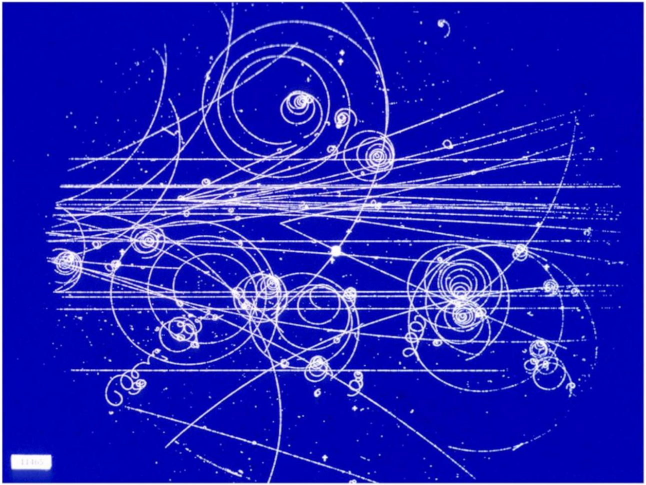
Dr C.G. Lester, 2023

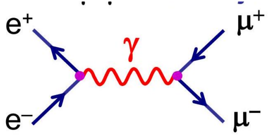


H4: Electron-Positron Annihilation

QED Calculations I

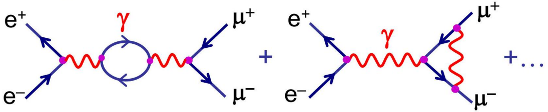
To calculate a cross section using QED (e.g. $e^+e^- \to \mu^+\mu^-$):

1. Draw all possible Feynman Diagrams
   - For $\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-$ there is just one lowest order diagram



$M \propto e^2 \propto \alpha_{em}$

   - There are many second order diagrams ...



$M \propto e^4 \propto \alpha_{em}^2$

2. For each diagram calculate the matrix element using Feynman rules derived in the previous handout.

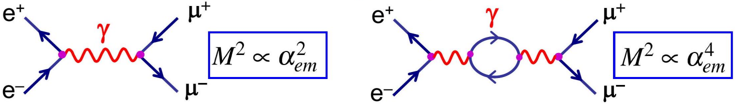
QED Calculations II

3. Sum the individual matrix elements (i.e. sum the amplitudes)

$$M_{fi} = M_1 + M_2 + M_3 + \ldots$$

Note: in this sum of amplitudes (every term of which has the same initial and the same final state) interference can be both constructive or destructive! On account of this interference one can no more ask ‘Which virtual particle was involved?’ than one can ask ‘Through which of Young’s slits did the photon pass?’

4. Find the square of the modulus:
$|M_{fi}|^2 = (M_1 + M_2 + M_3 + \ldots.)\,(M_1^* + M_2^* + M_3^* + \ldots.)$
$\rightarrow$ this gives the full perturbation expansion in α_{em}.
   - For QED $\alpha_{em} \sim 1/137$ the lowest order diagram dominates and for most purposes it is sufficient to neglect higher order diagrams.

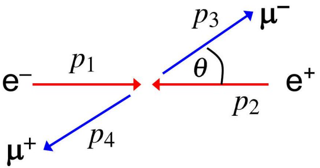


QED Calculations III

5. Calculate decay rate/cross section using formulae from Handout 1. E.g.:
   - For a decay
$$\Gamma = \frac{p^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 \, \mathrm{d}\Omega .$$
   - For scattering in the centre-of-mass frame
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s} \frac{\left|\vec{p}_f^*\right|}{\left|\vec{p}_i^*\right|} |M_{fi}|^2 .$$
   - For scattering in lab. frame (neglecting mass of scattered particle)
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{M E_1}\right)^2 |M_{fi}|^2 .$$

Electron Positron Annihilation

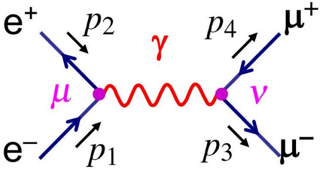
Consider the process: $e^+e^- \to \mu^+\mu^-$



Work in C.o.M. frame (this is appropriate for most e^+e^-colliders).

$$p_1 = (E, 0, 0, p) \qquad p_2 = (E, 0, 0, -p)$$
$$p_3 = (E, \vec{p}_f) \qquad p_4 = (E, -\vec{p}_f)$$

