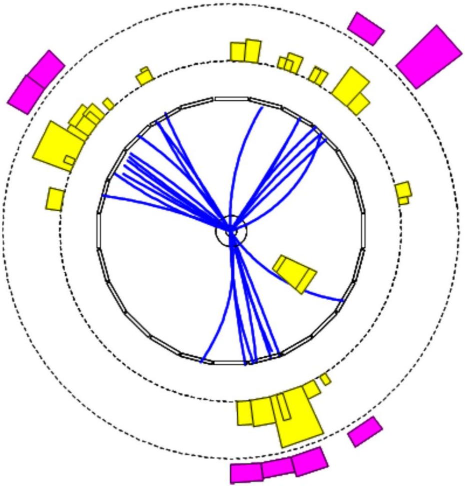
Dr C.G. Lester, 2023



H8: Quantum Chromodynamics

The Local Gauge Principle

(see Appendices IX, X, XIII and XIV for more non-examinable details)

- All the interactions between fermions and spin-1 bosons in the SM are specified by the principle of LOCAL GAUGE INVARIANCE.
- To arrive at QED, we **require** physics to be invariant under the local phase transformation of particle wave-functions:

$$\psi \to \psi' = e^{iq\chi(x)}\psi.$$

- Note that the change of phase depends on the space-time coordinate: $\chi(t, \vec{x})$.

Under this transformation the Dirac Equation transforms as

$$\boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0} \longrightarrow \boxed{i\gamma^\mu \left(\partial_\mu + iq\partial_\mu \chi\right)\psi - m\psi = 0}.$$

The above is is bad news! We want everything physical (and thus the Dirac equation too) to be invariant under local gauge transformations.

- We are FORCED to introduce a massless gauge boson, A_μ, and the Dirac equation has to be modified to include this new field:

$$\boxed{i\gamma^\mu \left(\partial_\mu + iqA_\mu\right)\psi - m\psi = 0}.$$

- The modified Dirac equation is invariant under local phase transformations if:

$$\boxed{A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi}.$$

The Local Gauge Group ($U(1)$ for QED) specifies the Gauge Boson Interactions and thus the Feynman Rules

Thus the principle of invariance under local phase transformations completely specifies the interaction between a fermion and the gauge boson (in this case the photon) once the gauge group is chosen.

For example, is from the iqA_μ term (which was created to keep everything gauge invariant) that the $ie\gamma^\mu$ vertex factor of QED's Feynman rules can be derived.

The local phase transformation of QED is a unitary $U(1)$ transformation:

$$\psi \to \psi' = \hat{U}\psi \quad \text{i.e.} \quad \psi \to \psi' = e^{iq\chi(x)}\psi \quad \text{with} \quad U^\dagger U = 1$$

We will now extend this idea ...

From QED ($U(1)$) to QCD ($SU(3)$-colour)

- Suppose there is another fundamental symmetry of the universe, say 'invariance under SU(3) local phase transformations'
- i.e. require invariance under $\boxed{\psi \to \psi' = e^{ig\vec{\lambda}\cdot\vec{\theta}(x)}\psi}$ where $\vec{\lambda}$ are the eight 3×3 Gell-Mann matrices introduced in Handout 7, and where $\vec{\theta}(x)$ are 8 functions taking different values at each point in space-time.
- **Unavoidably**, the wave function is now a vector in colour space: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$.

The QCD Lagrangian is created by requiring invariance under local $SU(3)$ gauge tfm.

From that Lagrangian, the Feynman Rules and all other properties may be derived:

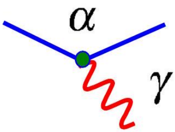
- the interaction vertex is: $-\frac{1}{2} i g_s \lambda^a \gamma^\mu$,
- there are 8 massless gauge bosons – the gluons – one for each λ, and
- there are 3 and 4 gluon vertices but no others. (The details are beyond the level of this course. See Gauge Field Theory course in Lent!)

ASIDE: Our $SU(3)$-colour gauge tfm rotates states in colour space about and angle and an axis which is different at every space-time point. Why might this be desirable in a theory with multiple observers?

Colour in QCD

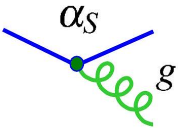
The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with three conserved 'colour' charges.

In QED:



- the electron carries one unit of charge $-e$,
- the anti-electron carries one unit of anti-charge $+e$,
- the force is mediated by a massless "gauge boson" - the photon.

In QCD:



- quarks carry colour charge: r, g, b,
- anti-quarks carry anti-charge: $\bar{r}, \bar{g}, \bar{b}$,
- The force is mediated by massless gluons.

The strong interaction is invariant under rotations in colour space

$$r \leftrightarrow b; r \leftrightarrow g; b \longleftrightarrow g.$$

$SU(3)$-colour symmetry is exact

This $SU(3)$-colour symmstry is an exact symmetry, unlike the approximate *uds* $SU(3)$-flavour symmetry discussed previously.

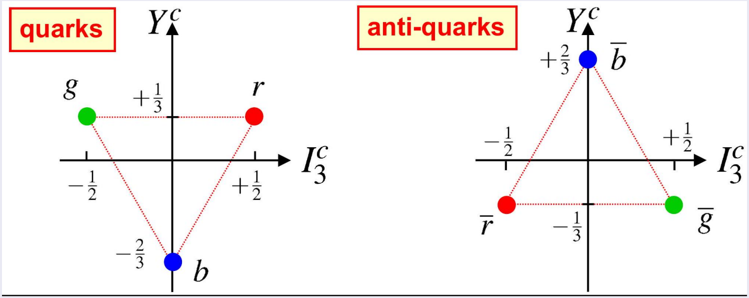
Represent $SU(3)$ colour states r, g, b by:

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Analogous to labelling u, d, s flavour states by I_3 and Y, Colour states can be labelled by two quantum numbers:

- I_3^c colour isospin, and
- Y^c colour hypercharge.

Each quark (anti-quark) can have the following colour quantum numbers:



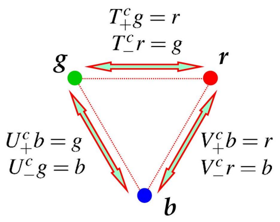
The Colour Confinement Hypothesis

The Colour Confinement Hypothesis is that 'all free particles have colour singlet wavefunctions'.

- Sometimes abbreviated to 'all free particles are colourless'.
- It is suspected that one day this hypothesis will be shown to be derivable from the other principles of the Standard Model.
- If true, then we will never observe free quarks.

We can re-interpret our $SU(3)$ flavour results via an $SU(3)$ colour lens – $(u, d, s) \to (r, g, b)$, but this time we can treat the symmetry as exact rather than approximate!

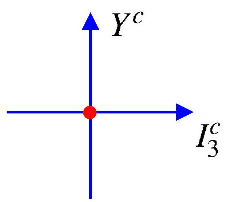
Just as for *uds* flavour symmetry can define colour ladder operators:



Colour Singlets

Reminder of what a colour singlet is:

- colour singlet states have zero colour quantum numbers $I_3^c = 0$, $Y^c = 0$,
- colour singlet states are invariant under SU(3) colour transformations, and
- all ladder operators $T_\pm$, $U_\pm$, $V_\pm$ yield zero when applied to a colour singlet state.

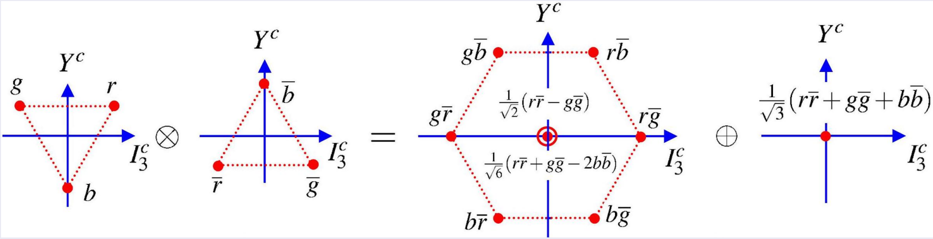


It is not sufficient to have just $I_3^c = 0$, $Y^c = 0$.

This alone does not signal a colour singlet state.

