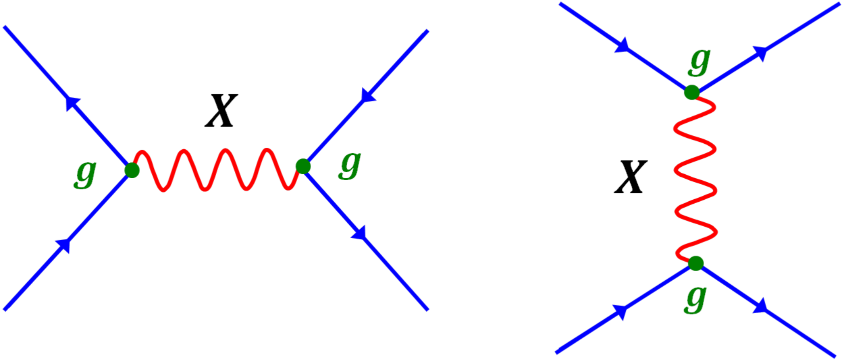
Dr C.G. Lester, 2023

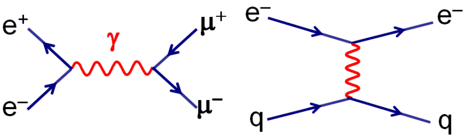


H3: Interaction by Particle Exchange and QED

Reminder, and plan for this handout

We are working towards a proper calculation of decay and scattering processes Initially concentrate on:

- $e^+e^- \to \mu^+\mu^-$
- $e^-q \to e^-q$



This handout concentrates on the Lorentz Invariant Matrix Element M_{fi}.

- Considerably more has to be taken on trust in this handout than in the previous handouts. All motivational information has the status of plausibility arguments, at best. It is not a substitute for the QFT course which is lectured in parallel.

The main areas which will be covered in this handout are:

- the meaning of the phrases 'interaction by particle exchange' and 'virtual particle';
- a take-it-on-trust introduction to Feynman diagrams;
- a description of the Feynman rules for tree-level QED; and
- use of those rules to work out scattering cross sections for processes like those shown above.

Interaction by Particle Exchange

- We previously noted that we **calculate transition rates from Fermi's Golden Rule**

$$\Gamma_{fi} = 2\pi \left| T_{fi} \right|^2 \rho(E_f)$$

where T_{fi} is perturbation expansion for the Transition Matrix Element

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \ldots .$$

- A relativistic normalisation of states was defined on page 36: $\psi' = \sqrt{2E}\sqrt{V}\psi$.
- On page 37 a Lorentz Invariant matrix element M_{fi} was defined in terms of the non-Lorentz Inviariabnt matrix element T_{fi} needed by Fermi's Golden Rule:

$$M_{fi} = \frac{1}{V} \left\langle \psi'_1 \psi'_2 \ldots \middle| \hat{H}_{\text{int}} \middle| \ldots \psi'_{N-1} \psi'_N \right\rangle$$

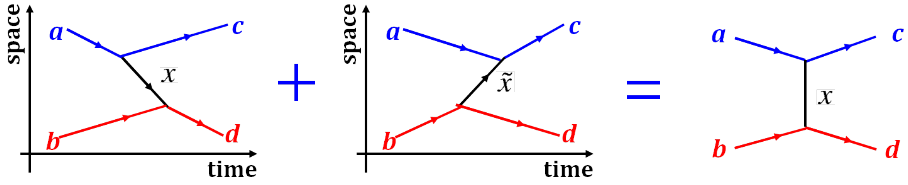
- We provide some (non-examinable) evidence in `https://www.hep.phy.cam.ac.uk/~lester/teaching/partIIIparticles/Propagators.pdf` as to why the above ingredients tends to result in M_{fi} containing terms resembling

$$M_{fi} \sim \frac{g_a g_b}{q^2 - m_X^2}$$

when the scattering between two particles a and b is caused by the 'exchange' of a virtual particle whose non-virtual mass (i.e. if it were it on shell) is m_X, and if q^μ is the four-momentum of the virtual particle.

Summary: the sum of all time-ordered momentum-non-conserving ‘exchanged’ real particles looks like one momentum-conserving ‘virtual’ particle.