- Only consider the lowest order Feynman diagram:
    - Feynman rules give: $-iM = [\bar{v}(p_2)\, ie\gamma^\mu u(p_1)]\, \frac{-ig_{\mu\nu}}{q^2}\, [\bar{u}(p_3)\, ie\gamma^\nu v(p_4)]$



NOTE: Incoming anti-particle $\bar{v}$
Incoming particle u
Adjoint spinor written first

- In the C.o.M. frame have

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f|}{|\vec{p}_i|} |M_{fi}|^2 \quad \text{with} \quad s = (p_1 + p_2)^2 = (E+E)^2 = 4E^2$$

Electron and Muon Currents

Here $q^2 = (p_1 + p_2)^2 = s$ and matrix element

$$-iM = [\bar{v}(p_2)\, ie\gamma^\mu u(p_1)]\, \frac{-ig_{\mu\nu}}{q^2}\, [\bar{u}(p_3)\, ie\gamma^\nu v(p_4)]$$

$$\Rightarrow \quad M = -\frac{e^2}{s} g_{\mu\nu} [\bar{v}(p_2)\, \gamma^\mu u(p_1)]\, [\bar{u}(p_3)\, \gamma^\nu v(p_4)]$$

- In Handout 2 we introduced the four-vector current:

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

  which has same form as the two terms in [ ] in the matrix element
- The matrix element can be written in terms of the electron and muon currents

$$(j_e)^\mu = \bar{v}(p_2)\, \gamma^\mu u(p_1) \quad \text{and} \quad (j_\mu)^\nu = \bar{u}(p_3)\, \gamma^\nu v(p_4)$$

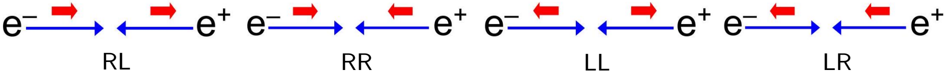
$$\Rightarrow \quad M = -\frac{e^2}{s} g_{\mu\nu} (j_e)^\mu (j_\mu)^\nu$$

$$M = -\frac{e^2}{s} j_e \cdot j_\mu$$

- Matrix element is a four-vector scalar product - confirming it is Lorentz Invariant

Spin in e^+e^- Annihilation

- In general the electron and positron will not be polarized, i.e. there will be equal numbers of positive and negative helicity states
- There are four possible combinations of spins in the initial state:



- Similarly there are four possible helicity combinations in the final state
- In total there are 16 combinations e.g. $RL \to RR, RL \to RL, \ldots$
- To account for these states we need to sum over all 16 possible helicity combinations and then average over the number of initial helicity states:

$$\left\langle |M|^2 \right\rangle = \frac{1}{4} \sum_{\text{spins}} |M_i|^2 = \frac{1}{4} \left(|M_{LL \to LL}|^2 + |M_{LL \to LR}|^2 + \ldots \right)$$

i.e. need to evaluate:

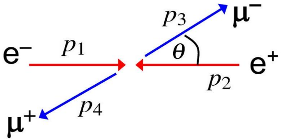
$$M = -\frac{e^2}{s} j_e \cdot j_\mu$$

for all 16 helicity combinations !

- Fortunately, in the limit $E \gg m_\mu$ only 4 helicity combinations give non-zero matrix elements - we will see that this is an important feature of QED/QCD.

- In the C.o.M. frame in the limit $E \gg m$

$$p_1 = (E, 0, 0, E);\ p_2 = (E, 0, 0, -E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta);$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$



- Left- and right-handed helicity spinors (Handout 2) for particles/anti-particles are:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix} \quad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

where $s = \sin\frac{\theta}{2}$; $\quad c = \cos\frac{\theta}{2}$ and $N = \sqrt{E+m}$

- In the limit $E \gg m$ these become:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix};\ u_\downarrow = \sqrt{E}\begin{pmatrix} -S \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix};\ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix};\ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

- The initial-state electron can either be in a left- or right-handed helicity state

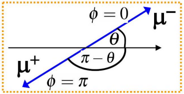
$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix};$$

- For the initial state positron $(\theta = \pi)$ can have either:

$$v_{\uparrow}(p_2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix};\, v_{\downarrow}(p_2) = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- Similarly for the final state μ^- which has polar angle θ and choosing $\phi = 0$