Meson (i.e. $q\bar{q}$) Colour Wave-functions

The combination of colour with anti-colour is mathematically identical to construction of meson wave-functions with *uds* flavour symmetry.



i.e. we get a COLOURED OCTET and a COLOURLESS SINGLET.

- Colour confinement implies that hadrons only exist in colour singlet states so the colour singlet wave-function for mesons is:

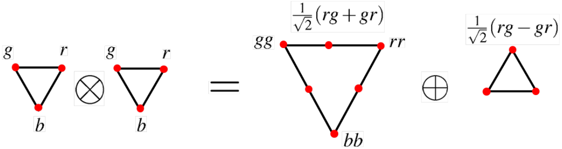
$$\psi_c^{q\bar{q}} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b}).$$

$qq\bar{q}$ bound states do not exist in nature . . .

. . . because there are no $qq\bar{q}$ state with $Y^c = 0; I_3^c = 0$, let alone a colour singlet!

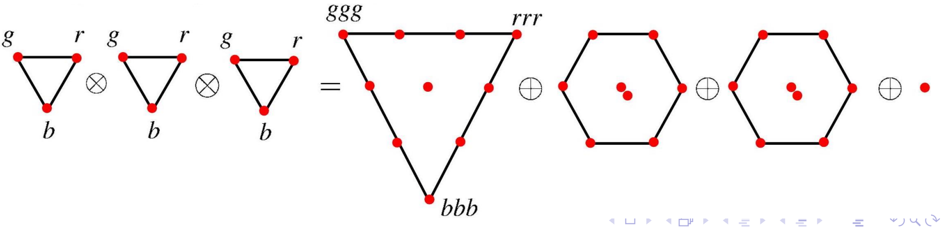
Baryon Colour Wave-Function

- Do qq bound states exist? This is equivalent to asking whether it possible to form a colour SINGLET from two colour TRIPLETs.
- Following the discussion of construction of baryon wave-functions in $SU(3)$-flavour symmetry obtain



- No qq colour SINGLET state!
- Colour confinement $\implies$ bound states of qq do not exist!

BUT: combination of three quarks (three colour TRIPLETs) gives a colour SINGLET state (pages 285-287).



The COLOUR SINGLET wave-function is:

$$\boxed{\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)}.$$

Make absolutely sure that this is a COLOUR SINGLET:

- It has $I_3^c = 0$, $Y^c = 0$: a necessary but not sufficient condition.
- Apply ladder operators, e.g. T_+ (recall $T_+ g = r$)

$$T_+\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rrb - rbr + rbr - rrb + brr - brr) = 0.$$

- Similarly $T_-\psi_c^{qqq} = 0$; $V_\pm\psi_c^{qqq} = 0$; $U_\pm\psi_c^{qqq} = 0$.

ϕ_c^{qqq} definitely is a colourless singlet!

- qqq bound states can exist, and
- The qqq colour singlet wave-function is anti-symmetric.

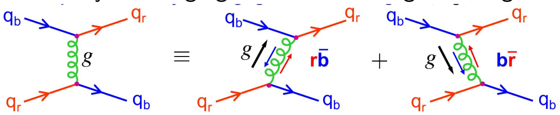
The possible hadrons (i.e. the possible colour singlet states) are therefore:

- $q\bar{q}, qqq$: **Mesons** and **Baryons**
- $q\bar{q}q\bar{q}, qqqq\bar{q}, \ldots$: i.e. **Tetraquarks**, **Pentaquarks**, ...

Until 2015, all discovered hadrons had been mesons or baryons. The first pentaquarks were discovered in 2015, 2019 and 2022 by LHCb. No tetraquark has yet been found.

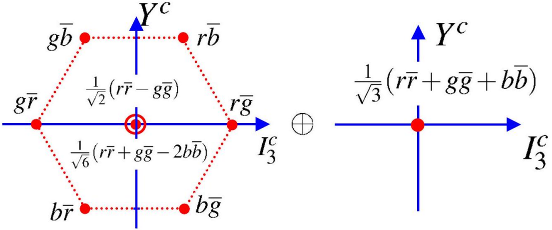
Gluons

* In QCD quarks interact by exchanging virtual massless gluons, e.g.:



- Gluons carry colour and anti-colour, e.g.:

$\star$ Gluon colour wave-functions (colour + anti-colour) are the same as those obtained for mesons (also colour + anti-colour) where we saw an OCTET and a COLOURLESS SINGLET:



So we might expect 9 physical gluons:

- OCTET: $r\bar{g}, r\bar{b}, g\bar{r}, g\bar{b}, b\bar{r}, b\bar{g}, \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g}), \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$
- SINGLET: $\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$

But, colour confinement hypothesis says that:

Only colour singlet states can exist as free particles $\Longrightarrow$ Colour singlet gluon would be unconfined $\Longrightarrow$ it would behave like a strongly interacting photon $\Longrightarrow$ **infinite-range strong force**.

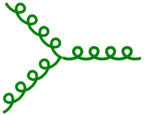
Empirically, the strong force is short range and therefore know that the physical gluons are confined. The colour singlet state does not exist in nature !

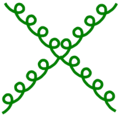
- The strong interaction arises from a fundamental $SU(3)$ symmetry, and the gluons arise from the generators of the symmetry group (the Gell-Mann λ matrices). There are 8 such matrices and so 8 gluons.
- If nature had 'chosen' a $U(3)$ symmetry, we would have 9 gluons. The additional gluon would be the colour singlet state and QCD would be an unconfined long-range force.
- The gauge symmetry determines the exact nature of the interaction and thus the Feynman Rules.

Gluon-Gluon Interactions

- In QED the photon does not carry the charge of the EM interaction (photons are electrically neutral).
- In contrast, in QCD the gluons do carry colour charge $\Longrightarrow$ Gluon Self-Interactions!
- Two new vertices (no QED analogues):

triple-gluon vertex

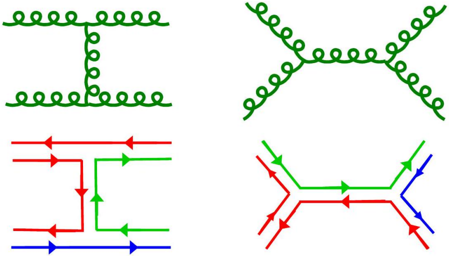




quartic-gluon vertex

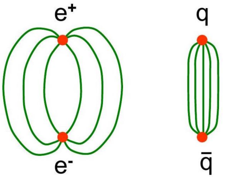
- In addition to quark-quark scattering, therefore can have gluon-gluon scattering:

e.g. possible way of arranging the colour flow:

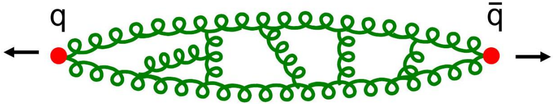


Gluon self-Interactions and Confinement

- Gluon self-interactions are believed to give rise to colour confinement.
- Unlike QCD in QCD "gluon self-interactions squeeze lines of force into a 'flux tube'.



- What happens when try to separate two coloured objects e.g. $q\bar{q}$?



- Form a flux tube of interacting gluons of approximately constant energy density $\sim 1\mathrm{GeV/fm}$

$$\Rightarrow V(r) \sim \lambda r$$

- Require infinite energy to separate coloured objects to infinity
- Coloured quarks and gluons are always confined within colourless states
- In this way QCD provides a plausible explanation of confinement - but not yet proven (although there has been recent progress with Lattice QCD)

Hadronisation and Jets

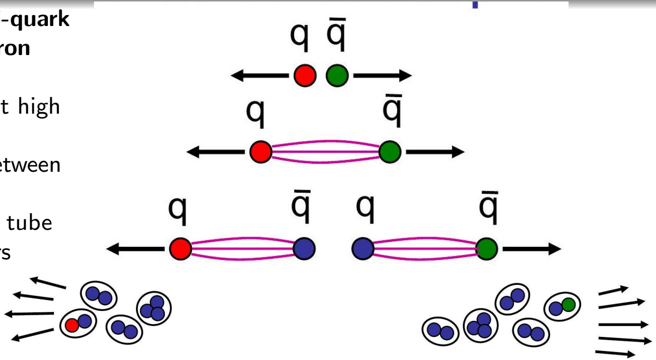
Consider a quark and anti-quark produced in electron positron annihilation:

i) Initially Quarks separate at high velocity

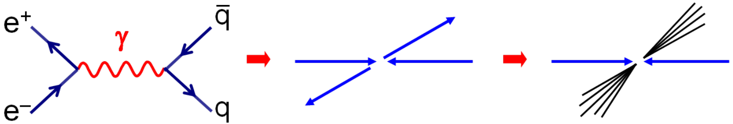
ii) Colour flux tube forms between quarks

iii) Energy stored in the flux tube sufficient to produce $q\bar{q}$ pairs

iv) Process continues until quarks pair up into jets of colourless hadrons

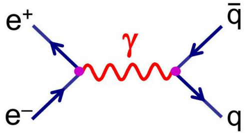


- This process is called hadronisation. It is not (yet) calculable (first principles).
- The main consequence is that at collider experiments quarks and gluons observed as jets of particles.



