# Appendix III: Dimensions of the Dirac Matrices I



In a $d$-dimensional spacetime there will always be $d$ gamma matrices, as one is associated with each spacetime derivative in the Hamiltonian. That is why in 4-dimensional spacetime we have four gamma matrices: $\gamma_0$, $\gamma_1$, $\gamma_2$ and $\gamma_3$.

But why does $d = 4$ force those matrices to be $(4 \times 4)$-matrices ?

Rather than answer the above question, we instead state (and later prove) the more general result (47) linking the $(n \times n)$ size of gamma matrices to the number $d$ of spacetime dimension with which they are associated:

$$n = 2^{\lfloor \frac{d}{2} \rfloor}. \tag{47}$$

The result (47) is a direct consequence of the gamma matrices having to satisfy (as we already saw in (27)) the defining property of a (so called) 'Clifford Algebra', namely that:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} 1_{n \times n}. \tag{48}$$

**Warning:** the proof we provide for the above statement relies on Schur's Lemma. This may be a source of dissatisfaction for some persons taking the course because **Schur's Lemma**, although stated in the Groups and Representations section of the Part IB Mathematics course within Natural Sciences Tripos, was stated in that coure without proof. If you find that annoying, you will have to find an alternative proof.

# Appendix III: Dimensions of the Dirac Matrices II



## Aside on size of Pauli matrices:

Although we are mainly interested in proving (47) to substantiate the claim that each $\gamma^\mu$ is a $(4 \times 4)$-matrix, we note that the same result can be used to explain why the Pauli matrices are $(2 \times 2)$-matrices. The reason is that the three ($d = 3$) Pauli matrices satisfy their own equivalent of (48), namely: $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$. Hence $n = 2^{\lfloor 3/2 \rfloor} = 2^1 = 2$.

We wish to prove the result stated in (47) is the relationship between the dimension $d$ of spacetime and the dimension $n$ of the (irreducible) $(n \times n)$ irreducible matrices $\gamma_\mu$ satisfying (48) with $\mu, \nu = 0, 1, \cdots, d-1$. Conveniently, the relationship (47) between $n$ and $d$ which we seek to prove does not depend on the signature of the metric since it is possible to convert a representation designed for one signature (say $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$) to another (say $g_{\mu\nu} = \mathrm{diag}(+,+,+,+)$) without changing $n$ by multiplying appropriate $\gamma$-matrices by $i = \sqrt{-1}$.
Therefore, without loss of generality, we actually take as our start point the simplest possibility, namely:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \cdot 1_{n \times n}. \qquad (49)$$

We nonetheless demand that the $\gamma$-matrices are irreducible – i.e. that there is not a similarity transformation that would reduce them all to a (non-trivial) block diagonal form. We start by noting that with those assumptions:

# Appendix III: Dimensions of the Dirac Matrices III



- **Every $\gamma^\mu$ is invertible**. [To prove this simply set $\mu = \nu$ in (49) and take the determinant of both sides.]
- **For the matrix $\gamma^* \equiv \gamma^0\gamma^1\cdots\gamma^{d-1}$ we have**

$$\gamma^*\gamma^\mu = (-1)^{d-1}\gamma^\mu\gamma^*. \tag{50}$$

[Proof: When $\gamma^\mu$ commutes with $\gamma^*$ it must pass $d-1$ dissimilar $\gamma$-matrices and a single 'identical' $\gamma$-matrix. Given (49) there are therefore $d-1$ anti-commutations and a single commutation. $\square$]

- **The matrix $\gamma^* \equiv \gamma^0\gamma^1\cdots\gamma^{d-1}$ squares to either +1 or -1 depending on $d$.** [Proof: it takes $\frac{1}{2}(d-1)d$ flips of adjacent pairs to reverse the order of $d$ objects, and since all the $\gamma$-matrices in $\gamma^*$ are dissimilar and thus anti-commute we can deduce that

$$\gamma^* \equiv \gamma^0\gamma^1\cdots\gamma^{d-1} = (-1)^{\frac{1}{2}(d-1)d}\cdot\gamma^{d-1}\cdots\gamma^1\gamma^0$$

# Appendix III: Dimensions of the Dirac Matrices IV



and so

$$\begin{aligned}(\gamma^*)^2 &= (-1)^{\frac{1}{2}(d-1)d} \cdot (\gamma^{d-1} \cdots \gamma^1 \gamma^0) \cdot (\gamma^0 \gamma^1 \cdots \gamma^{d-1}) \\ &= (-1)^{\frac{1}{2}(d-1)d} \prod_{\mu=0}^{d-1} \delta^{\mu\mu} \\ &= (-1)^{\frac{1}{2}(d-1)d} \\ &= s(d) \qquad (51)\end{aligned}$$

in which $s(d) \equiv (-1)^{\frac{1}{2}(d-1)d}$ is a $d$-dependent sign in $\{+1, -1\}$.]

- **If $d > 1$ then $n$ must be even.** [To prove this, consider $\mu \neq \nu$ (which requires $d > 1$) in (49). In this case (49) becomes $\gamma^\mu \gamma^\nu = -\gamma^\mu \gamma^\mu$ which implies that $\det\{\gamma^\mu\} \det\{\gamma^\nu\} = (-1)^n \det\{\gamma^\nu\} \det\{\gamma^\mu\}$ which (since every $\gamma^\mu$ is invertible) implies that $1 = (-1)^n$ and thus that $n$ is even. ]

# Appendix III: Dimensions of the Dirac Matrices V



- **Theorem A: Any product of any number of $\gamma$-matrices may (up to a sign) be written as a product of at most $d$ gamma matrices in strictly ascending order of their indices.** [This is because (49) states that dissimilar $\gamma$-matrices anti-commute, and that individual $\gamma$-matrices square to $\pm 1$'. Therefore, an arbitrary product of $\gamma$-matrices can always have its $\gamma$-matrices permuted into numerical order (with a sign change if an odd number of permutations is required) leaving at most one copy of each $\gamma$-matrix as repeats will disappear (up to a sign) on account of the squaring property.]

The last result above motivates the following definition.

