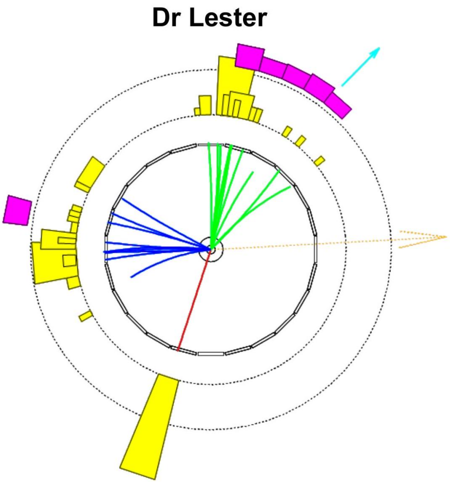
Dr C.G. Lester, 2023



H13: Electroweak Unification and the W and Z Bosons

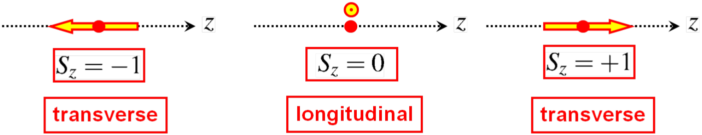
Boson Polarization Vectors

- Boson wave-functions are written in terms of **polarization four-vectors** ε^μ:

$$B^\mu = \varepsilon^\mu e^{-ip\cdot x} = \varepsilon^\mu e^{i(\vec{p}\cdot\vec{x}-Et)}.$$

- These **polarization four-vectors** ε^μ (in addition to p^μ) are the quantities used by Feynman rules to describe the state of spin-1 bosons entering or leaving a process.
- **Massless** spin-1 boson can exist in only two transverse polarization states (c.f. left and right circularly polarised light) – at least when they are not virtual.
- **Massive** spin-1 boson also can be longitudinally polarized.
- For a spin-1 boson travelling along the z-axis, the three **polarization four-vectors** are:

$$\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_L = \frac{1}{m}\left(p_z,0,0,E\right); \quad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0,1,i,0) \tag{180}$$

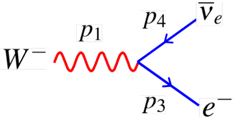


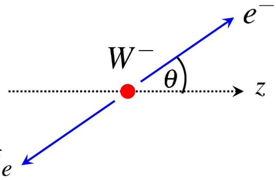
- All three polarisations satisfy $\varepsilon^\mu p_\mu = 0$ and $\varepsilon^\mu \varepsilon^*_\mu = -1$.

Results on this page are stated without proof. Nevertheless, some (non-examinable) partial justification can be found in Appendices IX, X, XI and XII.

W-boson decay to fermions in centre of mass frame

- For each polarisation ε^{μ} we will calculate the matrix element for this process:





so parameterised by $\left\{\begin{aligned} p_1 &= (m_W, \quad 0, 0, 0) \\ p_3 &= (E, +E\sin\theta, 0, +E\cos\theta) \\ p_4 &= (E, -E\sin\theta, 0, -E\cos\theta) \end{aligned}\right\}$ with $E = \frac{1}{2} m_W$.

- Feynman rules give:

$$-iM_{fi} = \varepsilon_{\mu}(p_1) \cdot \bar{u}(p_3) \cdot -i\frac{g_W}{\sqrt{2}}\gamma^{\mu}\frac{1}{2}\left(1-\gamma^5\right) \cdot v(p_4)$$

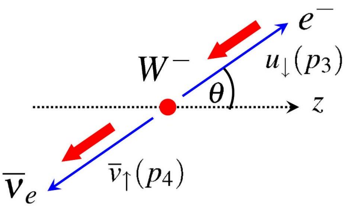
so $\boxed{M_{fi} = \frac{1}{\sqrt{2}} g_W \varepsilon_{\mu}(p_1) \cdot j^{\mu}}$ with $\boxed{j^{\mu} = \bar{u}(p_3)\gamma^{\mu}\frac{1}{2}\left(1-\gamma^5\right)v(p_4)}$.

W-decay lepton current in ultra-relativistic limit

- Our leptons are in the ultra-relativistic limit ($m_w \gg m_e$).
- In this limit we know that the weak interaction only couples to left handed particles and right handed anti-particles. Hence

$$j^{\mu}_{\uparrow\uparrow} = j^{\mu}_{\downarrow\downarrow} = j^{\mu}_{\uparrow\downarrow} = 0$$

and it is only necessary to calculate $j^{\mu}_{\downarrow\uparrow}$ for the process shown to the right.