QCD and Colour in e^+e^- Collisions

e^+e^--colliders are an excellent place to study QCD:

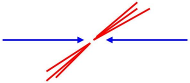


- QED process well-understood.
- No need to know parton structure functions.
- Experimentally very clean - no proton remnants.

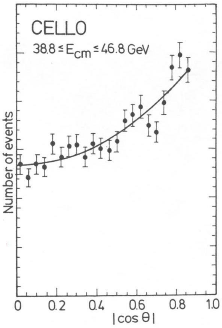
⋆ In Handout 5 obtained expressions for the $e^+e^- \to \mu^+\mu^-$ cross-section:

$$\sigma = \frac{4\pi\alpha^2}{3s} \qquad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right)$$

- In e^+e^- collisions produce all quark flavours for which $\sqrt{s} > 2m_q$
- In general, i.e. unless producing a $q\bar{q}$ bound state, produce jets of hadrons.
- Usually can't tell which jet came from the quark and came from anti-quark.



- Angular distribution of jets $\propto \left(1 + \cos^2\theta\right) \implies$ Quarks are spin-$\frac{1}{2}$.



H.J.Behrend et al., Phys Lett 183B (1987) 400

- Colour is conserved and quarks are produced as $r\bar{r}, g\bar{g}, b\bar{b}$.
- For a single quark flavour and single colour:

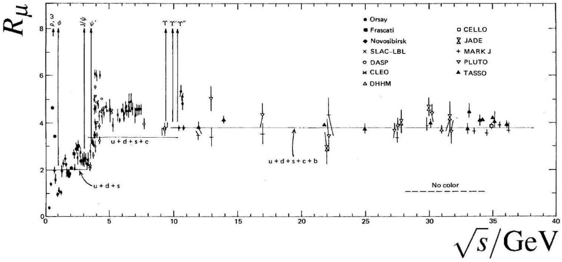
$$\sigma\left(e^+e^- \to q_i\bar{q}_i\right) = \frac{4\pi\alpha^2}{3s}Q_q^2.$$

- Experimentally observe jets of hadrons: (factor 3 is from colour!!)

$$\sigma\left(e^+e^- \to \text{ hadrons }\right) = 3\sum_{u,d,s,\dots}\frac{4\pi\alpha^2}{3s}Q_q^2.$$

- Usual to express as ratio compared to $\sigma\left(e^+e^- \to \mu^+\mu^-\right)$:

$$R_\mu = \frac{\sigma\left(e^+e^- \to \text{ hadrons }\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = 3\sum_{u,d,s,\dots}Q_q^2$$



u,d,s: $R_\mu = 3\times\left(\frac{1}{9}+\frac{4}{9}+\frac{1}{9}\right) = 2$

u,d,s,c: $R_\mu = \frac{10}{3}$

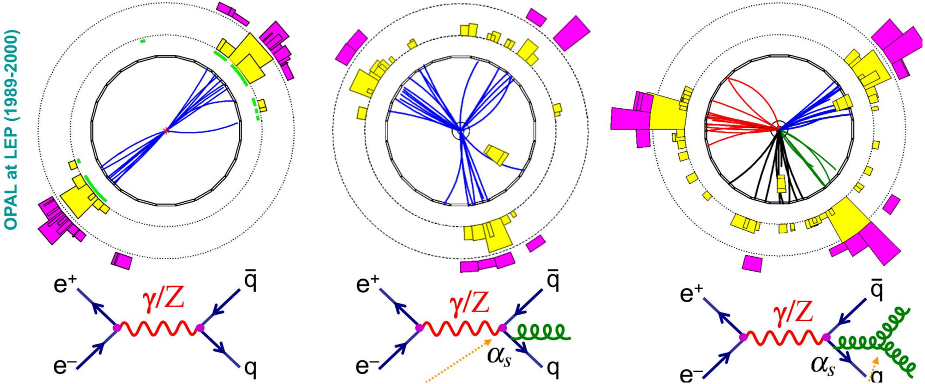
u,d,s,c,b: $R_\mu = \frac{11}{3}$

- Data are consistent with expectation provided there is a factor 3 from colour.

Jet production in e+e-Collisions

e^+e^--colliders are also a good place to study gluons.

$e^+e^- \to q\bar{q} \to 2\text{jets}$ $\quad$ $e^+e^- \to q\bar{q}g \to 3\text{jets}$ $\quad$ $e^+e^- \to q\bar{q}gg \to 4\text{jets}$



Experimentally:

- Three jet rate → measurement of α_s.
- Angular distributions ⇒ gluons are spin-1.
- Four-jet rate and distributions → QCD has an underlying $SU(3)$ symmetry.

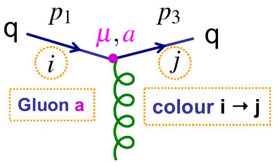
The Quark-Gluon Interaction

- The colour part of the fermion wave-functions are represented by:

$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$.
- The QCD qqg vertex is written:

$$\bar{u}(p_3)\, c_j^\dagger \left\{-\tfrac{1}{2} i g_s \lambda^a \gamma^\mu\right\} c_i\, u(p_1)$$



- Only difference w.r.t. QED is the insertion of the 3×3 SU(3) Gell-Mann matrices.
- Isolating the colour part:

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix}\lambda^a_{1i}\\ \lambda^a_{2i}\\ \lambda^a_{3i}\end{pmatrix} = \lambda^a_{ji}$$

- Hence the fundamental quark-gluon QCD interaction can be written:

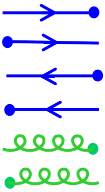
$$\bar{u}(p_3)\, c_j^\dagger \left\{-\tfrac{1}{2} i g_s \lambda^a \gamma^\mu\right\} c_i\, u(p_1) \equiv \boxed{\bar{u}(p_3) \left\{-\tfrac{1}{2} i g_s \lambda^a_{ji} \gamma^\mu\right\} u(p_1)}.$$

Feynman Rules for QCD

- **External Lines**

spin 1/2
- incoming quark: $u(p)$
- outgoing quark: $\overline{u}(p)$
- incoming anti-quark: $\overline{v}(p)$
- outgoing anti-quark: $v(p)$

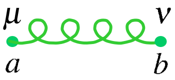
spin 1
- incoming gluon: $\varepsilon^{\mu}(p)$
- outgoing gluon: $\varepsilon^{\mu}(p)^*$



- **Internal Lines (propagators)**

spin 1 gluon

$$\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}$$

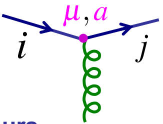


a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark

$$-ig_s\frac{1}{2}\lambda^a_{ji}\gamma^{\mu}$$

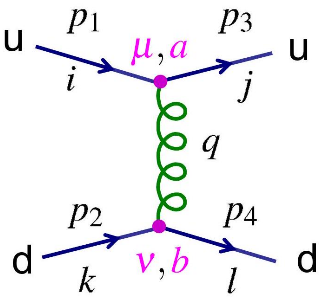


i, j = 1,2,3 are quark colours,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element** $-iM$ = **product of all factors**

Matrix Element for *ud* quark-quark scattering



- The incoming and out-going quark colours are labelled by $i, j, k, l = \{1, 2, 3\}$ (or $\{r, g, b\}$)
- In terms of colour this scattering is

$$ik \to jl.$$

- The 8 different gluons are accounted for by the colour indices $a, b = 1, 2, \ldots, 8$.
- NOTE: the δ-function in the propagator ensures $a = b$, i.e. the gluon "emitted" at a is the same as that "absorbed" at b.

