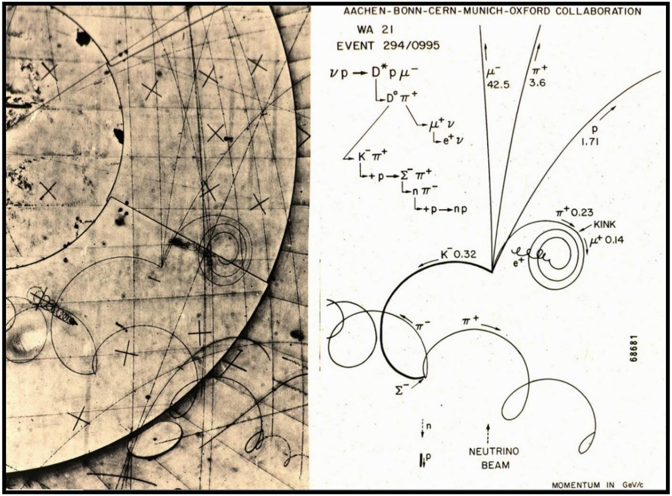
Dr C.G. Lester, 2023



H7: Symmetries and the Quark Model

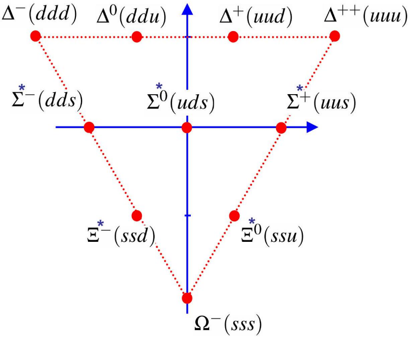
Symmetries in this handout

Symmetries play a central role in particle physics; one aim of particle physics is to discover the fundamental symmetries of our universe.

In this handout will apply the idea of symmetry to the quark model with the aim of:

- deriving hadron wave-functions,
- providing an introduction to the more abstract ideas of colour and QCD (Handout 8), and
- ultimately explaining why hadrons only exist as $\bar{q}q$ (mesons) qqq (baryons) or $\bar{q}\bar{q}\bar{q}$ (anti-baryons).

En route we will see some early usage of $SU(2)$ and $SU(3)$ symmetry groups which play a role both hear and later on (e.g. see Handout 13).



Symmetries and Conservation Laws

- Suppose physics is invariant under the transformation

$$\psi \to \psi' = \hat{U}\psi \qquad \text{e.g. rotation of the coordinate axes}$$

- If U is a symmetry which preserves state normalisations then for all $|\psi >$ we require:

$$\langle \psi \mid \psi \rangle = \langle \psi' \mid \psi' \rangle = \langle \hat{U}\psi \mid \hat{U}\psi \rangle = \left\langle \psi \middle| \hat{U}^\dagger \hat{U} \middle| \psi \right\rangle$$

$$\Longrightarrow \boxed{\hat{U}^\dagger \hat{U} = 1} \qquad \text{i.e. } \hat{U} \text{ has to be unitary.} \tag{120}$$

- For physical predictions to be unchanged by the symmetry transformation, we also require (for all $|\psi >$) that

$$\langle \psi | \hat{H} | \psi \rangle = \left\langle \psi' | \hat{H} | \psi' \right\rangle .$$

Since $\left\langle \psi' | \hat{H} | \psi' \right\rangle = \left\langle \psi \middle| \hat{U}^\dagger \hat{H} \hat{U} \middle| \psi \right\rangle$ we need $\hat{U}^\dagger \hat{H} \hat{U} = \hat{H}$ which (using (120)) says:

$$\boxed{[\hat{H}, \hat{U}] = 0} \qquad \text{i.e. } \hat{U} \text{ commutes with the Hamiltonian.}$$

- If the symmetry can be small (i.e. almost the identity) then an infinitesimal transformation could be wrtten (in terms of $\varepsilon \ll 1$) as:

$$\hat{U} = 1 + i\varepsilon \hat{G}.$$

$\hat{G}$ is called the generator of the transformation.

- For $\hat{U}$ to be unitary

$$\hat{U}\hat{U}^\dagger = (1 + i\varepsilon\hat{G})\left(1 - i\varepsilon\hat{G}^\dagger\right) = 1 + i\varepsilon\left(\hat{G} - \hat{G}^\dagger\right) + O\left(\varepsilon^2\right)$$

  neglecting terms in ε^2 $\quad UU^\dagger = 1 \quad \Rightarrow \quad \hat{G} = \hat{G}^\dagger$ i.e. $\hat{G}$ is Hermitian and therefore corresponds to an observable quantity G !
- Furthermore, $\quad [\hat{H}, \hat{U}] = 0 \quad \Rightarrow [\hat{H}, 1 + i\varepsilon\hat{G}] = 0 \quad \Rightarrow [\hat{H}, \hat{G}] = 0.$
  But from Ehrenfest Theorem in QM:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{G}\rangle = i\langle[\hat{H}, \hat{G}]\rangle = 0$$

  i.e. G is a conserved quantity.

Symmetry $\Longleftrightarrow$ Conservation Law

- Each such symmetry of nature therefore has an observable conserved quantity.
  Example: Infinitesimal spatial translation $x \to x + \varepsilon$ i.e. expect physics to be invariant under $\psi(x) \to \psi' = \psi(x + \varepsilon)$:

$$\psi'(x) = \psi(x + \varepsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\varepsilon = \left(1 + \varepsilon\frac{\partial}{\partial x}\right)\psi(x)$$

  but

$$\hat{p}_x = -i\frac{\partial}{\partial x} \Rightarrow \psi'(x) = (1 + i\varepsilon\hat{p}_x)\,\psi(x).$$

  The generator of the symmetry transformation is $\hat{p}_x$ and so p_x is conserved.
- Translational invariance of physics implies momentum conservation!

- In general the symmetry operation may depend on more than one parameter:

$$\hat{U} = 1 + i\vec{\varepsilon} \cdot \vec{G}$$

For example for an infinitesimal 3D linear translation: $\vec{r} \longrightarrow \vec{r} + \vec{\varepsilon}$ with $\vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$ we have

$$\hat{U} = 1 + i\vec{\varepsilon} \cdot \vec{p}.$$

- So far have only considered an infinitesimal transformation. Fortunately, finite transformations can be expressed as a series of infinitesimal transformations:

$$\hat{U}(\vec{\varepsilon}) = \lim_{n \to \infty} \left(1 + i\frac{\vec{\varepsilon}}{n} \cdot \vec{G}\right)^n = e^{i\vec{\varepsilon} \cdot \vec{G}}.$$

- Example: Finite spatial translation in 1D: $x \to x + x_0$ with $\hat{U}(x_0) = e^{ix_0 \hat{p}_x}$

$$\psi'(x) = \psi(x + x_0) = \hat{U}\psi(x) = \exp\left(x_0 \frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) \quad \left(p_x = -i\frac{\partial}{\partial x}\right)$$
$$= \left(1 + x_0\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x_0^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \ldots\right)\psi(x)$$
$$= \psi(x) + x_0\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{x_0^2}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \ldots$$

confirming that one obtains the expected Taylor expansion for a translated field.