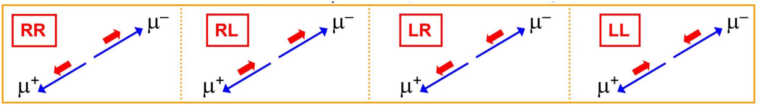
$$u_{\uparrow}(p_3) = \sqrt{E}\begin{pmatrix}c\\s\\c\\s\end{pmatrix};\, u_{\downarrow}(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix}$$



- And for the final state μ^+ replacing $\theta \to \pi - \theta; \quad \phi \to \pi$ obtain

$$v_{\uparrow}(p_4) = \sqrt{E}\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix};\, v_{\downarrow}(p_4) = \sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix}; \qquad \text{using}\begin{cases}\sin\left(\frac{\pi-\theta}{2}\right) = \cos\frac{\theta}{2}\\ \cos\left(\frac{\pi-\theta}{2}\right) = \sin\frac{\theta}{2}\\ e^{i\pi} = -1\end{cases}$$

- We wish to calculate the matrix element $M = -\frac{e^2}{s} j_e \cdot j_\mu$
- first consider the muon current j_μ for 4 possible helicity combinations



The Muon Current

- We want to evaluate $(j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$ for all four helicity combinations.
- For arbitrary spinors ψ, ϕ one may show that the components of $\bar{\psi}\gamma^\mu\phi$ are:

$$\bar{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4$$
$$\bar{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1$$
$$\bar{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i\left(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1\right)$$
$$\bar{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2$$

- Consider the $\mu_R^-\mu_L^+$ combination using $\psi = u_\uparrow$ and $\phi = v_\downarrow$ with

$$v_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix};\; u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}.$$

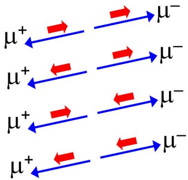
From these we can compute:

$$\bar{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) = E(cs - sc + cs - sc) = 0,$$
$$\bar{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) = E\left(-c^2 + s^2 - c^2 + s^2\right) = 2E\left(s^2 - c^2\right) = -2E\cos\theta,$$
$$\bar{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) = -iE\left(-c^2 - s^2 - c^2 - s^2\right) = 2iE, \qquad \text{and}$$
$$\bar{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) = E(cs + sc + cs + sc) = 4Esc = 2E\sin\theta.$$

- Hence the four-vector muon current for the RL combination is

$$\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta).$$

- The results for the 4 helicity combinations (obtained in the same manner) are:



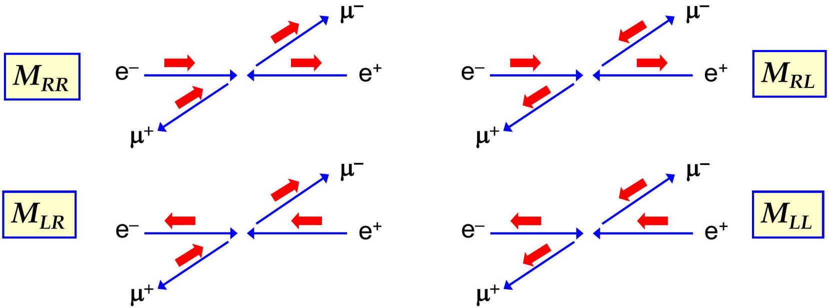
$$\begin{aligned}
\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) &= 2E(0, -\cos\theta, i, \sin\theta)\\
\bar{u}_\uparrow(p_3)\gamma^\nu v_\uparrow(p_4) &= (0,0,0,0)\\
\bar{u}_\downarrow(p_3)\gamma^\nu v_\downarrow(p_4) &= (0,0,0,0)\\
\bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) &= 2E(0, -\cos\theta, -i, \sin\theta).
\end{aligned}$$

... in the limit $E \gg m$ which was used above only two helicity combinations are non-zero!

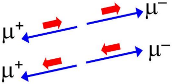
- This is an important feature of QED. It applies equally to QCD.
- In the Weak interaction only **one** helicity combination contributes.
- The origin of this will be discussed later.
- As a consequence: of the 16 possible helicity combinations only 4 give non-zero matrix elements ((2 initial) × (2 final)).

Electron Positron Annihilation cont.