## Definition

If $A$ is any integer whose binary representation modulo $2^d$ is $\vec{A}$, i.e. if ($A$ mod $2^d$) $= \sum_{i=0}^{d-1} A_i \cdot 2^i$ with each $A_i \in \{0, 1\}$, then define $\Gamma_A$ by

$$\Gamma_A = \prod_{i=0}^{d-1} \left\{ \begin{matrix} \gamma_i & \text{if } A_i = 1 \\ 1 & \text{otherwise} \end{matrix} \right\}. \tag{52}$$

For example, this definition would make $\Gamma_{13} = \gamma_0\gamma_2\gamma_3$ since $13 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3$.

# Appendix III: Dimensions of the Dirac Matrices VI



On account of the modulo $2^d$ part of the definition, any continuous range of indices of length $2^d$ would suffice to include every such Γ-matrix. Without loss of generality will always take indices $A$ to be in the set

$$\mathcal{A} = \{1, 2, \cdots, 2^d\},$$

and mapped into that range, if necessary, by an implicit modulo $2^d$ operation. We therefore define a complete list, $L$, of Γ-matrices as follows:

$$L = (\Gamma_1, \Gamma_2, \ldots, \Gamma_{2^d}) = (\Gamma_A \mid A \in \mathcal{A}). \tag{53}$$

Note that although we have defined $2^d$ quantities $\Gamma_A$ in the list $L$ we have not shown that they are all unique. In other words, we cannot assume '$(A \neq B) \implies (\Gamma_A \neq \Gamma_B)$' or '$(\Gamma_A = \Gamma_B) \implies (A = B)$' unless later proved.

We now state and prove two important properties of the Γ-matrices:
The most general form of this Lemma is

$$\mathrm{Tr}\left[\Gamma_A\right] = \begin{cases} n & \text{if } A = 0 \mod 2^n \\ 0 & \text{if } (A \neq 0 \mod 2^n) \text{ and } (d \text{ is even or } \sum_{i=1}^{d} A_i \text{ is even}) \\ Tr[\Gamma_A] & \text{otherwise.} \end{cases} \tag{54}$$

## Appendix III: Dimensions of the Dirac Matrices VII



Alternatively, a narrower form could be stated as follows

$$\text{When } d \text{ is even:} \qquad \mathrm{Tr}\left[\Gamma_A\right] = \begin{cases} n & \text{if } A = 0 \mod 2^n \\ 0 & \text{otherwise.} \end{cases} \tag{55}$$

The trace of $\Gamma_0$ is always trivially $n$ as $\Gamma_0 = 1_{n\times n}$. Every other $\Gamma_A$ is the product of one or more dissimilar $\gamma$-matrices. We split the remainder of the proof into two parts: part (i) shows that traces of products are zero where the remaining products contain an **even** number of $\gamma$-matrices, while part (ii) shows the same for products containing any **odd** number of $\gamma$-matrices. Note the subtle differences between these two parts of of the proof: the first needs to assume that the multiplied gammas are **distinct** but does not need to worry about whether $d$ is even or odd. In contrast the second does not care about distinctness in the gammas but **needs to assume that $d$ is even**. If $k$ is an integer

## Appendix III: Dimensions of the Dirac Matrices VIII



greater than zero, and if $a_1, a_2, \ldots, a_k$ are $k$ **distinct** integers in $[0, d-1]$ and if $T = \mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right]$ then

$$\begin{aligned} T &= \mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] \\ &= (-1)^{k-1}\cdot\mathrm{Tr}\left[\gamma_{a_k}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\right] \\ &\qquad\qquad (\text{after } k-1 \text{ anti-commutations using (49) and } k>0) \\ &= (-1)^{k-1}\cdot\mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] \qquad (\text{trace cyclicity}) \\ &= (-1)^{k-1}\cdot T \end{aligned}$$

therefore:

$$\text{“The trace of the product of an } \textbf{even} \text{ number of } \textbf{distinct } \gamma\text{-matrices} \ldots$$
$$\ldots \text{is zero provided the even number is greater than or equal to two”.} \qquad (56)$$

## Appendix III: Dimensions of the Dirac Matrices IX



If $k$ is an integer greater than zero, and if $a_1, a_2, \ldots, a_k$ are $k$ integers in $[0, d-1]$ and if $T = \mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right]$ then

$$\begin{aligned}
& & T &= \mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] & \\
\implies & & s(d)\cdot T &= \mathrm{Tr}\left[(\gamma^*\gamma^*)\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] & \text{(by (51))} \\
\implies & & s(d)\cdot T &= \mathrm{Tr}\left[\gamma^*\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\gamma^*\right] & \text{(trace cyclicity)} \\
\implies & & s(d)\cdot T &= ((-1)^{d-1})^k\cdot\mathrm{Tr}\left[\gamma^*\gamma^*\gamma_{a_k}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\right] & \text{(after } k \text{ uses of (50))} \\
\implies & & \cdot T &= (-1)^{k(d-1)}\cdot\mathrm{Tr}\left[\gamma_{a_k}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\right] & \text{(by (51) again)} \\
\implies & & \cdot T &= (-1)^{k(d-1)}\cdot T &
\end{aligned}$$

therefore:

"when $d$ is even, the trace of the product of an odd number of $\gamma$-matrices is zero". (57)

This concludes our proof of Lemma 1. □

$$\Gamma_A\Gamma_B = s(A,B)\cdot\Gamma_{A\oplus B} \tag{58}$$

## Appendix III: Dimensions of the Dirac Matrices X



in which '$\oplus$' represents 'BITWISE EXCLUSIVE OR' and $s(A, B)$ is a function mapping pairs of indices to the set $\{+1, -1\}$.

$$\begin{aligned}
\Gamma_A \Gamma_B &= \prod_{i=0}^{d-1} \left\{ \begin{matrix} \gamma_i & \text{if } A_i = 1 \\ 1 & \text{otherwise} \end{matrix} \right\} \prod_{i=0}^{d-1} \left\{ \begin{matrix} \gamma_i & \text{if } B_i = 1 \\ 1 & \text{otherwise} \end{matrix} \right\} \\
&= s_1(A, B) \prod_{i=0}^{d-1} \left( \left\{ \begin{matrix} \gamma_i & \text{if } A_i = 1 \\ 1 & \text{otherwise} \end{matrix} \right\} \left\{ \begin{matrix} \gamma_i & \text{if } B_i = 1 \\ 1 & \text{otherwise} \end{matrix} \right\} \right)
\end{aligned}$$