- By re-using the calculation of tha ($\mu^-_L \mu^+_R$)-current on page 178 which told us that:

$$j^{\mu}_{\uparrow\downarrow} = 2E(0, -\cos\theta, -i, \sin\theta)$$

we can substitute since $E = m_W/2$ to see that:

$$\boxed{j^{\mu}_{\uparrow\downarrow} = m_W(0, -\cos\theta, -i, \sin\theta)}. \tag{181}$$

- For a W-boson at rest, the three poloarisation vectors of (180) on page 524 become:

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_L = (0,0,0,1); \quad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0).$$

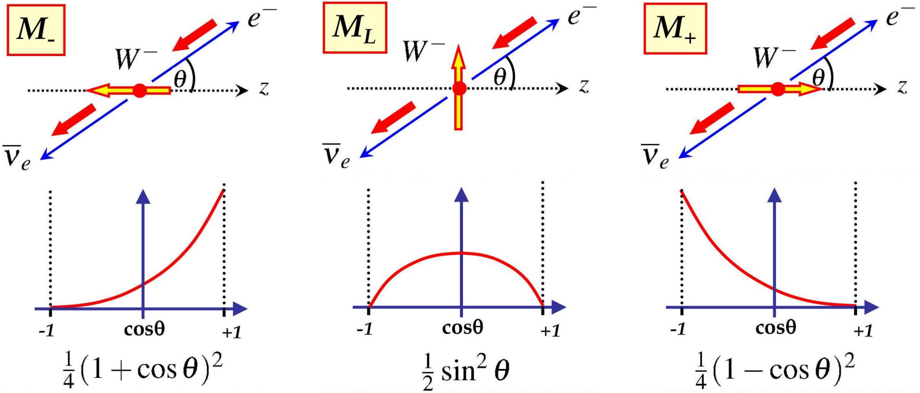
We can now calculate the three W-boson decay matrix elements $M_{fi} = \frac{g_W}{\sqrt{2}}\varepsilon_\mu(p_1)j^\mu$ by dotting the current (181) with each of the three polarization vectors shown above:

$$\begin{aligned}
M_- &= \frac{g_W}{\sqrt{2}} \times \tfrac{1}{\sqrt{2}}(0,1,-i,0)\cdot m_W(0,-\cos\theta,-i,\sin\theta) = \frac{1}{2}g_W m_W(1+\cos\theta)\\
M_L &= \frac{g_W}{\sqrt{2}} \times (0,0,0,1)\cdot m_W(0,-\cos\theta,-i,\sin\theta) = -\frac{1}{\sqrt{2}}g_W m_W \sin\theta\\
M_+ &= \frac{g_W}{\sqrt{2}} \times \tfrac{-1}{\sqrt{2}}(0,1,i,0)\cdot m_W(0,-\cos\theta,-i,\sin\theta) = \frac{1}{2}g_W m_W(1-\cos\theta).
\end{aligned}$$

The W-boson decay matrix elements are therefore:

$$\left\{\begin{aligned}
|M_-|^2 &= g_W^2 m_W^2 \frac{1}{4}(1+\cos\theta)^2\\
|M_L|^2 &= g_W^2 m_W^2 \frac{1}{2}\sin^2\theta\\
|M_+|^2 &= g_W^2 m_W^2 \frac{1}{4}(1-\cos\theta)^2
\end{aligned}\right\} \tag{182}$$

The angular distributions can be understood in terms of the spin of the particles:



Differential decay distributions for W-bosons of **definite polarization**

- The differential decay rate can be found using the expression given in (4):

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}|M|^2$$

where $p^* = \frac{m_W}{2}$ is the C.O.M. momentum of the final state particles.

- For the three different polarisations we obtain:

$$\frac{\mathrm{d}\Gamma_-}{\mathrm{d}\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{4}(1+\cos\theta)^2; \quad \frac{\mathrm{d}\Gamma_L}{\mathrm{d}\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{2}\sin^2\theta; \quad \frac{\mathrm{d}\Gamma_+}{\mathrm{d}\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{4}(1-\cos\theta)^2.$$

- Integrating over all angles using

$$\int \frac{1}{4}(1 \pm \cos\theta)^2 \ \mathrm{d}\phi \mathrm{d}\cos\theta = \int \frac{1}{2}\sin^2\theta \mathrm{d}\phi \mathrm{d}\cos\theta = \frac{4\pi}{3}$$

gives

$$\boxed{\Gamma_- = \Gamma_L = \Gamma_+ = \frac{g_W^2 m_W}{48\pi}}.$$

$\Longrightarrow$ The decay rate for a W-boson does not depend on its direction of polarization.

This has to be the case! The decay rate should not depend on the arbitrary definition of the z-axis, even if the direction of the decay produces does.