Symmetries in Particle Physics : Isospin

- The proton and neutron have very similar masses and the nuclear force is found to be approximately charge-independent, i.e.

$$V_{pp} \approx V_{np} \approx V_{nn}.$$

- To reflect this symmetry, Heisenberg (1932) proposed that:

> If you could "switch off" the electric charge of the proton there would be no way to distinguish between a proton and neutron.

- More specifically he proposed that the neutron and proton should be considered as two states of a single entity; the nucleon:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- Symmetry ended up being called isospin since maths is very similar that of spin-up/spin-down states of spin-$\frac{1}{2}$ particles.
- Expect physics to be invariant under continuous (ahem) 'rotations' in isospin space just as the axis against which ordinary spin is measured can be continuously varied.
- The neutron and proton form an isospin doublet with total isospin $I = \frac{1}{2}$ and third component $I_3 = \pm\frac{1}{2}$.

Flavour Symmetry of the Strong Interaction

We can extend this idea to the quarks: we can assume that assume the strong interaction treats all quark flavours equally (which it does!).

- Because $m_u \approx m_d$ the strong interaction possesses an approximate flavour symmetry i.e. from the point of view of the strong interaction nothing changes if all up quarks are replaced by down quarks and vice versa.

- Choose the basis

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- Express the invariance of the strong interaction under $u \leftrightarrow d$ as invariance under "rotations" in an abstract isospin space:

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}.$$

In general a 2×2 complex matrix has 4 complex (i.e. 8 real) degrees of freedom. However, when such a matrix is unitary it has only 4 real degrees of freedom as $\hat{U}^\dagger \hat{U} = 1$ imposes 4 real constraints (since no matter what $\hat{U}$ is, $\hat{U}^\dagger \hat{U}$ is always Hermitian).

$$\Longrightarrow \text{ 4 independent matrices}$$

- In the language of group theory, the four matrices form the $U(2)$ group.

Generators for $SU(2)$-flavour a.k.a. 'isospin'

- One of the four available degrees of freedom (d.o.f.) corresponds to multiplying by a phase factor. This is not a **flavour transformation** and so is not relevant.

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

- The remaining three d.o.f. parameterise the $SU(2)$ group of 'special' (i.e. $\det U = 1$) unitary matrices. For infinitessimal transformations, place these d.o.f. in ε_i and write:

$$\hat{U}(\vec{\varepsilon}) = 1 + i\sum_{i=1}^{3} \varepsilon_i \hat{G}_i$$

where the three $\hat{G}_i$ are called the **generators** of the symmetry. $[\vec{G} \equiv (\hat{G}_1, \hat{G}_2, \hat{G}_3)]$

- **EXERCISE:** check that $(\det U = 1) \iff (\mathrm{Tr}(\hat{G}_i) = 0)$ for infinitessimal tfms.
- **EXERCISE:** (re)check that $(\hat{U}^\dagger \hat{U} = 1) \iff (\hat{G}_i^\dagger = \hat{G}_i)$ for infinitessimal tfms.
- The Pauli Matrices are three linearly independent traceless and Hermitian matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- The three generators $\hat{G}_i$ of $SU(2)$-flavour (a.k.a. isospin) symmetry are traditionally denoted $\hat{T}_i$ and are defined by $\hat{T}_i = \frac{1}{2}\sigma_i$ making the general tfm.: $U(\vec{\varepsilon}) = e^{i\vec{\varepsilon}\cdot\vec{T}}$.

(The $\frac{1}{2}$ does not stop these $\hat{T}_i$ being traceless, Hermitian and linearly independent!)

Aside: a check that isospin has claimed properties

Not examinable

For any $SU(2)$-flavour (i.e. isospin) transformation $U(\varepsilon)$ we have:

$$\begin{aligned}\hat{U}(\varepsilon) &= 1 + \frac{1}{2} i\vec{\varepsilon}\cdot\vec{\sigma} + O(\varepsilon^2)\\ &= 1 + \frac{i}{2}\left(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3\right) + O(\varepsilon^2)\\ &= \begin{pmatrix} 1+\frac{1}{2}i\varepsilon_3 & \frac{1}{2}i\left(\varepsilon_1 - i\varepsilon_2\right)\\ \frac{1}{2}i\left(\varepsilon_1 + i\varepsilon_2\right) & 1-\frac{1}{2}i\varepsilon_3\end{pmatrix} + O(\varepsilon^2).\end{aligned}$$

$\hat{U}(\vec{\varepsilon})$ is evidently unitary (at least for infinitessimal transformations) because:

$$\begin{aligned}U^\dagger U &= \begin{pmatrix} 1-\frac{1}{2}i\varepsilon_3 & -\frac{1}{2}i\left(\varepsilon_1 - i\varepsilon_2\right)\\ -\frac{1}{2}i\left(\varepsilon_1 + i\varepsilon_2\right) & 1+\frac{1}{2}i\varepsilon_3\end{pmatrix}\begin{pmatrix} 1+\frac{1}{2}i\varepsilon_3 & \frac{1}{2}i\left(\varepsilon_1 - i\varepsilon_2\right)\\ \frac{1}{2}i\left(\varepsilon_1 + i\varepsilon_2\right) & 1-\frac{1}{2}i\varepsilon_3\end{pmatrix} + O(\varepsilon^2)\\ &= I + O(\varepsilon^2) \qquad \text{(multiply the terms above to see why!)}.\end{aligned}$$

$\hat{U}(\vec{\varepsilon})$ also has determinant 1 (at least for infinitessimal transformations) because:

$$\begin{aligned}\det U &= \left|\begin{pmatrix} 1+\frac{1}{2}i\varepsilon_3 & \frac{1}{2}i\left(\varepsilon_1 - i\varepsilon_2\right)\\ \frac{1}{2}i\left(\varepsilon_1 + i\varepsilon_2\right) & 1-\frac{1}{2}i\varepsilon_3\end{pmatrix} + O(\varepsilon^2)\right|\\ &= (1+\frac{1}{2}i\varepsilon_3)(1-\frac{1}{2}i\varepsilon_3) - O(\varepsilon^2)\\ &= 1 + O(\varepsilon^2).\end{aligned}$$

Not examinable

Mathematical similarities between Isopin and Spin

- Isospin generators have exactly the same properties as those of spin:

$$[T_1, T_2] = iT_3, \quad [T_2, T_3] = iT_1, . \quad [T_3, T_1] = iT_2$$
$$\left[T^2, T_3\right] = 0, \quad T^2 = T_1^2 + T_2^2 + T_3^2.$$

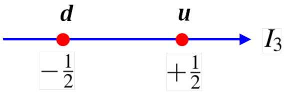
As in the case of spin, have three non-commuting Hermitian operators, T_1, T_2, T_3, so even though all three correspond to observables, we can't measure them simultaneously.