# Appendix III: Dimensions of the Dirac Matrices XI



where $s_1(A,B) \in \{+1,-1\}$ is a sign which will depend on how many anti-commutations deriving from (49) were needed to re-order the matrices, and so

$$\begin{aligned}
\Gamma_A\Gamma_B &= s_1(A,B)\prod_{i=0}^{d-1}\left\{\begin{array}{ll}(\gamma_i)^2 & \text{if } A_i = B_i = 1\\ \gamma_i & \text{if } A_i \oplus B_i = 1\\ 1 & \text{otherwise}\end{array}\right\}\\
&= s_1(A,B)\prod_{i=0}^{d-1}\left\{\begin{array}{lll}g_{ii} & \text{(no sum } i) & \text{if } A_i = B_i = 1\\ \gamma_i & & \text{if } A_i \oplus B_i = 1\\ 1 & & \text{otherwise}\end{array}\right\} \qquad \text{(by (49))}\\
&= s(A,B)\prod_{i=0}^{d-1}\left\{\begin{array}{ll}1 & \text{if } A_i = B_i = 1\\ \gamma_i & \text{if } A_i \oplus B_i = 1\\ 1 & \text{otherwise}\end{array}\right\}
\end{aligned}$$

# Appendix III: Dimensions of the Dirac Matrices XII



where $s(A,B)$ is a new sign function that accounts for our having replaced $g_{ii}$ with 1, and so

$$\begin{aligned}\Gamma_A\Gamma_B &= s(A,B)\prod_{i=0}^{d-1}\left\{\begin{matrix}\gamma_i & \text{if } A_i \oplus B_i = 1\\ 1 & \text{otherwise}\end{matrix}\right\}\\ &= s(A,B)\Gamma_{A\oplus B} \qquad \square.\end{aligned}$$

A corollary of (58) is that every $\Gamma$-matrix is invertible. [Proof: setting $B$ equal to $A$ in (58) tells us that $(\Gamma_A)^2 = s(A,A)\cdot\Gamma_0 = s(A,A)\cdot 1_{n\times n} = \pm 1_{n\times n}$ and so

$$(\Gamma_A)^{-1} \text{ is either } \Gamma_A \text{ or } -\Gamma_A. \tag{59}$$

]
Perhaps we can do better. Suppose $A$ has $a$ ones in its binary representation (i.e. $a = \sum_{i=0}^{d-1} A_i$ so that $\Gamma_A$ is a product of $a$ gamma matrices in ascending order of index). If we then square $\Gamma_A$ we could attempt to permute adjacent gamma matrices within the product so as to annihilate every identical pairing, leaving behind only a sign. This process would require $a-1$ anticommutations to annihilate the first pair, $a-2$ the

## Appendix III: Dimensions of the Dirac Matrices XIII



second, *etc*, and none for the last. This is a total of $\frac{1}{2}(a-1)a$ anticommutations, and so we can make the very specific claim that

$$(\Gamma_A)^2 = (-1)^{\frac{1}{2}(a-1)a} \tag{60}$$

or equivalently

$$(\Gamma_A)^{-1} = (-1)^{\frac{1}{2}(a-1)a} \cdot \Gamma_A. \tag{61}$$

Indeed, we see that the already derived result (51) could be viewed with hindsight as a simple corollary of (60).

Knowing that the Γ-matrices are all invertible we may define a matrix $S$ as follows:

$$S = \sum_{X \in \mathcal{A}} (\Gamma_X)^{-1} \cdot Y \cdot \Gamma_X \tag{62}$$

## Appendix III: Dimensions of the Dirac Matrices XIV



where $Y$ is an arbitrary $(n \times n)$-matrix whose value we will fix later. It is follows that for any integer $A$ (not summed) in the usual range $\mathcal{A}$:

$$
\begin{aligned}
(\Gamma_A)^{-1} \cdot S \cdot \Gamma_A &= \sum_{X \in \mathcal{A}} (\Gamma_X \Gamma_A)^{-1} \cdot Y \cdot (\Gamma_X \Gamma_A) \\
&= \sum_{X \in \mathcal{A}} (s_X \Gamma_{A \oplus X})^{-1} \cdot Y \cdot (s_X \Gamma_{A \oplus X}) \qquad \text{(using (58))} \\
&= \sum_{X \in \mathcal{A}} (\Gamma_{A \oplus X})^{-1} \cdot Y \cdot (\Gamma_{A \oplus X}) \\
&= \sum_{X \in A \oplus \mathcal{A}} (\Gamma_X)^{-1} \cdot Y \cdot (\Gamma_X) \\
&= \sum_{X \in \mathcal{A}} (\Gamma_X)^{-1} \cdot Y \cdot (\Gamma_X) \qquad \text{(since } A \oplus \mathcal{A} \equiv \{A \oplus B, B \in \mathcal{A}\} = \mathcal{A}) \\
&= S
\end{aligned}
$$

and thus $S \cdot \Gamma_A = \Gamma_A \cdot S$.
Having found a matrix $S$ which commutes with every element $\Gamma_A$ of a list $L$ of matrices, one might hope to use Schur's Lemma to claim that $S$ is some multiple of $1_{n \times n}$. However, a precondition of the only version of Schur's Lemma which I understand and which also

## Appendix III: Dimensions of the Dirac Matrices XV



allows that conclusion to be drawn requires the elements of $L$ to form an irreducible representation of some group $G$. Not only have we not yet shown that this precondition is satisfied, it actually looks likely to be false! For example, for the usual $\gamma$-matrices in $d = 4$ dimensions we would have $\Gamma_1\Gamma_2 = \gamma_1\gamma_2 = - \gamma_2\gamma_1 = -\Gamma_2\Gamma_1$ and so for $L$ to be closed under multiplication it would need to contain both $+\Gamma_2\Gamma_1$ and $-\Gamma_2\Gamma_1$. This seems unlikely as we did not set up $L$ to contain negated copies of every element. It therefore seems unlikely that $L$ is closed under multiplication and so it seems unlikely that $L$ represents a group.
It could be argued that the source of the problem is the annoying sign $s(A, B)$ in (58). If that pesky sign were not there and the constant '$+1$' were always in its place, products of $\Gamma$-matrices would be closed. We cannot arbitrarily dispose of that pesky sign, but it does suggest a resolution: we could double the length of our list $L$ by adding to it another copy of itself but with the sign of every matrix reversed in the second half. The elements of this list will then be closed under multiplication, which is would be a requirement for them to be any kind of representation. We shall call the set containing all those elements $G$:

$$G = \{+\Gamma_A \mid A \in \mathcal{A}\} \cup \{-\Gamma_A \mid A \in \mathcal{A}\}. \tag{63}$$