- Applying the Feynman rules:

$$-iM = \left[\bar{u}_u(p_3)\left\{-\frac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\right\}u_u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\bar{u}_d(p_4)\left\{-\frac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\right\}u_d(p_2)\right]$$

where summation over a and b (and μ and v) is implied.

- Summing over a and b using the δ-function gives:

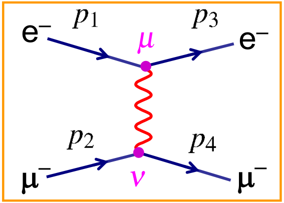
$$M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}\left[\bar{u}_u(p_3)\gamma^\mu u_u(p_1)\right]\left[\bar{u}_d(p_4)\gamma^\nu u_d(p_2)\right].$$

QCD vs QED

QED

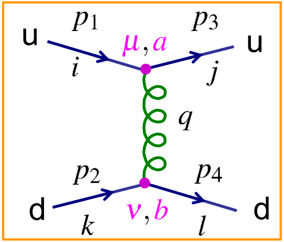
$$-iM = [\overline{u}(p_3)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^\nu u(p_2)]$$

$$M = -e^2\frac{1}{q^2}g_{\mu\nu}[\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\gamma^\nu u(p_2)]$$



QCD

$$M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]$$



★ **QCD Matrix Element = QED Matrix Element with:**

- $e^2 \to g_s^2$ **or equivalently** $\alpha = \frac{e^2}{4\pi} \to \alpha_s = \frac{g_s^2}{4\pi}$

✚ **QCD Matrix Element includes an additional "colour factor"**

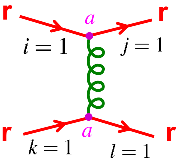
$$C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}$$

Evaluation of QCD Colour Factors

•**QCD colour factors reflect the gluon states that are involved**

$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\bar{g}, g\bar{r}$	$r\bar{b}, b\bar{r}$	$g\bar{b}, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

❶ Configurations involving a single colour



•**Only matrices with non-zero entries in 11 position are involved**

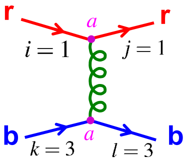
$$C(rr \to rr) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{11}\lambda^a_{11} = \frac{1}{4}(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11})$$

$$= \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3}$$

Similarly find

$$\boxed{C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \frac{1}{3}}$$

❷ Other configurations where quarks don't change colour   e.g. $rb \to rb$



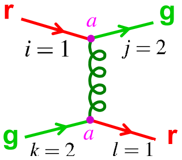
•Only matrices with non-zero entries in **11** and **33** position are involved

$$C(rb \to rb) = \frac{1}{4}\sum_{a=1}^{8} \lambda_{11}^{a}\lambda_{33}^{a} = \frac{1}{4}(\lambda_{11}^{8}\lambda_{33}^{8})$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}.\frac{-2}{\sqrt{3}}\right) = -\frac{1}{6}$$

Similarly $C(rb \to rb) = C(rg \to rg) = C(gr \to gr) = C(gb \to gb) = C(br \to br) = C(bg \to bg) = -\frac{1}{6}$

❸ Configurations where quarks swap colours   e.g. $rg \to gr$



•Only matrices with non-zero entries in **12** and **21** position are involved

Gluons $r\bar{g}$, $g\bar{r}$

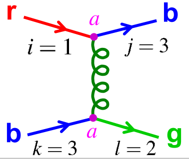
$$C(rg \to gr) = \frac{1}{4}\sum_{a=1}^{8} \lambda_{21}^{a}\lambda_{12}^{a} = \frac{1}{4}(\lambda_{21}^{1}\lambda_{12}^{1} + \lambda_{21}^{2}\lambda_{12}^{2})$$

$$= \frac{1}{4}(i(-i)+1) = \frac{1}{2}$$

$\hat{T}_{+}^{(ij)}\hat{T}_{-}^{(kl)}$

$$C(rb \to br) = C(rg \to gr) = C(gr \to rg) = C(gb \to bg) = C(br \to rb) = C(bg \to gb) = \frac{1}{2}$$

❹ Configurations involving 3 colours   e.g. $rb \to bg$



•Only matrices with non-zero entries in the **13** and **32** position

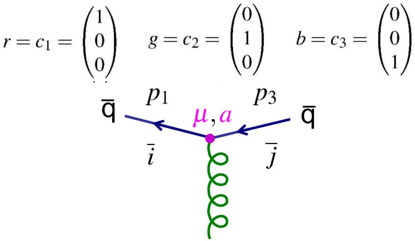
•But none of the λ matrices have non-zero entries in the **13** and **32** positions. Hence the colour factor is zero

★ colour is conserved

Colour Factors : Quarks vs Anti-Quarks

- The QCD qqg vertex was written

$$\bar{u}(p_3)\, c_j^\dagger \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_i\, u(p_1)$$



- Now consider the anti-quark vertex $\bar{q}\bar{q}g$:

$$\bar{v}(p_1)\, c_i^\dagger \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_j\, v(p_3)$$

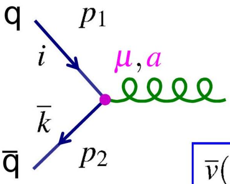
Note that the incoming anti-particle now enters on the LHS of the expression!

The colour part is $c_i^\dagger \lambda^a c_j = c_i^\dagger \begin{pmatrix} \lambda^a_{1j} \\ \lambda^a_{2j} \\ \lambda^a_{3j} \end{pmatrix} = \lambda^a_{ij}$ i.e. indices ij are swapped with respect to the quark case.

- Hence $\boxed{\bar{v}(p_1)\, c_i^\dagger \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_j\, v(p_3) \equiv \bar{v}(p_1) \left\{ -\frac{1}{2} i g_s \lambda^a_{ij} \gamma^\mu \right\} v(p_3)}$.
- Compare with the quark-gluon QCD interaction:

$$\bar{u}(p_3)\, c_j^\dagger \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_i\, u(p_1) \equiv \bar{u}(p_3) \left\{ -\frac{1}{2} i g_s \lambda^a_{ji} \gamma^\mu \right\} u(p_1)$$

Finally we can consider the quark - anti-quark annihilation vertex:

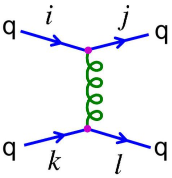


QCD vertex: $\overline{v}(p_2) c_k^\dagger \{-\frac{1}{2} i g_s \lambda^a \gamma^\mu\} c_i u(p_1)$

with $c_k^\dagger \lambda^a c_i = \lambda_{ki}^a$

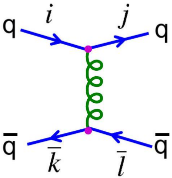
$$\boxed{\overline{v}(p_2) c_k^\dagger \{-\tfrac{1}{2} i g_s \lambda^a \gamma^\mu\} c_i u(p_1) \equiv \overline{v}(p_2) \{-\tfrac{1}{2} i g_s \lambda_{ki}^a \gamma^\mu\} u(p_1)}$$

- Consequently the colour factors for the different diagrams are:



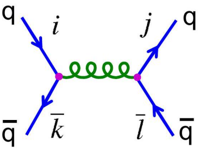
$$C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}$$

e.g.
$$C(rr \to rr) = \tfrac{1}{3}$$
$$C(rg \to rg) = -\tfrac{1}{6}$$
$$C(rg \to gr) = \tfrac{1}{2}$$



$$C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{kl}$$

e.g.
$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = -\tfrac{1}{6}$$
$$C(r\bar{r} \to g\bar{g}) = \tfrac{1}{2}$$



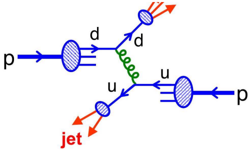
$$C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ki}\lambda^a_{jl}$$

e.g.
$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = \tfrac{1}{2}$$
$$C(r\bar{r} \to g\bar{g}) = -\tfrac{1}{6}$$

Colour index of adjoint spinor comes first.

