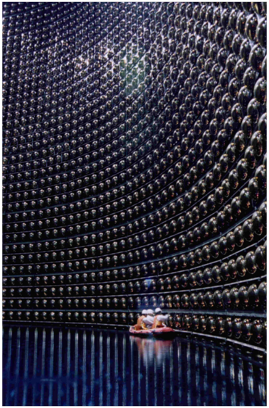
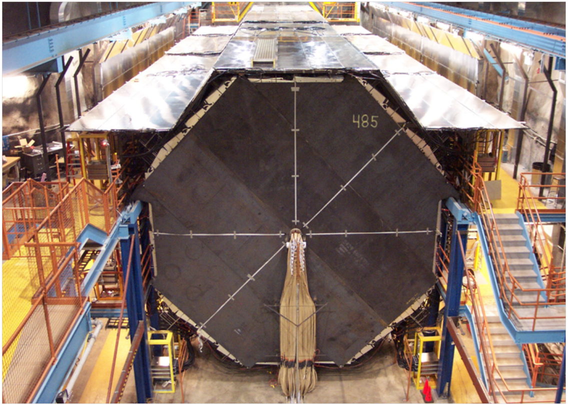
Dr C.G. Lester, 2023



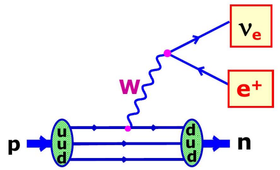


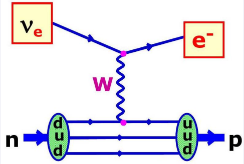
H11: Neutrino Oscillations

Neutrino Flavours Revisited

'Weak' eigenstates: ν_e, ν_μ, ν_τ

Earlier we defined the neutrino weak eigenstates to be the states produced or absorbed in association with an electron, muon or tau lepton respectively.





Mass eigenstates: ν_1, ν_2, ν_3

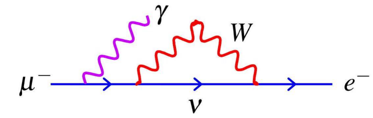
We can define the neutrino mass eigenstates to be the states which have a well defined mass. These propagate without changing as eigenstates of the free Hamiltonian.

Are the mass eigenstates the same as the weak eigenstates?

Evidence at short length-scales appeared to suggest that the mass and flavour eigenstates were identical.

Evidence in support of that historical view included:

- The non-observation of the decay $\mu \to e\gamma$.
  (observations suggest that $\mathrm{BR}\left(\mu^- \to e^-\gamma\right) < 10^{-12}$)



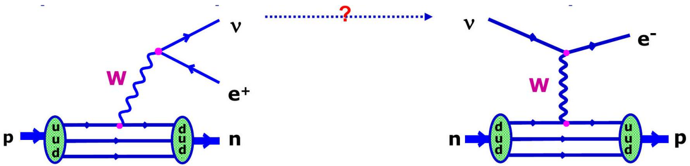
- Neutrinos produced with a given flavour (e.g. ν_e produce with e) that were later observed to undergo a nearby weak interaction with matter were seen to generate leptons which were always the same flavour as the progenitor.

Alas, the mass and weak eigenstates are not the same!

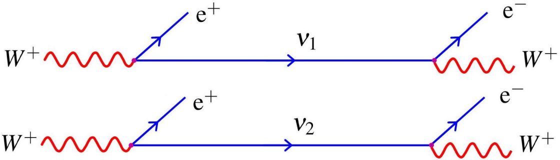
We will later see that **neutrinos change flavour on longer length-scales**, and so we will need a framework in which to account for that difference.

Mass Eigenstates and Weak Eigenstates

- The neutrino 'crossing the gap' between production and destruction remains unobserved while in transit.
- Just as the total amplitude the **double-slit** experiment is an **interference** between ampltides for transists of each slit, the amplitude for a neutrino production-and-destruction process like the one shown below is an interference between amplitudes for the mass eigenstates that could have 'crossed the gap'.
- In other words, a process like this:



can be thought of as an intereference between competing processes like these:



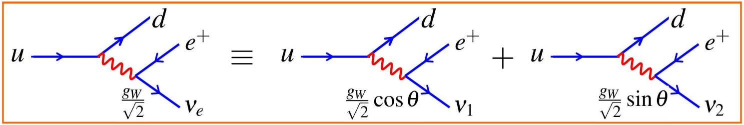
Neutrino Oscillations for Two Flavours

- Neutrinos are produced and interact as weak eigenstates, ν_e, ν_μ.
- The weak eigenstates are coherent linear combinations of the fundamental mass eigenstates ν_1, ν_2.
- There must be a 2×2 unitary change-of-basis matrix that relates the weak and mass eigenstates:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{136}$$

and so

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}. \tag{137}$$



- The mass eigenstates are the free particle solutions to the wave-equation and evolve as plane waves in the usual way since they have well defined masses:

$$|\nu_1(t)\rangle = |\nu_1\rangle\, e^{i\vec{p}_1\cdot\vec{x} - iE_1 t} \qquad |\nu_2(t)\rangle = |\nu_2\rangle\, e^{i\vec{p}_2\cdot\vec{x} - iE_2 t}.$$

-Suppose at time $t = 0$ a neutrino is produced in a pure ν_e state in a decay $u \to de^+\nu_e$. Then:

$$|\psi(0)\rangle = |\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle\,.$$

-Take the z-axis to be along the neutrino direction.

-The wave-function evolves according to the time-evolution of the mass eigenstates (free particle solutions to the wave equation):

$$|\psi(t)\rangle = \cos\theta\,|\nu_1\rangle\,e^{-ip_1\cdot x} + \sin\theta\,|\nu_2\rangle\,e^{-ip_2\cdot x}$$

where $p_i \cdot x = E_i t - \vec{p}_i \cdot \vec{x} = E_i t - |\vec{p}_i|\,z$.

-Suppose the neutrino interacts in a detector at a distance L and at a time T. Then

$$\boxed{|\psi(L,T)\rangle = \cos\theta\,|\nu_1\rangle\,e^{-i\phi_1} + \sin\theta\,|\nu_2\rangle\,e^{-i\phi_2}}$$

for $\phi_i = p_i \cdot x = E_i T - |\vec{p}_i|\,L$.