This set of matrices is: (i) closed under multiplication, (ii) contains the identit $\Gamma_{2^d} = 1_{n\times n}$, (iii) contains an inverse for every element (see proof in (59)). Finally (iv) matrix multiplication is associative. Therefore $G$ together with the operation of matrix multiplication forms a group. As it is a finite matrix group it is also representation of

# Appendix III: Dimensions of the Dirac Matrices XVI



itself. This representation must be irreducible since the representation contains elements which are copies of the original $\gamma$-matrices (e.g. $\Gamma_1 = \gamma_0$, $\Gamma_2 = \gamma_1$, ... $\Gamma_{2^d} = \gamma_d$), and those original $\gamma$-matrices were taken to be be irreducible at the outset by assumption (see paragraph containing (49)). Although we have increased the number of elements in $G$ relative to $L$, we can be sure that our old $S$ will commute with every element of the new $G$ because

$$([S, +\Gamma_A] = 0) \iff ([S, -\Gamma_A] = 0).$$

We have thus established all the preconditions necessary to allow us to use Schur's Lemma to state that $S$ is a multiple of the identity, or more specifically:

$$\lambda \cdot 1_{n\times n} = \sum_{X\in\mathcal{A}} (\Gamma_A)^{-1} \cdot Y \cdot \Gamma_A \tag{64}$$

## Appendix III: Dimensions of the Dirac Matrices XVII



for some scalar $\lambda$ that will depend on $Y$. Taking the trace of both sides of (64) and using the cyclicity of the trace gives us:

$$\begin{aligned} n\lambda &= \sum_{X\in\mathcal{A}} \mathrm{Tr}\left[(\Gamma_A)^{-1}\cdot Y\cdot\Gamma_A\right] \\ &= \sum_{X\in\mathcal{A}} \mathrm{Tr}\left[Y\cdot\Gamma_A\cdot(\Gamma_A)^{-1}\right] \\ &= \sum_{X\in\mathcal{A}} \mathrm{Tr}\, Y \\ &= 2^d\cdot\mathrm{Tr}\, Y \end{aligned}$$

and thus

$$\lambda = \frac{2^d}{n}\cdot\mathrm{Tr}\, Y. \tag{65}$$

Putting this value for $\lambda$ back into (64) yields

$$\frac{2^d}{n}\cdot\mathrm{Tr}\, Y\cdot 1_{n\times n} = \sum_{X\in\mathcal{A}}(\Gamma_A)^{-1}\cdot Y\cdot\Gamma_A. \tag{66}$$

## Appendix III: Dimensions of the Dirac Matrices XVIII



We now exercise our remaining freedom to choose $Y$ to be any $(n \times n)$-matrix we wish, deciding to let

$$[Y]_{ij} = \delta_{is}\delta_{jt}$$

where $s$ and $t$ are integers in $[1, n]$ which we may choose to fix later. With that choice in mind, and with $i$ and $j$ being other arbitrary integers also in $[1, n]$, (66) can be expanded as:

$$\left[\frac{2^d}{n} \cdot \mathrm{Tr}\, Y \cdot 1_{n\times n}\right]_{ij} = \left[\sum_{X\in\mathcal{A}} (\Gamma_A)^{-1} \cdot Y \cdot \Gamma_A\right]_{ij}$$

or equivalently

$$\frac{2^d}{n} \cdot (\delta_{ms}\delta_{mt}) \cdot \delta_{ij} = \sum_{X\in\mathcal{A}} ((\Gamma_A)^{-1})_{im} \cdot (\delta_{ms}\delta_{nt}) \cdot (\Gamma_A)_{nj}$$

which simplifies to

$$\frac{2^d}{n} \cdot \delta_{st} \cdot \delta_{ij} = \sum_{X\in\mathcal{A}} ((\Gamma_A)^{-1})_{is} \cdot (\Gamma_A)_{tj}. \tag{67}$$

## Appendix III: Dimensions of the Dirac Matrices XIX



Since (67) is true for any $i$,$j$,$s$,$t$ in $[1, n]$, let us set $s \to i$ and $t \to j$ and then sum over $i$ and $j$. Making use of the summation convention over $i$ and $j$ we find that:

$$\frac{2^d}{n} \cdot \delta_{ij} \cdot \delta_{ij} = \sum_{A \in \mathcal{A}} ((\Gamma_A)^{-1})_{ii} \cdot (\Gamma_A)_{jj}$$

which simplifies to

$$\frac{2^d}{n} \cdot n = \sum_{A \in \mathcal{A}} \mathrm{Tr}\left[(\Gamma_A)^{-1}\right] \cdot \mathrm{Tr}[\Gamma_A]$$

or

$$2^d = \sum_{A \in \mathcal{A}} \mathrm{Tr}\left[(\Gamma_A)^{-1}\right] \cdot \mathrm{Tr}[\Gamma_A]. \tag{68}$$

# Appendix III: Dimensions of the Dirac Matrices XX



For the case that $d$ is even we may now use (55) to simplify (68) to

$$\begin{aligned} 2^d &= \mathrm{Tr}\left[(\Gamma_0)^{-1}\right] \cdot \mathrm{Tr}[\Gamma_0] \\ &= \mathrm{Tr}\left[(1_{n\times n})^{-1}\right] \cdot \mathrm{Tr}[1_{n\times n}] \\ &= \mathrm{Tr}[1_{n\times n}] \cdot \mathrm{Tr}[1_{n\times n}] \\ &= n \cdot n = n^2 \\ \implies \qquad n &= 2^{d/2} \qquad \text{(but only for } d \text{ even!).} \end{aligned} \tag{69}$$