- For $e^+e^- \to \mu^+\mu^-$ now only have to consider the 4 matrix elements:



- Previously we derived the muon currents for the allowed helicities:



$$\begin{aligned} \mu_R^- \mu_L^+ &: \quad \overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta) \\ \mu_L^- \mu_R^+ &: \quad \overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) = 2E(0, -\cos\theta, -i, \sin\theta) \end{aligned}$$

- Now need to consider the electron current

The Electron Current

- The incoming electron and positron spinors (L and R helicities) are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix};\, u_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}; \quad v_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix};\, v_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- The electron current can either be obtained from equations (3)-(6) as before or it can be obtained directly from the expressions for the muon current.

$$(j_e)^\mu = \bar{v}(p_2)\gamma^\mu u(p_1) \quad (j_\mu)^\mu = \bar{u}(p_3)\gamma^\mu v(p_4)$$

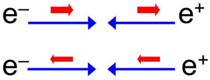
- Taking the Hermitian conjugate of the muon current gives

$$\begin{aligned}
[\bar{u}(p_3)\gamma^\mu v(p_4)]^\dagger &= \left[u(p_3)^\dagger \gamma^0\gamma^\mu v(p_4)\right]^\dagger \\
&= v(p_4)^\dagger \gamma^{\mu\dagger}\gamma^{0\dagger} u(p_3) && (AB)^\dagger = B^\dagger A^\dagger \\
&= v(p_4)^\dagger \gamma^{\mu\dagger}\gamma^0 u(p_3) && \gamma^{0\dagger} = \gamma^0 \\
&= v(p_4)^\dagger \gamma^0\gamma^\mu u(p_3) && \gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu \\
&= \bar{v}(p_4)\gamma^\mu u(p_3)
\end{aligned}$$

- Taking the complex conjugate of the muon currents for the two non-zero helicity configurations:

$$\bar{v}_{\downarrow}(p_4)\gamma^{\mu}u_{\uparrow}(p_3) = [\bar{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4)]^* = 2E(0, -\cos\theta, -i, \sin\theta)$$
$$\bar{v}_{\uparrow}(p_4)\gamma^{\mu}u_{\downarrow}(p_3) = [\bar{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4)]^* = 2E(0, -\cos\theta, i, \sin\theta)$$

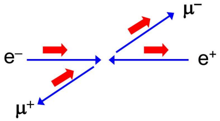
To obtain the electron currents we simply need to set $\theta = 0$



$$e_R^- e_L^+ : \quad \bar{v}_{\downarrow}(p_2)\gamma^{\nu}u_{\uparrow}(p_1) = 2E(0, -1, -i, 0)$$
$$e_L^- e_R^+ : \quad \bar{v}_{\uparrow}(p_2)\gamma^{\nu}u_{\downarrow}(p_1) = 2E(0, -1, i, 0)$$

Matrix Element Calculation

- We can now calculate $M = -\frac{e^2}{s} j_e \cdot j_\mu$ for the four possible helicity combinations M_{RR}, M_{RL}, M_{LR} and M_{LL}.
- In the above matrix element names, the first subscript refers to the helicity of the e^- and the second to the helicity of the μ^-. We don't need to specify other helicities due to 'helicity conservation', only certain chiral combinations are non-zero.
- E.g. the matrix element for $e^-_R e^+_L \to \mu^-_R \mu^+_L$ will be denoted by M_{RR}



Using:

$$e^-_R e^+_L : \quad (j_e)^\mu = \bar{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1) = 2E(0, -1, -i, 0)$$

$$\mu^-_R \mu^+_L : \quad (j_\mu)^\nu = \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta)$$

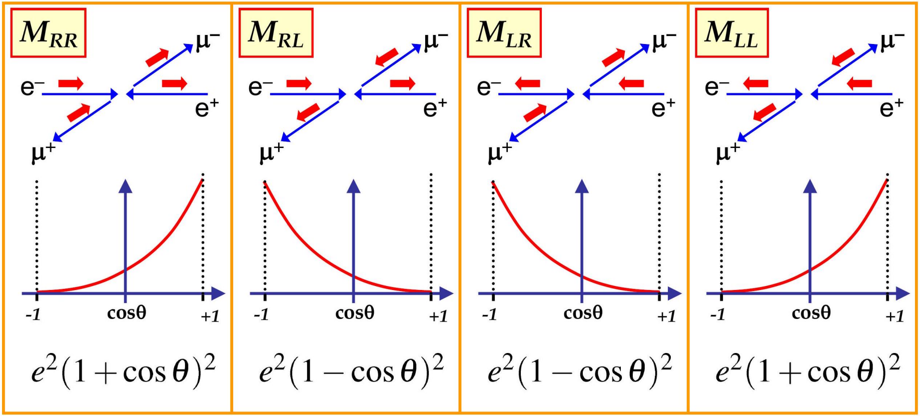
gives

$$\begin{aligned} M_{RR} &= -\frac{e^2}{s}[2E(0,-1,-i,0)] \cdot [2E(0, -\cos\theta, i, \sin\theta)] \\ &= e^2(1+\cos\theta) \\ &= 4\pi\alpha(1+\cos\theta) \end{aligned}$$

where $\alpha = e^2/4\pi \approx 1/137$.