Unpolarised W-boson decays

- For an unpolarized W-bosons, each of the three polarization states is equally likely.
- Thus, for the **average matrix element** sum over all possible matrix elements and average over the three initial polarization states:

$$\left\langle |M_{fi}|^2 \right\rangle = \frac{1}{3}\left(|M_-|^2 + |M_L|^2 + |M_+|^2\right)$$
$$= \frac{1}{3} g_W^2 m_W^2 \left[\frac{1}{4}(1+\cos\theta)^2 + \frac{1}{2}\sin^2\theta + \frac{1}{4}(1-\cos\theta)^2\right] = \boxed{\frac{1}{3} g_W^2 m_W^2}.$$

As one would expect, unpolarized W-bosons decay isotropically.

Such bosons have no special directions along which to favour cerain decays!
For such decays, the differential decay rate is:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}\left\langle |M|^2 \right\rangle = \frac{\frac{1}{2} m_W}{32\pi^2 m_W^2}\frac{1}{3} g_W^2 m_W^2 = \frac{g_W^2 m_W}{192\pi^2}$$

so the total unploarised decay rate to this flavour of lepton is:

$$\Gamma\left(W^- \to e^- \bar{\nu}\right) = 4\pi \cdot \frac{g_W^2 m_W}{192\pi^2} = \frac{g_W^2 m_W}{48\pi}.$$

W-boson width to all final states (i.e. total width or total decay rate)

- The calculation for the other decay modes (neglecting final state particle masses) is same. For quarks need to account for colour and CKM matrix. No decays to the top quark as its mass (173 GeV) is greater than the W-boson mass (80 GeV).
- Defining $X = \frac{g_W^2 m_W}{48\pi} = \Gamma(W^- \to e^- \bar{\nu}_e)$ from last page we therefore have:

$$\begin{array}{lll}
\Gamma(W^- \to e^- \bar{\nu}_e) = X & \Gamma(W^- \to d\bar{u}) = 3\,|V_{ud}|^2 X & \Gamma(W^- \to d\bar{c}) = 3\,|V_{cd}|^2 X \\
\Gamma(W^- \to \mu^- \bar{\nu}_\mu) = X & \Gamma(W^- \to s\bar{u}) = 3\,|V_{us}|^2 X & \Gamma(W^- \to s\bar{c}) = 3\,|V_{cs}|^2 X \\
\Gamma(W^- \to \tau^- \bar{\nu}_\tau) = X & \Gamma(W^- \to b\bar{u}) = 3\,|V_{ub}|^2 X & \Gamma(W^- \to b\bar{c}) = 3\,|V_{cb}|^2 X
\end{array}$$

- Unitarity of CKM matrix gives, e.g. $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$, hence $\sum_q BR\left(W^- \to qq'\right) = 3X + 3X = 6X$.
Separately we note $\sum_l BR\left(W^- \to l\bar{\nu}_l\right) = 3X$.
- Thus our leading-order prediction for the total W-boson decay rate is:

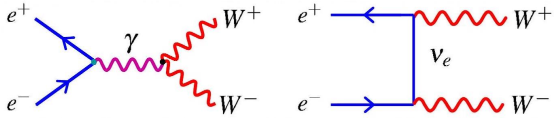
$$\Gamma_W \equiv \Gamma(W^- \to \text{anything}) = 6X + 3X = 9X = \frac{3g_W^2 m_W}{16\pi} \approx \boxed{2.07\ \mathrm{GeV}}.$$

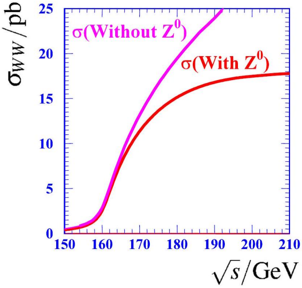
Experiment measures the W-boson width to be $\boxed{\Gamma_W = 2.085 \pm 0.042\ \mathrm{GeV}}$.

Our leading-order calculation is remarkably good. Probably better than it deserves to be!

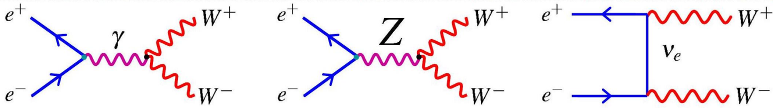
From W to Z

- W-bosons can be produced in e^+e^- annihilation:





- With just these two diagrams there is a problem: the cross section increases with C.o.M. energy and at some point **violates QM unitarity**
  [i.e. calculation would predict larger larger flux of W-bosons than incoming flux of electrons/positrons!]
- Problem could be 'fixed' by introducing a new boson, the Z. The new diagram could interferes negatively with the above two diagrams fixing the unitarity problem:



$$|M_{\gamma WW} + M_{ZWW} + M_{\nu WW}|^2 < |M_{\gamma WW} + M_{\nu WW}|^2$$

- Idea would only work if Z, γ and W couplings were appropriately related. ELECTROWEAK UNIFICATION will achieve this.