This is a bit of a trick. One may always generate an irreducible representation of the gamma matrices for an **odd** spacetime dimension $d+1$ from an irreducible representation valid for an **even** number of spacetime dimensions $d$. The way to do this is surprisingly simple: if

$$\left\{\gamma^0, \gamma^1, \ldots, \gamma^{d-1}\right\}$$

is an irrep of (49) for an **even** number of spacetime dimensions $d$, and if we define

$$\gamma^* \equiv \gamma^0 \gamma^1 \cdots \gamma^{d-1}$$

## Appendix III: Dimensions of the Dirac Matrices XXI

Not examinable

and if we recall the definition of $s(d)$ from (51), then

$$\left\{\gamma^0,\gamma^1,\ldots,\gamma^{d-1}\right\}\cup\left\{\sqrt{s(d)}\cdot\gamma^*\right\} \tag{70}$$

will be an irrep of (49) valid for dimension $d+1$ spacetime dimensions. That (70) is the irrep it is claimed to be is a consequence of three things: (i) $\gamma^*$ was proved in (50) to anticommute with all the other gamma matrices when $d$ is **even** and this anti-commutation is the property enforced/required by (49) whenever $\mu\neq\nu$, (ii) that $\sqrt{s(d)}\gamma^*$ squares to 1 was proved in (51), and this is the property enforced/required by (49) whenever $\mu=\nu$, and (iii) the representation (70) is an irrep as the first $d$ gammas formed an irrep by themselves (i.e. as there was no transformation which could 'reduce' them, there cannot be an irrep that could 'reduce' both then and $\gamma^*$). It may be observed that this argument cannot be used to grow irreps without limit, since once an irrep for even $d$ is grown to an irrep for odd $d$, the 'next' $\gamma^*$ would fail to anticommute as desired. Nonetheless, the clear message is that the dimension of the gamma matrices for odd spacetime dimension $d$ is always the same as the even dimension $d-1$, and so (69) now informs us that

$$n=2^{(d-1)/2} \qquad \text{(but only when } d \text{ is odd!).} \tag{71}$$

Not examinable

# Appendix III: Dimensions of the Dirac Matrices XXII



There result (69) for even $d$ can be merged with the result (71) for odd $d$ into a single expression valid for any $d$:

$$\begin{aligned} n &= \begin{cases} 2^{d/2} & \text{(when } d \text{ is even)} \\ 2^{(d-1)/2} & \text{(when } d \text{ is odd)} \end{cases} \\ \implies \qquad n &= 2^{\lfloor d/2 \rfloor} \qquad \text{(for any } d\text{)}. \end{aligned} \tag{72}$$

This concludes the proof of (47) which is also a proof of the lesser claim that Dirac Spinors have four components in the usual 4-dimensional spacetime.

# Appendix IV: Magnetic Moment I



- In the part II Relativity and Electrodynamics course it was shown that the motion of a charged particle in an electromagnetic field $A^{\mu} = (\phi, \vec{A})$ can be obtained by making the minimal substitution $\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi$
- Applying this to the equations in (**??**)

$$\begin{aligned}(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_B &= (E - m - q\phi)u_A \\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (E + m - q\phi)u_B \qquad (73)\end{aligned}$$

Multiplying (73) by $(E + m - q\phi)$

$$\begin{aligned}(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_B &= (E - m - q\phi)u_A \\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (E + m - q\phi)u_B \qquad (74)\end{aligned}$$

where kinetic energy $T = E - m$

- In the non-relativistic limit $T \ll m$ (74) becomes

$$\begin{aligned}(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &\approx 2m(T - q\phi)u_A \\ \left[(\vec{\sigma}.\vec{p})^2 - q(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p}) - q(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{A}) + q^2(\vec{\sigma}.\vec{A})^2\right] u_A &\approx 2m(T - q\phi)u_A \qquad (75)\end{aligned}$$

Not examinable

# Appendix IV: Magnetic Moment II



- Now $\vec{\sigma}.\vec{A} = \begin{pmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{pmatrix}$ ; $\vec{\sigma}.\vec{B} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$ ; which leads to $(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = \vec{A}.\vec{B} + i\vec{\sigma}.(\vec{A} \wedge \vec{B})$
  and $(\vec{\sigma}.\vec{A})^2 = |\vec{A}|^2$
- The operator on the LHS of (75):

$$
\begin{aligned}
&= \vec{p}^2 - q\left[\vec{A}.\vec{p} + i\vec{\sigma}.\vec{A} \wedge \vec{p} + \vec{p}.\vec{A} + i\vec{\sigma}.\vec{p} \wedge \vec{A}\right] + q^2\vec{A}^2 \\
&= (\vec{p} - q\vec{A})^2 - iq\vec{\sigma}.\left[\vec{A} \wedge \vec{p} + \vec{p} \wedge \vec{A}\right] \\
&= (\vec{p} - q\vec{A})^2 - q^2\vec{\sigma}.\left[\vec{A}.\vec{\nabla} + \vec{\nabla}.\vec{A}\right] \qquad (\text{since } \vec{p} = -i\vec{\nabla}) \\
&= (\vec{p} - q\vec{A})^2 - q\vec{\sigma}.(\vec{\nabla} \wedge \vec{A}) \qquad (\text{since } (\vec{\nabla} \wedge \vec{A})\psi = \vec{\nabla} \wedge (\vec{A}\psi) + \vec{A} \wedge (\vec{\nabla}\psi)) \\
&= (\vec{p} - q\vec{A})^2 - q\vec{\sigma}.\vec{B} \qquad (\text{since } \vec{B} = \vec{\nabla} \wedge \vec{A})
\end{aligned}
$$

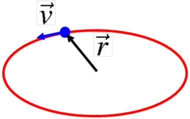
Substituting back into (75) gives the Schrödinger-Pauli equation for the motion of a non-relativisitic spin $\frac{1}{2}$ particle in an EM field:

$$
\left[\frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{\sigma}.\vec{B} + q\phi\right] u_A = Tu_A.
$$

# Appendix IV: Magnetic Moment III



$$\left[\frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{\sigma}.\vec{B} + q\phi\right] u_A = T u_A$$