-Expressing the mass eigenstates, $|\nu_1\rangle\,, |\nu_2\rangle$, in terms of weak eigenstates using (137) gives:

$$|\psi(L,T)\rangle = \cos\theta\,(\cos\theta\,|\nu_e\rangle - \sin\theta\,|\nu_\mu\rangle)\,e^{-i\phi_1} + \sin\theta\,(\sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle)\,e^{-i\phi_2}$$

and so

$$\boxed{|\psi(L,T)\rangle = |\nu_e\rangle\left(\cos^2\theta e^{-i\phi_1} + \sin^2\theta e^{-i\phi_2}\right) + |\nu_\mu\rangle\sin\theta\cos\theta\left(-e^{-i\phi_1} + e^{-i\phi_2}\right)}.$$

- If the masses of $|\nu_1\rangle, |\nu_2\rangle$ are the same, the mass eigenstates remain in phase, $\phi_1 = \phi_2$, and the state remains the linear combination corresponding to $|\nu_e\rangle$ and in a weak interaction will produce an electron
- If the masses are different, the wave-function no longer remains a pure $|\nu_e\rangle$:

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= |\langle \nu_\mu \mid \psi(L, T)\rangle|^2 \\ &= \cos^2\theta \sin^2\theta \left(-e^{-i\phi_1} + e^{-i\phi_2}\right)\left(-e^{+i\phi_1} + e^{+i\phi_2}\right) \\ &= \frac{1}{4}\sin^2 2\theta \,(2 - 2\cos(\phi_1 - \phi_2)) \\ &= \sin^2 2\theta \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right). \end{aligned}$$

- Evaluation of the phase difference:

$$\Delta\phi_{12} = \phi_1 - \phi_2 = (E_1 - E_2)\,T - (|p_1| - |p_2|)\,L$$

needs care ...

- (i) One could assume $|p_1| = |p_2| = p$ in which case

$$\Delta\phi_{12} = (E_1 - E_2)\,T = \left[\left(p^2 + m_1^2\right)^{1/2} - \left(p^2 + m_2^2\right)^{1/2}\right] L \quad \text{using } L \approx (c)\,T.$$

Thus:

$$\Delta\phi_{12} = p\left[\left(1+\frac{m_1^2}{p^2}\right)^{1/2} - \left(1+\frac{m_2^2}{p^2}\right)^{1/2}\right] L \approx \frac{m_1^2 - m_2^2}{2p} L.$$

- However we have neglected that fact that for the same momentum, different mass eigenstates will propagate at different velocities and be observed at different times. A full derivation requires a wave-packet treatment which we will not provide – but the same result obtained above would still be found.
- The last statement may be partly by justified by noting that the phase difference can **also** be written:

$$\Delta\phi_{12} = (E_1 - E_2)\, T - \left(\frac{|p_1|^2 - |p_2|^2}{|p_1| + |p_2|}\right) L$$

i.e. as:

$$\Delta\phi_{12} = (E_1 - E_2)\left[T - \left(\frac{E_1 + E_2}{|p_1| + |p_2|}\right) L\right] + \left(\frac{m_1^2 - m_2^2}{|p_1| + |p_2|}\right) L$$

and the first term on the RHS **vanishes if we assume** (ii) $E_1 = E_2$ **or** (iii) $\beta_1 = \beta_2$.

- **In all three eases:**

$$\boxed{\Delta\phi_{12} \approx \frac{m_1^2 - m_2^2}{2\mathrm{p}} L \approx \frac{\Delta m^2}{2E} L}.$$

⋆ Hence the two-flavour oscillation probability is:

$$P(\nu_e \to \nu_\mu) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right)$$

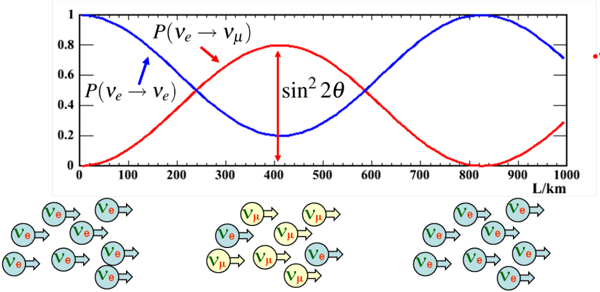
with

$$\Delta m^2_{21} = m^2_2 - m^2_1.$$

⋆ The corresponding two-flavour survival probability is:

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right).$$

For example, with $\Delta m^2 = 0.003\ \mathrm{eV}^2$, $\sin^2 2\theta = 0.8$ and $E_\nu = 1\ \mathrm{GeV}$ get:



• wavelength

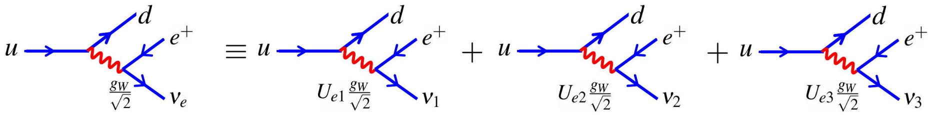
$$\boxed{\lambda_{osc} = \frac{4\pi E}{\Delta m^2}}$$

Neutrino Oscillations for Three Flavours

⋆ It is simple to extend this treatment to three generations of neutrinos.

⋆ In this case we have:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$



⋆ This 3×3 Unitary matrix U is known as the Pontecorvo-Maki-Nakagawa-Sakata matrix, usually abbreviated 'PMNS Matrix'.

⋆ It has to be unitary to conserve state normalizations.