Quark-Quark Scattering

- Consider the process $u + d \to u + d$ which can occur in the high energy proton-proton scattering
- There are nine possible colour configurations of the colliding quarks which are all equally likely.
- Need to determine the average matrix element which is the sum over all possible colours divided by the number of possible initial colour states



$$\left\langle |M_{fi}|^2 \right\rangle = \frac{1}{3} \cdot \frac{1}{3} \sum_{i,j,k,l=1}^{3} |M_{fi}(ij \to kl)|^2$$

- The colour average matrix element contains the average colour factor

$$\left\langle |C|^2 \right\rangle = \frac{1}{9} \sum_{i,j,k,l=1}^{3} |C(ij \to kl)|^2$$

•For $qq \to qq$

$$\left\langle |C|^2 \right\rangle = \frac{1}{9} \left[3 \times \underbrace{\left(\frac{1}{3}\right)^2}_{rr \to rr, \ldots} + 6 \times \underbrace{\left(-\frac{1}{6}\right)^2}_{rb \to rb, \ldots} + 6 \times \underbrace{\left(\frac{1}{2}\right)^2}_{rb \to br, \ldots} \right] = \frac{2}{9}$$

• Previously derived the Lorentz Invariant cross section for $e^-\mu^- \to e^-\mu^-$ elastic scattering in the ultra-relativistic limit (Handout 6).

QED

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s}\right)^2\right]$$

• For ud → ud in QCD replace $\alpha \to \alpha_s$ and multiply by $\langle |C|^2 \rangle$

QCD

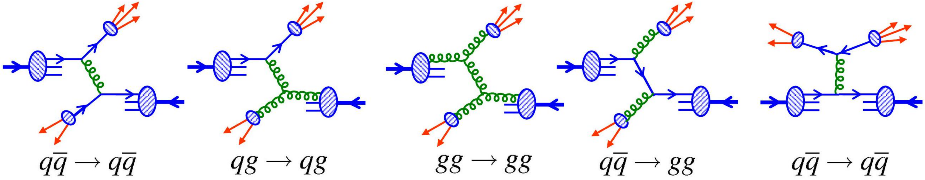
$$\frac{d\sigma}{dq^2} = \frac{2}{9}\frac{2\pi\alpha_S^2}{q^4}\left[1 + \left(1 + \frac{q^2}{\hat{s}}\right)^2\right].$$

Never see colour, but enters through colour factors. Can tell QCD is $SU(3)$

- Here $\hat{S}$ is the centre-of-mass energy of the quark-quark collision

-The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions

e.g. two jet production in proton-antiproton collisions

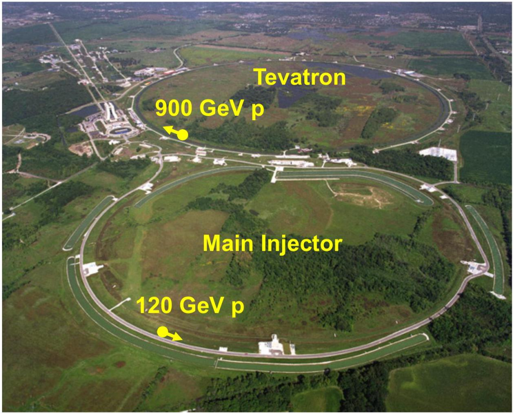


e.g. $p\bar{p}$ collisions at the Tevatron

⋆ Tevatron collider at Fermi National Laboratory (FNAL)

- located $\sim$ 40 miles from Chigaco, US
- started operation in 1987 (ran until 2010)

⋆$\mathrm{p\bar{p}}$ collisions at $\sqrt{\mathrm{s}} = 1.8$ TeV c.f. 14 TeV at the LHC



Two main accelerators:

⋆ Main Injector

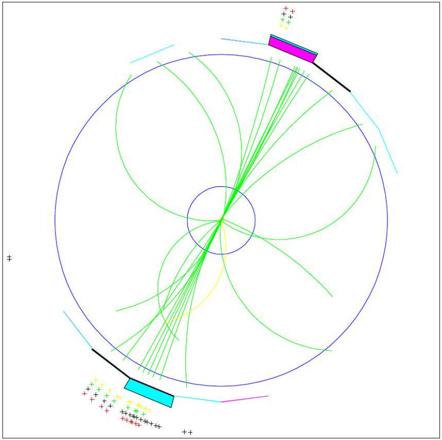
- Accelerated $8\mathrm{GeV}\, p$ to $120\mathrm{GeV}$
- also $\bar{p}$ to $120\mathrm{GeV}$
- Protons sent to Tevatron & MINOS
- $\bar{p}$ all went to Tevatron

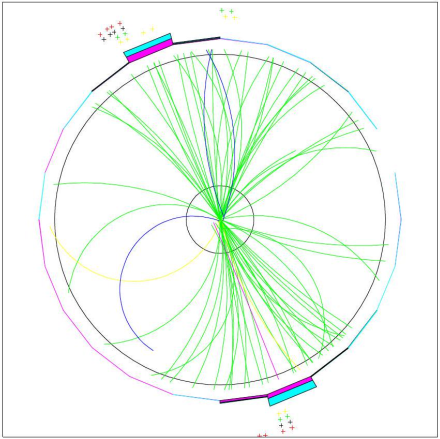
⋆ Tevatron

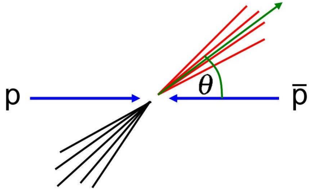
- 4 mile circumference
- accelerated $p/\bar{p}$ from $120\mathrm{GeV}$ to $900\mathrm{GeV}$

Can test QCD predictions by looking at production of pairs of high energy jets

$p\bar{p} \to$ jet jet $+X$:



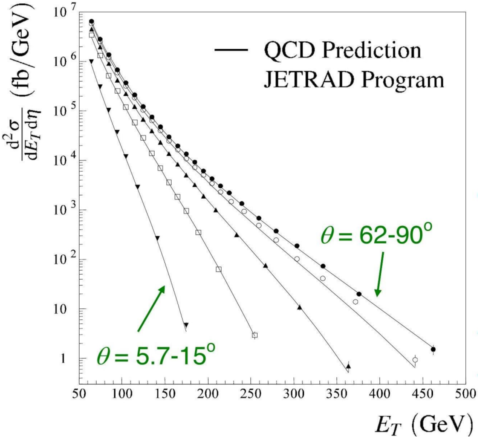




★ **Measure cross-section in terms of**

- **"transverse energy"** $E_T = E_{\text{jet}} \sin\theta$
- **"pseudorapidity"** $\eta = \ln\left[\cot\left(\frac{\theta}{2}\right)\right]$

...don't worry too much about the details here, what matters is that...



D0 Collaboration, Phys. Rev. Lett. 86 (2001)

★ **QCD predictions provide an excellent description of the data**

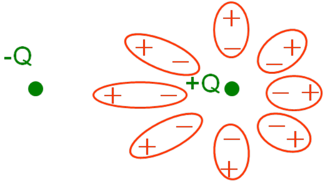
★ **NOTE:**

- **at low E_T cross-section is dominated by low x partons**
  i.e. gluon-gluon scattering
- **at high E_T cross-section is dominated by high x partons**
  i.e. quark-antiquark scattering

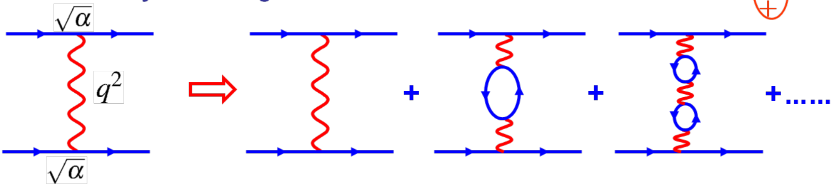
Running Coupling Constants

QED

- **"bare" charge of electron screened by virtual e^+e^- pairs**
- **behaves like a polarizable dielectric**



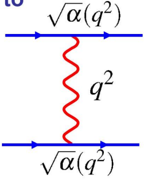
★ **In terms of Feynman diagrams:**



⋆ Same final state so add matrix element amplitudes: $M = M_1 + M_2 + M_3 + \ldots$

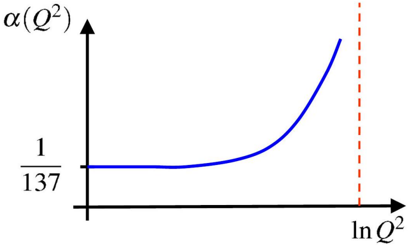
⋆ Giving an infinite series which can be summed and is equivalent to a single diagram with "running" coupling constant

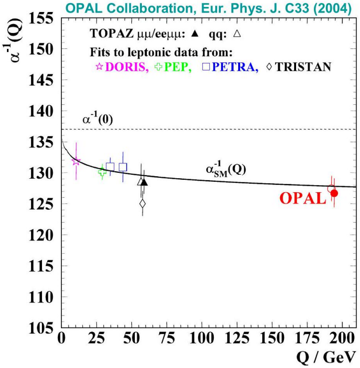
to



$$\alpha\left(Q^2\right) = \alpha\left(Q_0^2\right) / \left[1 - \frac{\alpha\left(Q_0^2\right)}{3\pi} \ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

for $Q^2 \gg Q_0^2$.





