The colour factors can be obtained (more intuitively) as follows :

-Write $C(ik \to jl) = \frac{1}{2} c_1 c_2$

-Where the colour coefficients at the two vertices depend on the quark and gluon colours



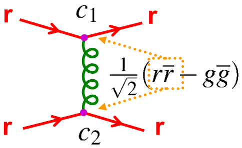


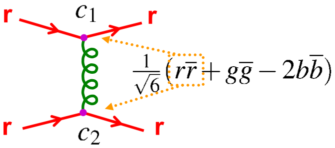


-Sum over all possible exchanged gluons conserving colour at both vertices

① Configurations involving a single colour

e.g. $rr \to rr$ **: two possible exchanged gluons**

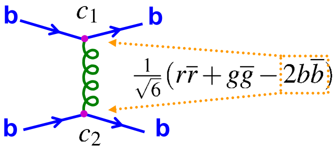




$$c_1 = c_2 = \frac{1}{\sqrt{2}} \qquad c_1 = c_2 = \frac{1}{\sqrt{6}}$$

$$C(rr \to rr) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{3}$$

e.g. $bb \to bb$ **: only one possible exchanged gluon**

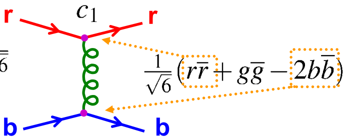


$$c_1 = c_2 = -\frac{2}{\sqrt{6}}$$

$$\Rightarrow \quad C(bb \to bb) = \frac{1}{2}\left(\frac{2}{\sqrt{6}}\frac{2}{\sqrt{6}}\right) = \frac{1}{3}$$

② Other configurations where quarks don′t change colour

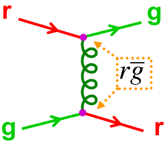
$c_1 = \frac{1}{\sqrt{6}}$



$c_2 = -\frac{2}{\sqrt{6}}$

$$C(rb \to rb) = \frac{1}{2}\left(-\frac{1}{\sqrt{6}}\frac{2}{\sqrt{6}}\right) = -\frac{1}{6}$$

③ Configurations where quarks swap colours



$$c_1 = c_2 = 1$$

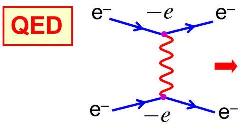
$$C(rg \to gr) = \frac{1}{2}$$

Appendix XVI: Colour Potentials

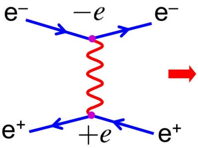
-Previously argued that gluon self-interactions lead to a $+\lambda r$ long-range potential and that this is likely to explain colour confinement

- Have yet to consider the short range potential - i.e. for quarks in mesons and baryons does QCD lead to an attractive potential?

-Analogy with QED: (NOTE this is very far from a formal proof)



$$V(r) = +\frac{\alpha}{r}$$

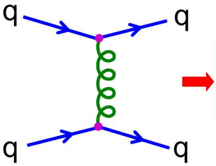


$$V(r) = -\frac{\alpha}{r}$$

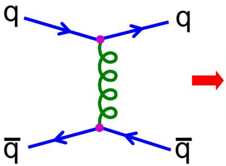
Repulsive Potential

Attractive Potential

⋆ by analogy with QED expect potentials of form



$$V(r) = +C\frac{\alpha_S}{r}$$



$$V(r) = -C\frac{\alpha_S}{r}$$

⋆ Whether it is a attractive or repulsive potential depends on sign of colour factor

Not examinable

⋆ Consider the colour factor for a q $\bar{q}$ system in the colour singlet state:

$$\psi = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

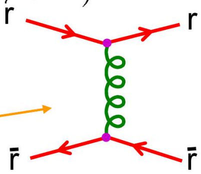
with colour potential $\langle V_{q\bar{q}}\rangle = \langle\psi\,|V_{\mathrm{QCD}}|\,\psi\rangle$

$$\Rightarrow \quad \langle V_{q\bar{q}}\rangle = \frac{1}{3}\left(\langle r\bar{r}|V_{\mathrm{QCD}}|r\bar{r}\rangle + \ldots + \langle r\bar{r}|V_{\mathrm{QCD}}|b\bar{b}\rangle + \ldots\right)$$

owing the QED analogy:

$$\langle r\bar{r}|V_{\mathrm{QCD}}|r\bar{r}\rangle = -C(r\bar{r} \to r\bar{r})\frac{\alpha_S}{r}$$

hich is the term arising from $r\bar{r} \to r\bar{r}$



-Have 3 terms like $r\bar{r} \to r\bar{r}, b\bar{b} \to b\bar{b}, \ldots$ and 6 like $r\bar{r} \to g\bar{g}, r\bar{r} \to b\bar{b}, \ldots$

$$\langle V_{q\bar{q}}\rangle = -\tfrac{1}{3}\tfrac{\alpha_S}{r}[3 \times C(r\bar{r} \to r\bar{r}) + 6 \times C(r\bar{r} \to g\bar{g})] = -\tfrac{1}{3}\tfrac{\alpha_S}{r}\left[3 \times \tfrac{1}{3} + 6 \times \tfrac{1}{2}\right]$$
$$\longrightarrow \langle V_{q\bar{q}}\rangle = -\tfrac{4}{3}\tfrac{\alpha_S}{r} \quad \text{NEGATIVE} \Rightarrow \text{ATTRACTIVE}$$

-The same calculation for a q $\bar{q}$ colour octet state, e.g. $r\bar{g}$ gives a positive repulsive potential: $C(r\bar{g} \longrightarrow r\bar{g}) = -\frac{1}{6}$

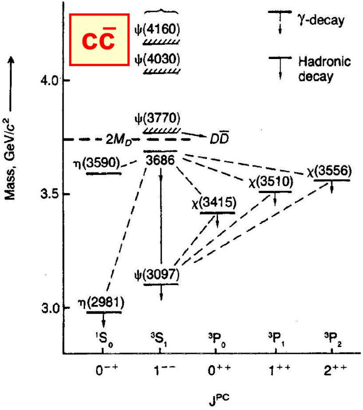
Not examinable

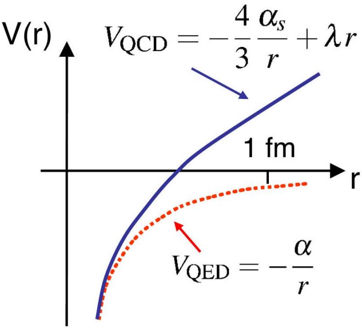
⋆ Whilst not a formal proof, it is comforting to see that in the colour singlet $q\bar{q}$ state the QCD potential is indeed attractive.