-Using $\quad U^\dagger U = I \quad \Rightarrow \quad U^{-1} = U^\dagger = (U^*)^T$

gives
$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1} \\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2} \\ U^*_{e3} & U^*_{\mu 3} & U^*_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

Unitarity Relations

⋆ The unitarity of the PMNS matrix gives several useful relations:

$$UU^\dagger = I$$

implies that

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1} \\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2} \\ U^*_{e3} & U^*_{\mu 3} & U^*_{\tau 3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so

$$U_{e1}U^*_{e1} + U_{e2}U^*_{e2} + U_{e3}U^*_{e3} = 1 \tag{138}$$
$$U_{\mu 1}U^*_{\mu 1} + U_{\mu 2}U^*_{\mu 2} + U_{\mu 3}U^*_{\mu 3} = 1 \tag{139}$$
$$U_{\tau 1}U^*_{\tau 1} + U_{\tau 2}U^*_{\tau 2} + U_{\tau 3}U^*_{\tau 3} = 1 \tag{140}$$
$$U_{e1}U^*_{\mu 1} + U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3} = 0 \tag{141}$$
$$U_{e1}U^*_{\tau 1} + U_{e2}U^*_{\tau 2} + U_{e3}U^*_{\tau 3} = 0 \tag{142}$$
$$U_{\mu 1}U^*_{\tau 1} + U_{\mu 2}U^*_{\tau 2} + U_{\mu 3}U^*_{\tau 3} = 0. \tag{143}$$

Three flavour oscillations (cont.) I

$\star$ To calculate the oscillation probability proceed as before by first considering a state which is produced at $t = 0$ as a $|\nu_e\rangle$:

$$|\psi(t=0)\rangle = |\nu_e\rangle = U_{e1}\,|\nu_1\rangle + U_{e2}\,|\nu_2\rangle + U_{e3}\,|\nu_3\rangle$$

-The wave-function evolves as:

$$|\psi(t)\rangle = U_{e1}\,|\nu_1\rangle\, e^{-ip_1\cdot x} + U_{e2}\,|\nu_2\rangle\, e^{-ip_2\cdot x} + U_{e3}\,|\nu_3\rangle\, e^{-ip_3\cdot x}$$

where

$$p_i \cdot x = E_i t - \vec{p}_i \cdot \vec{x} = E_i t - |\vec{p}|z.$$

-After travelling a distance L in the z-direction:

$$|\psi(L)\rangle = U_{e1}\,|\nu_1\rangle\, e^{-i\phi_1} + U_{e2}\,|\nu_2\rangle\, e^{-i\phi_2} + U_{e3}\,|\nu_3\rangle\, e^{-i\phi_3}$$

where $\phi_i = p_i.x = E_i t - |\vec{p}|L = (E_i - |\vec{p}_i|)\,L$.
-As before we can approximate

$$\phi_i \approx \frac{m_i^2}{2E_i} L.$$

Three flavour oscillations (cont.) II

-Expressing the mass eigenstates in terms of the weak eigenstates we find

$$\begin{aligned}|\psi(L)\rangle = & \, U_{e1}\left(U_{e1}^{*}\left|\nu_{e}\right\rangle + U_{\mu 1}^{*}\left|\nu_{\mu}\right\rangle + U_{\tau 1}^{*}\left|\nu_{\tau}\right\rangle\right) e^{-i\phi_{1}} \\ & + U_{e2}\left(U_{e2}^{*}\left|\nu_{e}\right\rangle + U_{\mu 2}^{*}\left|\nu_{\mu}\right\rangle + U_{\tau 2}^{*}\left|\nu_{\tau}\right\rangle\right) e^{-i\phi_{2}} \\ & + U_{e3}\left(U_{e3}^{*}\left|\nu_{e}\right\rangle + U_{\mu 3}^{*}\left|\nu_{\mu}\right\rangle + U_{\tau 3}^{*}\left|\nu_{\tau}\right\rangle\right) e^{-i\phi_{3}}\end{aligned}$$

which can be rearranged to give

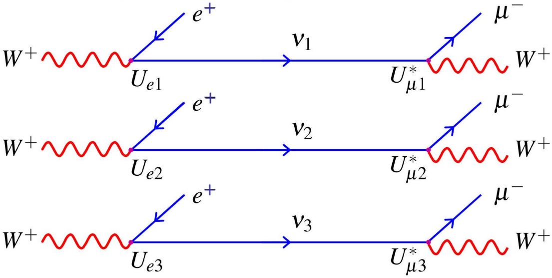
$$\begin{aligned}|\psi(L)\rangle = & \left(U_{e1} U_{e1}^{*} e^{-i\phi_{1}} + U_{e2} U_{e2}^{*} e^{-i\phi_{2}} + U_{e3} U_{e3}^{*} e^{-i\phi_{3}}\right)\left|\nu_{e}\right\rangle \\ & + \left(U_{e1} U_{\mu 1}^{*} e^{-i\phi_{1}} + U_{e2} U_{\mu 2}^{*} e^{-i\phi_{2}} + U_{e3} U_{\mu 3}^{*} e^{-i\phi_{3}}\right)\left|\nu_{\mu}\right\rangle \\ & + \left(U_{e1} U_{\tau 1}^{*} e^{-i\phi_{1}} + U_{e2} U_{\tau 2}^{*} e^{-i\phi_{2}} + U_{e3} U_{\tau 3}^{*} e^{-i\phi_{3}}\right)\left|\nu_{\tau}\right\rangle . \end{aligned} \tag{144}$$

-Therefore

$$\begin{aligned}P\left(\nu_{e} \rightarrow \nu_{\mu}\right) & = \left|\left\langle\nu_{\mu} \mid \psi(L)\right\rangle\right|^{2} \\ & = \left|U_{e1} U_{\mu 1}^{*} e^{-i\phi_{1}} + U_{e2} U_{\mu 2}^{*} e^{-i\phi_{2}} + U_{e3} U_{\mu 3}^{*} e^{-i\phi_{3}}\right|^{2} .\end{aligned} \tag{145}$$

Three flavour oscillations (cont.) III

-The terms in the last expression (145) can be depicted as follows:



-Because of the unitarity of the PMNS matrix we have (141):

$$U_{e1}U^*_{\mu 1} + U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3} = 0.$$

Consequently, unless the phases of the different components are different, the sum of these three diagrams (i.e. the expression (145) for $P(\nu_e \to \nu_\mu)$) is zero.

Conclusion (as before)

We require different neutrino masses for oscillations!