- So label states in terms of total isospin I and the third component of isospin I_3. These eigenstates are exact analogues of the eigenstates of ordinary angular momentum $|s, m\rangle \rightarrow |I, I_3\rangle$:

$$T^2 |I, I_3\rangle = I(I+1) |I, I_3\rangle \quad T_3 |I, I_3\rangle = I_3 |I, I_3\rangle$$

- In terms of isospin:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left|\frac{1}{2}, +\frac{1}{2}\right\rangle, \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle.$$



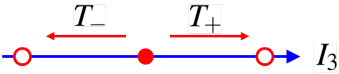
$$\boxed{I = \frac{1}{2}, I_3 = \pm\frac{1}{2}}$$

- In general $\quad I_3 = \frac{1}{2}(N_u - N_d)$.

Can define isospin ladder operators - analogous to spin ladder operators

$T_- \equiv T_1 - iT_2$

$u \to d$



$T_+ \equiv T_1 + iT_2$

$d \to u$

$$T_+ \, |I, I_3\rangle = \sqrt{I(I+1) - I_3\,(I_3+1)}\,|I, I_3+1\rangle$$
$$T_- \, |I, I_3\rangle = \sqrt{I(I+1) - I_3\,(I_3-1)}\,|I, I_3-1\rangle$$

These ops step up/down in I_3 until reach end of multiplet: $T_+|I, +I\rangle = 0 \quad T_-|I, -I\rangle = 0.$

$$\boxed{T_+ u = 0, \quad T_+ d = u, \quad T_- u = d, \quad T_- d = 0}$$

What is isospin of a sytemt of two quarks?

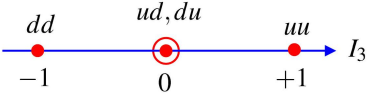
- Combining/adding isospin works same as combining/adding angular momentum:

$$\left|I^{(1)}, I_3^{(1)}\right\rangle \left|I^{(2)}, I_3^{(2)}\right\rangle \to |I, I_3\rangle \, .$$

- I_3 additive : $\quad I_3 = I_3^{(1)} + I_3^{(2)}$,
- I in integer steps from $\left|I^{(1)} - I^{(2)}\right|$ to $\left|I^{(1)} + I^{(2)}\right|$.
- In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum.

Combining Quarks

Isospin starts to become useful in defining states of more than one quark.
Two quarks states could use a $\{uu, ud, du, dd\}$ basis:



This is not a good basis, though, becayse two of these states are not eigenstates of total isospin. Here (and on next slide) we will derive a better basis whose elements have well defined I.

- We can immediately identify the extremes since I_3 is additive:

$$uu \equiv \left|\frac{1}{2}, \frac{1}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle = |1, +1\rangle \quad dd \equiv \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = |1, -1\rangle.$$

- To obtain the $|1, 0\rangle$ state use ladder operators

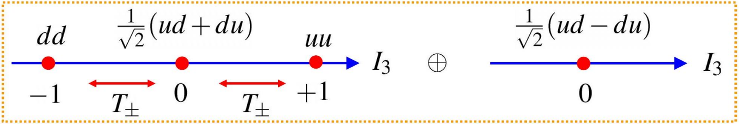
$$T_{-}|1, +1\rangle = \sqrt{2}|1, 0\rangle = T_{-}(uu) = ud + du$$

$$\Longrightarrow \quad |1, 0\rangle = \frac{1}{\sqrt{2}}(ud + du)$$

- The final (fourth) basis state, $|0, 0\rangle$, can be found from orthogonality with $|1, 0\rangle$

$$\Longrightarrow \quad |0, 0\rangle = \frac{1}{\sqrt{2}}(ud - du).$$

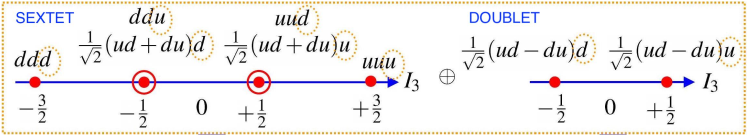
- From four possible combinations of isospin doublets obtain a triplet of isospin 1 states and a singlet isospin 0 state $2 \otimes 2 = 3 \oplus 1$



- Can move around within multiplets using ladder operators.
- As anticipated $I_3 = \frac{1}{2}(N_u - N_d)$.
- States with different total isospin are physically different — the isospin 1 triplet is symmetric under interchange of quarks 1 and 2 whereas singlet is anti-symmetric.

Now add an additional up or down quark:

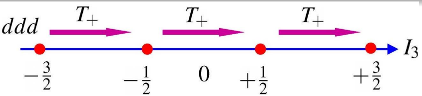
From each of the above 4 states get two new isospin states with $I_3' = I_3 \pm \frac{1}{2}$:



- As for two quark case, the **extremal** states of a given multiplet have well defined isospin, but we must use ladder operators and orthogonality find the states which shares (or have different) properties to the extremal states. E.g. to obtain the $I = \frac{3}{2}$ states, step up from *ddd* (or down from *uuu*) within the SEXTET. (See next page!)