$SU(2)_L$: The Gauge Principle of the Weak Interaction

- The Weak Interaction arises from SU(2) local phase transformations:

$$\psi \to \psi' = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}}\psi$$

where the $\vec{\sigma}$ are the three generators of the $SU(2)$, i.e. the three Pauli spin matrices.

Three generators imply three gauge bosons : $\boxed{W_1^\mu, W_2^\mu, W_3^\mu}$.

- The **left-chiral fermion** (and **right-chiral anti-fermion**) components are placed in **'weak isospin'** DOUBLETS. The local gauge transformation therefore corresponds to:

$$\begin{pmatrix}\nu_{eL}\\ e_L^-\end{pmatrix} \longrightarrow \begin{pmatrix}\nu_{eL}\\ e_L^-\end{pmatrix}' = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}}\begin{pmatrix}\nu_{eL}\\ e_L^-\end{pmatrix}.$$

- The **right-chiral fermion** (and **left-chiral anti-fermion**) components are placed in **'weak isospin'** SINGLETS. Singlets are always unmodified by gauge transformations!

Summary of fermion weak isospin assignments (anti-fermion assignments are opposite) :

DOUBLETs $I_W = \frac{1}{2}$: $\begin{pmatrix}\nu_{eL}\\ e_L^-\end{pmatrix}, \begin{pmatrix}\nu_{\mu L}\\ \mu_L^-\end{pmatrix}, \begin{pmatrix}\nu_{\tau L}\\ \tau_L^-\end{pmatrix}, \begin{pmatrix}u_L\\ d_L'\end{pmatrix}, \begin{pmatrix}c_L\\ s_L'\end{pmatrix}, \begin{pmatrix}t_L\\ b_L'\end{pmatrix}$ $\begin{matrix}\leftarrow I_W^3 = +\frac{1}{2}\\ \leftarrow I_W^3 = -\frac{1}{2}\end{matrix}$

SINGLETs $I_W = 0$: $(\nu_{eR}), (e_R^-), \ldots\ldots\ldots\ldots\ldots\ldots, (t_R), (b_R)$ $\leftarrow I_W^3 = 0.$

This gauge symmetry reproduces the W-boson currents:

- $SU(2)$ has three generators (Pauli matrices) so has these three currens (one per generator) which we associate with the W_1, W_2, W_3 bosons respectively:

$$j_1^\mu = g_W \bar{\chi}_L \gamma^\mu \frac{1}{2}\sigma_1 \chi_L; \quad j_2^\mu = g_W \bar{\chi}_L \gamma^\mu \frac{1}{2}\sigma_2 \chi_L; \quad j_3^\mu = g_W \bar{\chi}_L \gamma^\mu \frac{1}{2}\sigma_3 \chi_L.$$

- Define the $j_\pm^\mu$ currents as the following linear combinations of currents:

$$j_\pm^\mu \equiv \frac{1}{\sqrt{2}}(j_1^\mu \mp i\, j_2^\mu) \qquad \text{so} \qquad j_\pm^\mu = \frac{g_W}{\sqrt{2}} \bar{\chi}_L \gamma^\mu \frac{1}{2}(\sigma_1 \mp i\sigma_2)\chi_L.$$

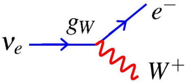
- Therefore, taking χ_L to be the weak isospin DOUBLET $\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$ we would find:

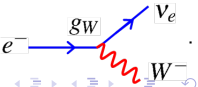
$$j_+^\mu = \frac{g_W}{\sqrt{2}} \begin{pmatrix} \bar{\nu}_{eL} & \bar{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} = \frac{g_W}{\sqrt{2}} \bar{e}_L \gamma^\mu \nu_{eL} = \frac{g_W}{\sqrt{2}} \bar{e} \gamma^\mu \frac{1}{2}(1-\gamma^5)\nu_e$$

and

$$j_-^\mu = \frac{g_W}{\sqrt{2}} \begin{pmatrix} \bar{\nu}_{eL} & \bar{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} = \frac{g_W}{\sqrt{2}} \bar{\nu}_{eL} \gamma^\mu e_L = \frac{g_W}{\sqrt{2}} \bar{\nu}_e \gamma^\mu \frac{1}{2}(1-\gamma^5) e$$

and so

j_+^μ is the W^+-current: 

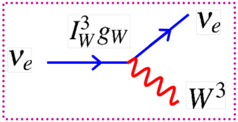
; j_-^μ is the W^--current: 

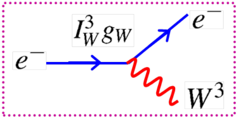
.