* Combining the short-range QCD potential with the linear long-range term discussed previously:

$$V_{\mathrm{QCD}} = -\frac{4}{3}\frac{\alpha_s}{r} + \lambda r$$

⋆ This potential is found to give a good description of the observed charmonium (cc) and bottomonium (bb) bound states





NOTE:

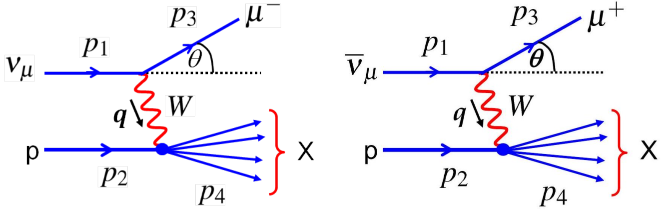
$\cdot$c, b are heavy quarks

non-relativistic - orbit

- probe $1/\mathrm{r}$ part of V

Agreement of data with prediction provides strong evidence that V_{QCD} has the Expected

Appendix XVII: Deep-Inelastic Neutrino Scattering



- Two steps:
  - First write down most general cross section in terms of structure functions.
  - Then evaluate expressions in the quark-parton model.
- QED Revisited:
  - In the limit $s \gg M^2$the most general electro-magnetic deep-inelastic cross section (from single photon exchange) can be written (Eq. (90) of Handout 6) as

$$\frac{d^2\sigma_{e^\pm p}}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\,\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right].$$

  - For neutrino scattering typically measure the energy of the produced muon $E_\mu = E_\nu(1-y)$ and differential cross-sections expressed in terms of $dx\,dy$
  - $Q^2 = (s - M^2)xy \approx sxy \;\; \rightarrow$

$$\frac{d^2\sigma}{dx\,dy} = \left|\frac{dQ^2}{dy}\right| \frac{d^2\sigma}{dx\,dQ^2} = sx\frac{d^2\sigma}{dx\,dQ^2}.$$

- In the limit $s \gg M^2$ the general Electro-magnetic DIS cross section can be written

$$\frac{d^2\sigma^{e^\pm p}}{dx\,dy} = \frac{4\pi\alpha^2 s}{Q^4}\left[(1-y)\,F_2(x,Q^2) + y^2 x F_1(x,Q^2)\right]. \tag{238}$$

- NOTE: This is the most general Lorentz Invariant parity conserving expression
- For neutrino DIS parity is violated and the general expression includes an additional term to allow for parity violation. New structure function: $F_3(x,Q^2)$:

  $\nu_\mu p \to \mu^- X$

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[(1-y)\,F_2^{\nu p}(x,Q^2) + y^2 x F_1^{\nu p}(x,Q^2) + y\left(1-\frac{y}{2}\right) x F_3^{\nu p}(x,Q^2)\right]$$

- For anti-neutrino scattering new structure function enters with opposite sign

  $\bar{\nu}_\mu p \to \mu^+ X$

$$\frac{d^2\sigma^{\bar{\nu} p}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[(1-y)\,F_2^{\bar{\nu} p}(x,Q^2) + y^2 x F_1^{\bar{\nu} p}(x,Q^2) - y\left(1-\frac{y}{2}\right) x F_3^{\bar{\nu} p}(x,Q^2)\right]$$

- Similarly for neutrino-neutron scattering

  $\nu_\mu n \to \mu^- X$

$$\frac{d^2\sigma^{\nu n}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[(1-y)\,F_2^{\nu n}(x,Q^2) + y^2 x F_1^{\nu n}(x,Q^2) + y\left(1-\frac{y}{2}\right) x F_3^{\nu n}(x,Q^2)\right]$$

  $\bar{\nu}_\mu n \to \mu^+ X$

$$\frac{d^2\sigma^{\bar{\nu} n}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[(1-y)\,F_2^{\bar{\nu} n}(x,Q^2) + y^2 x F_1^{\bar{\nu} n}(x,Q^2) - y\left(1-\frac{y}{2}\right) x F_3^{\bar{\nu} n}(x,Q^2)\right]$$

Neutrino Interaction Structure Functions

- In terms of the parton distribution functions we found (106):

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2}{\pi} sx \left[d(x) + (1-y)^2 \bar{u}(x)\right]$$

- Compare coefficients of y with the general Lorentz Invariant form (238) and assume Bjorken scaling, i.e. $F(x, Q^2) \rightarrow F(x)$

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2 s}{2\pi} \left[(1-y)\,F_2^{\nu p}(x) + y^2 x F_1^{\nu p}(x) + y\left(1 - \frac{y}{2}\right) x F_3^{\nu p}(x)\right]$$

- Re-writing (106):

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2}{2\pi} s \left[2xd(x) + 2x\bar{u}(x) - 4xy\bar{u}(x) + 2xy^2\bar{u}(x)\right]$$

and equating powers of y

$$\begin{aligned} 2xd + 2x\bar{u} &= F_2 \\ -4x\bar{u} &= -F_2 + xF_3 \\ 2\bar{u} &= F_1 - xF_3/2 \end{aligned}$$

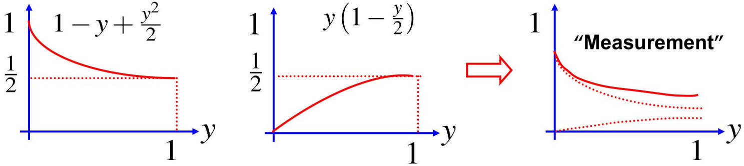
gives:

$$F_2^{\nu p} = 2xF_1^{\nu p} = 2x[d(x) + \bar{u}(x)]$$
$$xF_3^{\nu p} = 2x[d(x) - \bar{u}(x)].$$

- NOTE: again we get the Callan-Gross relation $F_2 = 2xF_1$.
- No surprise, underlying process is scattering from point-like spin-1/2 quarks

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[\left(1 - y + \frac{y^2}{2}\right) F_2^{\nu p}(x) + y\left(1 - \frac{y}{2}\right) xF_3^{\nu p}(x)\right]$$

- Experimentally measure F_2 and F_3 from y distributions at fixed x
  - Different y dependencies (from different rest frame angular distributions) allow contributions from the two structure functions to be measured



  - Then use $F_2^{\nu p} = 2x[d(x) + \bar{u}(x)]$ and $F_3^{\nu p} = 2[d(x) - \bar{u}(x)]$ →$d(x)$ and $\bar{u}(x)$ separately