-Simplify

$$P\left(\nu_e \to \nu_\mu\right) = \left| U_{e1} U_{\mu 1}^* e^{-i\phi_1} + U_{e2} U_{\mu 2}^* e^{-i\phi_2} + U_{e3} U_{\mu 3}^* e^{-i\phi_3} \right|^2$$

using the identity

$$\boxed{\left|z_1 + z_2 + z_3\right|^2 \equiv \left|z_1\right|^2 + \left|z_2\right|^2 + \left|z_3\right|^2 + 2\Re\left(z_1 z_2^* + z_1 z_3^* + z_2 z_3^*\right)} \tag{146}$$

giving:

$$P\left(\nu_e \to \nu_\mu\right) = \left| U_{e1} U_{\mu 1}^* \right|^2 + \left| U_{e2} U_{\mu 2}^* \right|^2 + \left| U_{e3} U_{\mu 3}^* \right|^2 + \tag{147}$$
$$2\Re\left(U_{e1} U_{\mu 1}^* U_{e2}^* U_{\mu 2} e^{-i(\phi_1 - \phi_2)} + U_{e1} U_{\mu 1}^* U_{e3}^* U_{\mu 3} e^{-i(\phi_1 - \phi_3)} + U_{e2} U_{\mu 2}^* U_{e3}^* U_{\mu 3} e^{-i(\phi_2 - \phi_3)} \right).$$

-But by applying identity (146) to $|(141)|^2$ we see that:

$$\left| U_{e1} U_{\mu 1}^* + U_{e2} U_{\mu 2}^* + U_{e3} U_{\mu 3}^* \right|^2 = 0$$

and so

$$\left| U_{e1} U_{\mu 1}^* \right|^2 + \left| U_{e2} U_{\mu 2}^* \right|^2 + \left| U_{e3} U_{\mu 3}^* \right|^2 =$$
$$- 2\Re\left(U_{e1} U_{\mu 1}^* U_{e2}^* U_{\mu 2} + U_{e1} U_{\mu 1}^* U_{e3}^* U_{\mu 3} + U_{e2} U_{\mu 2}^* U_{e3}^* U_{\mu 3} \right)$$

which can be substituted into equation (147) to give:

$$\boxed{\begin{aligned} P\left(\nu_e \to \nu_\mu\right) = {} & 2\mathfrak{R}\left\{U_{e1}U_{\mu 1}^{*}U_{e2}^{*}U_{\mu 2}\left[e^{-i(\phi_1-\phi_2)}-1\right]\right\} \\ & + 2\mathfrak{R}\left\{U_{e1}U_{\mu 1}^{*}U_{e3}^{*}U_{\mu 3}\left[e^{-i(\phi_1-\phi_3)}-1\right]\right\} \\ & + 2\mathfrak{R}\left\{U_{e2}U_{\mu 2}^{*}U_{e3}^{*}U_{\mu 3}\left[e^{-i(\phi_2-\phi_3)}-1\right]\right\} \end{aligned}}. \tag{148}$$

$\star$ The corresponding expression for the electron survival probability (below) may be obtained from the coefficient for $|\nu_e\rangle$ in (144):

$$\begin{aligned} P\left(\nu_e \to \nu_e\right) &= \left|\langle \nu_e \mid \psi(L)\rangle\right|^2 \\ &= \left| U_{e1} U_{e1}^* e^{-i\phi_1} + U_{e2} U_{e2}^* e^{-i\phi_2} + U_{e3} U_{e3}^* e^{-i\phi_3} \right|^2 . \end{aligned}$$

which using the unitarity relation (138)

$$\left| U_{e1} U_{e1}^* + U_{e2} U_{e2}^* + U_{e3} U_{e3}^* \right|^2 = 1.$$

can be written

$$\begin{aligned} P\left(\nu_e \to \nu_e\right) = 1 &+ 2\left|U_{e1}\right|^2 \left|U_{e2}\right|^2 \Re\{[e^{-i(\phi_1-\phi_2)} - 1]\} \\ &+ 2\left|U_{e1}\right|^2 \left|U_{e3}\right|^2 \Re\{[e^{-i(\phi_1-\phi_3)} - 1]\}. \\ &+ 2\left|U_{e2}\right|^2 \left|U_{e3}\right|^2 \Re\{[e^{-i(\phi_2-\phi_3)} - 1]\} \end{aligned} \tag{149}$$

The expression (149) can be simplified using

$$\begin{aligned}\Re\left\{e^{-i(\phi_1-\phi_2)}-1\right\} &= \cos\left(\phi_2-\phi_1\right)-1 \\ &= -2\sin^2\left(\frac{\phi_2-\phi_1}{2}\right) \quad \text{with } \phi_i \approx \frac{m_i^2}{2E}L \\ &= -2\sin^2\left(\frac{\left(m_2^2-m_1^2\right)L}{4E}\right) &\quad (150) \\ &= -2\sin^2\left(\Delta_{21}\right). &\quad (151)\end{aligned}$$

where we have defined the dimensionless phase difference:

$$\boxed{\Delta_{ij}=\frac{\left(m_i^2-m_j^2\right)L}{4E}=\frac{\Delta m_{ij}^2 L}{4E}} \quad (152)$$

with $\Delta m_{21}^2 = m_2^2 - m_1^2$ as before.
-Using the above the electron neutrino survival probability becomes:

$$\boxed{P\left(\nu_e\to\nu_e\right)=1-4\left|U_{e1}\right|^2\left|U_{e2}\right|^2\sin^2\Delta_{21}-4\left|U_{e1}\right|^2\left|U_{e3}\right|^2\sin^2\Delta_{31}-4\left|U_{e2}\right|^2\left|U_{e3}\right|^2\sin^2\Delta_{32}}.$$

Similar expressions can be obtained for the muon and tau neutrino survival probabilities for muon and tau neutrinos.

Note that since we only have three neutrino generations there are only two independent mass-squared differences, i.e.:

$$m_3^2 - m_1^2 = \left(m_3^2 - m_2^2\right) + \left(m_2^2 - m_1^2\right)$$

so in

$$\boxed{P\left(\nu_e \to \nu_e\right) = 1 - 4\left|U_{e1}\right|^2\left|U_{e2}\right|^2 \sin^2\Delta_{21} - 4\left|U_{e1}\right|^2\left|U_{e3}\right|^2 \sin^2\Delta_{31} - 4\left|U_{e2}\right|^2\left|U_{e3}\right|^2 \sin^2\Delta_{32}}. \tag{153}$$

only two of the Δ_{ij} are independent.

