Appendix IX: Electromagnetism

⋆ In Heaviside-Lorentz units $\varepsilon_0 = \mu_0 = c = 1$ Maxwell's equations in the vacuum become

$$\vec{\nabla}\cdot\vec{E} = \rho; \quad \vec{\nabla}\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t}; \quad \vec{\nabla}\cdot\vec{B} = 0; \quad \vec{\nabla}\wedge\vec{B} = \vec{J} + \frac{\partial\vec{E}}{\partial t}$$

⋆ The electric and magnetic fields can be expressed in terms of scalar and vector potentials

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi; \quad \vec{B} = \vec{\nabla}\wedge\vec{A}$$

⋆ In terms of the 4-vector potential $A^\mu = (\phi, \vec{A})$ and the 4-vector current $j^\mu = (\rho, \vec{J})$ Maxwell's equations can be expressed in the covariant form:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{123}$$

where $F^{\mu\nu}$ is the anti-symmetric field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{124}$$

-Combining (123) and (124)

$$\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = j^\nu$$

which can be written

$$\Box^2 A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = j^\mu \qquad (125)$$

where the D'Alembertian operator

$$\Box^2 = \partial_\nu \partial^\nu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

-Acting on (125) with ∂_V gives

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial_\nu \partial^\nu A^\mu = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{Conservation of Electric Charge}$$

Appendix X: Gauge Invariance

- Conservation laws are associated with symmetries. Here the symmetry is the GAUGE INVARIANCE of electro-magnetism

⋆ The electric and magnetic fields are unchanged for the gauge transformation:

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi; \quad \phi \to \phi' = \phi - \frac{\partial \chi}{\partial t}$$

where $\chi = \chi(t, \vec{x})$ is any finite differentiable function of position and time

⋆ In 4-vector notation the gauge transformation can be expressed as:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$$

- Using the fact that the physical fields are gauge invariant, choose χ to be a solution of

⋆ In this case we have

$$\partial^\mu A'_\mu = \partial^\mu \left(A_\mu + \partial_\mu \chi\right) = \partial^\mu A_\mu + \Box^2 \chi = 0$$

⋆ Dropping the prime we have a chosen a gauge in which

$$\partial_\mu A^\mu = 0 \qquad \text{The Lorentz Condition}$$

- With the Lorentz condition, equation (125) becomes:

$$\boxed{\Box^2 A^\mu = j^\mu}. \tag{126}$$

- Having imposed the Lorentz condition we still have freedom to make a further gauge transformation, i.e.

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$$

where $\Lambda(t, \vec{x})$ is any function that satisfies

$$\Box^2 \Lambda = 0 \tag{127}$$

⋆ Clearly (126) remains unchanged, in addition the Lorentz condition still holds:

Not examinable

$$\partial^\mu A'_\mu = \partial^\mu \left(A_\mu + \partial_\mu \Lambda\right) = \partial^\mu A_\mu + \Box^2 \Lambda = \partial^\mu A_\mu = 0$$

Appendix XI: Photon Polarization

- For a free photon (i.e. $j^\mu = 0$) equation (126) becomes

$$\boxed{\Box^2 A^\mu = 0} \tag{128}$$

(note have chosen a gauge where the Lorentz condition is satisfied)

- Equation (127) has solutions (i.e. the wave-function for a free photon)

$$A^\mu = \varepsilon^\mu(q)e^{-iq\cdot x}$$

where ε^μ is the four-component polarization vector and q is the photon four-momentum

$$0 = \Box^2 A^\mu = -q^2\varepsilon^\mu e^{-iq\cdot x}$$
$$\Rightarrow q^2 = 0$$

- Hence equation (128) describes a massless particle.
- But the solution has four components - might ask how it can describe a spin-1 particle which has three polarization states?
- But for (127) to hold we must satisfy the Lorentz condition:

$$0 = \partial_\mu A^\mu = \partial_\mu \left(\varepsilon^\mu e^{-iq\cdot x}\right) = \varepsilon^\mu \partial_\nu \left(e^{-iq\cdot x}\right) = -i\varepsilon^\mu q_\mu e^{-iq\cdot x}$$