Four states that are symmetric under exchange of first two quarks

Derive the $I = \frac{3}{2}$ states from $ddd \equiv \left|\frac{3}{2}, -\frac{3}{2}\right\rangle$:



$$\boxed{\left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = ddd}$$

$$\Longrightarrow \quad T_+ \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = T_+(ddd) = (T_+ d)\, dd + d\, (T_+ d)\, d + dd\, (T_+ d)$$

$$\Longrightarrow \quad \sqrt{3}\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = udd + dud + ddu$$

$$\Longrightarrow \quad \boxed{\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(udd + dud + ddu)}$$

$$\Longrightarrow \quad T_+ \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}} T_+(udd + dud + ddu)$$

$$\Longrightarrow \quad 2\left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(uud + udu + uud + duu + udu + duu)$$

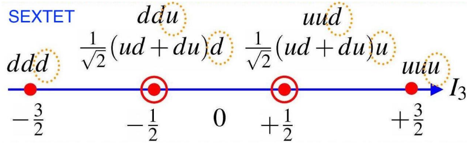
$$\Longrightarrow \quad \boxed{\left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(uud + udu + duu)}$$

$$\Longrightarrow \quad T_+ \left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}} T_+(uud + udu + duu)$$

$$\Longrightarrow \quad \sqrt{3}\left|\tfrac{3}{2}, +\tfrac{3}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(\;\mathrm{uuu}\; + \;\mathrm{uuu}\; + \;\mathrm{uuu}\;)$$

$$\Longrightarrow \quad \boxed{\left|\tfrac{3}{2}, +\tfrac{3}{2}\right\rangle = \;\mathrm{uuu}}\,.$$

Two more states that are symmetric under exchange of first two quarks



- The SEXTET (above) contained six states.
  All were symmetric under exchange of the first two quarks, $q_1 \leftrightarrow q_2$.
- On last page we found four states with the same $q_1 \leftrightarrow q_2$ symmetry (but with well defined isopin). There must therefore be two more states which are orthogonal to the four just found, and which are symmetric under $q_1 \leftrightarrow q_2$ but which have well defined isospin.
- The two missing orthogonal states hidden within the SEXTET are:

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = -\frac{1}{\sqrt{6}}(2ddu - udd - dud), \qquad \text{and}$$
$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle = +\frac{1}{\sqrt{6}}(2uud - udu - duu).$$

[EXERCISE: Check that these states are mapped to each other by $T_\pm$, that they have $I = 1/2$, and that they are orthogonal to each other and to the other states.]

Flavour basis for states containing three quarks

We still have the two DOUBLET states from page 265 – which are (conveniently) already an iso-doublet. Thus, we have managed to replace the 8-dimensional basis $\{uuu, uud, udu, udd, duu, dud, ddu, ddd\}$ with one formed from **one iso-quadruplet** and **two iso-doublets**:

$$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = (2 \otimes 3) \oplus (2 \otimes 1) = 4 \oplus 2 \oplus 2$$

Each multiplet has its own distinct symmetry:

$$\left.\begin{aligned} \left|\tfrac{3}{2}, +\tfrac{3}{2}\right\rangle &= \text{uuu} \\ \left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(uud + udu + duu) \\ \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(ddu + dud + udd) \\ \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle &= ddd \end{aligned}\right\} S;$$

$S =$ A quadruplet of states which are Symmetric under the interchange of any two quarks.

$$\left.\begin{aligned} \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle &= +\tfrac{1}{\sqrt{6}}(2uud - udu - duu) \\ \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle &= -\tfrac{1}{\sqrt{6}}(2ddu - udd - dud) \end{aligned}\right\} M_S;$$

$M_S =$ Mixed, Symmetric under interchange of quarks $1 \leftrightarrow 2$.

$$\left.\begin{aligned} \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}(udu - duu) \\ \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}(udd - dud) \end{aligned}\right\} M_A;$$

$M_A =$ Mixed, Antisymmetric under interchange of quarks $1 \leftrightarrow 2$.

The states in M_S and M_A have no definite symmetry under interchange of the third quark with either of the others. This will change when we add spin later.

Spin basis for states containing three spin-half particles

Since the maths for spin is the same as the maths for isospin, we can replace the 8-dimensional basis $\{\uparrow\uparrow\uparrow, \uparrow\uparrow\downarrow, \uparrow\downarrow\uparrow, \uparrow\downarrow\downarrow, \downarrow\uparrow\uparrow, \downarrow\uparrow\downarrow, \downarrow\downarrow\uparrow, \downarrow\downarrow\downarrow\}$ with one formed from **one iso-quadruplet** and **two iso-doublets**:

$$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = (2 \otimes 3) \oplus (2 \otimes 1) = 4 \oplus 2 \oplus 2$$

Each multiplet has its own distinct symmetry:

$$\left.\begin{aligned} \left|\tfrac{3}{2}, +\tfrac{3}{2}\right\rangle &= \uparrow\uparrow\uparrow \\ \left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \\ \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow) \\ \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle &= \downarrow\downarrow\downarrow \end{aligned}\right\} S;$$

$S =$ A quadruplet of states which are Symmetric under the interchange of any two quarks.

$$\left.\begin{aligned} \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle &= +\tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\ \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle &= -\tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow) \end{aligned}\right\} M_S;$$

$M_S =$ Mixed, Symmetric under interchange of quarks $1 \leftrightarrow 2$.

$$\left.\begin{aligned} \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\ \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow) \end{aligned}\right\} M_A;$$

$M_A =$ Mixed, Antisymmetric under interchange of quarks $1 \leftrightarrow 2$.

As was the case for the flavour basis, the states in M_S and M_A have no definite symmetry under interchange of the third quark with either of the others.

Baryon Wave-functions

Quarks are fermions so the total wave-function must be anti-symmetric under the interchange of any two quarks.

- The total wave-function can be factorised into:

$$\psi = \phi_{\text{flavour}}\, \chi_{\text{spin}}\, \xi_{\text{colour}}\, \eta_{\text{space}}$$

- ξ_{colour} is anti-symmetric as it is a bound qqq states (see Handout 8).
- η_{space} is symmetric (for us) as we will only consider the lowest mass, ground state baryons ($L = 0$). [Symmetry is $(-1)^L$ in general.]

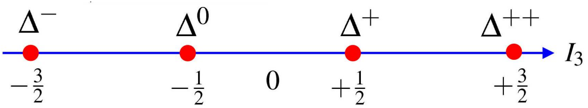
Thus:

> $\phi_{flavour}\chi_{spin}$ must be symmetric under the interchange of any two quarks

- Two ways to form a totally symmetric wave-function from spin and isospin states:

(1) combine totally symmetric spin and isospin wave-functions $\phi(S)\chi(S)$

$ddd \quad \frac{1}{\sqrt{3}}(ddu + dud + udd) \quad \frac{1}{\sqrt{3}}(uud + udu + duu) \quad uuu$



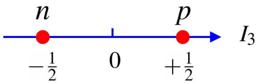
Spin 3/2
Isospin 3/2

(2) combine mixed symmetry spin and mixed symmetry isospin states:

- Both $\phi(M_S)\chi(M_S)$ and $\phi(M_A)\chi(M_A)$ are sym. under inter-change of quarks $1 \leftrightarrow 2$
- Not sufficient, these combinations have no definite symmetry under $1 \leftrightarrow 3, \ldots$
- However, it is not difficult to show that the (normalised) linear combination:

$$\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S) + \frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)$$

is totally symmetric (i.e. symmetric under $q_1 \leftrightarrow q_2; q_1 \leftrightarrow q_3; q_2 \leftrightarrow q_3$).