Similarly

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$$
$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$$

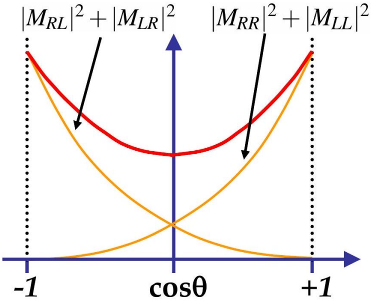


- Assuming that the incoming electrons and positrons are unpolarized, all 4 possible initial helicity states are equally likely.

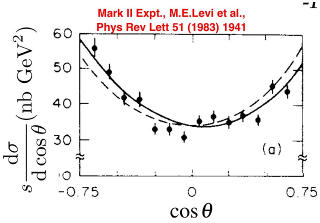
Differential Cross Section

- The cross section is obtained by averaging over the initial spin states and summing over the final spin states:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \times \frac{1}{64\pi^2 s}\left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2\right.$$
$$= \frac{(4\pi\alpha)^2}{256\pi^2 s}\left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2\right)$$
$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$



Example: $e^+e^- \to \mu^+\mu^- \quad \sqrt{s} = 29\text{GeV}$



- dashed line = pure QED
- solid line = QED plus Z contribution
- Angular distribution becomes slightly asymmetric in higher order QED or when Z contribution is included

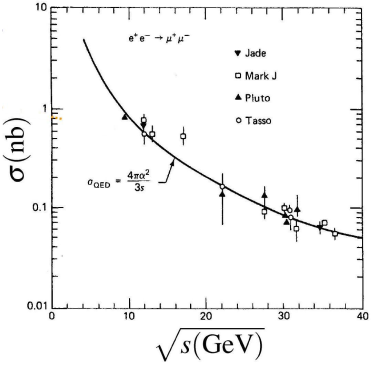
- The total cross section is obtained by integrating over θ, ϕ using

$$\int \left(1+\cos^2\theta\right) \mathrm{d}\Omega = 2\pi \int_{-1}^{+1} \left(1+\cos^2\theta\right) \mathrm{d}\cos\theta = \frac{16\pi}{3}$$

giving the QED total cross-section for the process $\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-$

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

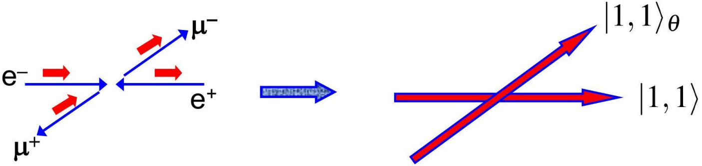
- This is an impressive result. From first principles we have arrived at an expression for the electron-positron annihilation cross section which is good to 1%



Spin Considerations ($E \gg m$)

- The angular dependence of the QED electron-positron matrix elements can be understood in terms of angular momentum
- Because of the allowed helicity states, the electron and positron interact in a spin state with $S_z = \pm 1$, i.e. in a total spin 1 state aligned along the z axis: $|1, +1\rangle$ or $|1, -1\rangle$
- Similarly the muon and anti-muon are produced in a total spin 1 state aligned along an axis with polar angle θ

e.g. $\mathbf{M_{RR}}$



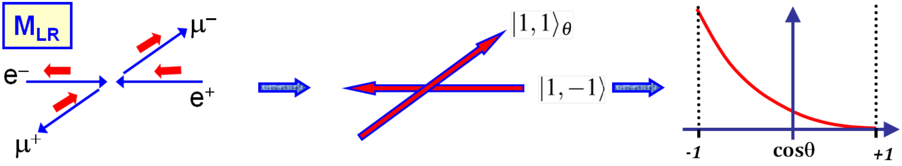
- Hence $M_{\rm RR} \propto \langle \psi \mid 1, 1\rangle$ where ψ corresponds to the spin state, $|1, 1\rangle_\theta$, of the muon pair.
- To evaluate this need to express $|1, 1\rangle_\theta$ in terms of eigenstates of S_z
- In Appendix VII (and also in IB QM) it is shown that:

$$|1, 1\rangle_\theta = \frac{1}{2}(1 - \cos\theta)|1, -1\rangle + \frac{1}{\sqrt{2}} \sin\theta |1, 0\rangle + \frac{1}{2}(1 + \cos\theta)|1, +1\rangle$$