Aside

Conversion to more units commonly used in neutrino physics gives:

$$\Delta_{21} = 1.27\frac{\Delta m_{21}^2\left(\mathrm{eV}^2\right)\ L(\mathrm{km})}{E(\mathrm{GeV})}$$

and

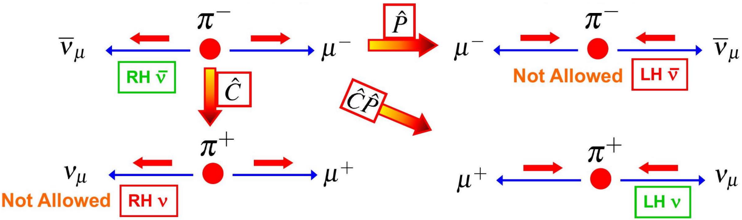
$$\lambda_{\mathrm{osc}}(\mathrm{km}) = 2.47\frac{E(\mathrm{GeV})}{\Delta m^2\left(\mathrm{eV}^2\right)}.$$

CP and CPT in the Weak Interaction

⋆ In addition to parity there are two other important discrete symmetries:

Parity	$\hat{P}$:	$\boxed{\vec{r} \to -\vec{r}}$
Time Reversal	$\hat{T}$:	$\boxed{t \to -t}$
Charge Conjugation	$\hat{C}$:	$\boxed{\text{Particle} \longleftrightarrow \text{Anti-particle}}$

- The weak interaction violates parity conservation, but what about $\hat{C}$?
  Consider pion decay remembering that the neutrino is ultra-relativistic and only left-handed neutrinos and right-handed anti-neutrinos participate in weak interaction.



- Hence weak interaction also violates charge conjugation symmetry but appears to be invariant under combined effect of $\hat{C}$ and $\hat{P}$.

- CP transforms:

$$\boxed{\begin{array}{ccc} \text{RH Particles} & \longleftrightarrow & \text{LH Anti-particles} \\ \text{LH Particles} & \longleftrightarrow & \text{RH Anti-particles} \end{array}}.$$

- If the weak interaction were invariant under CP expect

$$\boxed{\Gamma\left(\pi^+ \to \mu^+ \nu_\mu\right) = \Gamma\left(\pi^- \to \mu^- \bar{\nu}_\mu\right)}.$$

- The CPT-Theorem says that all Lorentz invariant Quantum Field Theories make invariant predictions under (charge conjugation) + (parity) + (time reversal).
- The CPT-Theorem says that particles/anti-particles have identical mass, lifetime, magnetic moments, ...
- The best current experimental test of the CPT-Theorem is perhaps the measurement of $m_{K^0} - m_{\bar{K}^0} < 6 \times 10^{-19} m_{K^0}$ so we currently believe that CPT is a good symmetry.
- Therefore, T-violation implies CP-violation!
- We are interested in CP violation as the small excess of matter over anti-matter in the universe may have been generated by it!
- CP violation can arise in the weak interaction (see also Handout 12).

CP and T Violation in Neutrino Oscillations

-Previously derived the oscillation probability for $\nu_e \to \nu_\mu$:

$$
\begin{aligned}
P\left(\nu_e \to \nu_\mu\right) = {} & 2\,\mathrm{Re}\left\{U_{e1}U_{\mu 1}^{*}U_{e2}^{*}U_{\mu 2}\left[e^{-I(\phi_1-\phi_2)}-1\right]\right\} \\
& + 2\,\mathrm{Re}\left\{U_{e1}U_{\mu 1}^{*}U_{e3}^{*}U_{\mu 3}\left[e^{-i(\phi_1-\phi_3)}-1\right]\right\} \\
& + 2\,\mathrm{Re}\left\{U_{e2}U_{\mu 2}^{*}U_{e3}^{*}U_{\mu 3}\left[e^{-i(\phi_2-\phi_3)}-1\right]\right\}
\end{aligned}
$$

-The oscillation probability for $\nu_\mu \to \nu_e$ can be obtained in the same manner or by simply exchanging the labels $(e) \leftrightarrow (\mu)$:

$$
\begin{aligned}
P\left(\nu_\mu \to \nu_e\right) = {} & 2\,\mathrm{Re}\left\{U_{\mu 1}U_{e1}^{*}U_{\mu 2}^{*}U_{e2}\left[e^{-i(\phi_1-\phi_2)}-1\right]\right\} \\
& + 2\,\mathrm{Re}\left\{U_{\mu 1}U_{e1}^{*}U_{\mu 3}^{*}U_{e3}\left[e^{-i(\phi_1-\phi_3)}-1\right]\right\} \\
& + 2\,\mathrm{Re}\left\{U_{\mu 2}U_{e2}^{*}U_{\mu 3}^{*}U_{e3}\left[e^{-i(\phi_2-\phi_3)}-1\right]\right\}
\end{aligned}
\tag{154}
$$

$\star$ Unless the elements of the PMNS matrix are real (ignore any overall complex phase)

$$
P\left(\nu_e \to \nu_\mu\right) \neq P\left(\nu_\mu \to \nu_e\right). \tag{155}
$$

so **three-flavour** neutrino oscillations are not invariant under time reversal $t \to -t$.

- Consider the effects of T, CP and CPT on neutrino oscillations:

T	$\nu_e \to \nu_\mu$	$\overset{\hat{T}}{\longleftrightarrow}$	$\nu_\mu \to \nu_e$
CP	$\nu_e \to \nu_\mu$	$\overset{\hat{C}\hat{P}}{\longleftrightarrow}$	$\overline{\nu}_e \to \overline{\nu}_\mu$
CPT	$\nu_e \to \nu_\mu$	$\overset{\hat{C}\hat{P}\hat{T}}{\longleftrightarrow}$	$\overline{\nu}_\mu \to \overline{\nu}_e$

(C alone is not sufficient as it transforms LH neutrinos into LH anti-neutrinos (not involved in Weak Interaction).)