- Neutrino experiments require large detectors (often iron) i.e. isoscalar target $F_2^{\nu N} = 2xF_1^{\nu N} = \frac{1}{2}\left(F_2^{\nu p} + F_2^{\nu n}\right) = x[u(x) + d(x) + \bar{u}(x) + \bar{d}(x)]$ $xF_3^{\nu N} = \frac{1}{2}\left(xF_3^{\nu p} + xF_3^{\nu n}\right) = x[u(x) + d(x) - \bar{u}(x) - \bar{d}(x)]$
- For electron – nucleon scattering: $F_2^{ep} = 2xF_1^{ep} = x[\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\bar{u}(x) + \frac{1}{9}\bar{d}(x)]$ $F_2^{en} = 2xF_1^{en} = x[\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\bar{d}(x) + \frac{1}{9}\bar{u}(x)]$ $F_2^{\nu N} = \frac{18}{5}F_2^{eN}$
- For an isoscalar target

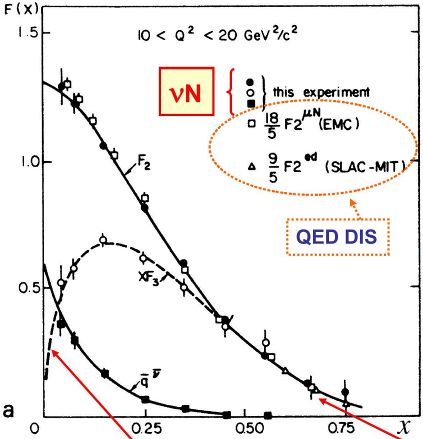
$$F_2^{eN} = \frac{1}{2}\left(F_2^{ep} + F_2^{en}\right) = \frac{5}{18}x[u(x) + d(x) + \bar{u}(x) + \bar{d}(x)]$$

- Note that the factor $\frac{5}{18} = \frac{1}{2}\left(q_u^2 + q_d^2\right)$ and by comparing neutrino to electron scattering structure functions measure the sum of quark charges

Experiment: 0.29 ± 0.02

Measurements of F2(x) and F3(x)

- CHDS Experiment $\nu_\mu + \mathrm{Fe} \rightarrow \mu^- + X$



H. Abramowicz et al., Z.Phys. C17 (1983) 283

$$F_2^{\nu N} = x[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)]$$
$$xF_3^{\nu N} = x[u(x) + d(x) - \overline{u}(x) - \overline{d}(x)]$$

$$\Rightarrow \quad F_2^{\nu N} - xF_3^{\nu N} = 2x[\overline{u} + \overline{d}]$$

✶ **Difference in neutrino structure functions measures anti-quark (sea) parton distribution functions**

Sea dominates so expect xF_3 to go to zero as q(x) = q̄(x)

Sea contribution goes to zero

Valence Contribution

- Separate parton density functions into sea and valence components

$$u(x) = u_V(x) + u_S(x) = u_V(x) + S(x)$$

$$d(x) = d_V(x) + d_S(x) = d_V(x) + S(x)$$

$$\bar{u}(x) = \bar{u}_S(x) = S(x)$$

$$\bar{d}(x) = \bar{d}_S(x) = S(x)$$

$$\rightarrow F_3^{\nu N} = [u(x) + d(x) - \bar{u}(x) - \bar{d}(x)] = u_V(x) + d_V(x) \ \rightarrow$$

$$\int_0^1 F_3^{\nu N}(x)\, dx = \int_0^1 (u_V(x) + d_V(x))\ dx = N_u^V + N_d^V$$

- Area under measured function gives a measurement of the total number of valence quarks in a nucleon! Expect

$$\int_0^1 F_3^{\nu N}(x)\, dx = 3$$

  "Gross–Llewellyn-Smith sum rule" Experiment: 3.0 ± 0.2.

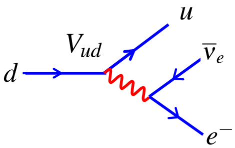
- Note: $F_2^{\bar{\nu} p} = F_2^{\nu n}$; $F_2^{\bar{\nu} n} = F_2^{\nu p}$; $F_3^{\bar{\nu} p} = F_3^{\nu n}$; $F_3^{\bar{\nu} n} = F_3^{\nu p}$ and anti-neutrino structure functions contain same pdf information.

Appendix XVIII: Determination of the CKM Matrix

- The experimental determination of the CKM matrix elements comes mainly from measurements of leptonic decays (the leptonic part is well understood).
- It is easy to produce/observe meson decays, however theoretical uncertainties associated with the decays of bound states often limits the precision
- Contrast this with the measurements of the PMNS matrix, where there are few theoretical uncertainties and the experimental difficulties in dealing with neutrinos limits the precision.

❶ **$|V_{ud}|$** **from nuclear beta decay**

$$\begin{pmatrix} \times & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$



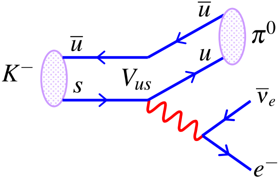
Super-allowed $0^+ \to 0^+$ beta decays are relatively free from theoretical uncertainties

$$\Gamma \propto |V_{ud}|^2$$

$$\boxed{|V_{ud}| = 0.97377 \pm 0.00027} \qquad (\approx \cos\theta_c)$$

❷ **$|V_{us}|$** **from semi-leptonic kaon decays**

$$\begin{pmatrix} \cdot & \times & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$



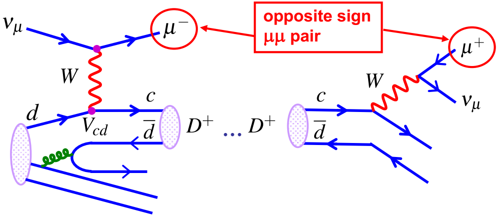
$$\Gamma \propto |V_{us}|^2$$

$$\boxed{|V_{us}| = 0.2257 \pm 0.0021} \qquad (\approx \sin\theta_c)$$

❸ **$|V_{cd}|$** **from neutrino scattering** $\nu_\mu + N \rightarrow \mu^+\mu^- X$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \times & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

Look for opposite charge di-muon events in ν_μ scattering from production and decay of a $D^+(c\bar{d})$ meson



opposite sign μμ pair

$$\text{Rate} \propto |V_{cd}|^2 \underbrace{\text{Br}(D^+ \rightarrow X\mu^+\nu_\mu)}$$

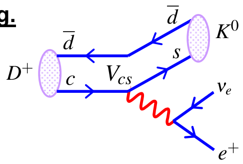
Measured in various collider experiments

$$\Rightarrow \quad \boxed{|V_{cd}| = 0.230 \pm 0.011}$$

❹ **$|V_{cs}|$** **from semi-leptonic charmed meson decays**

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \times & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

e.g.



$$\Gamma \propto |V_{cs}|^2$$

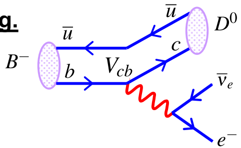
- **Precision limited by theoretical uncertainties**

$$\boxed{|V_{cs}| = 0.957 \pm \underbrace{0.017}_{\text{experimental error}} \pm \underbrace{0.093}_{\text{theory uncertainty}}}$$

❺ **$|V_{cb}|$** **from semi-leptonic B hadron decays**

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \times \\ \cdot & \cdot & \cdot \end{pmatrix}$$

e.g.