What about the remaining third current $j_3^\mu = g_W \bar{\chi}_L \gamma^\mu \frac{1}{2} \sigma_3 \chi_L$?

$$
\begin{aligned}
j_3^\mu &= g_W \begin{pmatrix} \bar{\nu}_{eL} & \bar{e}_L \end{pmatrix} \gamma^\mu \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \\
&= g_W \left[\tfrac{1}{2} \bar{\nu}_{eL} \gamma^\mu \nu_{eL} - \tfrac{1}{2} \bar{e}_L \gamma^\mu e_L \right] \\
&= g_W \left[\left(+\tfrac{1}{2}\right) \cdot \bar{\nu}_{eL} \gamma^\mu \nu_{eL} + \left(-\tfrac{1}{2}\right) \cdot \bar{e}_L \gamma^\mu e_L + (0) \cdot \bar{e}_R \gamma^\mu e_R + (0) \cdot \bar{\nu}_{eR} \gamma^\mu \nu_{eR} \right] \\
&= g_W \left[I_W^3(\nu_{eL}) \cdot \bar{\nu}_{eL} \gamma^\mu \nu_{eL} + I_W^3(e_L) \cdot \bar{e}_L \gamma^\mu e_L + I_W^3(e_R) \cdot \bar{e}_R \gamma^\mu e_R + I_W^3(\nu_{eR}) \cdot \bar{\nu}_{eR} \gamma^\mu \nu_{eR} \right]
\end{aligned}
$$

- The above describes a **flavour-conserving NEUTRAL CURRENT INTERACTION**.
- The coupling strength is g_W but multiplied by the third component of weak isospin:





- Technically this interaction only affects members of the weak isospin DOUBLETs, (i.e. left chiral particles and right chiral anti-particles).
- However, since members of weak isospin SINGLETs have $I_W^3 = 0$, **one may regard the interaction as applying to everything in proportion to** I_W^3**.** In effect, the SINGLETs are just 'neutral' so far as I_W^3 is concerned.

Electroweak Unification and $U(1)_Y$ Weak Hypercharge

- It would be tempting **(and wrong!)** to identify the W^3 as the Z.
- There are two physical neutral spin-1 gauge bosons: the photon (A) and the Z. By definition these are mass eigenstates. There is no reason that the **gauge bosons should correspond to mass eigenstates**. They can mix!

The Z and A (photon) are written in terms of the W^3 and a new neutral spin-1 'B-boson'.

In terms of an *a-priori* undetermined '**W**eak' (or '**W**einberg'?) mixing angle θ_W we define:

$$A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W, \qquad \text{and} \tag{183}$$

$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W \tag{184}$$

where the B is associated with a new $U(1)_Y$ gauge symmetry called **weak hypercharge**.

Hindsight is a wonderful thing ...

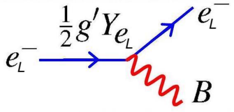
We realise that the $U(1)_{\text{em}}$ gauge symmetry we explored on page 295 was an emergent rather than a fundamental gauge symmetry of The Standard Model, which actually lives in:

$$U(1)_Y \times SU(2)_L \times SU(3)_C.$$

EM's (apparent) $U(1)_{\text{em}}$ symmetry, and the Z-boson, emerge from within $U(1)_Y \times SU(2)_L$.

Relationship of Weak Hypercharge Y to electric charge Q and I_W^3

- The coupling strength for Weak Hypercharge is denoted g'.
- By convention the coupling of any fermion to the B-boson is



$$\frac{1}{2}g'Y$$

where Y is the weak hypercharge of the fermion in question.

- Like I_W^3 (but unlike electric charge Q) **the weak hypercharge Y can be different for left and right chiral fermions**.
- We will shortly see that $\boxed{Y = 2Q - 2I_W^3}$

where Q is the EM charge and I_W^3 the third component of weak isospin for the (left or right chiral) particle in question.

Example Weak Hypercharges:

$$e_L : Y = 2(-1) - 2\left(-\tfrac{1}{2}\right) = -1 \qquad \nu_{eL} : Y = 2(0) - 2\left(\tfrac{1}{2}\right) = -1$$
$$e_R : Y = 2(-1) - 2\,(0) = -2 \qquad \nu_{eR} : Y = 2(0) - 2\,(0) = 0$$

- The neutral-current for electrons (or indeed any fermion) take the form:

$$\begin{array}{llll} A\ (\gamma): & j_\mu^{em} = eQ_e \cdot \bar{\psi}\gamma_\mu\psi = & eQ_e \cdot \bar{e}_L\gamma_\mu e_L + & eQ_e \cdot \bar{e}_R\gamma_\mu e_R \\ B\ : & j_\mu^{Y} = \frac{g'}{2} Y_e\bar{\psi}\gamma_\mu\psi = & \frac{g'}{2} Y_{e_L} \cdot \bar{e}_L\gamma_\mu e_L + & \frac{g'}{2} Y_{e_R} \cdot \bar{e}_R\gamma_\mu e_R \\ W^3\ : & j_\mu^{W^3} = & = g_W I_W^3(e_L) \cdot \bar{e}_L\gamma_\mu e_L + & g_W I_W^3(e_R) \cdot \bar{e}_R\gamma_\mu e_R \end{array}$$