-Thus if the weak interactions is invariant under CPT then

$$P\left(\nu_\mu \to \nu_e\right) = P\left(\bar{\nu}_e \to \bar{\nu}_\mu\right)$$

which, if combined with (155) tells us that if the PMNS matrix is not purely real, then

$$\boxed{P\left(\nu_e \to \nu_\mu\right) \neq P\left(\bar{\nu}_e \to \bar{\nu}_\mu\right)}.$$

Hence: unless the PMNS matrix is real, CP will be violated in neutrino oscillations!

Future experiments, e.g. 'a neutrino factory', are being considered as a way to investigate CP violation in neutrino oscillations. However, CP violating effects are well below the current experimental sensitivity. In the following discussion we will take the PMNS matrix to be real.

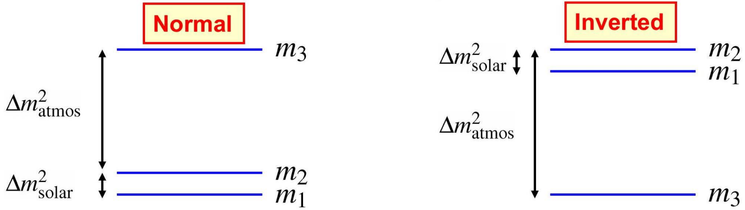
(Now try question 22.)

Neutrino Mass Hierarchy

- To date, results on neutrino oscillations only determine

$$\left|\Delta m_{ji}^2\right| = \left|m_j^2 - m_i^2\right|$$

- Two distinct and very different mass scales:
  - Atmospheric neutrino oscillations : $\left|\Delta m^2\right|_{\text{atmos}} \sim 2.5 \times 10^{-3}\ \mathrm{eV}^2$
  - Solar neutrino oscillations: $\left|\Delta m^2\right|_{\text{solar}} \sim 8 \times 10^{-5}\ \mathrm{eV}^2$
- Two possible assignments of mass hierarchy:



In both cases: $\Delta m_{21}^2 \sim 8 \times 10^{-5}\ \mathrm{eV}^2$ (solar)

$$\left|\Delta m_{31}^2\right| \approx \left|\Delta m_{32}^2\right| \sim 2.5 \times 10^{-3}\ \mathrm{eV}^2 \qquad \text{(atmospheric)}$$

hence using (152) we approximate $\boxed{\left|\Delta m_{31}^2\right| \approx \left|\Delta m_{32}^2\right|}$ and $\boxed{\left|\Delta_{31}\right| \approx \left|\Delta_{32}\right|}$. (156)

Approx/Simplified Three Flavour Oscillations, neglecting CP Violation

- Taking the PMNS matrix to be real (i.e. neglecting CP violation) considerably simplifies the algebra of three flavour oscillations: E.g. osc. prob. (148) becomes:

$$P(\nu_e \to \nu_\mu) = -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} - 4U_{e1}U_{\mu 1}U_{e3}U_{\mu 3}\sin^2\Delta_{31} - 4U_{e2}U_{\mu 2}U_{e3}U_{\mu 3}\sin^2\Delta_{32}$$

Then using (156) ($|\Delta_{31}| \approx |\Delta_{32}|$) we can further simplify the ν_e **oscillation probablility**:

$$P(\nu_e \to \nu_\mu) \approx -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} - 4\left(U_{e1}U_{\mu 1} + U_{e2}U_{\mu 2}\right)U_{e3}U_{\mu 3}\sin^2\Delta_{32}$$

which can be simplified again using (141) $(U_{e1}U^*_{\mu 1} + U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3} = 0)$ to give

$$\boxed{P(\nu_e \to \nu_\mu) \approx -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} + 4U^2_{e3}U^2_{\mu 3}\sin^2\Delta_{32}}.$$

- Can apply same approximations to the ν_e **survival probability** (153):

$$\begin{aligned} P(\nu_e \to \nu_e) &= 1 - 4U^2_{e1}U^2_{e2}\sin^2\Delta_{21} - 4U^2_{e1}U^2_{e3}\sin^2\Delta_{31} - 4U^2_{e2}U^2_{e3}\sin^2\Delta_{32} \\ &\approx 1 - 4U^2_{e1}U^2_{e2}\sin^2\Delta_{21} - 4\left(U^2_{e1} + U^2_{e2}\right)U^2_{e3}\sin^2\Delta_{32} \end{aligned}$$

which can be simplified using (138) $U^2_{e1} + U^2_{e2} + U^2_{e3} = 1$ to give:

$$\boxed{P(\nu_e \to \nu_e) \approx 1 - 4U^2_{e1}U^2_{e2}\sin^2\Delta_{21} - 4\left(1 - U^2_{e3}\right)U^2_{e3}\sin^2\Delta_{32}}.$$

Summary of approx three flavour oscillation probabilities neglecting CP-violation and using $|\Delta m^2_{31}| \approx |\Delta m^2_{32}|$

Survival probabilities

$$P(\nu_e \to \nu_e) \approx 1 - 4U^2_{e1}U^2_{e2}\sin^2\Delta_{21} - 4\left(1 - U^2_{e3}\right)U^2_{e3}\sin^2\Delta_{32} \tag{157}$$

$$P(\nu_\mu \to \nu_\mu) \approx 1 - 4U^2_{\mu1}U^2_{\mu2}\sin^2\Delta_{21} - 4\left(1 - U^2_{\mu3}\right)U^2_{\mu3}\sin^2\Delta_{32} \tag{158}$$

$$P(\nu_\tau \to \nu_\tau) \approx 1 - 4U^2_{\tau1}U^2_{\tau2}\sin^2\Delta_{21} - 4\left(1 - U^2_{\tau3}\right)U^2_{\tau3}\sin^2\Delta_{32} \tag{159}$$

Oscillation probabilities

$$P(\nu_e \to \nu_\mu) = P(\nu_\mu \to \nu_e) \approx -4U_{e1}U_{\mu1}U_{e2}U_{\mu2}\sin^2\Delta_{21} + 4U^2_{e3}U^2_{\mu3}\sin^2\Delta_{32} \tag{160}$$

$$P(\nu_e \to \nu_\tau) = P(\nu_\tau \to \nu_e) \approx -4U_{e1}U_{\tau1}U_{e2}U_{\tau2}\sin^2\Delta_{21} + 4U^2_{e3}U^2_{\tau3}\sin^2\Delta_{32} \tag{161}$$

$$P(\nu_\mu \to \nu_\tau) = P(\nu_\tau \to \nu_\mu) \approx -4U_{\mu1}U_{\tau1}U_{\mu2}U_{\tau2}\sin^2\Delta_{21} + 4U^2_{\mu3}U^2_{\tau3}\sin^2\Delta_{32} \tag{162}$$