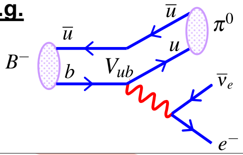
$$\Gamma \propto |V_{cb}|^2$$

$$\boxed{|V_{cb}| = 0.0416 \pm 0.0006}$$

❻ **$|V_{ub}|$** **from semi-leptonic B hadron decays**

$$\begin{pmatrix} \cdot & \cdot & \times \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

e.g.



$$\Gamma \propto |V_{ub}|^2$$

$$\boxed{|V_{ub}| = 0.0043 \pm 0.0003}$$

Appendix XIX: Particle–AntiParticle Mixing

-The wave-function for a single particle with lifetime $\tau = 1/\Gamma$ evolves with time as:

$$\psi(t) = Ne^{-\Gamma t/2}e^{-iMt}$$

which gives the appropriate exponential decay of

$$\langle\psi(t) \mid \psi(t)\rangle = \langle\psi(0) \mid \psi(0)\rangle e^{-t/\tau}$$

-The wave-function satisfies the time-dependent wave equation:

$$\hat{H}|\psi(t)\rangle = \left(M - \frac{1}{2}i\Gamma\right)|\psi(t)\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle$$

-For a bound state such as a K^0 the mass term includes the "mass" from the weak interaction "potential" $\hat{H}_{\text{weak}}$

$$M = m_{K^0} + \left\langle K^0 \left| \hat{H}_{\text{weak}} \right| K^0 \right\rangle + \sum_j \frac{\left|\left\langle K^0 \left| \hat{H}_{\text{weak}} \right| j \right\rangle\right|^2}{m_{K^0} - E_j} \leftarrow$$

Sum over
intermediate
states j

The third term is the 2nd order term in the perturbation expansion corresponding to box diagrams resulting in $K^0 \rightarrow \bar{K}^0$

Not examinable

- The total decay rate is the sum over all possible decays $K^0 \to f$

$$\Gamma = 2\pi \sum_f \left| \left\langle f \left| \hat{H}_{\text{weak}} \right| K^0 \right\rangle \right|^2 \rho_F \longleftarrow \text{ Density of final states}$$

⋆ Because there are also diagrams which allow $K^0 \leftrightarrow \bar{K}^0$ mixing need to consider the time evolution of a mixed stated

$$\psi(t) = a(t)K^0 + b(t)\bar{K}^0$$

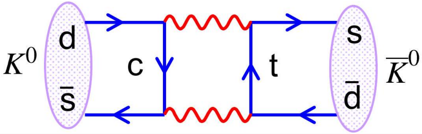
⋆ The time dependent wave-equation of (A1) becomes

$$\begin{pmatrix} M_{11} - \frac{1}{2}i\Gamma_{11} & M_{12} - \frac{1}{2}i\Gamma_{12} \\ M_{21} - \frac{1}{2}i\Gamma_{21} & M_{22} - \frac{1}{2}i\Gamma_{22} \end{pmatrix} \begin{pmatrix} \left|K^0(t)\right\rangle \\ \left|\bar{K}^0(t)\right\rangle \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} \left|K^0(t)\right\rangle \\ \left|\bar{K}^0(t)\right\rangle \end{pmatrix}$$

the diagonal terms are as before, and the off-diagonal terms are due to mixing.

$$M_{11} = m_{K^0} + \left\langle K^0 \left| \hat{H}_{\text{weak}} \right| K^0 \right\rangle + \sum_n \frac{\left| \left\langle K^0 \left| \hat{H}_{\text{weak}} \right| K^0 \right\rangle \right|^2}{m_{K^0} - E_n}$$

$$M_{12} = \sum_j \frac{\langle K^0|\hat{H}_{\text{weak}}|j\rangle^* \langle j|\hat{H}_{\text{weak}}|\overline{K}^0\rangle}{m_{K^0} - E_j}$$



-The off-diagonal decay terms include the effects of interference between decays to a common final state

$$\Gamma_{12} = 2\pi \sum_f \left\langle f \left| \hat{H}_{\text{weak}} \right| K^0 \right\rangle^* \left\langle f \left| \hat{H}_{\text{weak}} \right| \bar{K}^0 \right\rangle \rho_F$$

-In terms of the time dependent coefficients for the kaon states, (A3) becomes

$$\left[\mathbf{M} - i\frac{1}{2}\Gamma \right] \begin{pmatrix} a \\ b \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix}$$

where the Hamiltonian can be written:

$$\mathbf{H} = \mathbf{M} - i\frac{1}{2}\Gamma = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}$$

-Both the mass and decay matrices represent observable quantities and are Hermitian

$$M_{11} = M_{11}^*, \quad M_{22} = M_{22}^*, \quad M_{12} = M_{21}^*$$
$$\Gamma_{11} = \Gamma_{11}^*, \quad \Gamma_{22} = \Gamma_{22}^*, \quad \Gamma_{12} = \Gamma_{21}^*$$

-Furthermore, if CPT is conserved then the masses and decay rates of the $\bar{K}^0$ and K^0 are identical:

$$M_{11} = M_{22} = M; \quad \Gamma_{11} = \Gamma_{22} = \Gamma$$

-Hence the time evolution of the system can be written:

$$\begin{pmatrix} M-\frac{1}{2}i\Gamma & M_{12}-\frac{1}{2}i\Gamma_{12} \\ M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*} & M-\frac{1}{2}i\Gamma \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=i\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix}$$

- To solve the coupled differential equations for $a(t)$ and $b(t)$, first find the eigenstates of the Hamiltonian (the K_L and K_S) and then transform into this basis. The eigenvalue equation is:

$$\begin{pmatrix} M-\frac{1}{2}i\Gamma & M_{12}-\frac{1}{2}i\Gamma_{12} \\ M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*} & M-\frac{1}{2}i\Gamma \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

-Which has non-trivial solutions for

$$|\mathbf{H}-\lambda I|=0$$

$$\Rightarrow\left(M-\frac{1}{2}i\Gamma-\lambda\right)^{2}-\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)=0$$

with eigenvalues

$$\lambda=M-\frac{1}{2}i\Gamma\pm\sqrt{\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)}$$