Might worry that coupling becomes infinite at

$$\ln\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$$

i.e. at

$$Q \sim 10^{26}\ \mathrm{GeV}.$$

- But quantum gravity effects would come in way below this energy and it is highly unlikely that QED 'as is' would be valid in this regime.

In QED, running coupling increases very slowly:

- Atomic physics: $Q^2 \sim 0$

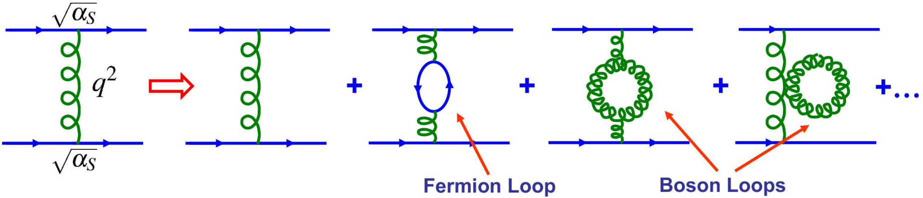
$$1/\alpha = 137.03599976(50)$$

- High energy physics:

$$1/\alpha(193\ \mathrm{GeV}) = 127.4 \pm 2.1$$

Running of α_s

QCD Similar to QED but also have gluon loops



- Remembering adding amplitudes, so can get negative interference and the sum can be smaller than the original diagram alone
- Bosonic loops "interfere negatively

$$\alpha_S\left(Q^2\right) = \alpha_S\left(Q_0^2\right) / \left[1 + B\alpha_S\left(Q_0^2\right)\ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

$$\text{with} \quad B = \frac{11N_c - 2N_f}{12\pi} \quad \begin{cases} N_c = & \text{no. of colours} \\ N_f = & \text{no. of quark flavours} \end{cases}$$

$$N_c = 3; N_f = 6 \implies B > 0 \quad \text{so}$$

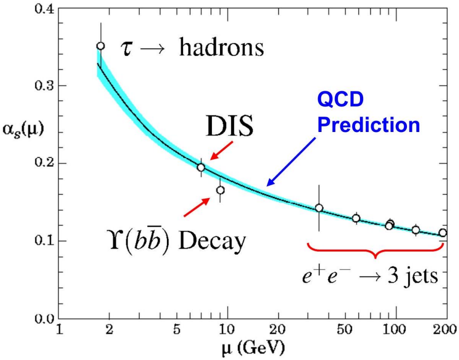
α_S decreases with Q^2

Nobel Prize for Physics 2004 (Gross, Politzer, Wilczek)

Measure α_s in many ways:

- jet rates
- DIS
- tau decays
- bottomonium decays
- ...

As predicted by QCD, α_s decreases with Q^2.



$\mu \approx Q$

At low Q^2 : α_s is large, e.g. at $Q^2 = 1\ \mathrm{GeV}^2$ find $\alpha_\mathrm{s} \sim 1$.

- , Can't use perturbation theory! This is the reason why QCD calculations at low energies are so difficult, e.g. properties hadrons, hadronisation of quarks to jets,...

At high Q^2 : α_s is rather small, e.g. at $Q^2 = M_\mathrm{Z}^2$ find $\alpha_\mathrm{s} \sim 0.12$.

- Can use perturbation theory and this is the reason that in DIS at high Q^2 quarks behave as if they are quasi-free (i.e. only weakly bound within hadrons).

Summary

- Superficially QCD is very similar to QED.
- Gluon self-interactions are believed to result in colour confinement.
- All hadrons are colour singlets which explains why only observe Mesons and Baryons in certain specific flavour combinations.
- At low energies $\alpha_S \sim 1$ so can't use perturbation theory!
- Coupling constant 'runs' becoming smaller at higher energy scales ('Asymptotic Freedom'):

$$\alpha_S(100\ \mathrm{GeV}) \sim 0.1\ .$$

  so at higher energies we can use perturbation theory.
- Where calculations can be performed, QCD provides a good description of relevant experimental data.

Appendix IX: Electromagnetism

Not examinable

⋆ In Heaviside-Lorentz units $\varepsilon_0 = \mu_0 = c = 1$ Maxwell's equations in the vacuum become

$$\vec{\nabla} \cdot \vec{E} = \rho; \quad \vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}; \quad \vec{\nabla} \cdot \vec{B} = 0; \quad \vec{\nabla} \wedge \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}$$

⋆ The electric and magnetic fields can be expressed in terms of scalar and vector potentials

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi; \quad \vec{B} = \vec{\nabla} \wedge \vec{A}$$

⋆ In terms of the 4-vector potential $A^\mu = (\phi, \vec{A})$ and the 4-vector current $j^\mu = (\rho, \vec{J})$ Maxwell's equations can be expressed in the covariant form:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{123}$$

where $F^{\mu\nu}$ is the anti-symmetric field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{124}$$

Not examinable

- Combining (123) and (124)

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = j^\nu$$

Not examinable

which can be written

$$\Box^2 A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = j^\mu \qquad (125)$$

where the D'Alembertian operator

$$\Box^2 = \partial_\nu \partial^\nu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

-Acting on (125) with ∂_V gives

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial_\nu \partial^\nu A^\mu = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{Conservation of Electric Charge}$$

Not examinable

Appendix X: Gauge Invariance

Not examinable

- Conservation laws are associated with symmetries. Here the symmetry is the GAUGE INVARIANCE of electro-magnetism

⋆ The electric and magnetic fields are unchanged for the gauge transformation:

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\chi; \quad \phi \rightarrow \phi' = \phi - \frac{\partial\chi}{\partial t}$$

where $\chi = \chi(t, \vec{x})$ is any finite differentiable function of position and time

⋆ In 4-vector notation the gauge transformation can be expressed as:

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu\chi$$

Not examinable

Not examinable

- Using the fact that the physical fields are gauge invariant, choose χ to be a solution of

⋆ In this case we have

$$\partial^\mu A'_\mu = \partial^\mu \left(A_\mu + \partial_\mu \chi\right) = \partial^\mu A_\mu + \Box^2 \chi = 0$$

⋆ Dropping the prime we have a chosen a gauge in which

$$\partial_\mu A^\mu = 0 \qquad \text{The Lorentz Condition}$$

- With the Lorentz condition, equation (125) becomes:

$$\boxed{\Box^2 A^\mu = j^\mu}. \tag{126}$$

- Having imposed the Lorentz condition we still have freedom to make a further gauge transformation, i.e.