- The relation $A_\mu = B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W$ is equivalent to:

$$\boxed{j_\mu^{em} = j_\mu^Y \cos\theta_W + j_\mu^{W^3} \sin\theta_W}$$

and so consistency with earlier results will be achieved only if, for all particles:

$$\boxed{eQ_x = \cos\theta_W \frac{g'}{2} Y_{x_\bullet} + \sin\theta_W g_W I_W^3(x_\bullet)} \tag{185}$$

where $x \in \{e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, s, c, b, t\}$ labels the fermion, and '$\bullet$' $\in \{L, R\}$.

- Divide our parameters up according to whether they are old or new:

$$\begin{aligned} \text{pre-existing/known quantities are:} \quad & \{e, Q_x, g_W, I_W^3(x_\bullet)\}, \\ \text{new quantities are:} \quad & \{g', \theta_W, Y_{x_\bullet}\}. \end{aligned}$$

- We 'win' this 'unification game' if we can find values for our **new** parameters in terms of our **old** parameters that cause (185) to be satisfied.

Can we win the unification game? Yes!

- Given fixed values of our pre-existing/known quantities $\{e, Q_x, g_W, I^3_W(x_\bullet)\}$ we can easliy ensure that the following relationship:

$$\boxed{eQ_x = \cos\theta_W \frac{g'}{2} Y_{x_\bullet} + \sin\theta_W g_W I^3_W(x_\bullet)}$$

is always satisfied **by requiring** the **three** new quantities $\{g', \theta_W, Y_{x_\bullet}\}$ to take the values which solve the following **three** simultaneous conditions:

$$\boxed{\begin{aligned} e &= g' \cos\theta_W \\ e &= g_W \sin\theta_W \\ Q_x &= \frac{1}{2} Y_{x_\bullet} + I^3_W(x_\bullet) \end{aligned}} \qquad (186)$$

Aside

The relations in (186) can be solved by taking $\theta_W = \arcsin(e/g_W)$ and thence by $g' = \frac{e}{\cos\theta_W}$ together with

$$\boxed{Y_{x_\bullet} = 2Q_x - 2I^3_W(x_\bullet)} \qquad (187)$$

which is the weak hypercharge assignment claimed earlier on page 537 !

In this model, we can now derive the couplings of the Z Boson.

- In (184) we learned that: $\boxed{Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W}$ which is equivalent to

$$j^Z_\mu = -j^Y_\mu \sin\theta_W + j^{W^3}_\mu \cos\theta_W$$

which, using our results from page 538, becomes

$$\begin{aligned} j^Z_\mu(x_\bullet) &= -\sin\theta_W \left(\frac{g'}{2} Y_{x_\bullet} \cdot \overline{x}_\bullet \gamma_\mu x_\bullet\right) + \cos\theta_W \left(g_W I^3_W(x_\bullet) \cdot \overline{x}_\bullet \gamma_\mu x_\bullet\right) \\ &= \left(\cos\theta_W g_W I^3_W(x_\bullet) - \sin\theta_W \frac{g'}{2} Y_{x_\bullet}\right) \cdot \overline{x}_\bullet \gamma_\mu x_\bullet \end{aligned} \tag{188}$$

if $x \in \{e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, s, c, b, t\}$ labels the fermion, and '$\bullet$' $\in \{L, R\}$.

- Using the conditions in (186) we can re-write (188) as

$$j^Z_\mu(x_\bullet) = \left[g' \frac{(I^3_W - Q \sin^2\theta_W)}{\sin\theta_W}\right] \cdot \overline{x}_\bullet \gamma_\mu x_\bullet$$

or

$$j^Z_\mu(x_\bullet) = g_Z \left(I^3_W(x_\bullet) - Q_x \sin^2\theta_W\right) \cdot \overline{x}_\bullet \gamma_\mu x_\bullet \tag{189}$$

if we define g_Z by $\boxed{e = g_Z \cos\theta_W \sin\theta_W}$; *i.e.* $\boxed{g_Z = \frac{g_W}{\cos\theta_W}}$. [Check as exercise!]