-The eigenstates can be obtained by substituting back into (A5)

$$\left(M-\tfrac{1}{2}i\Gamma\right)x_1+\left(M_{12}-\tfrac{1}{2}i\Gamma_{12}\right)=\left(M-\tfrac{1}{2}i\Gamma\pm\sqrt{\left(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*\right)\left(M_{12}-\tfrac{1}{2}i\Gamma_{12}\right)}\right)x_1$$

$$\Rightarrow\quad\frac{x_2}{x_1}=\pm\sqrt{\frac{M_{12}^*-\frac{1}{2}i\Gamma_{12}^*}{M_{12}-\frac{1}{2}i\Gamma_{12}}}$$

⋆ Define

$$\eta=\sqrt{\frac{M_{12}^*-\frac{1}{2}i\Gamma_{12}^*}{M_{12}-\frac{1}{2}i\Gamma_{12}}}$$

- Hence the normalised eigenstates are

$$|K_\pm\rangle=\frac{1}{\sqrt{1+|\eta|^2}}\begin{pmatrix}1\\ \pm\eta\end{pmatrix}=\frac{1}{\sqrt{1+|\eta|^2}}\left(\left|K^0\right\rangle\pm\eta\left|\bar{K}^0\right\rangle\right)$$

⋆ Note, in the limit where M_{12},Γ_{12} are real, the eigenstates correspond to the CP eigenstates K_1 and K_2. Hence we can identify the general eigenstates as as the long and short lived neutral kaons:

$$|K_L\rangle=\frac{1}{\sqrt{1+|\eta|^2}}\left(\left|K^0\right\rangle+\eta\left|\bar{K}^0\right\rangle\right)\qquad|K_S\rangle=\frac{1}{\sqrt{1+|\eta|^2}}\left(\left|K^0\right\rangle-\eta\left|\bar{K}^0\right\rangle\right)$$

Substituting these states back into (A2):

$$\begin{aligned}|\psi(t)\rangle &= a(t)\left|K^0\right\rangle + b(t)\left|\bar{K}^0\right\rangle \\ &= \sqrt{1+|\eta|^2}\left[\frac{a(t)}{2}(K_L+K_S) + \frac{b(t)}{2\eta}(K_L-K_S)\right] \\ &= \sqrt{1+|\eta|^2}\left[\left(\frac{a(t)}{2}+\frac{b(t)}{2\eta}\right)K_L + \left(\frac{a(t)}{2}-\frac{b(t)}{2\eta}\right)K_S\right] \\ &= \frac{\sqrt{1+|\eta|^2}}{2}\left[a_L(t)K_L + a_S(t)K_S\right]\end{aligned}$$

with

$$a_L(t) \equiv a(t) + \frac{b(t)}{\eta} \quad a_S(t) \equiv a(t) - \frac{b(t)}{\eta}$$

- Now consider the time evolution of $a_L(t)$

$$i\frac{\partial a_L}{\partial t} = i\frac{\partial a}{\partial t} + \frac{i}{\eta}\frac{\partial b}{\partial t}$$

$\star$ Which can be evaluated using (A4) for the time evolution of $a(t)$ and $b(t)$:

$$
\begin{aligned}
i\frac{\partial a_L}{\partial t} &= \left[\left(M-\frac{1}{2}i\Gamma_{12}\right)a+\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)b\right]+\frac{1}{\eta}\left[\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)a+\left(M-\frac{1}{2}i\Gamma\right)b\right] \\
&= \left(M-\frac{1}{2}i\Gamma\right)\left(a+\frac{b}{\eta}\right)+\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)b+\frac{1}{\eta}\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)a \\
&= \left(M-\frac{1}{2}i\Gamma\right)a_L+\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)b+\left(\sqrt{\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)}\right)a \\
&= \left(M-\frac{1}{2}i\Gamma\right)a_L+\left(\sqrt{\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)}\right)\left(a+\frac{b}{\eta}\right) \\
&= \left(M-\frac{1}{2}i\Gamma\right)a_L+\left(\sqrt{\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)}\right)a_L \\
&= \left(m_L-\frac{1}{2}i\Gamma_L\right)a_L
\end{aligned}
$$

⋆ Hence:

$$
i\frac{\partial a_L}{\partial t} = \left(m_L-\frac{1}{2}i\Gamma_L\right)a_L
$$

with $m_L = M + \mathrm{Re}\left\{\sqrt{\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)}\right\}$

⋆ Following the same procedure obtain:

$$i\frac{\partial a_S}{\partial t} = \left(m_S - \frac{1}{2}i\Gamma_S\right)a_S$$

with $m_S = M - \Re\left\{\sqrt{\left(M_{12}^* - \frac{1}{2}i\Gamma_{12}^*\right)\left(M_{12} - \frac{1}{2}i\Gamma_{12}\right)}\right\}$

and $\quad \Gamma_S = \Gamma + 2\Im\left\{\sqrt{\left(M_{12}^* - \frac{1}{2}i\Gamma_{12}^*\right)\left(M_{12} - \frac{1}{2}i\Gamma_{12}\right)}\right\}$

⋆ In matrix notation we have

⋆ Solving we obtain

$$\begin{pmatrix} M_L - \frac{1}{2}i\Gamma_L & 0 \\ 0 & M_S - \frac{1}{2}i\Gamma_S \end{pmatrix}\begin{pmatrix} a_L \\ a_S \end{pmatrix} = i\frac{\partial}{\partial t}\begin{pmatrix} a_L \\ a_S \end{pmatrix}$$

$$a_L(t) \propto e^{-im_L t - \Gamma_L t/2} \quad a_S(t) \propto e^{-im_S t - \Gamma_S t/2}$$

⋆ Hence in terms of the $\mathrm{K_L}$ and $\mathrm{K_S}$ basis the states propagate as independent particles with definite masses and lifetimes (the mass eigenstates). The time evolution of the neutral kaon system can be written

$$|\psi(t)\rangle = A_L e^{-im_L t - \Gamma_L t/2}|K_L\rangle + A_S e^{-im_S t - \Gamma_S t/2}|K_S\rangle$$

where A_L and A_S are constants

Appendix XX: CP Violation : $\pi\pi$ decays

⋆ Consider the development of the $K^0 - \bar{K}^0$ system now including CP violation

⋆ Repeat previous derivation using

$$|K_S\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_1\rangle + \varepsilon\,|K_2\rangle\right] \quad |K_L\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_2\rangle + \varepsilon\,|K_1\rangle\right]$$

-Writing the CP eigenstates in terms of $K^0, \bar{K}^0$

$$|K_L\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)\,|K_0\rangle + (1-\varepsilon)\left|\bar{K}^0\right\rangle\right]$$

$$|K_S\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)\,|K_0\rangle - (1-\varepsilon)\left|\bar{K}^0\right\rangle\right]$$