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda$$

where $\Lambda(t, \vec{x})$ is any function that satisfies

$$\Box^2 \Lambda = 0 \tag{127}$$

⋆ Clearly (126) remains unchanged, in addition the Lorentz condition still holds:

Not examinable

$$\partial^\mu A'_\mu = \partial^\mu \left(A_\mu + \partial_\mu \Lambda\right) = \partial^\mu A_\mu + \Box^2 \Lambda = \partial^\mu A_\mu = 0$$

Appendix XI: Photon Polarization

Not examinable

- For a free photon (i.e. $j^\mu = 0$) equation (126) becomes

$$\boxed{\Box^2 A^\mu = 0} \tag{128}$$

(note have chosen a gauge where the Lorentz condition is satisfied)

- Equation (127) has solutions (i.e. the wave-function for a free photon)

$$A^\mu = \varepsilon^\mu(q) e^{-iq\cdot x}$$

where ε^μ is the four-component polarization vector and q is the photon four-momentum

$$0 = \Box^2 A^\mu = -q^2 \varepsilon^\mu e^{-iq\cdot x}$$

$$\Rightarrow q^2 = 0$$

- Hence equation (128) describes a massless particle.
- But the solution has four components - might ask how it can describe a spin-1 particle which has three polarization states?
- But for (127) to hold we must satisfy the Lorentz condition:

$$0 = \partial_\mu A^\mu = \partial_\mu \left(\varepsilon^\mu e^{-iq\cdot x}\right) = \varepsilon^\mu \partial_\nu \left(e^{-iq\cdot x}\right) = -i\varepsilon^\mu q_\mu e^{-iq\cdot x}$$

Hence the Lorentz condition gives

Not examinable

$$q_\mu \varepsilon^\mu = 0 \tag{129}$$

Not examinable

⋆ However, in addition to the Lorentz condition still have the addional gauge freedom of $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda$ with (127) $\Box^2 \Lambda = 0$
-Choosing $\Lambda = iae^{-iq\cdot x}$ which has $\Box^2 \Lambda = q^2 \Lambda = 0$

$$\begin{aligned} A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda &= \varepsilon_\mu e^{-iq\cdot x} + ia\partial_\mu e^{-iq\cdot x} \\ &= \varepsilon_\mu e^{-iq\cdot x} + ia\left(-iq_\mu\right) e^{-iq\cdot x} \\ &= \left(\varepsilon_\mu + aq_\mu\right) e^{-iq\cdot x} \end{aligned}$$

⋆ Hence the electromagnetic field is left unchanged by

$$\varepsilon_\mu \rightarrow \varepsilon'_\mu = \varepsilon_\mu + aq_\mu$$

⋆ Hence the two polarization vectors which differ by a mulitple of the photon four-momentum describe the same photon. Choose *a* such that the time-like component of ε_μ is zero, i.e. $\varepsilon_0 \equiv 0$
⋆ With this choice of gauge, which is known as the COULOMB GAUGE, the Lorentz condition (129) gives

$$\vec{\varepsilon} \cdot \vec{q} = 0$$

i.e. only 2 independent components, both transverse to the photons momentum

Not examinable

Not examinable

⋆ A massless photon has two transverse polarisation states. For a photon travelling in the z direction these can be expressed as the transversly polarized states:

$$\varepsilon_1^\mu = (0,1,0,0); \quad \varepsilon_2^\mu = (0,0,1,0)$$

⋆ Alternatively take linear combinations to get the circularly polarized states

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

- It can be shown that the ε_+state corresponds the state in which the photon spin is directed in the $+\mathbf{z}$ direction, i.e. $S_z = +1$

Not examinable

Appendix XII: Massive Spin-1 particles

Not examinable

- For a massless photon we had (before imposing the Lorentz condition) we had from equation (125):

$$\Box^2 A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = j^\mu$$

⋆ The Klein-Gordon equation for a spin-0 particle of mass m is

$$\left(\Box^2 + m^2\right)\phi = 0$$

suggestive that the appropriate equations for a massive spin-1 particle can be obtained by replacing $\Box^2 \to \Box^2 + m^2$

- This is indeed the case, and from QFT it can be shown that for a massive spin 1 particle equation (125): becomes

$$\left(\Box^2 + m^2\right) B^\mu - \partial^\mu \left(\partial_\nu B^\nu\right) = j^\mu$$

- Therefore a free particle must satisfy

$$\left(\Box^2 + m^2\right) B^\mu - \partial^\mu \left(\partial_\nu B^\nu\right) = 0 \qquad (130)$$

Not examinable

Not examinable

- Acting on equation (130) with ∂_ν gives

$$\left(\Box^2 + m^2\right)\partial_\mu B^\mu - \partial_\mu\partial^\mu\left(\partial_\nu B^\nu\right) = 0$$
$$\left(\Box^2 + m^2\right)\partial_\mu B^\mu - \Box^2\left(\partial_\nu B^\nu\right) = 0$$
$$m^2\partial_\mu B^\mu = 0 \qquad (131)$$

- Hence, for a massive spin-1 particle, unavoidably have $\partial_\mu B^\mu = 0$; note this is not a relation that reflects to choice of gauge.

-Equation (130) becomes

$$\boxed{\left(\Box^2 + m^2\right) B^\mu = 0} : \qquad (132)$$

⋆ For a free spin-1 particle with 4-momentum, p^μ, equation (132): admits solutions

$$B_\mu = \varepsilon_\mu e^{-ip.x}$$

- Substituting into equation (131) gives

$$p_\mu \varepsilon^\mu = 0$$

⋆ The four degrees of freedom in ε^μ are reduced to three, but for a massive particle, equation (132) does not allow a choice of gauge and we can not reduce the number of degrees of freedom any further.

Not examinable

Not examinable

⋆ Hence we need to find three orthogonal polarisation states satisfying

$$p_\mu \varepsilon^\mu = 0 \tag{133}$$

⋆ For a particle travelling in the z direction, can still admit the circularly polarized states.

$$\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

⋆ Writing the third state as

$$\varepsilon^\mu_L = \frac{1}{\sqrt{\alpha^2+\beta^2}}(\alpha,0,0,\beta)$$

equation (133) gives $\alpha E - \beta p_z = 0$

$$\Rightarrow \quad \varepsilon^\mu_L = \frac{1}{m}(p_z,0,0,E)$$

- This longitudinal polarisation state is only present for massive spin-1 particles, i.e. there is no analogous state for a free on-shell photon.

Not examinable

Appendix XIII: Local Gauge Invariance

Not examinable

⋆ The Dirac equation for a charged particle in an electro-magnetic field can be obtained from the free particle wave-equation by making the minimal substitution

$$\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi \quad (q = \text{ charge })$$

In QM: $i\partial_\mu \to i\partial_\mu - qA_\mu$ and the Dirac equation becomes

$$\gamma^\mu \left(i\partial_\mu - qA_\mu\right)\psi - m\psi = 0$$

- In Appendix X: saw that the physical EM fields where invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$$

⋆ Under this transformation the Dirac equation becomes

$$\gamma^\mu \left(i\partial_\mu - qA_\mu + q\partial_\mu\chi\right)\psi - m\psi = 0$$

which is not the same as the original equation. If we require that the Dirac equation is invariant under the Gauge transformation then under the gauge transformation we need to modify the wave-functions

Not examinable

$$\psi \to \psi' = \psi e^{iq\chi}$$

Not examinable

⋆ To prove this, applying the gauge transformation :

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi \quad \psi \to \psi' = \psi e^{iq\chi}$$

to the original Dirac equation gives

$$\gamma^\mu \left(i\partial_\mu - qA_\mu + q\partial_\mu \chi \right) \psi e^{iq\chi} - m\psi e^{iq\chi} = 0 \tag{134}$$

⋆ But

$$i\partial_\mu \left(\psi e^{iq\chi} \right) = ie^{iq\chi} \partial_\mu \psi - q \left(\partial_\mu \chi \right) e^{iq\chi} \psi$$

⋆ Equation (134) becomes

$$\gamma^\mu e^{iq\chi} \left(i\partial_\mu - qA_\mu + q\partial_\mu \chi - q\partial_\mu \chi \right) \psi - m\psi e^{iq\chi} = 0$$

$$\Rightarrow \quad \gamma^\mu e^{iq\chi} \left(i\partial_\mu - qA_\mu \right) \psi - m\psi e^{iq\chi} = 0$$

$$\Longrightarrow$$

$$\gamma^\mu \left(i\partial_\mu - qA_\mu \right) \psi - m\psi = 0$$

which is the original form of the Dirac equation

Not examinable

Appendix XIV : Local Gauge Invariance 2

Not examinable

⋆ Reverse the argument of Appendix XIII. Suppose there is a fundamental symmetry of the universe under local phase transformations

$$\psi(x) \to \psi'(x) = \psi(x)e^{iq\chi(x)}$$

- Note that the local nature of these transformations: the phase transformation depends on the space-time coordinate $x = (t, \vec{x})$