Spin 1/2
Isospin 1/2

- The spin-up proton wave-function is therefore:

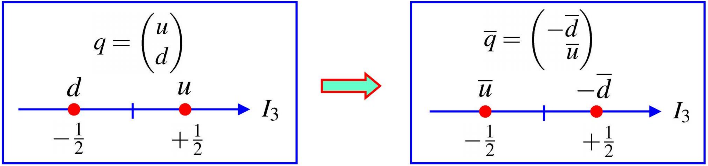
$$|p\uparrow\rangle = \frac{1}{6\sqrt{2}}(2uud - udu - duu)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \frac{1}{2\sqrt{2}}(udu - duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$\Longrightarrow |p\uparrow\rangle = \left\{ \begin{array}{ll} \frac{1}{\sqrt{18}}(& 2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow \\ & +2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow \\ & +2d\downarrow u\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - d\uparrow u\uparrow u\downarrow \quad) \end{array} \right\}.$$

NOTE: it is not always necessary to use the fully symmetrised proton wave-function. E.g. the first three terms are sufficient for calculating the proton magnetic moment.

Anti-quarks and Mesons (u and d)

⋆ The **u**, **d** quarks and $\bar{\mathbf{u}}, \bar{\mathbf{d}}$ anti-quarks are represented as isospin doublets



$$\bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{d} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- The ordering and the minus sign in the anti-quark doublet ensures that anti-quarks and quarks transform in the same way (see Appendix VIII). This is necessary if we want physical predictions to be invariant under $u \leftrightarrow d; \bar{u} \leftrightarrow \bar{d}$
- For anti-quarks the ladder operators introduce an extra minus sign not seen for quarks. E.g.:

$$T_+\bar{u} = T_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\bar{d}$$

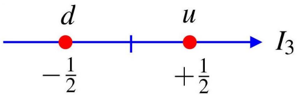
The effect of the ladder operators on all isospin states are:

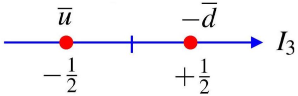
$$\text{anti-quarks}: \quad T_+\bar{u} = -\bar{d}, \quad T_+\bar{d} = 0, \quad T_-\bar{u} = 0, \quad T_-\bar{d} = -\bar{u}, \text{ and}$$

$$\text{quarks}: \quad T_+d = +u, \quad T_+u = 0, \quad T_-d = 0, \quad T_-u = +d.$$

Light ud Mesons

- Can now construct meson states from combinations of up/down quarks





- Consider the $q\bar{q}$ combinations in terms of isospin

$$|1,+1\rangle = \left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle \overline{\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle} = -u\bar{d}$$
$$|1,-1\rangle = \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \overline{\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle} = \quad d\bar{u}$$

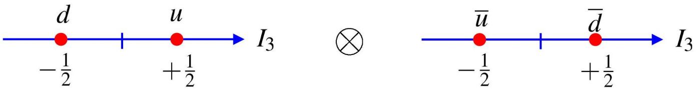
(The bar indicates this is the isospin representation of an anti-quark.)

- To obtain the $I_3 = 0$ states use ladder operators and orthogonality:

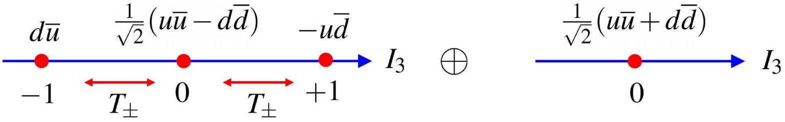
$$T_-|1,+1\rangle = T_-[-u\bar{d}]$$
$$\implies \quad \sqrt{2}|1,0\rangle = -T_-[u]\bar{d} - uT_-[\bar{d}]$$
$$= -d\bar{d} + u\bar{u}$$
$$\implies \quad \boxed{|1,0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})}.$$

Orthogonality gives: $\boxed{|0,0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})}.$

- To summarise:



decomposes into a triplet of $I = 1$ states and a singlet $I = 0$ state:



- You will see this written as $2 \otimes \bar{2} = 3 \oplus 1$ with 2 being the quark doublet and $\bar{2}$ being the anti-quark doublet.
- To show the state obtained from orthogonality with $|1,0\rangle$ is a singlet use ladder operators

$$T_+|0,0\rangle = T_+\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) = \frac{1}{\sqrt{2}}(-u\bar{d} + u\bar{d}) = 0.$$

Silimiarly

$$T_-|0,0\rangle = 0.$$

- A singlet state is a ‘dead end’ from the point of view of ladder operators.

$SU(3)$-flavour: the extension to the strange quark

- Since $m_s > m_u, m_d$ have only an approximate symmetry. Nonetheless, m_s is not 'very' different from m_u, m_d, and so the strong interaction acts as if its states were approximately symmetric under $u \leftrightarrow d \leftrightarrow s$.
- The assumed *uds* flavour symmetry can timesbe expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3×3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters There are 9 constraints from $\hat{U}^\dagger \hat{U} = 1$ so the $U(3)$ group is $18 - 9 = 9$-dimesional.
- As before, one d.o.f. simply allows for matrices in $U(3)$ to be multiplied by a complex phase and is of no interest in the context of flavour symmetry.
- The remaining 8 d.o.f. control variation between matrices with the 'special' property $\det U = 1$. The group which these d.o.f. parameterise is therefore called $SU(3)$.
- The eight traceless and Hermitian generators G_i for $SU(3)$-flavour are denoted:

$\boxed{\vec{G} = \frac{1}{2}\vec{\lambda}}$ making a general element of the group: $\boxed{\hat{U}(\vec{\varepsilon}) = e^{i\vec{\varepsilon}\cdot\vec{G}} = e^{i\frac{1}{2}\vec{\varepsilon}\cdot\vec{\lambda}}}$.