The wavelengths associated with $\sin^2\Delta_{21}$ and $\sin^2\Delta_{32}$ are:

$$\boxed{\begin{array}{c}\text{“ATMOSPHERIC”} \\ \lambda_{32} = \frac{4\pi E}{\Delta m^2_{32}} \\ \text{“Short”-Wavelength}\end{array}} \quad \text{and} \quad \boxed{\begin{array}{c}\text{“SOLAR”} \quad \lambda_{21} = \frac{4\pi E}{\Delta m^2_{21}} \\ \text{“Long”-Wavelength}\end{array}}.$$

PMNS Matrix

- The PMNS matrix is usually expressed in terms of 3 rotation angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a complex phase δ, using the notation $s_{ij} = \sin\theta_{ij}, \quad c_{ij} = \cos\theta_{ij}$

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{“Atmospheric”}} \times \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{+i\delta} & 0 & c_{13} \end{pmatrix} \times \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{“Solar”}}$$

Dominates: **“Atmospheric”** **“Solar”**

- Writing this out in full:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.$$

- So there are six SM parameters that can be measured in ν oscillation experiments:

$\lvert\Delta m_{21}\rvert^2 = \lvert m_2^2 - m_1^2\rvert$	θ_{12}	Solar and reactor neutrino experiments
$\lvert\Delta m_{32}\rvert^2 = \lvert m_3^2 - m_2^2\rvert$	θ_{23}	Atmospheric and beam neutrino experiments
	θ_{13}	Reactor neutrino experiments + future beam
	δ	Future beam experiments

Neutrino Experiments

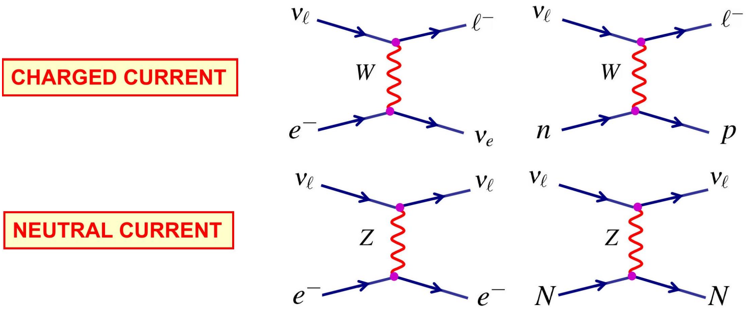
Before discussing current experimental data, need to consider how neutrinos interact in matter (i.e. our detectors)

Two processes:

- Charged current (CC) interactions (via a W-boson) $\Rightarrow$ charged lepton
- Neutral current (NC) interactions (via a Z-boson)

Two possible "targets": can have neutrino interactions with:

- atomic electrons
- nucleons within the nucleus



Neutrino Interaction Thresholds I

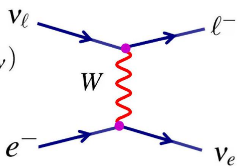
Neutrino detection method depends on the neutrino energy and (weak) flavour

- Neutrinos from the sun and nuclear reactions have $E_V \sim 1$ MeV.
- Atmospheric neutrinos have $E_\nu \sim 1$ GeV.

These energies are relatively low and not all interactions are kinematically allowed, i.e. there is a threshold energy before an interaction can occur. Require sufficient energy in the centre-of-mass frame to produce the final state particles

1. Charged current interactions **on atomic electrons** (in laboratory frame)

$$p_\nu = (E_\nu, 0, 0, E_\nu)$$
$$p_e = (m_e, 0, 0, 0)$$



$s = (p_\nu + p_e)^2 = (E_\nu + m_e)^2 - E_\nu^2$
Interaction requires $s > m_\ell^2$ so

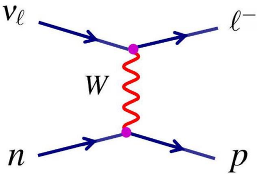
$$\boxed{E_\nu > \left[\left(\frac{m_\ell}{m_e}\right)^2 - 1\right]\frac{m_e}{2}}.$$

Putting in the numbers, *CC* interactions **with atomic electrons** require:

$$E_{\nu_e} > 0 \qquad \underbrace{E_{\nu_\mu} > 11\ \mathrm{GeV} \quad E_{\nu_\tau} > 3090\ \mathrm{GeV}}_{\text{These are high energy thresholds compared to most neutrino energies we will consider.}} .$$

Neutrino Interaction Thresholds II

2. Charged Current interactions **on nucleons** (in laboratory frame):



$$s = (p_\nu + p_n)^2 = (E_\nu + m_n)^2 - E_\nu^2$$

Interaction requires $s > (m_\ell + m_p)^2$ so

$$\boxed{E_\nu > \frac{(m_p^2 - m_n^2) + m_\ell^2 + 2m_p m_\ell}{2m_n}}$$

Therefore, *CC* interactions **with neutrons** require:

$$E_{\nu_e} > 0 \quad E_{\nu_\mu} > 110\ \mathrm{MeV} \quad E_{\nu_\tau} > 3.5\ \mathrm{GeV}.$$

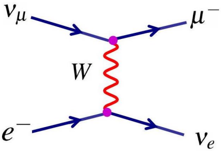
Some neutrinos therefore effectively 'disappear'.
From point of view of CC, these cannot interact:

Electron neutrinos from the sun and nuclear reactors $E_\nu \sim 1\ \mathrm{MeV}$ which oscillate **into muon or tau** neutrinos.
Atmospheric muon neutrinos $E_\nu \sim 1\ \mathrm{GeV}$ which oscillate **into tau** neutrinos.

Therefore, many experimental signatures for neutrino oscillation are **deficits** of neutrino interactions (a notable exception is SNO (pg 449), which makes use of the Neutral Current) because most neutrinos are below threshold for production of leptons of different flavour to the original neutrino!