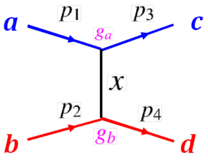
- The sum over all possible time-orderings is represented by a Feynman diagram



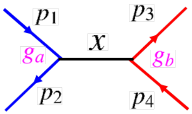
- Momentum is not conserved at any vertex in the **time-ordered diagram** (FGR did not ask for this in its sums over states!), but the exchanged particles there all have their ‘real’ masses, ($E_X^2 - p_X^2 = m_X^2$ or ‘on mass shell’) but . . .
- . . . the **virtual particles in the Feynman Diagrams** have ended up conserving momentum at each vertex, albeit at the cost of having the ‘wrong’ masses ($E_X^2 - p_X^2 = q^2 \neq m_X^2$ or “off mass shell”).
- A ‘propagator’, i.e. a factor like $\frac{1}{q^2 - m_X^2}$ arises naturally in association with each virtual particle in a Feynman diagram.

Sign of q^2

A matrix element like $M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$ depends on the four-momentum, q, carried by the (virtual) particle which is determined from energy/momentum conservation at the vertices. Note that q^2 can be either positive or negative:



Here $q^2 = (p_1 - p_3)^2 = (p_4 - p_2)^2 = t$.
For elastic scattering: $p_1 = (E, \vec{p}_1)$ and $p_3 = (E, \vec{p}_3)$ so
$q^2 = (E - E)^2 - (\vec{p}_1 - \vec{p}_3)^2 < 0$ and so this is termed 'space-like' t-channel scattering.

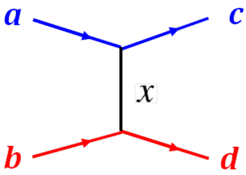


Here $q^2 = (p_1 + p_2)^2 = (p_3 + p_4)^2 = s$.
In C.o.M. $p_1 = (E, \vec{p})$ and $p_2 = (E, -\vec{p})$ so
$q^2 = (E + E)^2 - (\vec{p} - \vec{p})^2 = 4E^2 > 0$ and so this is termed 'time-like' s-channel annihilation.

Aside: V(r) from Particle Exchange

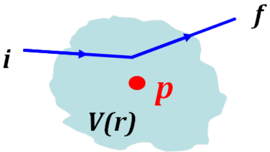
One can view the scattering of an electron by a proton at rest in two ways:

(1) As an interaction by particle exchange in 2nd order perturbation theory.



$$M_{fi} = \frac{g_a g_b}{q^2 - m_X^2}.$$

(2) As a process in first order perturbation theory treating proton as a fixed source of a field which gives rise to a potential V(r).



$$M_{fi} = \langle \psi_f | V(r) | \psi_i \rangle .$$

One obtains the same expression for M_{fi} in both cases if one uses a ‘Yukawa Potential’:

$$V(r) = g_a g_b \frac{e^{-mr}}{r}.$$

In this way, one can relate potential and forces to the particle exchange picture. However, scattering from a fixed potential $V(r)$ is not a relativistic invariant view!

Quantum Electrodynamics (QED) from semi-classical / historical perspective

The basic interaction between a photon and a charged particle may have been introduced by making the minimal substitution mentioned earlier (see Part II Electrodynamics) i.e. via:

$$i\partial_\mu \to i\partial_\mu - qA_\mu$$

where $A_\mu = (\phi, -\vec{A})$. As we saw ealier, this leads to the Dirac equation changing from

$$\gamma^\mu \partial_\mu \psi + im\psi = 0$$

to

$$\gamma^\mu \partial_\mu \psi + iq\gamma^\mu A_\mu \psi + im\psi = 0$$

or equivalently

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\vec{\gamma} \cdot \vec{\nabla}\psi - q\gamma^\mu A_\mu \psi - m\psi = 0$$

thus

$$i\gamma^0 \frac{\partial \psi}{\partial t} = -i\vec{\gamma} \cdot \vec{\nabla}\psi + q\gamma^\mu A_\mu \psi + m\psi$$

and so $(\times\gamma^0)$

$$i\frac{\partial \psi}{\partial t} = -i\gamma^0\vec{\gamma} \cdot \vec{\nabla}\psi + m\gamma^0\psi. + q\gamma^0\gamma^\mu A_\mu \psi. \tag{95}$$