- In SU(3) flavour, the three quark states are represented by:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- In SU(3) uds flavour symmetry contains SU(2) ud flavour symmetry which allows us to write the first three matrices:

$$\lambda_1 = \begin{pmatrix} \sigma_1 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e. $\boxed{\mathbf{u} \leftrightarrow \mathbf{d} \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$

- The third component of isospin is now written $I_3 = \frac{1}{2}\lambda_3$ (instead of $\frac{1}{2}\sigma_3$) thus

$$I_3 u = +\tfrac{1}{2}u, \quad I_3 d = -\tfrac{1}{2}d, \quad \text{and} \quad I_3 s = 0.$$

- I_3 counts the number of up quarks minus number of down quarks in a state.
- As before, ladder operators $T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$ achieve $\boxed{d \longleftarrow T_\pm \longrightarrow u}$.

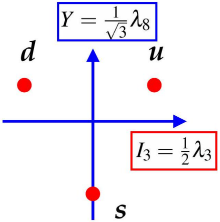
- Now consider the matrices corresponding to the **u** $\leftrightarrow$ **s** and **d** $\leftrightarrow$ **s**

u ↔ s $\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

d ↔ s $\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- Hence in addition to $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have two other traceless diagonal matrices.
- However, these three diagonal matrices are not independent, so define the eighth matrix, λ_8, as the following linear combination:

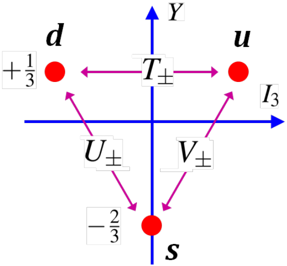
$$\lambda_8 = \tfrac{1}{\sqrt{3}} \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$



which specifies the 'vertical position' in the 2D plane.

Only need two axes (quantum numbers) to specify a state in the 2D plane: (I_3, Y), i.e. third component of isospin and hypercharge, Y.

There are now six ladder operators (built from Gell-Mann matrices λ_i) which step between the states:



$$T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7)$$

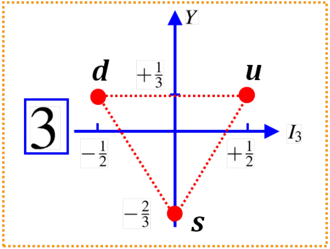
$$I_3 = \tfrac{1}{2}\lambda_3$$
$$Y = \tfrac{1}{\sqrt{3}}\lambda_8$$

u ↔ d $\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

u ↔ s $\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$

d ↔ s $\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Quarks and anti-quarks in SU(3) Flavour

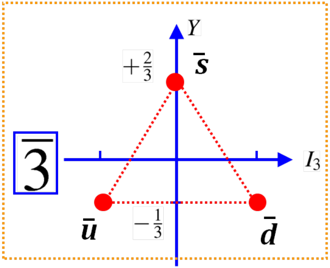


Quarks

$$I_3 u = +\tfrac{1}{2} u; \qquad I_3 d = -\tfrac{1}{2} d; \qquad I_3 s = 0$$

$$Y u = +\tfrac{1}{3} u; \qquad Y d = +\tfrac{1}{3} d; \qquad Y s = -\tfrac{2}{3} s$$

- The anti-quarks have opposite **SU**(3) flavour quantum numbers



Anti-Quarks

$$I_3 \bar{u} = -\tfrac{1}{2} \bar{u}; \quad I_3 \bar{d} = +\tfrac{1}{2} \bar{d}; \quad I_3 \bar{s} = 0$$

$$Y \bar{u} = -\tfrac{1}{3} \bar{u}; \quad Y \bar{d} = -\tfrac{1}{3} \bar{d}; \quad Y \bar{s} = +\tfrac{2}{3} \bar{s}$$

$SU(3)$ Ladder Operators

The *uds* $SU(3)$**-flavour symmetry contains** *ud*, *us* **and** *ds* $SU(2)$**-flavour symmetries.**

Consider the $u \leftrightarrow s$ symmetry 'V-spin' which has the associated $s \to u$ ladder operator

$$V_+ = \tfrac{1}{2}(\lambda_4 + i\lambda_5) = \tfrac{1}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix} + \tfrac{i}{2}\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

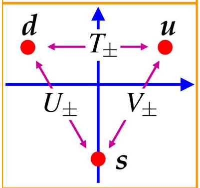
so $$V_+ s = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix} = +u.$$

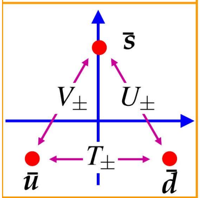
- The actions of all six ladder operators are:

$T_+ d = u;$	$T_- u = d;$	$T_+ \bar{u} = -\bar{d};$	$T_- \bar{d} = -\bar{u}$
$V_+ s = u;$	$V_- u = s;$	$V_+ \bar{u} = -\bar{s};$	$V_- \bar{s} = -\bar{u}$
$U_+ s = d;$	$U_- d = s;$	$U_+ \bar{d} = -\bar{s};$	$U_- \bar{s} = -\bar{d}$

and all other unlisted actions give zero.

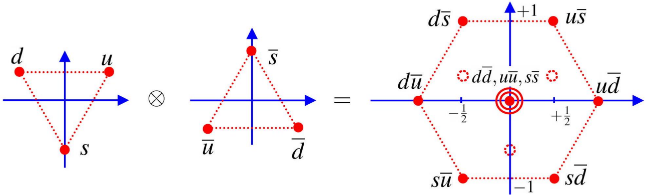
$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$





Light (uds) Mesons

- Use ladder operators to construct uds mesons from the nine possible $q\bar{q}$ states

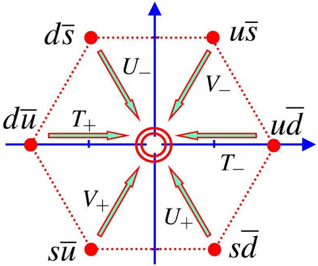


- The central states with $Y = I_3 = 0$ can be obtained using the ladder operators and orthogonality. Starting from the outer states can reach the centre in six ways:

$$T_+|d\bar{u}\rangle = |u\bar{u}\rangle - |d\bar{d}\rangle \quad T_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle$$

$$V_+|s\bar{u}\rangle = |u\bar{u}\rangle - |s\bar{s}\rangle \quad V_-|u\bar{s}\rangle = |s\bar{s}\rangle - |u\bar{u}\rangle$$

$$U_+|s\bar{d}\rangle = |d\bar{d}\rangle - |s\bar{s}\rangle \quad U_-|d\bar{s}\rangle = |s\bar{s}\rangle - |d\bar{d}\rangle$$



- Only two of these six states are linearly independent,
- but there are three original states with $Y = I_3 = 0$.
- Therefore one state is not part of this first multiplet, which is therefore an octet. The ninth state cannot be reached with ladder ops as it is orthogonal.