Hence the Lorentz condition gives

$$q_\mu \varepsilon^\mu = 0 \tag{129}$$

⋆ However, in addition to the Lorentz condition still have the addional gauge freedom of $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$ with (127) $\Box^2 \Lambda = 0$
-Choosing $\Lambda = iae^{-iq\cdot x}$ which has $\Box^2 \Lambda = q^2 \Lambda = 0$

$$\begin{aligned} A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda &= \varepsilon_\mu e^{-iq\cdot x} + ia\partial_\mu e^{-iq\cdot x} \\ &= \varepsilon_\mu e^{-iq\cdot x} + ia\left(-iq_\mu\right) e^{-iq\cdot x} \\ &= \left(\varepsilon_\mu + aq_\mu\right) e^{-iq\cdot x} \end{aligned}$$

⋆ Hence the electromagnetic field is left unchanged by

$$\varepsilon_\mu \to \varepsilon'_\mu = \varepsilon_\mu + aq_\mu$$

⋆ Hence the two polarization vectors which differ by a mulitple of the photon four-momentum describe the same photon. Choose a such that the time-like component of ε_μ is zero, i.e. $\varepsilon_0 \equiv 0$
⋆ With this choice of gauge, which is known as the COULOMB GAUGE, the Lorentz condition (129) gives

$$\vec{\varepsilon} \cdot \vec{q} = 0$$

i.e. only 2 independent components, both transverse to the photons momentum

⋆ A massless photon has two transverse polarisation states. For a photon travelling in the z direction these can be expressed as the transversly polarized states:

$$\varepsilon_1^\mu = (0,1,0,0); \quad \varepsilon_2^\mu = (0,0,1,0)$$

⋆ Alternatively take linear combinations to get the circularly polarized states

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

- It can be shown that the ε_+state corresponds the state in which the photon spin is directed in the $+\mathbf{z}$ direction, i.e. $S_z = +1$

Appendix XII: Massive Spin-1 particles

- For a massless photon we had (before imposing the Lorentz condition) we had from equation (125):

$$\Box^2 A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = j^\mu$$

$\star$ The Klein-Gordon equation for a spin-0 particle of mass m is

$$\left(\Box^2 + m^2\right)\phi = 0$$

suggestive that the appropriate equations for a massive spin-1 particle can be obtained by replacing $\Box^2 \rightarrow \Box^2 + m^2$

- This is indeed the case, and from QFT it can be shown that for a massive spin 1 particle equation (125): becomes

$$\left(\Box^2 + m^2\right) B^\mu - \partial^\mu \left(\partial_\nu B^\nu\right) = j^\mu$$

- Therefore a free particle must satisfy

$$\left(\Box^2 + m^2\right) B^\mu - \partial^\mu \left(\partial_\nu B^\nu\right) = 0 \qquad (130)$$

- Acting on equation (130) with ∂_ν gives

$$\left(\Box^2 + m^2\right) \partial_\mu B^\mu - \partial_\mu \partial^\mu \left(\partial_\nu B^\nu\right) = 0$$
$$\left(\Box^2 + m^2\right) \partial_\mu B^\mu - \Box^2 \left(\partial_\nu B^\nu\right) = 0$$
$$m^2 \partial_\mu B^\mu = 0 \qquad (131)$$

- Hence, for a massive spin-1 particle, unavoidably have $\partial_\mu B^\mu = 0$; note this is not a relation that reflects to choice of gauge.