- Inverting these expressions obtain

$$\left|K^0\right\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1+\varepsilon}\left(|K_L\rangle + |K_S\rangle\right) \quad \left|\bar{K}^0\right\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1-\varepsilon}\left(|K_L\rangle - |K_S\rangle\right)\,|$$

-Hence a state that was produced as a K^0 evolves with time as:

$$|\psi(t)\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1+\varepsilon}\left(\theta_L(t)\,|K_L\rangle + \theta_S(t)\,|K_S\rangle\right)$$

where as before $\theta_S(t) = e^{-\left(im_S+\frac{\Gamma_S}{2}\right)t}$ and $\theta_L(t) = e^{-\left(im_L+\frac{\Gamma_L}{2}\right)t}$

-If we are considering the decay rate to $\pi\pi$ need to express the wave-function in terms of the CP eigenstates (remember we are neglecting CP violation in the decay)

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\frac{1}{1+\varepsilon}\left[\left(|K_2\rangle + \varepsilon\,|K_1\rangle\right)\theta_L(t) + \left(|K_1\rangle + \varepsilon\,|K_2\rangle\right)\theta_S(t)\right]$$

$$= \frac{1}{\sqrt{2}}\frac{1}{1+\varepsilon}\left[\left(\theta_S + \varepsilon\theta_L\right)|K_1\rangle + \left(\theta_L + \varepsilon\theta_S\right)|K_2\rangle\right]$$

CP Eigenstates

-Two pion decays occur with $\mathrm{CP} = +1$ and therefore arise from decay of the $\mathrm{CP} = +1$ kaon eigenstate, i.e. K_1

$$\Gamma\left(K^0_{t=0} \to \pi\pi\right) \propto |\langle K_1 \mid \psi(t)\rangle|^2 = \frac{1}{2}\left|\frac{1}{1+\varepsilon}\right|^2 |\theta_S + \varepsilon\theta_L|^2$$

- Since $|\varepsilon| \ll 1$

$$\left|\frac{1}{1+\varepsilon}\right|^2 = \frac{1}{(1+\varepsilon^*)(1+\varepsilon)} \approx \frac{1}{1+2\Re\{\varepsilon\}} \approx 1 - 2\Re\{\varepsilon\}$$

- Now evaluate the $|\theta_S + \varepsilon\theta_L|^2$ term again using

$$|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\Re\left(z_1 z_2^*\right)$$

Not examinable

$$
\begin{aligned}
|\theta_S + \varepsilon\theta_L|^2 &= \left| e^{-im_S t - \frac{\Gamma_S}{2} t} + \varepsilon e^{-im_L t - \frac{\Gamma_L}{2} t} \right|^2 \\
&= e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2\,\mathrm{Re}\left\{ e^{-im_S t - \frac{\Gamma_S}{2} t} \cdot \varepsilon^* e^{+im_L t - \frac{\Gamma_L}{2} t} \right\}
\end{aligned}
$$

-Writing $\varepsilon = |\varepsilon| e^{i\phi}$

$$
\begin{aligned}
|\theta_S + \varepsilon\theta_L|^2 &= e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2}\,\mathrm{Re}\left\{ e^{i(m_L - m_S)t - \phi} \right\} \\
&= e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2} \cos(\Delta m.t - \phi)
\end{aligned}
$$

-Putting this together we obtain:

$$
\Gamma\left(K^0_{t=0} \to \pi\pi\right) = \frac{1}{2}(1 - 2\,\mathrm{Re}\{\varepsilon\}) N_{\pi\pi} \left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2} \cos(\Delta m.t - \phi) \right]
$$

Short lifetime

component

$\mathrm{K_S} \to \pi\pi$

CP violating long lifetime component KL¦p

Interference term

-In exactly the same manner obtain for a beam which was produced as $\bar{K}^0$

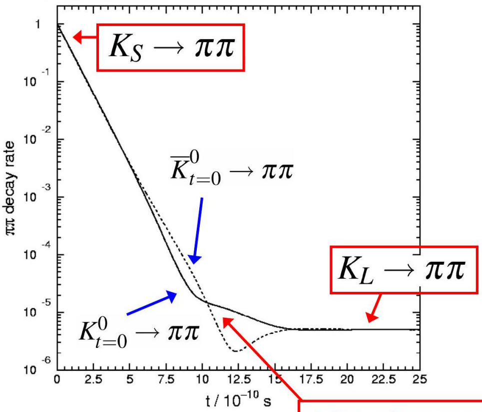
Not examinable

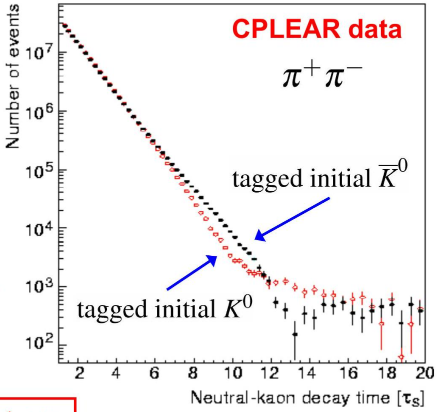
Not examinable

$$\Gamma\left(K^0_{t=0}\to\pi\pi\right)\to\frac{1}{2}(1-2\,\mathrm{Re}\{\varepsilon\})N_{\pi\pi}\cdot|\varepsilon|^2e^{-\Gamma_L t}$$

i.e. CP violating $K_L\to\pi\pi$ decays

⋆ Since CPLEAR can identify whether a K^0 or $\bar{K}^0$ was produced, able to measure $\Gamma\left(K^0_{t=0}\to\pi\pi\right)$ and $\Gamma\left(\bar{K}^0_{t=0}\to\pi\pi\right)$





± interference term

$\star$ The CPLEAR data shown previously can be used to measure $\varepsilon = |\varepsilon|e^{i\phi}$ -Define the asymmetry: $A_{+-} = \frac{\Gamma\left(\bar{K}^0_{t=0}\to\pi\pi\right)-\Gamma\left(K^0_{t=0}\to\pi\pi\right)}{\Gamma\left(\bar{K}^0_{t=0}\to\pi\pi\right)+\Gamma\left(K^0_{t=0}\to\pi\pi\right)}$