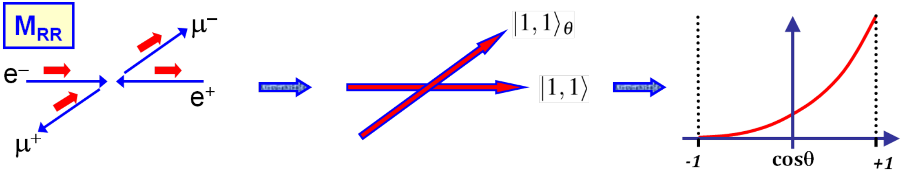
- Using the wave-function for a spin 1 state along an axis at angle θ

$$\psi = |1,1\rangle_\theta = \frac{1}{2}(1-\cos\theta)|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \frac{1}{2}(1+\cos\theta)|1,+1\rangle$$

can immediately understand the angular dependence



$$|M_{\rm RR}|^2 \propto |\langle\psi \mid 1,+1\rangle|^2 = \frac{1}{4}(1+\cos\theta)^2$$

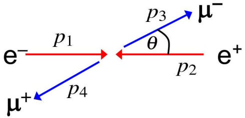


$$|M_{\rm LR}|^2 \propto |\langle\psi \mid 1,-1\rangle|^2 = \frac{1}{4}(1-\cos\theta)^2$$

Lorentz Invariant form of Matrix Element

- Before concluding this discussion, note that the spin-averaged Matrix Element derived above is written in terms of the muon angle in the C.o.M. frame.

$$\begin{aligned}\left\langle |M_{fi}|^2 \right\rangle &= \frac{1}{4} \times \left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + \left|M_{LL}^2\right| \right) \\ &= \frac{1}{4} e^4 \left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2 \right) \\ &= e^4 \left(1 + \cos^2\theta \right)\end{aligned}$$



- The matrix element is Lorentz Invariant (scalar product of 4-vector currents) and it is desirable to write it in a frame-independent form, i.e. express in terms of Lorentz Invariant 4-vector scalar products
- In the C.o.M. $\quad p_1 = (E, 0, 0, E) \quad p_2 = (E, 0, 0, -E)$

$$p_3 = (E, E\sin\theta, 0, E\cos\theta) \quad \text{and} \quad p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

giving: $\quad p_1 \cdot p_2 = 2E^2; \quad p_1 \cdot p_3 = E^2(1-\cos\theta); \quad p_1 \cdot p_4 = E^2(1+\cos\theta)$

- Hence we can write

$$\left\langle |M_{fi}|^2 \right\rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} \equiv 2e^4 \left(\frac{t^2 + u^2}{s^2} \right)$$

⋆ Valid in any frame!

CHIRALITY

The helicity eigenstates for a particle/anti-particle for $E \gg m$ (using $s = \sin\frac{\theta}{2}$ and $c = \cos\frac{\theta}{2}$) are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix};\ u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix};\ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix};\ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

- **Define** the matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- Note that in the limit $E \gg m$ the helicity states are also eigenstates of γ^5

$$\gamma^5 u_\uparrow = +u_\uparrow; \quad \gamma^5 u_\downarrow = -u_\downarrow; \quad \gamma^5 v_\uparrow = -v_\uparrow; \quad \gamma^5 v_\downarrow = +v_\downarrow$$

- For E of any size **define u_R, u_L, v_R and v_L to be the** 'LEFT AND RIGHT CHIRAL EIGENSTATES OF γ^5' by requiring that they satisfy:

$$\gamma^5 u_R = +u_R; \quad \gamma^5 u_L = -u_L; \quad \gamma^5 v_R = -v_R; \quad \gamma^5 v_L = +v_L$$

together with:

$$(u_R, u_L, v_R, v_L) = \lim_{E\to\infty}(u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow).$$

- In general the HELICITY and CHIRAL eigenstates are not the same. It is only in the ultra-relativistic limit that the chiral eigenstates correspond to the helicity eigenstates.
- Chirality is an import concept in the structure of QED, and any interaction of the form $\bar{u}\gamma^{v}u$
- Since the eigenstates of the chirality operator are:

$$\gamma^5 u_R = +u_R; \qquad \gamma^5 u_L = -u_L; \qquad \gamma^5 v_R = -v_R; \qquad \gamma^5 v_L = +v_L$$

define the projection operators:

$$P_R = \frac{1}{2}\left(1+\gamma^5\right); \quad P_L = \frac{1}{2}\left(1-\gamma^5\right).$$