We recognise

$$i\frac{\partial \psi}{\partial t} = -i\gamma^0\vec{\gamma}\cdot\vec{\nabla}\psi + m\gamma^0\psi. + q\gamma^0\gamma^\mu A_\mu\psi \qquad (95)$$

as the Schroedinger Equation:

$$i\frac{\partial \psi}{\partial t} = \hat{H}\psi$$

with

$$\hat{H} = \underbrace{\left(\gamma^0 m - i\gamma^0\vec{\gamma}\cdot\vec{\nabla}\right)}_{\text{Combined rest mass + K.E.}} + \underbrace{q\gamma^0\gamma^\mu A_\mu}_{\text{Potential or interaction energy}}$$

i.e. we can infer that the operator associated with the interaction/potential energy of a charged spin-half particle in an electromagnetic field, $\hat{H}_{\text{int}}$, might resemble:

$$\hat{H}_{\text{int}} \sim q\gamma^0\gamma^\mu A_\mu.$$

i.e. we expect the $\left\langle i\middle|\hat{H}_{\text{int}}\middle|j\right\rangle$ terms in F.G.R. to contain expressions like:

$$\left\langle \Psi_i\middle|q\gamma^0\gamma^\mu A_\mu\middle|\Psi_j\right\rangle$$

or, using the definition of the adjoint spinor $\bar{\psi} = \psi^\dagger\gamma^0$:

$$A_\mu \boxed{\left\langle \bar{\Psi}_i\middle|q\gamma^\mu\middle|\Psi_j\right\rangle}.$$

Alas, there are many complications concerning A_μ that are beyond this course (but which relate to the fact that A_μ is able/needed to encode photon polarisations, but can also encode unphysical things on account of gauge invariance). E.g. for a real photon propagating in the z direction

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(p_z z - Et)}$$

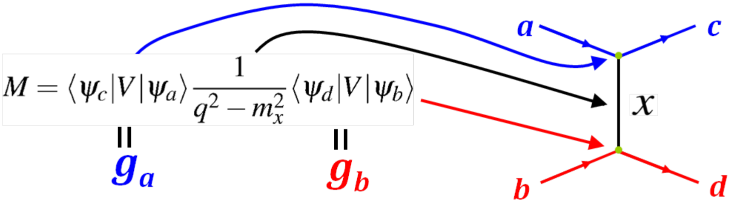
we have two orthogonal transverse polarization states in some 'gauges' (see Appendix XI much later) :

$$\boldsymbol{\varepsilon}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \varepsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The area concerning photon polarisation spin-sums $\sum_\lambda \varepsilon_\mu^\lambda \left(\varepsilon_\nu^\lambda\right)^*$ is very complicated and needs an entire course on **Gauge Field Theories**. (Consider reading around eq (4.66) in Michio Kaku's "Quantum Field Theory: a modern introduction" if you want an inkling of what is involved ...)

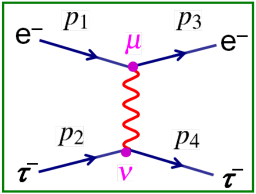
Suffice it to say that after the dust has settled, terms like those shown on the next slide are those which appear in expansions of the M_{fi} terms needed by Fermi's Golden Rule ...

That which schematically was like this:



becomes like this for the case of electron-tau scattering via a photon:

$$M = \underbrace{\left[\bar{u}_e\left(p_3\right) q_e \gamma^{\mu} u_e\left(p_1\right)\right]}_{\text{Interaction of } e \text{ with photon}} \frac{-g_{\mu\nu}}{q^2 - 0^2} \underbrace{\left[\bar{u}_\tau\left(p_4\right) q_\tau \gamma^{\nu} u_\tau\left(p_2\right)\right]}_{\text{Interaction of } \tau \text{ with photon}}$$