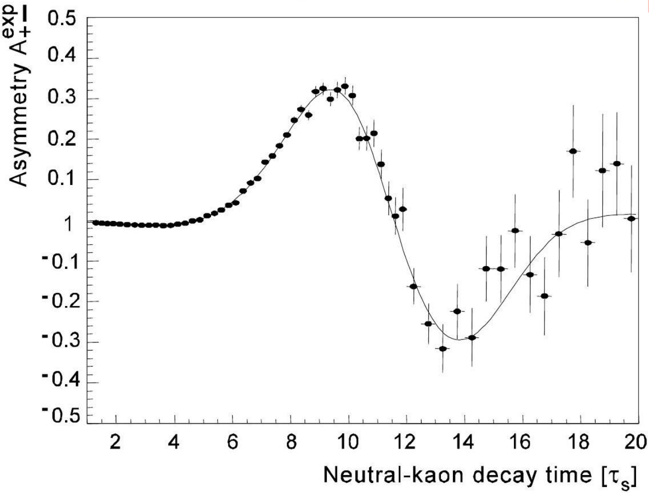
-Using expressions on page 443

$$A_{+-} = \frac{4\mathfrak{R}\{\varepsilon\}\left[e^{-\Gamma_S t}+|\varepsilon|^2e^{-\Gamma_L t}\right]-4|\varepsilon|e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t-\phi)}{2\left[e^{-\Gamma_S t}+|\varepsilon|^2e^{-\Gamma_L t}\right]-\underbrace{8\mathfrak{R}\{\varepsilon\}|\varepsilon|e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t-\phi)}_{\substack{\propto\,|\varepsilon|\mathfrak{R}\{\varepsilon\}\text{ i.e. two small quantities}\\ \text{can safely be neglected}}}}$$

$$A_{+-} \approx \frac{2\mathfrak{R}\{\varepsilon\}\left[e^{-\Gamma_S t}+|\varepsilon|^2e^{-\Gamma_L t}\right]-2|\varepsilon|e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t-\phi)}{e^{-\Gamma_S t}+|\varepsilon|^2e^{-\Gamma_L t}}$$

$$= 2\mathfrak{R}\{\varepsilon\} - \frac{2|\varepsilon|e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t-\phi)}{e^{-\Gamma_S t}+|\varepsilon|^2e^{-\Gamma_L t}}$$

$$= 2\mathfrak{R}\{\varepsilon\} - \frac{2|\varepsilon|e^{(\Gamma_S-\Gamma_L)t/2}\cos(\Delta m.t-\phi)}{1+|\varepsilon|^2e^{(\Gamma_S-\Gamma_L)t}}$$



Best fit to the data:

$$|\varepsilon| = (2.264 \pm 0.035) \times 10^{-3}$$
$$\phi = (43.19 \pm 0.73)^\circ$$

Appendix XXI: CP Violation via Mixing

- A full description of the SM origin of CP violation in the kaon system is beyond the level of this course, nevertheless, the relation to the box diagrams is illustrated below

⋆ The K-long and K-short wave-functions depend on η

$$|K_L\rangle = \frac{1}{\sqrt{1+|\eta|^2}}\left(\left|K^0\right\rangle + \eta\left|\bar{K}^0\right\rangle\right) ||K_S\Big\rangle = \frac{1}{\sqrt{1+|\eta|^2}}\left(\left|K^0\right\rangle - \eta\left|\bar{K}^0\right\rangle\right)$$

$$\text{with} \qquad \eta = \sqrt{\frac{M_{12}^* - \frac{1}{2}i\Gamma_{12}^*}{M_{12} - \frac{1}{2}i\Gamma_{12}}}$$

⋆ If $\quad M_{12}^* = M_{12}; \quad \Gamma_{12}^* = \Gamma_{12}$ then the K-long and K-short correspond to the CP eigenstates K_1 and K_2

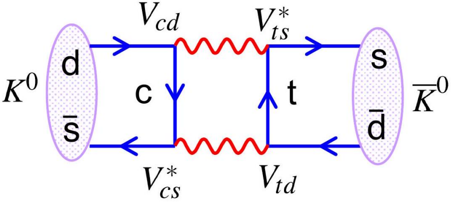
-CP violation is therefore associated with imaginary off-diagonal mass and decay elements for the neutral kaon system

-Experimentally, CP violation is small and $\eta \approx 1$

-Define: $\varepsilon = \frac{1-\eta}{1+\eta} \quad \Rightarrow \quad \eta = \frac{1-\varepsilon}{1+\varepsilon}$

- Consider the mixing term M_{12} which arises from the sum over all possible intermediate states in the mixing box diagrams

e.g.



- In the Standard Model, CP violation is associated

with the imaginary components of the CKM matrix, and it can be shown that mixing leads to CP violation with

$$|\varepsilon| \propto \Im\left\{M_{12}\right\}$$

-The differences in masses of the mass eigenstates can be shown to be:

$$\Delta m_K = m_{K_L} - m_{K_S} \approx \sum_{q,q'} \frac{G_{\mathrm{F}}^2}{3\pi^2} f_K^2 m_K \left| V_{qd} V_{qs}^* V_{q'd} V_{q's}^* \right| m_q m_{q'}$$

where q and q' are the quarks in the loops and f_K is a constant

- In terms of the small parameter ε

$$|K_L\rangle = \frac{1}{2\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)\left|K^0\right\rangle + (1-\varepsilon)\left|\bar{K}^0\right\rangle\right]$$

$$|K_S\rangle = \frac{1}{2\sqrt{1+|\varepsilon|^2}}\left[(1-\varepsilon)\left|K^0\right\rangle + (1+\varepsilon)\left|\bar{K}^0\right\rangle\right]$$

- If epsilon is non-zero we have CP violation in the neutral kaon system

$$\text{Writing} \quad \eta = \sqrt{\frac{M_{12}^* - \frac{1}{2}i\Gamma_{12}^*}{M_{12} - \frac{1}{2}i\Gamma_{12}}} = \sqrt{\frac{z^*}{z}} \quad \text{and} \quad z = ae^{i\phi}$$

$$\text{gives} \quad \eta = e^{-i\phi}$$

- From which we can find an expression for ε

$$\varepsilon \cdot \varepsilon^* = \frac{1-e^{-i\phi}}{1+e^{-i\phi}} \cdot \frac{1-e^{+i\phi}}{1+e^{i\phi}} = \frac{2-\cos\phi}{2+\cos\phi} = \tan^2\frac{\phi}{2}$$

$$|\varepsilon| = \left|\tan\frac{\phi}{2}\right|$$

Experimentally we know ε is small, hence ϕ is small

$$|\varepsilon| \approx \frac{1}{2}\phi = \frac{1}{2}\arg z \approx \frac{1}{2}\frac{\Im\left\{M_{12} - \frac{1}{2}i\Gamma_{12}\right\}}{\left|M_{12} - \frac{1}{2}i\Gamma_{12}\right|}$$

Appendix XXII: Time Reversal Violation

-Previously in equations (142) and (143) we obtained expressions for strangeness oscillations in the absence of CP violation, e.g.:

$$\Gamma\left(K^0_{t=0}\to K^0\right)=\frac{1}{4}\left[e^{-\Gamma_S t}+e^{-\Gamma_L t}+2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