- The projection operators, project out the chiral eigenstates

$$\begin{array}{llll} P_R u_R = u_R; & P_R u_L = 0; & P_L u_R = 0; & P_L u_L = u_L \\ P_R v_R = 0; & P_R v_L = v_L; & P_L v_R = v_R; & P_L v_L = 0 \end{array}$$

- Note P_R projects out right-handed particle states and left-handed anti-particle states
- We can then write any spinor in terms of it left and right-handed chiral components:

$$\psi = \frac{1}{2}\left(1+\gamma^5\right)\psi + \frac{1}{2}\left(1-\gamma^5\right)\psi = P_R\psi + P_L\psi = \psi_R + \psi_L$$

Chirality in QED

- In QED the basic interaction between a fermion and photon is:

$$ie\bar{\psi}\gamma^\mu\phi$$

We can decompose the spinors in terms of Left and Right-handed chiral components:

$$\begin{aligned} ie\bar{\psi}\gamma^\mu\phi &= ie\left(\bar{\psi}_L + \bar{\psi}_R\right)\gamma^\mu\left(\phi_R + \phi_L\right) \\ &= ie\left(\bar{\psi}_R\gamma^\mu\phi_R + \bar{\psi}_R\gamma^\mu\phi_L + \bar{\psi}_L\gamma^\mu\phi_R + \bar{\psi}_L\gamma^\mu\phi_L\right). \end{aligned}$$

- Using the properties of γ^5

$$\left(\gamma^5\right)^2 = 1; \quad \gamma^{5\dagger} = \gamma^5; \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$$

it is straightforward to show (ex. sheet Q9) that $\bar{\psi}_R\gamma^\mu\phi_L = 0$ and $\bar{\psi}_L\gamma^\mu\phi_R = 0$ so

$$ie\bar{\psi}\gamma^\mu\phi = ie\left(\bar{\psi}_R\gamma^\mu\phi_R + \bar{\psi}_L\gamma^\mu\phi_L\right).$$

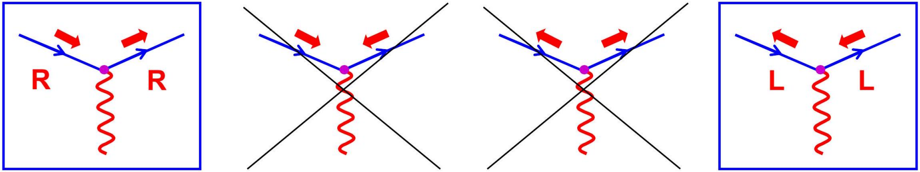
- Hence only certain combinations of chiral eigenstates contribute to the interaction. This statement is ALWAYS true.
- For $E \gg m$, the chiral and helicity eigenstates are equivalent. This implies that for $E \gg m$ only certain helicity combinations contribute to the QED vertex! This is why previously we found that for two of the four helicity combinations for the muon current were zero.

Allowed QED Helicity Combinations

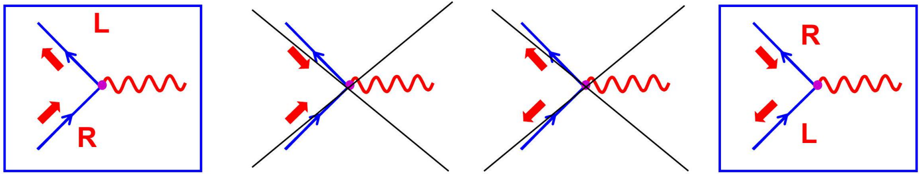
- In the ultra-relativistic limit the helicity eigenstates $\equiv$ chiral eigenstates
- In this limit, the only non-zero helicity combinations in QED are:

"Helicity conservation"

Scattering:



Annihilation:



Summary

- In the centre-of-mass frame the $\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-$ differential cross-section is

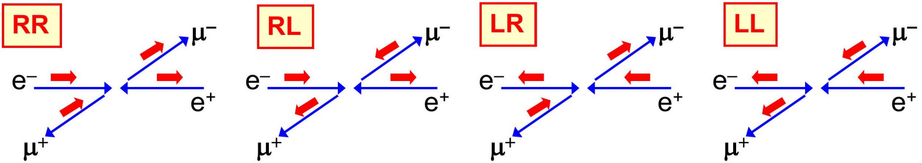
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right)$$