- In contrast to the $W^{\pm}$, the Z-boson couples to both LH and RH chiral components, **but not equally ...**
- Illustrate the above by defining:

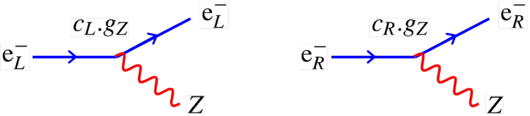
$$\boxed{c_{\bullet}(x) = \left(I_W^3(x_{\bullet}) - Q_x \sin^2\theta_W \right)} \tag{190}$$

- With L and R replacing $\bullet$ we see that:

$$c_L(x) = \left(I_W^3(x_L) - Q_x \sin^2\theta_W \right) \qquad \text{whereas} \tag{191}$$

$$c_R(x) = \left(\qquad\qquad - Q_x \sin^2\theta_W \right) \tag{192}$$

because right chiral particles (and left chiral anti-particles) have $I_W^3 = 0$.



$\boxed{c_L = I_W^3 - Q\sin^2\theta_W}$ $\boxed{c_R = -Q\sin^2\theta_W}$

W³ part of Z couples only to LH components (like W±)

B$_\mu$ part of Z couples equally to LH and RH components

Use projection operators to obtain vector and axial vector couplings.

- From last two pages:

$$j_\mu^Z(x_\bullet) = g_Z c_\bullet(x) \cdot \bar{x}_\bullet \gamma_\mu x_\bullet.$$

- Also (and renaming $x \to u$):

$$\bar{u}_L \gamma_\mu u_L = \bar{u}\gamma_\mu \frac{1}{2}(1-\gamma_5)\, u \qquad \text{and} \qquad \bar{u}_R \gamma_\mu u_R = \bar{u}\gamma_\mu \frac{1}{2}(1+\gamma_5)\, u.$$

- Thus

$$\begin{aligned} j_\mu^Z &= g_Z \bar{u}\gamma_\mu \left[c_L \frac{1}{2}(1-\gamma_5) + c_R \frac{1}{2}(1+\gamma_5) \right] u \\ &= \frac{g_Z}{2} \bar{u}\gamma_\mu \left[(c_L + c_R) + (c_R - c_L)\gamma_5 \right] u \\ &= \frac{g_Z}{2} \bar{u}\gamma_\mu \left[c_V - c_A \gamma_5 \right] u \end{aligned}$$

if one defines

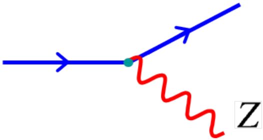
$$\boxed{c_V(x) = c_L(x) + c_R(x)} = I_W^3(x_L) - 2Q_x \sin^2\theta_W \qquad \text{and} \tag{193}$$

$$\boxed{c_A(x) = c_L(x) - c_R(x)} = I_W^3(x_L). \tag{194}$$

Summary of Z-boson couplings

- The vertex factor for the Z boson is:

$$-ig_Z \frac{1}{2}\gamma_\mu \left[c_V - c_A\gamma_5\right].$$



- Using the experimentally determined value of the weak mixing angle: $\sin^2\theta_W \approx 0.23$

Fermion x	Q_x	$I_W^3(x_L)$	$I_W^3(x_R)$	$c_L(x)$	$c_R(x)$	$c_V(x)$	$c_A(x)$
ν_e, ν_μ, ν_τ	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$
e, μ, τ	-1	$-\frac{1}{2}$	0	-0.27	0.23	-0.04	$-\frac{1}{2}$
u, c, t	$+\frac{2}{3}$	$+\frac{1}{2}$	0	0.35	-0.15	+0.19	$+\frac{1}{2}$
d, s, b	$-\frac{1}{3}$	$-\frac{1}{2}$	0	-0.42	0.08	-0.35	$-\frac{1}{2}$

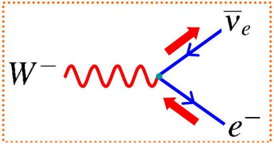
- Reminder:

$$\boxed{c_V(x) = c_L(x) + c_R(x)} = I_W^3(x_L) - 2Q_x \sin^2\theta_W \quad \text{and} \tag{195}$$

$$\boxed{c_A(x) = c_L(x) - c_R(x)} = I_W^3(x_L). \tag{196}$$

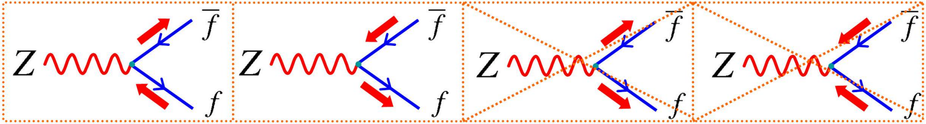
Z Boson Decay: Γ_Z

$\star$ In W-boson decay only had to consider one helicity combination of (assuming we can neglect final state masses: helicity states = chiral states)



W-boson couples: to LH particles and RH anti-particles

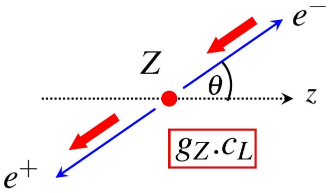
- But Z-boson couples to LH and RH particles (with different strengths)
- Need to consider only two helicity (or more correctly chiral) combinations:



This can be seen by considering either of the combinations which give zero. E.g.:

$$
\begin{aligned}
\bar{u}_R\gamma^\mu\left(c_V + c_A\gamma_5\right)v_R &= u^\dagger\frac{1}{2}\left(1+\gamma^5\right)\gamma^0\gamma^\mu\left(c_V + c_A\gamma^5\right)\frac{1}{2}\left(1-\gamma^5\right)v \\
&= \frac{1}{4}u^\dagger\gamma^0\left(1-\gamma^5\right)\gamma^\mu\left(1-\gamma^5\right)\left(c_V + c_A\gamma^5\right)v \\
&= \frac{1}{4}\bar{u}\gamma^\mu\left(1+\gamma^5\right)\left(1-\gamma^5\right)\left(c_V + c_A\gamma_5\right)v = 0.
\end{aligned}
$$

- In terms of left and right-handed combinations need to calculate:



- For unpolarized Z bosons: (Question 26):

$$\left\langle |M_{fi}|^2 \right\rangle = \frac{1}{3}\left[2c_L^2 g_Z^2 m_Z^2 + 2c_R^2 g_Z^2 m_Z^2\right] = \frac{2}{3} g_Z^2 m_Z^2 \left(c_L^2 + c_R^2\right)$$

(average over polarization)

- Using $c_V^2 + c_A^2 = 2\left(c_L^2 + c_R^2\right)$ and $\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}|M|^2$

  we get

$$\boxed{\Gamma\left(Z \to e^+ e^-\right) = \frac{g_Z^2 m_Z}{48\pi}\left(c_V^2 + c_A^2\right)}.$$

Z Branching Ratios

Question 27

- Neglecting fermion masses, we obtain the same expression for the other decays:

$$\Gamma(Z \to f\bar{f}) = \frac{g_Z^2 m_Z}{48\pi}\left(c_V^2 + c_A^2\right).$$

- Using values for c_V and c_A from page 543 we obtain:

$$\begin{aligned}
&\mathrm{Br}\left(Z \to e^+e^-\right) = \mathrm{Br}\left(Z \to \mu^+\mu^-\right) = \mathrm{Br}\left(Z \to \tau^+\tau^-\right) \approx 3.5\% \\
&\mathrm{Br}\left(Z \to \nu_1\bar{\nu}_1\right) = \mathrm{Br}\left(Z \to \nu_2\bar{\nu}_2\right) = \mathrm{Br}\left(Z \to \nu_3\bar{\nu}_3\right) \approx 6.9\% \\
&\mathrm{Br}\left(Z \to d\bar{d}\right) = \mathrm{Br}\left(Z \to s\bar{s}\right) = \mathrm{Br}\left(Z \to b\bar{b}\right) \approx 15\% \\
&\mathrm{Br}\left(Z \to u\bar{u}\right) = \mathrm{Br}\left(Z \to c\bar{c}\right) \approx 12\%
\end{aligned}$$

- The Z-boson therefore predominantly decays to hadrons:

$$Br(Z \to \text{ hadrons }) \approx 69\% \qquad \text{... mainly due to factor 3 from colour}$$

- Also predict total decay rate (total width)

$$\Gamma_Z = \sum_i \Gamma_i = 2.5\ \mathrm{GeV}$$

Experiment: $\Gamma_Z = 2.4952 \pm 0.0023\ \mathrm{GeV}$

Summary

- The Standard Model interactions are mediated by spin-1 gauge bosons
- The form of the interactions are completely specified by the assuming an underlying local phase transformation $\Rightarrow$ GAUGE INVARIANCE, initially trying:

$$
\begin{array}{rcl}
U(1)_{\text{em}} & \longrightarrow & QED \\
SU(2)_L & \longrightarrow & W^{\pm} \text{ and } W^3 \\
SU(3)_C & \longrightarrow & QCD
\end{array}
$$

- In order to 'unify' the electromagnetic and weak interactions, introduced a new symmetry gauge symmetry : $U(1)_Y$ (weak hypercharge) with an associated B-boson.
- The physical Z-boson and the photon are mixtures of the neutral W-boson and B determined by the Weak Mixing angle θ_W which is such that $\sin^2\theta_W \approx 0.23$.
- Have we really unified the EM and Weak interactions ? Well not really:
  - We started with two independent theories with coupling constants g_W, e.
  - We ended up with coupling constants which are related but at the cost of introducing a new parameter in the Standard Model, θ_W.
  - Interactions were not unified from any higher theoretical principle ... but it works!
  - Didn't discuss why photon ends up massless and Z-boson massive. (Relates to Higgs, but beyond course.)