- Since the energy of a magnetic moment in a field is we can identify the intrinsic magnetic moment of a spin $\frac{1}{2}$ particle to be:

$$\vec{\mu} = \frac{q}{2m}\vec{\sigma}$$

In terms of the spin: $\vec{S} = \frac{1}{2}\vec{\sigma}$

$$\vec{\mu} = \frac{q}{m}\vec{S}$$

- Classically, for a charged particle current loop

$$\mu = \frac{q}{2m}\vec{L}$$

- The intrinsic magnetic moment of a spin half Dirac particle is twice that expected from classical physics. This is often expressed in terms of the gyromagnetic ratio is g=2.

$$\vec{\mu} = g\frac{q}{}\vec{S}$$

## Appendix V: Generators of Lorentz Transformations I



It will shortly be seen that the quantities

$$(M^{\alpha\beta})^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta} \tag{76}$$

or the equivalent (but less symmetric) quantities

$$(M^{\alpha\beta})^{\mu}{}_{\nu} = g^{\mu\alpha} \delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu} g^{\mu\beta} \tag{77}$$

are generators of Lorentz Transformations. The indices $\alpha\beta$ choose between generators $M^{\alpha\beta}$, while ${}^{\mu}{}_{\nu}$ in $(M^{\alpha\beta})^{\mu}{}_{\nu}$ are there to act on vector indices. Evident antisymmetry in the $\alpha\beta$ of (76) means that there are only six independent non-zero generators. Suppressing

Not examinable

# Appendix V: Generators of Lorentz Transformations II



the vector indices (taken to be ${}^{\mu}{}_{\nu}$) and taking $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$ the six independent generators are:

$$K_1 = M^{01} = -M^{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_2 = M^{02} = -M^{20} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_3 = M^{03} = -M^{30} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

## Appendix V: Generators of Lorentz Transformations III



and

$$J_1 = M^{23} = -M^{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}$$

$$J_2 = M^{31} = -M^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J_3 = M^{12} = -M^{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

or, for short:

$$J_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$$
$$K_i = M^{0i}.$$

# Appendix V: Generators of Lorentz Transformations IV



[Aside: The generators obey commutation relations

$$[J_i, J_j] = \epsilon_{ijk} J_k, \qquad [J_i, K_j] = \epsilon_{ijk} K_k, \qquad [K_i, K_j] = -\epsilon_{ijk} J_k.$$

The first of these says that the J's generate rotations in three-dimensional space and fixes the overall sign of the $J$s. The second says the $K$s transform as a vector under rotations. End of aside]

With above definition[1] one could represent and arbitrary Lorentz transformation (boost, rotation or both) as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

with

$$\Lambda^{\mu}{}_{\nu} = \left(\exp\left[\frac{1}{2} w_{\alpha\beta} (M^{\alpha\beta})^{\bullet}{}_{\bullet}\right]\right)^{\mu}{}_{\nu} \tag{78}$$

$$= \delta^{\mu}_{\nu} + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}{}_{\nu} + O(\omega^2) \tag{79}$$

using a set of parameters $w_{\alpha\beta}$ which may as well be antisymmetric in $\alpha\beta$ (since any symmetric part would not participate in (79) on account of the ($\alpha \leftrightarrow \beta$)-antisymmetry of $M^{\alpha\beta}$) and so contain six independent degrees of freedom (controlling three boosts and

## Appendix V: Generators of Lorentz Transformations V



three rotations) as required. In most of the proofs which follow we use the infinitesimal transformations to first order in $\omega$ since if some properties can be proved for infinitesimal transformations then it is always be possible to generalise that result to the exponential form for a finite transformation.



[1] Compare to similar but slightly different sign/index conventions in `http://www.phys.ufl.edu/~fry/6607/lorentz.pdf`.

# Appendix V Why do $(M^{\alpha\beta})^{\mu}{}_{\nu}$ generate Lorentz transformations? I



Lorentz transformations should be continuously connected to the identity (which (79) is, when $\omega_{\alpha\beta} = 0$) and should preserve inner products. The transformation in Eq. (79) preserves inner products because:

$$
\begin{aligned}
x' \cdot y' &= g_{\mu\nu} x'^{\mu} y'^{\nu} \\
&= g_{\mu\nu} (\Lambda^{\mu}{}_{\sigma} x^{\sigma})(\Lambda^{\nu}{}_{\tau} y^{\tau}) \\
&= g_{\mu\nu} (\delta^{\mu}_{\sigma} + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}{}_{\sigma})(\delta^{\nu}_{\tau} + \frac{1}{2}\omega_{\bar\alpha\bar\beta}(M^{\bar\alpha\bar\beta})^{\nu}{}_{\tau}) x^{\sigma} y^{\tau} + O(\omega)^2 \\
&= \left[ g_{\sigma\tau} + \frac{1}{2}\left( \omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} + \omega_{\bar\alpha\bar\beta}(M^{\bar\alpha\bar\beta})_{\sigma\tau} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \\
&= \left[ g_{\sigma\tau} + \frac{1}{2}\left( \omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} + \omega_{\alpha\beta}(M^{\alpha\beta})_{\sigma\tau} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \qquad \text{relabelling} \\
&= \left[ g_{\sigma\tau} + \frac{1}{2}\left( \omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} - \omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \qquad \text{antisymmetry of } M \\
&= g_{\sigma\tau} x^{\sigma} y^{\tau} + O(\omega^2) \\
&= x \cdot y + O(\omega^2).
\end{aligned}
$$

Not examinable

## Appendix V Why do $(M^{\alpha\beta})^{\mu}{}_{\nu}$ generate Lorentz transformations? II



If the above argument seems too abstract, a more concrete way of checking that we have generators of Lorentz transformations might instead be to compute

$$\exp\{(\eta K_1)\} = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{80}$$

as this will be recognised by some as a boost in the positive $x$-direction with rapidity $\eta$ (that is with $\cosh\eta = \gamma$ and $\sinh\eta = \beta\gamma$) while

$$\exp\{(\theta J_1)\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{81}$$

will be recognised by most as a rotation by an angle $\theta$ about the $x$-axis.