-Equation (130) becomes

$$\boxed{\left(\Box^2 + m^2\right) B^\mu = 0} : \qquad (132)$$

⋆ For a free spin-1 particle with 4-momentum, p^μ, equation (132): admits solutions

$$B_\mu = \varepsilon_\mu e^{-ip.x}$$

- Substituting into equation (131) gives

$$p_\mu \varepsilon^\mu = 0$$

⋆ The four degrees of freedom in ε^μ are reduced to three, but for a massive particle, equation (132) does not allow a choice of gauge and we can not reduce the number of degrees of freedom any further

⋆ Hence we need to find three orthogonal polarisation states satisfying

$$p_\mu \varepsilon^\mu = 0 \tag{133}$$

⋆ For a particle travelling in the z direction, can still admit the circularly polarized states.

$$\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0, 1, -i, 0); \quad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0, 1, i, 0)$$

⋆ Writing the third state as

$$\varepsilon^\mu_L = \frac{1}{\sqrt{\alpha^2 + \beta^2}}(\alpha, 0, 0, \beta)$$

equation (133) gives $\alpha E - \beta p_z = 0$

$$\Rightarrow \quad \varepsilon^\mu_L = \frac{1}{m}\left(p_z, 0, 0, E\right)$$

- This longitudinal polarisation state is only present for massive spin-1 particles, i.e. there is no analogous state for a free on-shell photon.

Appendix XIII: Local Gauge Invariance

⋆ The Dirac equation for a charged particle in an electro-magnetic field can be obtained from the free particle wave-equation by making the minimal substitution

$$\vec{p} \rightarrow \vec{p} - q\vec{A}; \quad E \rightarrow E - q\phi \quad (q = \text{ charge })$$

In QM: $i\partial_\mu \rightarrow i\partial_\mu - qA_\mu$ and the Dirac equation becomes

$$\gamma^\mu \left(i\partial_\mu - qA_\mu\right)\psi - m\psi = 0$$

- In Appendix X: saw that the physical EM fields where invariant under the gauge transformation

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \chi$$

⋆ Under this transformation the Dirac equation becomes

$$\gamma^\mu \left(i\partial_\mu - qA_\mu + q\partial_\mu \chi\right)\psi - m\psi = 0$$

which is not the same as the original equation. If we require that the Dirac equation is invariant under the Gauge transformation then under the gauge transformation we need to modify the wave-functions

$$\psi \rightarrow \psi' = \psi e^{iq\chi}$$

Not examinable

⋆ To prove this, applying the gauge transformation :

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi \quad \psi \to \psi' = \psi e^{iq\chi}$$

to the original Dirac equation gives

$$\gamma^\mu \left(i\partial_\mu - qA_\mu + q\partial_\mu \chi\right) \psi e^{iq\chi} - m\psi e^{iq\chi} = 0 \tag{134}$$

⋆ But

$$i\partial_\mu \left(\psi e^{iq\chi}\right) = ie^{iq\chi}\partial_\mu \psi - q\left(\partial_\mu \chi\right) e^{iq\chi}\psi$$

⋆ Equation (134) becomes

$$\gamma^\mu e^{iq\chi} \left(i\partial_\mu - qA_\mu + q\partial_\mu \chi - q\partial_\mu \chi\right) \psi - m\psi e^{iq\chi} = 0$$

$$\Rightarrow \quad \gamma^\mu e^{iq\chi} \left(i\partial_\mu - qA_\mu\right) \psi - m\psi e^{iq\chi} = 0$$

$$\Longrightarrow$$

$$\gamma^\mu \left(i\partial_\mu - qA_\mu\right) \psi - m\psi = 0$$

which is the original form of the Dirac equation

Not examinable

Appendix XIV : Local Gauge Invariance 2

⋆ Reverse the argument of Appendix XIII. Suppose there is a fundamental symmetry of the universe under local phase transformations

$$\psi(x) \to \psi'(x) = \psi(x) e^{iq\chi(x)}$$

- Note that the local nature of these transformations: the phase transformation depends on the space-time coordinate $x = (t, \vec{x})$

⋆ Under this transformation the free particle Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

becomes $\quad i\gamma^\mu \partial_\mu \left(\psi e^{iq\chi}\right) - m\psi e^{iq\chi} = 0$

$$ie^{iq\chi}\gamma^\mu \left(\partial_\mu \psi + iq\psi\partial_\mu \chi\right) - m\psi e^{iq\chi} = 0$$

$$i\gamma^\mu \left(\partial_\mu + iq\partial_\mu \chi\right)\psi - m\psi = 0$$

Local phase invariance is not possible for a free theory, i.e. one without interactions