NOTE: neglected masses of the muons, i.e. assumed $E \gg m_\mu$

- In QED only certain combinations of LEFT- and RIGHT-HANDED CHIRAL states give non-zero matrix elements
- CHIRAL states defined by chiral projection operators

$$P_R = \frac{1}{2}\left(1 + \gamma^5\right); \quad P_L = \frac{1}{2}\left(1 - \gamma^5\right)$$

- In limit $E \gg m$ the chiral eigenstates correspond to the HELICITY eigenstates and only certain HELICITY combinations give non-zero matrix elements

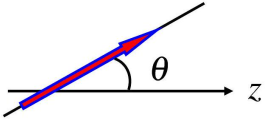


Appendix VI: Spin-1 Rotation Matrices I

- Consider the spin-1 state with spin $+1$ along the axis defined by unit vector

$$\vec{n} = (\sin\theta, 0, \cos\theta)$$

- Spin state is an eigenstate of $\vec{n}\cdot\vec{S}$ with eigenvalue $+1$



$$(\vec{n}.\vec{S})|\psi\rangle = +1|\psi\rangle \tag{96}$$

- Express in terms of linear combination of spin 1 states which are eigenstates of S_z

$$|\psi\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$$

with

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

Not examinable

Appendix VI: Spin-1 Rotation Matrices II

- (96) becomes:

$$(\sin\theta S_x + \cos\theta S_z)\left(\alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle\right) = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle \quad (97)$$

- Write S_x in terms of ladder operators $S_x = \frac{1}{2}(S_+ + S_-)$ where

$$S_+|1,1\rangle = 0 \quad S_+|1,0\rangle = \sqrt{2}|1,1\rangle \quad S_+|1,-1\rangle = \sqrt{2}|1,0\rangle$$

$$S_-|1,1\rangle = \sqrt{2}|1,0\rangle \quad S_-|1,0\rangle = \sqrt{2}|1,-1\rangle \quad S_-|1,-1\rangle = 0$$

- from which we find $S_x|1,1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle$
- (97) becomes

$$S_x|1,0\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$$
$$S_x|1,-1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle$$

$$\sin\theta\left[\frac{\alpha}{\sqrt{2}}|1,0\rangle + \frac{\beta}{\sqrt{2}}|1,-1\rangle + \frac{\beta}{\sqrt{2}}|1,1\rangle + \frac{\gamma}{\sqrt{2}}|1,0\rangle\right] +$$
$$\alpha\cos\theta|1,1\rangle - \gamma\cos\theta|1,-1\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$$

Appendix VI: Spin-1 Rotation Matrices III

- which gives

$$\left.\begin{array}{r} \beta\frac{\sin\theta}{\sqrt{2}} + \alpha\cos\theta = \alpha \\ (\alpha+\gamma)\frac{\sin\theta}{\sqrt{2}} = \beta \\ \beta\frac{\sin\theta}{\sqrt{2}} - \gamma\cos\theta = \gamma \end{array}\right\}.$$

- Using $\alpha^2 + \beta^2 + \gamma^2 = 1$ the above equations yield

$$\alpha = \frac{1}{\sqrt{2}}(1+\cos\theta) \quad \beta = \frac{1}{\sqrt{2}}\sin\theta \quad \gamma = \frac{1}{\sqrt{2}}(1-\cos\theta)$$

- hence

$$\psi = \frac{1}{2}(1-\cos\theta)|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \frac{1}{2}(1+\cos\theta)|1,+1\rangle.$$

- The coefficients α, β, γ are examples of what are known as quantum mechanical rotation matrices. The express how angular momentum eigenstate in a particular direction is expressed in terms of the eigenstates defined in a different direction $d^j_{m',m}(\theta)$.
- For spin-1 ($j = 1$) we have just shown that

$$d^1_{1,1}(\theta) = \frac{1}{2}(1+\cos\theta) \quad d^1_{0,1}(\theta) = \frac{1}{\sqrt{2}}\sin\theta \quad d^1_{-1,1}(\theta) = \frac{1}{2}(1-\cos\theta).$$

Appendix VI: Spin-1 Rotation Matrices IV

- For spin-1/2 it is straightforward to show

$$d^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta) = \cos\frac{\theta}{2} \quad d^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\theta) = \sin\frac{\theta}{2}.$$

Not examinable