Not examinable

# Appendix V: Lorentz covariance of the Dirac equation I



If the Dirac Equation:

$$i\gamma^\mu \partial_\mu \psi = m\psi \tag{82}$$

is to be Lorentz covariant, there would have to exist a matrix $S(\Lambda)$ such that $\psi' = S(\Lambda)\psi$ is the solution of the Lorentz transformed Dirac Equation

$$i\gamma^\mu \partial'_\mu \psi' = m\psi'. \tag{83}$$

Equation (83) implies

$$i\gamma_\mu \partial'^\mu \psi' = m\psi' \tag{84}$$

and so

$$i\gamma_\mu \Lambda^\mu{}_\nu \partial^\nu S(\Lambda)\psi = mS(\Lambda)\psi \tag{85}$$

and so since $S(\Lambda)$ is independent of position

$$i\gamma_\mu S(\Lambda)\Lambda^\mu{}_\nu \partial^\nu \psi = S(\Lambda)m\psi \tag{86}$$

which using (82) becomes



$$i\gamma_\mu S(\Lambda)\Lambda^\mu{}_\nu \partial^\nu \psi = S(\Lambda)i\gamma^\mu \partial_\mu \psi$$

## Appendix V: Lorentz covariance of the Dirac equation II



and hence

$$i\gamma^\mu S(\Lambda)\Lambda_\mu{}^\nu \partial_\nu \psi = S(\Lambda) i\gamma^\nu \partial_\nu \psi$$

or

$$i\left[\gamma^\mu S(\Lambda)\Lambda_\mu{}^\nu - S(\Lambda)\gamma^\nu\right]\partial_\nu\psi = 0. \tag{87}$$

Therefore, if we can show that there exists a matrix $S(\Lambda)$ satisfying

$$\gamma^\mu S(\Lambda)\Lambda_\mu{}^\nu = S(\Lambda)\gamma^\nu \tag{88}$$

we will have found a solution to (87) and thus will have found that the Dirac Equation is Lorentz covariant as desired. Thought it would be entirely possible to work directly with (88) it is perhaps nicer to bring both $S$ matrices to the left hand side

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\Lambda_\mu{}^\nu = \gamma^\nu$$

and then use the identity

$$\Lambda_\mu{}^\nu \Lambda^\sigma{}_\nu \equiv \delta_\mu^\sigma \tag{89}$$

Not examinable

# Appendix V: Lorentz covariance of the Dirac equation III



so that (88) ends up being written in the more common and (perhaps) more suggestive and useful form:

$$S^{-1}(\Lambda)\gamma^\sigma S(\Lambda) = \Lambda^\sigma{}_\nu \gamma^\nu. \tag{90}$$

[Aside: Here is (for infinitesimal Lorentz transformations) a proof of the identity (89):

$$\begin{aligned}
\Lambda_\mu{}^\nu \Lambda^\sigma{}_\nu &= \left(g_\mu{}^\nu + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})_\mu{}^\nu\right)\left(g^\sigma{}_\nu + \frac{1}{2}\omega_{\bar\alpha\bar\beta}(M^{\bar\alpha\bar\beta})^\sigma{}_\nu\right) + O(\omega^2)\\
&= \delta^\sigma_\mu + \frac{1}{2}\left[\omega_{\alpha\beta}(M^{\alpha\beta})_\mu{}^\sigma + \omega_{\bar\alpha\bar\beta}(M^{\bar\alpha\bar\beta})^\sigma{}_\mu\right] + O(\omega^2)\\
&= \delta^\sigma_\mu + \frac{1}{2}\left[\omega_{\alpha\beta}(M^{\alpha\beta})_\mu{}^\sigma + \omega_{\alpha\beta}(M^{\alpha\beta})^\sigma{}_\mu\right] + O(\omega^2) \qquad \text{(relabelling)}\\
&= \delta^\sigma_\mu + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})_\mu{}^\sigma + (M^{\alpha\beta})^\sigma{}_\mu\right] + O(\omega^2) \qquad \text{(factorising)}\\
&= \delta^\sigma_\mu + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})^{\tau\sigma} + (M^{\alpha\beta})^{\sigma\tau}\right] g_{\mu\tau} + O(\omega^2) \qquad \text{(tidying)}\\
&= \delta^\sigma_\mu + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})^{\tau\sigma} - (M^{\alpha\beta})^{\tau\sigma}\right] g_{\mu\tau} + O(\omega^2) \qquad \text{(antisymmetry of } M\text{)}\\
&= \delta^\sigma_\mu + O(\omega^2).
\end{aligned}$$

Not examinable

# Appendix V: Lorentz covariance of the Dirac equation IV



End of aside]

**Lemma**

*A valid choice of $S(\Lambda)$ (for an infinitesimal Lorentz transformation) is given by:*

$$S(\Lambda) = 1 + \frac{1}{4}\omega_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta} + O(\omega^2). \tag{91}$$

# Appendix V: Lorentz covariance of the Dirac equation V



**Proof.**

$$
\begin{aligned}
S^{-1}(\Lambda)\gamma^\sigma S(\Lambda) &= \left(1 - \frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\right)\gamma^\sigma\left(1 + \frac{1}{4}\omega_{\bar\alpha\bar\beta}\gamma^{\bar\alpha}\gamma^{\bar\beta}\right) + O(\omega^2) \\
&= \gamma^\sigma + \frac{1}{4}\left(\omega_{\bar\alpha\bar\beta}\gamma^\sigma\gamma^{\bar\alpha}\gamma^{\bar\beta} - \omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\gamma^\sigma\right) + O(\omega^2) \\
&= \gamma^\sigma + \frac{1}{4}\omega_{\alpha\beta}\left(\gamma^\sigma\gamma^\alpha\gamma^\beta - \gamma^\alpha\gamma^\beta\gamma^\sigma\right) + O(\omega^2) \\
&= \gamma^\sigma + \frac{1}{4}\omega_{\alpha\beta}\left((\gamma^\sigma\gamma^\alpha + \gamma^\alpha\gamma^\sigma)\gamma^\beta - \gamma^\alpha(\gamma^\sigma\gamma^\beta + \gamma^\beta\gamma^\sigma)\right) + O(\omega^2) \\
&= \gamma^\sigma + \frac{1}{4}\omega_{\alpha\beta}\left(2g^{\sigma\alpha}\gamma^\beta - \gamma^\alpha 2g^{\sigma\beta}\right) + O(\omega^2) \qquad \text{since } \{\gamma^\mu,\gamma^\nu\} \equiv 2g^{\mu\nu} \\
&= \left(\delta^\sigma_\nu + \frac{1}{2}\omega_{\alpha\beta}\left(g^{\sigma\alpha}\delta^\beta_\nu - \delta^\alpha_\nu g^{\sigma\beta}\right)\right)\gamma^\nu + O(\omega^2) \\
&= \left(\delta^\sigma_\nu + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^\sigma{}_\nu\right)\gamma^\nu + O(\omega^2) \qquad \text{using (77)} \\
&= \Lambda^\sigma{}_\nu\gamma^\nu + O(\omega^2) \qquad \text{using (79).}
\end{aligned}
$$