- To restore invariance under local phase transformations have to introduce a massless "gauge boson" A^μ which transforms as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$$

and make the substitution

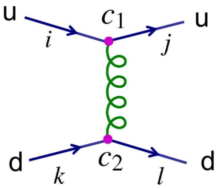
$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$$

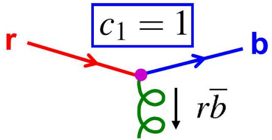
Appendix XV: Alternative evaluation of colour factors

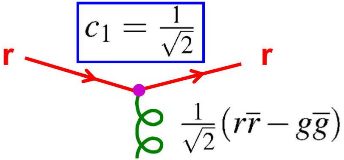
⋆ The colour factors can be obtained (more intuitively) as follows :

-Write $C(ik \to jl) = \frac{1}{2} c_1 c_2$

-Where the colour coefficients at the two vertices depend on the quark and gluon colours



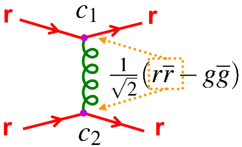


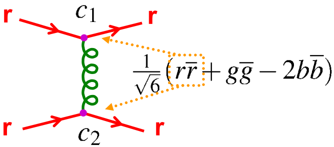


-Sum over all possible exchanged gluons conserving colour at both vertices

① Configurations involving a single colour

e.g. $rr \to rr$ **: two possible exchanged gluons**

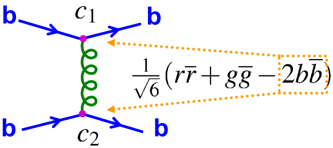




$$c_1 = c_2 = \frac{1}{\sqrt{2}} \qquad c_1 = c_2 = \frac{1}{\sqrt{6}}$$

$$C(rr \to rr) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{3}$$

e.g. $bb \to bb$ **: only one possible exchanged gluon**

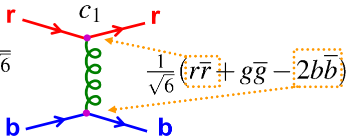


$$c_1 = c_2 = -\frac{2}{\sqrt{6}}$$

$$\Rightarrow \quad C(bb \to bb) = \frac{1}{2}\left(\frac{2}{\sqrt{6}}\frac{2}{\sqrt{6}}\right) = \frac{1}{3}$$

② Other configurations where quarks don't change colour

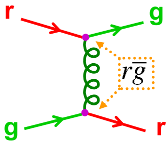
$$c_1 = \frac{1}{\sqrt{6}}$$



$$c_2 = -\frac{2}{\sqrt{6}}$$

$$C(rb \to rb) = \frac{1}{2}\left(-\frac{1}{\sqrt{6}}\frac{2}{\sqrt{6}}\right) = -\frac{1}{6}$$

③ Configurations where quarks swap colours



$$c_1 = c_2 = 1$$

$$C(rg \to gr) = \frac{1}{2}$$

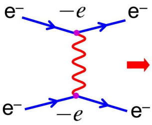
Appendix XVI: Colour Potentials

-Previously argued that gluon self-interactions lead to a $+\lambda r$ long-range potential and that this is likely to explain colour confinement

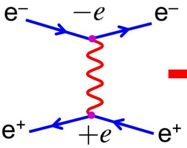
- Have yet to consider the short range potential - i.e. for quarks in mesons and baryons does QCD lead to an attractive potential?