⋆ Under this transformation the free particle Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

becomes $\quad i\gamma^\mu \partial_\mu \left(\psi e^{iq\chi}\right) - m\psi e^{iq\chi} = 0$

$$ie^{iq\chi}\gamma^\mu \left(\partial_\mu \psi + iq\psi\partial_\mu\chi\right) - m\psi e^{iq\chi} = 0$$
$$i\gamma^\mu \left(\partial_\mu + iq\partial_\mu\chi\right)\psi - m\psi = 0$$

Local phase invariance is not possible for a free theory, i.e. one without interactions

- To restore invariance under local phase transformations have to introduce a massless "gauge boson" A^μ which transforms as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi$$

and make the substitution

Not examinable

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$$

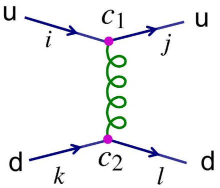
Appendix XV: Alternative evaluation of colour factors

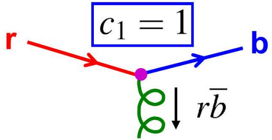
Not examinable

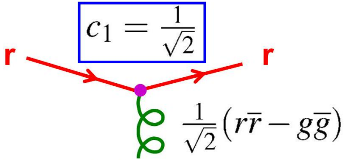
⋆ The colour factors can be obtained (more intuitively) as follows :

-Write $C(ik \to jl) = \frac{1}{2} c_1 c_2$

-Where the colour coefficients at the two vertices depend on the quark and gluon colours





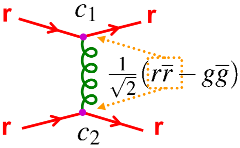


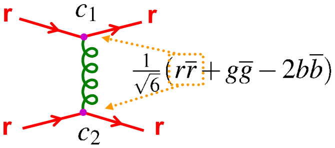
-Sum over all possible exchanged gluons conserving colour at both vertices

Not examinable

① Configurations involving a single colour

e.g. $rr \to rr$ **: two possible exchanged gluons**

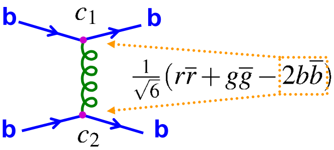




$$c_1 = c_2 = \frac{1}{\sqrt{2}} \qquad c_1 = c_2 = \frac{1}{\sqrt{6}}$$

$$C(rr \to rr) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{3}$$

e.g. $bb \to bb$ **: only one possible exchanged gluon**

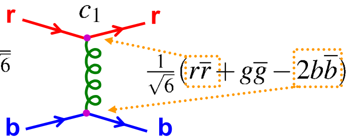


$$c_1 = c_2 = -\frac{2}{\sqrt{6}}$$

➡ $$C(bb \to bb) = \frac{1}{2}\left(\frac{2}{\sqrt{6}}\frac{2}{\sqrt{6}}\right) = \frac{1}{3}$$

② Other configurations where quarks don't change colour

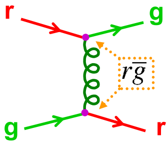
$$c_1 = \frac{1}{\sqrt{6}}$$



$$c_2 = -\frac{2}{\sqrt{6}}$$

$$C(rb \to rb) = \frac{1}{2}\left(-\frac{1}{\sqrt{6}}\frac{2}{\sqrt{6}}\right) = -\frac{1}{6}$$

③ Configurations where quarks swap colours



$$c_1 = c_2 = 1$$

$$C(rg \to gr) = \frac{1}{2}$$

Appendix XVI: Colour Potentials

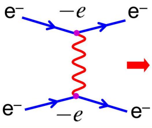
Not examinable

-Previously argued that gluon self-interactions lead to a $+\lambda r$ long-range potential and that this is likely to explain colour confinement

- Have yet to consider the short range potential - i.e. for quarks in mesons and baryons does QCD lead to an attractive potential?

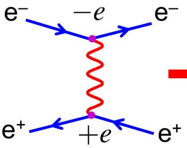
-Analogy with QED: (NOTE this is very far from a formal proof)

QED



$$V(r) = +\frac{\alpha}{r}$$

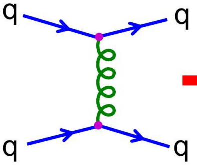
Repulsive Potential



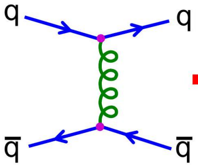
$$V(r) = -\frac{\alpha}{r}$$

Attractive Potential

⋆ by analogy with QED expect potentials of form



$$V(r) = +C\frac{\alpha_S}{r}$$



$$V(r) = -C\frac{\alpha_S}{r}$$

⋆ Whether it is a attractive or repulsive potential depends on sign of colour factor

Not examinable

⋆ Consider the colour factor for a q $\bar{q}$ system in the colour singlet state:

$$\psi = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

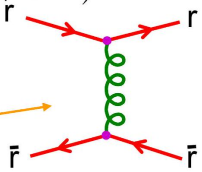
with colour potential $\langle V_{q\bar{q}} \rangle = \langle \psi | V_{\mathrm{QCD}} | \psi \rangle$

$$\Rightarrow \quad \langle V_{q\bar{q}} \rangle = \frac{1}{3}\left(\langle r\bar{r}|V_{\mathrm{QCD}}|r\bar{r}\rangle + \ldots + \langle r\bar{r}|V_{\mathrm{QCD}}|b\bar{b}\rangle + \ldots\right)$$

owing the QED analogy:

$$\langle r\bar{r}|V_{\mathrm{QCD}}|r\bar{r}\rangle = -C(r\bar{r} \to r\bar{r})\frac{\alpha_S}{r}$$

hich is the term arising from $\quad r\bar{r} \to r\bar{r}$



-Have 3 terms like $\quad r\bar{r} \to r\bar{r}, b\bar{b} \to b\bar{b}, \ldots$ and 6 like $\quad r\bar{r} \to g\bar{g}, r\bar{r} \to b\bar{b}, \ldots$

$$\langle V_{q\bar{q}} \rangle = -\tfrac{1}{3}\tfrac{\alpha_S}{r}[3 \times C(r\bar{r} \to r\bar{r}) + 6 \times C(r\bar{r} \to g\bar{g})] = -\tfrac{1}{3}\tfrac{\alpha_S}{r}\left[3 \times \tfrac{1}{3} + 6 \times \tfrac{1}{2}\right]$$

$$\longrightarrow \langle V_{q\bar{q}} \rangle = -\tfrac{4}{3}\tfrac{\alpha_S}{r} \quad \text{NEGATIVE} \Rightarrow \text{ATTRACTIVE}$$

-The same calculation for a q $\bar{q}$ colour octet state, e.g. $r\bar{g}$ gives a positive repulsive potential: $C(r\bar{g} \longrightarrow r\bar{g}) = -\frac{1}{6}$

Not examinable

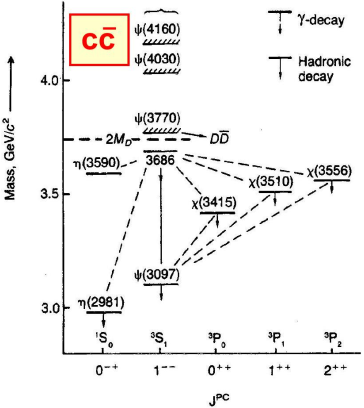
⋆ Whilst not a formal proof, it is comforting to see that in the colour singlet $q\bar{q}$ state the QCD potential is indeed attractive.

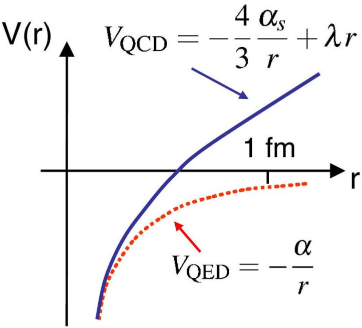
Not examinable

* Combining the short-range QCD potential with the linear long-range term discussed previously:

$$V_{\mathrm{QCD}} = -\frac{4}{3}\frac{\alpha_s}{r} + \lambda r$$

⋆ This potential is found to give a good description of the observed charmonium (cc) and bottomonium (bb) bound states





NOTE:
$\cdot$c, b are heavy quarks
non-relativistic - orbit
- probe $1/r$ part of V

Agreement of data with prediction provides strong evidence that V_{QCD} has the Expected