-This analysis can be extended to include the effects of CP violation to give the following rates (see Question 24):

$$\begin{aligned}
\Gamma\left(K^0_{t=0}\to K^0\right)&\propto\frac{1}{4}\left[e^{-\Gamma_S t}+e^{-\Gamma_L t}+2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]\\
\Gamma\left(\bar{K}^0_{t=0}\to \bar{K}^0\right)&\propto\frac{1}{4}\left[e^{-\Gamma_S t}+e^{-\Gamma_L t}+2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]\\
\Gamma\left(\bar{K}^0_{t=0}\to K^0\right)&\propto\frac{1}{4}(1+4\,\mathrm{Re}\{\varepsilon\})\left[e^{-\Gamma_S t}+e^{-\Gamma_L t}-2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]\\
\Gamma\left(K^0_{t=0}\to \bar{K}^0\right)&\propto\frac{1}{4}(1-4\,\mathrm{Re}\{\varepsilon\})\left[e^{-\Gamma_S t}+e^{-\Gamma_L t}-2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]
\end{aligned}$$

⋆ Including the effects of CP violation find that

$$\Gamma\left(\bar{K}^0_{t=0}\to K^0\right)\neq\Gamma\left(K^0_{t=0}\to \bar{K}^0\right)\qquad\text{Violation of time reversal symmetry !}$$

- No surprise, as CPT is conserved, CP violation implies T violation

Appendix XXIII: Non-relativistic Breit-Wigner

For energies close to the peak of the resonance, can write $\sqrt{s} = m_Z + \Delta$

$$s = m_Z^2 + 2m_Z\Delta + \Delta^2 \approx m_Z^2 + 2m_Z\Delta \quad \text{for } \Delta \ll m_Z$$

and with this approximation

$$\left(s - m_Z^2\right)^2 + m_Z^2\Gamma_Z^2 \approx (2m_Z\Delta)^2 + m_Z^2\Gamma_Z^2 = 4m_Z^2\left(\Delta + \frac{1}{4}\Gamma_Z^2\right) = 4m_Z^2\left[\left(\sqrt{s} - m_Z\right)^2 + \frac{1}{4}\Gamma_Z^2\right]$$

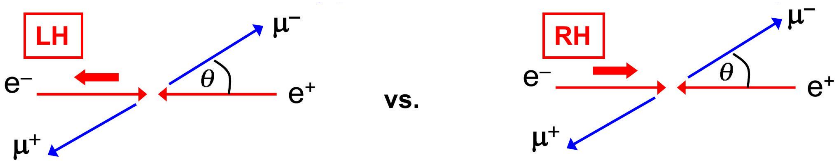
so that the relativistic Breith-Wigner formula of (165) can be approximated $\sigma\left(e^+e^- \to Z \to f\bar{f}\right) \approx \frac{3\pi}{m_Z^4}\frac{s}{\left(\sqrt{s}-m_Z\right)^2+\frac{1}{4}\Gamma_Z^2}\Gamma_e\Gamma_f$ which can be written:

$$\boxed{\sigma(E) = \frac{g\lambda_e^2}{4\pi}\frac{\Gamma_i\Gamma_f}{(E - E_0)^2 + \frac{1}{4}\Gamma^2}}.$$

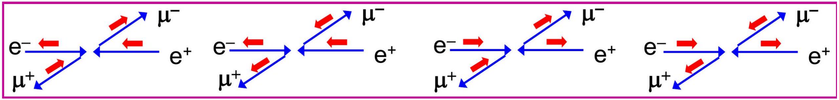
Γ_i and Γ_f are the partial decay widths of the initial and final state particles.
E and E_0 are the centre-of-mass energy and the energy of the resonance.
$g = \frac{(2J_Z+1)}{(2S_e+1)(2S_e+1)}$ is the spin counting factor $g = \frac{3}{2\times 2}$.
$\lambda_e = \frac{2\pi}{E}$ is the Compton wavelength (natural units) in the C.o.M of either initial particle.
The boxed equation is the non-relativistic form of the Breit-Wigner distribution first encountered in the Part II Particle and Nuclear Physics course (e.g. page 36 in Handout 2, "Kinematics, Decays and Recations", of the Part II course given in 2023).

Appendix XXIV: Left-Right Asymmetry, A_{LR}

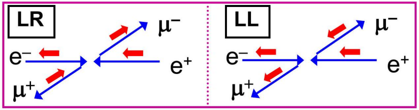
- At an $\mathbf{e}^+\mathbf{e}^-$ linear collider it is possible to produce polarized electron beams. E.g. Stanford Linear Collider (SLC; California), 1989-2000.
- At such a collider one could measure cross section for any process for LH and RH electrons separately

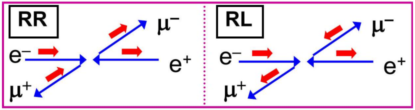


- At LEP one usually measured the total cross section: a sum of 4 helicity combinations:



- In contrast, at the SLC, tuning the polarization of the electron beams made it possble to measure cross sections separately for LH / RH electrons





Not examinable

- Define cross section asymmetry:

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R}$$

  where $\sigma_L = \sigma_{LL} + \sigma_{LR}$ and where $\sigma_R = \sigma_{RL} + \sigma_{RR}$ using the notation of page 456.
- Integrating the expressions for the differential cross sections on the same page gives:

$$\sigma_{LL} \propto \left(c_L^e\right)^2 \left(c_L^\mu\right)^2 \quad \sigma_{LR} \propto \left(c_L^e\right)^2 \left(c_R^\mu\right)^2 \quad \sigma_{RL} \propto \left(c_R^e\right)^2 \left(c_L^\mu\right)^2 \quad \sigma_{RR} \propto \left(c_R^e\right)^2 \left(c_R^\mu\right)^2$$

  and so

$$\sigma_L \propto \left(c_L^e\right)^2 \left[\left(c_L^\mu\right)^2 + \left(c_R^\mu\right)^2\right] \quad \sigma_R \propto \left(c_R^e\right)^2 \left[\left(c_L^\mu\right)^2 + \left(c_R^\mu\right)^2\right]$$

  and

$$A_{LR} = \frac{\left(c_L^e\right)^2 - \left(c_R^e\right)^2}{\left(c_L^e\right)^2 + \left(c_R^e\right)^2} = A_e$$

- Hence the Left-Right asymmetry for any cross section depends only on the couplings of the initial state electrons.
- Compare this to the Forward Backward asymmetry (see page 469) which depends on the couplings of the initial state electrons and the final state particles (muons, *etc*).

Not examinable