-Analogy with QED: (NOTE this is very far from a formal proof)

QED



$$V(r) = +\frac{\alpha}{r}$$

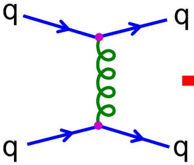


$$V(r) = -\frac{\alpha}{r}$$

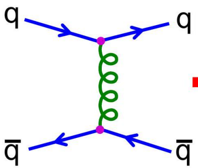
Repulsive Potential

Attractive Potential

⋆ by analogy with QED expect potentials of form



$$V(r) = +C\frac{\alpha_S}{r}$$



$$V(r) = -C\frac{\alpha_S}{r}$$

⋆ Whether it is a attractive or repulsive potential depends on sign of colour factor

⋆ Consider the colour factor for a q $\bar{q}$ system in the colour singlet state:

$$\psi = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

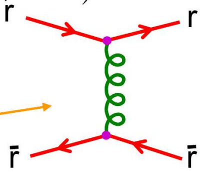
with colour potential $\langle V_{q\bar{q}}\rangle = \langle\psi\,|V_{\rm QCD}|\,\psi\rangle$

$$\langle V_{q\bar{q}}\rangle = \frac{1}{3}\left(\langle r\bar{r}|V_{\rm QCD}|r\bar{r}\rangle + \ldots + \langle r\bar{r}|V_{\rm QCD}|b\bar{b}\rangle + \ldots\right)$$

owing the QED analogy:

$$\langle r\bar{r}|V_{\rm QCD}|r\bar{r}\rangle = -C(r\bar{r}\rightarrow r\bar{r})\frac{\alpha_S}{r}$$

hich is the term arising from $r\bar{r}\rightarrow r\bar{r}$



-Have 3 terms like $r\bar{r}\rightarrow r\bar{r}, b\bar{b}\rightarrow b\bar{b},\ldots$ and 6 like $r\bar{r}\rightarrow g\bar{g}, r\bar{r}\rightarrow b\bar{b},\ldots$

$$\langle V_{q\bar{q}}\rangle = -\tfrac{1}{3}\tfrac{\alpha_S}{r}[3\times C(r\bar{r}\rightarrow r\bar{r}) + 6\times C(r\bar{r}\rightarrow g\bar{g})] = -\tfrac{1}{3}\tfrac{\alpha_S}{r}\left[3\times\tfrac{1}{3} + 6\times\tfrac{1}{2}\right]$$

$$\longrightarrow \langle V_{q\bar{q}}\rangle = -\tfrac{4}{3}\tfrac{\alpha_S}{r}\quad \text{NEGATIVE} \Rightarrow \text{ATTRACTIVE}$$

-The same calculation for a q $\bar{q}$ colour octet state, e.g. $r\bar{g}$ gives a positive repulsive potential: $C(r\bar{g}\longrightarrow r\bar{g}) = -\frac{1}{6}$

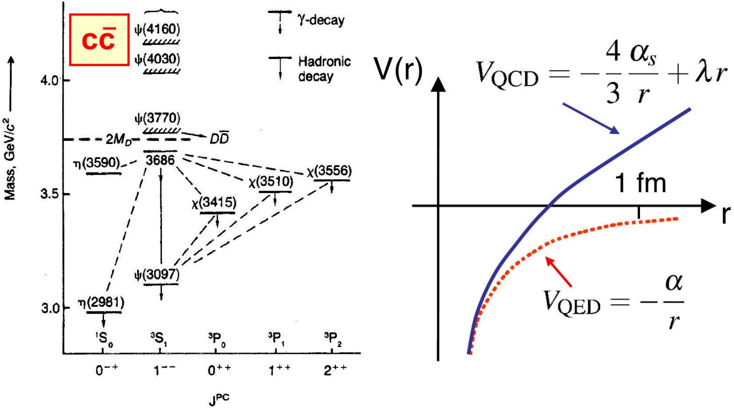
⋆ Whilst not a formal proof, it is comforting to see that in the colour singlet $q\bar{q}$ state the QCD potential is indeed attractive.

Not examinable

* Combining the short-range QCD potential with the linear long-range term discussed previously:

$$V_{\mathrm{QCD}} = -\frac{4}{3}\frac{\alpha_s}{r} + \lambda r$$

⋆ This potential is found to give a good description of the observed charmonium (cc) and bottomonium (bb) bound states



NOTE:
$\cdot$c, b are heavy quarks
non-relativistic - orbit
- probe $1/r$ part of V_{Q}

Agreement of data with prediction provides strong evidence that V_{QCD} has the Expected