- First form two linearly independent orthogonal states from:

$$|u\bar{u}\rangle - |d\bar{d}\rangle \quad |u\bar{u}\rangle - |s\bar{s}\rangle \quad |d\bar{d}\rangle - |s\bar{s}\rangle.$$

- If the SU(3) flavour symmetry were exact, the choice of states wouldn't matter. However, $m_S > m_{u,d}$ and the symmetry is only approximate.
- Experimentally observe three light mesons with m 140 MeV: π^+, π^0, π^-
- Identify one central octet state (the π^0) with the isospin triplet derived previously:

$$\boxed{\psi_1 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})}$$

- The second central octet state can be obtained as a linear combination of the other two states which is orthogonal to the π^0:

$$\psi_2 = \alpha(|u\bar{u}\rangle - |s\bar{s}\rangle) + \beta(|d\bar{d}\rangle - |s\bar{s}\rangle)$$

with orthonormality: $\langle\psi_1 \mid \psi_2\rangle = 0; \quad \langle\psi_2 \mid \psi_2\rangle = 1$

$$\Longrightarrow \boxed{\psi_2 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})}$$

- The remaining central state (**which is not part of the octet!**) is then whatever is orthogonal to ψ_1 and ψ_2:

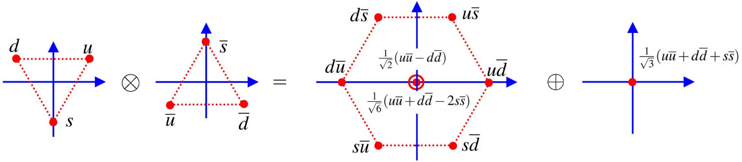
$$\Longrightarrow \boxed{\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})}$$ is the SINGLET!

- It is easy to check that

$$T_+\psi_3 = T_-\psi_3 = U_+\psi_3 = U_-\psi_3 = V_+\psi_3 = V_-\psi_3 = 0$$

thereby confirming that $\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ is a 'flavourless' SINGLET.

- Therefore the combination of a quark and anti-quark yields nine states which breakdown into an OCTET and a SINGLET:



- In the language of group theory: $3 \otimes \bar{3} = 8 \oplus 1$
- Compare with combination of two spin-half particles $2 \otimes 2 = 3 \oplus 1$

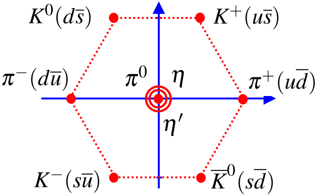
spin-1 TRIPLET: states: $|1,-1\rangle, |1,0\rangle, |1,+1\rangle$,

spin-0 SINGLET: state: $|0,0\rangle$.

- These spin triplet states are connected by ladder operators just as the meson uds octet states are connected by $SU(3)$ flavour ladder operators.
- The singlet state carries no angular momentum — in this sense the $SU(3)$ flavour singlet is flavourless.

Meson summary

PSEUDOSCALAR MESONS (L=0, S=0, J=0, P= –1)



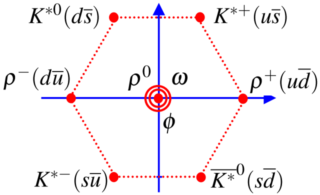
- Because SU(3) flavour is only approximate the physical states with $I_3 = 0,\ Y = 0$ can be mixtures of the octet and singlet states.

Empirically find:

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\eta \approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\eta' \approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

singlet

VECTOR MESONS (L=0, S=1, J=1, P= –1)



- For the vector mesons the physical states are found to be approximately "ideally mixed":

$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\omega \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$$\phi \approx s\bar{s}$$

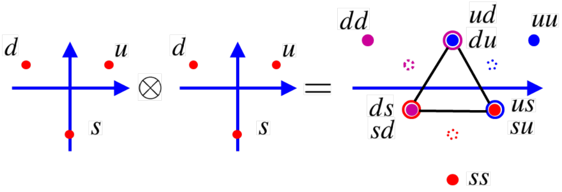
MASSES

$\pi^\pm$: 140 MeV	π^0 : 135 MeV
$K^\pm$: 494 MeV	$K^0/\overline{K}^0$: 498 MeV
η : 549 MeV	η' : 958 MeV

$\rho^\pm$: 770 MeV	ρ^0 : 770 MeV
$K^{*\pm}$: 892 MeV	$K^{*0}/\overline{K^*}^0$: 896 MeV
ω : 782 MeV	ϕ : 1020 MeV

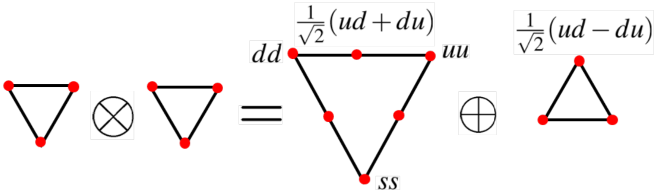
Combining *uds* quarks to form Baryons

- Have already seen that constructing Baryon states is a fairly tedious process when we derived the proton wave-function. Concentrate on multiplet structure rather than deriving all the wave-functions.
- Everything we do here is relevant to the treatment of colour
- First combine two quarks:

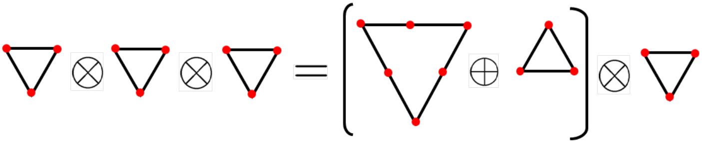


- Yields a SYMMETRIC SEXTET and ANTI-SYMMETRIC TRIPLET:

$3 \otimes 3 = 6 \oplus \bar{3}$:

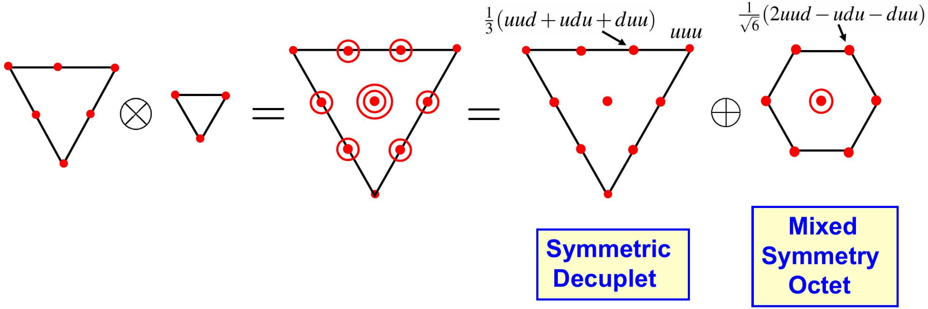


- Now add the third quark:

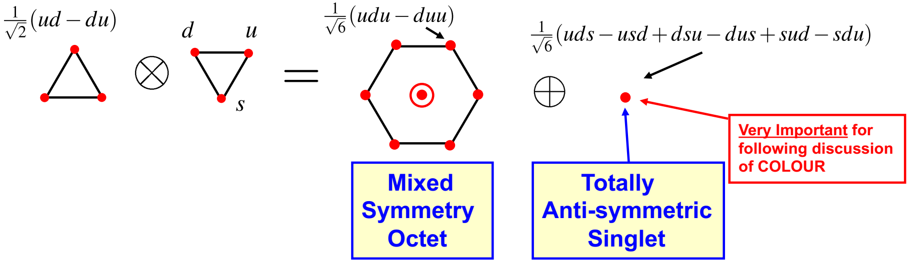


- Best considered in two parts, building on the sextet and triplet. Again concentrate on the multiplet structure (for the wave-functions refer to the discussion of proton wave-function).

(1) Building on the sextet: $6 \otimes 3 = 10 \oplus 8$:



(2) Building on the triplet: $\bar{3} \otimes 3 = 8 \oplus 1$:



- We also saw $\bar{3} \otimes 3 = 8 \oplus 1$ when we combind quarks with antiquarks to get mesons.
- Can verify the wave-function $\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$ is a singlet by using ladder operators, e.g.

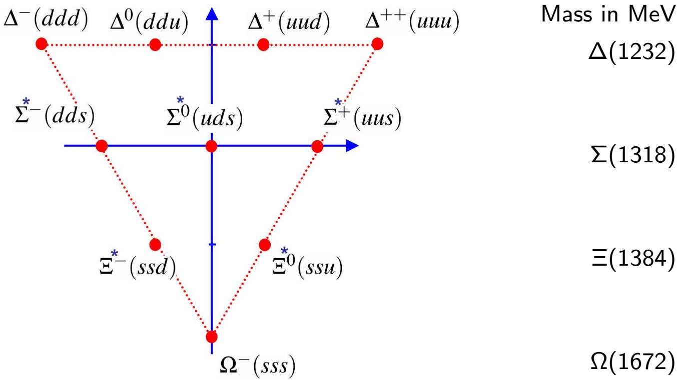
$$T_{+}\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uus - usu + usu - uus + suu - suu) = 0.$$

- In summary, the combination of three uds quarks decomposes into

$$3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \bar{3}) = 10 \oplus 8 \oplus 8 \oplus 1.$$

Baryon Decuplet ($L = 0, S = 3/2, J = 3/2, P = +1$)

- The spin-$\frac{3}{2}$ decuplet is formed from symmetric flavour and symmetric spin wave-functions $\phi_{\text{flavour}}(S)\chi_{\text{spin}}(S)$:



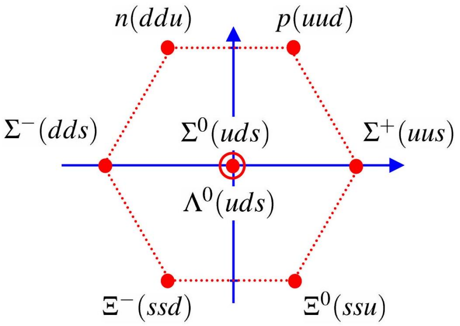
If $SU(3)$-flavour were an exact symmetry then all the masses would be the same. They are not, so $SU(3)$-flavour is a broken symmetry.

Baryon Octet ($L = 0, S = 1/2, J = 1/2, P = +1$)

- The spin 1/2 octet is formed from mixed symmetry flavour and mixed symmetry spin wave-functions

$$\alpha\phi\,(M_S)\,\chi\,(M_S) + \beta\phi\,(M_A)\,\chi\,(M_A)\,.$$

- Adapt previous discussion concerning proton to obtain wave-functions for all these:



Mass in MeV

n/p(939)

Σ(1193)

Λ(1116)

Ξ(1318)

Cannot form a totally symmetric wave-function based on the anti-symmetric flavour singlet as there no totally anti-symmetric spin wave-function for three quarks.

Summary

- We considered $SU(2)$ *ud* and $SU(3)$ *uds* flavour symmetries.
- Although these flavour symmetries are only approximate, they can still be used to explain observed multiplet structure for mesons/baryons.
- $SU(3)$ flavour symmetry results, e.g. predicted wave-functions, should be treated with a pinch of salt as $m_s \neq m_{u/d}$.
- We introduced the idea of singlet states being 'spinless' and/or 'flavourless'
- In the next handout apply these ideas to colour and QCD ...

Appendix VIII: the $SU(2)$ anti-quark representation

Not examinable

Define an anti-quark doublet

$$\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$$

from which it follows that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q} = \begin{pmatrix} u^* \\ d^* \end{pmatrix}. \quad (121)$$

The quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$ transforms as $\begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix}$ which complex conjugates to

$$\begin{pmatrix} u'^* \\ d'^* \end{pmatrix} = U^* \begin{pmatrix} u^* \\ d^* \end{pmatrix}$$

which using (121) can be re-written as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q}' = U^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q}.$$

Therefore, multiplying both sides of the last equation by the inverse of its left-most matrix, we see that $\bar{q}$ transforms as follows:

$$\bar{q}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q}. \quad (122)$$

Not examinable

Not examinable

An arbitrary 2×2 unitary matrix with unit determinant can always be written in the form

$$U = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix}$$

provided that one chooses c_{11} and c_{12} such that $|c_{11}|^2 + |c_{12}|^2 = 1$. Therefore, (122) can be re-written to express an arbitrary $SU(2)$ trasformation of $\bar{q}$ as:

$$\begin{aligned} \bar{q}' &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_{11}^* & c_{12}^* \\ -c_{12} & c_{11} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q} \\ &= \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix} \bar{q} \\ &= U\bar{q} \end{aligned}$$

which proves that the anti-quark doublet $\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$ transforms in the same way as the quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$ – thus allowing us to use the same ladder operators on q and $\bar{q}$.

This is a special property of SU(2). For $\mathrm{SU}(3)$ there is no analogous representation of the anti-quarks.

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