$\square$

## Appendix V: Lorentz covariance of the Dirac equation VI



[Aside: Since $\gamma^\alpha\gamma^\beta = \frac{1}{2}\{\gamma^\alpha, \gamma^\beta\} + \frac{1}{2}[\gamma^\alpha, \gamma^\beta]$ we can also rewrite (91) in the more frequently seen (conventional) form:

$$S(\Lambda) = 1 + \frac{1}{8}\omega_{\alpha\beta}[\gamma^\alpha, \gamma^\beta] + O(\omega^2). \tag{92}$$

End of aside]

# Appendix V: Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. I



Each of the expressions $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$ is of the form $\overline{\phi}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi$. To understand how any of them is affected by a Lorentz transformation it is therefore interesting to consider the following set of manipulations:[2]

$$\begin{aligned}
\overline{\phi'}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi' &= \overline{(S(\Lambda)\phi)}[\gamma^\mu\gamma^\nu\cdots\gamma^\tau](S(\Lambda)\psi)\\
&= \phi^\dagger S^\dagger(\Lambda)\gamma^0[\gamma^\mu S(\Lambda)S^{-1}(\Lambda)\gamma^\nu S(\Lambda)\cdots S^{-1}(\Lambda)\gamma^\tau]S(\Lambda)\psi\\
&= \phi^\dagger S^\dagger(\Lambda)\gamma^0 S(\Lambda)(S^{-1}(\Lambda)\gamma^\mu S(\Lambda))(S^{-1}(\Lambda)\gamma^\nu S(\Lambda))\cdots(S^{-1}(\Lambda)\gamma^\tau S(\Lambda))\psi\\
&= \phi^\dagger S^\dagger(\Lambda)\gamma^0 S(\Lambda)(\Lambda^\mu{}_\alpha\gamma^\alpha)(\Lambda^\nu{}_\beta\gamma^\gamma)\cdots(\Lambda^\tau{}_\lambda\gamma^\lambda)\psi \qquad \text{using (90)}
\end{aligned}$$

which itself suggests that if we can show that

$$S^\dagger(\Lambda)\gamma^0 S(\Lambda) = \gamma^0 \tag{93}$$

then we will have proved that

$$\overline{\phi'}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi' = \overline{\phi}(\Lambda^\mu{}_\alpha\gamma^\alpha)(\Lambda^\nu{}_\beta\gamma^\gamma)\cdots(\Lambda^\tau{}_\lambda\gamma^\lambda)\psi$$

which will itself have showed that each of the expressions under consideration transforms like a tensor of the appropriate rank.

Not examinable

## Appendix V: Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. II

Not examinable

We must therefore prove (93). To do so is a two-stage process. First we compute $S^\dagger(\Lambda)$. Then we combine it with $\gamma^0 S(\Lambda)$. Starting with (91):

$$\begin{aligned}
S^\dagger(\Lambda) &= \left[1 + \frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\right]^\dagger + O(\omega^2) \\
&= 1 + \frac{1}{4}\omega_{\alpha\beta}(\gamma^\alpha\gamma^\beta)^\dagger + O(\omega^2) \qquad (\omega_{\alpha\beta} \text{ are real}) \\
&= 1 + \frac{1}{4}\omega_{\alpha\beta}(\gamma^\beta)^\dagger(\gamma^\alpha)^\dagger + O(\omega^2) \\
&= 1 + \frac{1}{4}\omega_{\alpha\beta}(\gamma^0\gamma^\beta\gamma^0)(\gamma^0\gamma^\alpha\gamma^0) + O(\omega^2) \\
&= 1 + \frac{1}{4}\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0 + O(\omega^2) \qquad (94)
\end{aligned}$$

Not examinable

## Appendix V: Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. III

Not examinable

from which we can deduce (using (91)) that

$$
\begin{aligned}
S^\dagger(\Lambda)\gamma^0 S(\Lambda) &= \left(1+\frac{1}{4}\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0\right)\gamma^0\left(1+\frac{1}{4}\omega_{\bar{\alpha}\bar{\beta}}\gamma^{\bar{\alpha}}\gamma^{\bar{\beta}}\right)+O(\omega^2)\\
&= \gamma^0+\frac{1}{4}\left(\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0\gamma^0+\omega_{\bar{\alpha}\bar{\beta}}\gamma^0\gamma^{\bar{\alpha}}\gamma^{\bar{\beta}}\right)+O(\omega^2)\\
&= \gamma^0\left[1+\frac{1}{4}\left(\omega_{\alpha\beta}\gamma^\beta\gamma^\alpha+\omega_{\beta\alpha}\gamma^\beta\gamma^\alpha\right)\right]+O(\omega^2) \qquad ((\bar{\alpha},\bar{\beta})\to(\beta,\alpha))\\
&= \gamma^0\left[1+0\right]\psi+O(\omega^2) \qquad (\omega_{\alpha\beta}=-\omega_{\beta\alpha})\\
&= \gamma^0+O(\omega^2)
\end{aligned}
$$

verifying (93) as required. This completes our proof that:

- $\overline{\phi}\psi$ is Lorentz invariant scalar,
- $\overline{\phi}\gamma^\mu\psi$ transforms as a Lorentz vector, and
- $\overline{\phi}\gamma^\mu\gamma^\nu\psi$ transforms as a second-rank tensor, *etc.*

Not examinable

[2] These manipulations may look complex but they really only consist of inserting lots of 'ones' in form '$S(\Lambda)S^{-1}(\Lambda)$' at the right places, using $\overline{\phi}\equiv\phi^\dagger\gamma^0$ and using (90) many times.