Our claimed first-order Matrix Element for electron-tau scattering:

$$M = \left[\bar{u}_e\left(p_3\right) q_e \gamma^\mu u_e\left(p_1\right)\right] \frac{-g_{\mu\nu}}{q^2 - 0^2} \left[\bar{u}_\tau\left(p_4\right) q_\tau \gamma^\nu u_\tau\left(p_2\right)\right]$$

is a remarkably simple expression! It was shown in Appendix V that $\bar{u}_1 \gamma^\mu u_2$ transforms as a four vector, so writing

$$j_e^\mu = \bar{u}_e\left(p_3\right) \gamma^\mu u_e\left(p_1\right)$$

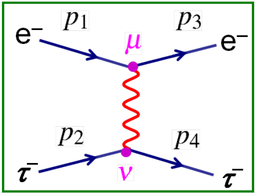
and

$$j_\tau^\nu = \bar{u}_\tau\left(p_4\right) \gamma^\nu u_\tau\left(p_2\right)$$

we have

$$M = -q_e q_\tau \frac{j_e \cdot j_\tau}{q^2}$$

making the Lorentz Invariance of M more easily visible.



Old Fashioned Time-Ordered Perturbation Theory vs Feynman Rules

- Even though we did not deduce the following leading-order expression very rigourously:

$$M = \left[\bar{u}_e\left(p_3\right) q_e \gamma^{\mu} u_e\left(p_1\right)\right] \frac{-g_{\mu\nu}}{q^2} \left[\bar{u}_\tau\left(p_4\right) q_\tau \gamma^{\nu} u_\tau\left(p_2\right)\right]$$

  we made attempts to illuatrate how it was obtained as a sum over all possible time orderings of the virtual photon – and as a sum over the photon polarisations (though these were discussed even less!). Calculations of that sort (but done rigorously) are now called calcuations in 'old fashioned time ordered perturbation theory'.
- Fortunately, an amazing result, first intuited by Feynman, then later proved by Schwinger and Dyson, is that there is a much simpler way that is provably equivalent albeit mysterious. **We can just write down any matrix element by using a set of simple rules, so called 'Feynman Rules' – see next page!**

Basic Feynman Rules for QED

External Lines

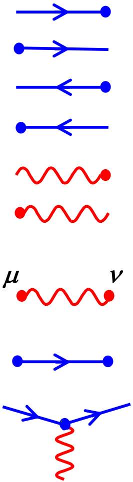
spin 1/2	incoming particle	$u(p)$	
	outgoing particle	$\bar{u}(p)$	
	incoming antiparticle	$\bar{v}(p)$	
	outgoing antiparticle	$v(p)$	
spin 1	incoming photon	$\varepsilon_\mu(p)$	
	outgoing photon	$\varepsilon_\mu(p)^*$	

Internal Lines (propagators)

spin 1	photon	$-\dfrac{ig_{\mu\nu}}{q^2}$	
spin 1/2	fermion	$\dfrac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$	

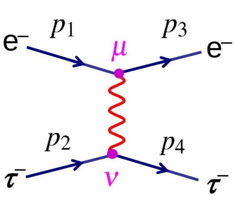
Vertex Factors

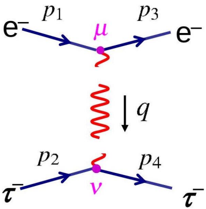
spin 1/2	fermion (charge -$\|e\|$)	$ie\gamma^\mu$	



Matrix Element $-iM$ = product of all factors

e.g.





$$\overline{u}_e(p_3)[ie\gamma^\mu]u_e(p_1)$$

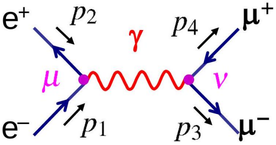
$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\overline{u}_\tau(p_4)[ie\gamma^\nu]u_\tau(p_2)$$

$$-iM = [\overline{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}_\tau(p_4)ie\gamma^\nu u_\tau(p_2)]$$

as we obtained previously. Or, we could look at an entirely different process,

e.g.



$$-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$$

NOTE:

- Each fermion line is traversed 'backwards' (i.e. against the arrows) from adjoint spinor to ordinary spinor via any vertices inbetween in the order they are encountered!

Summary

- Interaction by particle exchange naturally gives rise to Lorentz Invariant Matrix Element of the form

$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Derived the basic interaction in QED taking into account the spins of the fermions and polarization of the virtual photons:

$$-iM = [\bar{u}(p_3)\, ie\gamma^{\mu}\, u(p_1)]\, \frac{-ig_{\mu\nu}}{q^2}\, [\bar{u}(p_4)\, ie\gamma^{\nu}\, u(p_2)]$$

- We now have all the elements to perform proper calculations in QED!