In Handout 10 we derived expressions for CC neutrino-quark cross sections in ultra-relativistic limit (neglecting masses of neutrinos/quarks). Therefore ...

- for high energy **muon neutrinos** can directly use the result (128) namely $\sigma_{\nu_\mu e^-} = \frac{G_F^2 s}{\pi}$

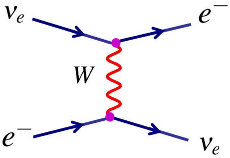


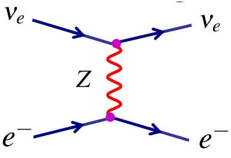
and so using $s = (E_\nu + m_e)^2 - E_\nu^2 \approx 2m_e E_\nu$ one obtains:

$$\boxed{\sigma_{\nu_\mu e^-} = \frac{2m_e G_F^2 E_\nu}{\pi}}.$$

$\Big\{$ Note that the neutrino scattering coss section increases linearly with lab. frame neutrino energy!

- For **electron neutrinos** there is another lowest order diagram with the same final state



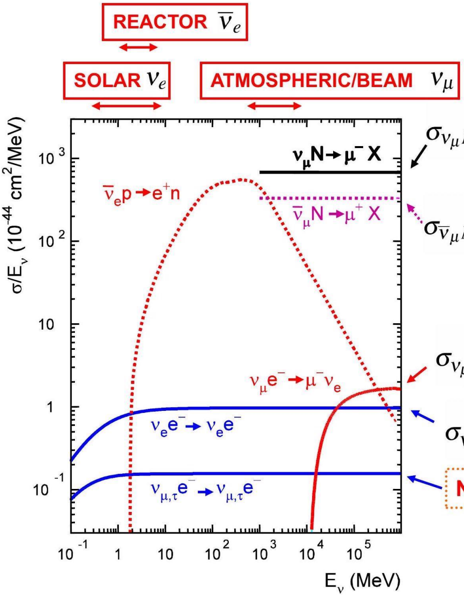


It turns out that the cross section is lower than the pure CC cross section due to negative interference when summing matrix elements $|M_{CC} + M_{NC}|^2 < |M_{CC}|^2$ so

$$\boxed{\sigma_{\nu_e e} \approx 0.6\sigma^{CC}_{\nu_e e}}.$$

- In the high energy limit the CC neutrino-nucleon cross sections are larqer due to the higher centre-of-mass energy: $s = (E_\nu + m_n)^2 - E_\nu^2 \approx 2m_n E_\nu$.

Neutrino scattering cross sections compared



$$\left.\begin{aligned}\sigma_{\nu_\mu N} &= 0.22\frac{2m_p G_F^2 E_\nu}{\pi}\\ \sigma_{\overline{\nu}_\mu N} &= 0.33\frac{2m_p G_F^2 E_\nu}{3\pi}\end{aligned}\right\}$$

Deep Inelastic Scattering

$$\sigma_{\nu_\mu e^-} = \frac{2m_e G_F^2 E_\nu}{\pi}$$

CC only
Threshold 11 GeV

$$\sigma_{\nu_e e^-} \approx 0.6\frac{2m_e G_F^2 E_\nu}{\pi}$$

NC + CC

NC only

Detector technology depends on process and neutrino type/energy

Atmospheric/Beam Neutrinos $\nu_e, \nu_\mu, \overline{\nu}_e, \overline{\nu}_\mu : E_\nu > 1\,\mathrm{GeV}$

❶ **Water Čerenkov:** **e.g. Super Kamiokande**

❷ **Iron Calorimeters:** **e.g. MINOS, CDHS**

- **Produce high energy charged lepton – relatively easy to detect**

Solar Neutrinos $\nu_e \; : \; E_\nu < 20\,\mathrm{MeV}$

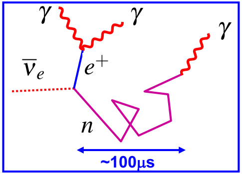
❶ **Water Čerenkov:** **e.g. Super Kamiokande**

- **Detect Čerenkov light from electron produced in** $\nu_e + e^- \to \nu_e + e^-$
- **Because of background from natural radioactivity limited to** $E_\nu > 5\,\mathrm{MeV}$
- **Because Oxygen is a doubly magic nucleus don't get** $\nu_e + n \to e^- + p$

❷ **Radio-Chemical:** **e.g. Homestake, SAGE, GALLEX**

- **Use inverse beta decay process, e.g.** $\nu_e + {}^{71}\mathrm{Ga} \to e^- + {}^{71}\mathrm{Ge}$
- **Chemically extract produced isotope and count decays (only gives a rate)**

Reactor Neutrinos $\overline{\nu}_e \; : \; E_{\overline{\nu}} < 5\,\mathrm{MeV}$



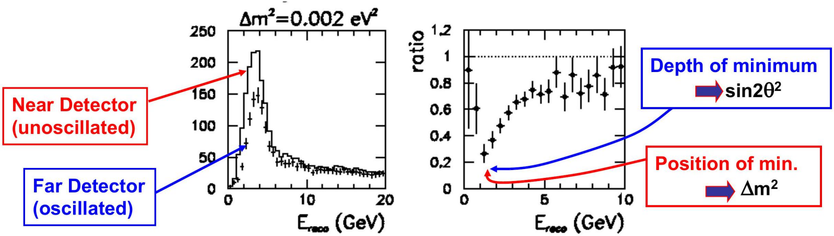
❶ **Liquid Scintillator:** **e.g. KamLAND**

- **Low energies ➡ large radioactive background**
- **Dominant interaction:** $\overline{\nu}_e + p \to e^+ + n$
- **Prompt positron annihilation signal + delayed signal from *n* (space/time correlation reduces background)**
- **electrons produced by photons excite scintillator which produces light**

1a: Long Baseline Neutrino Experiments

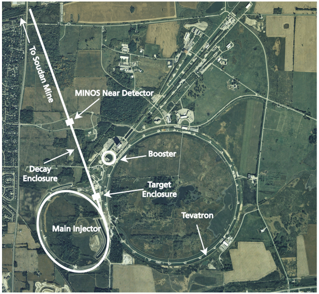
Basic Idea:

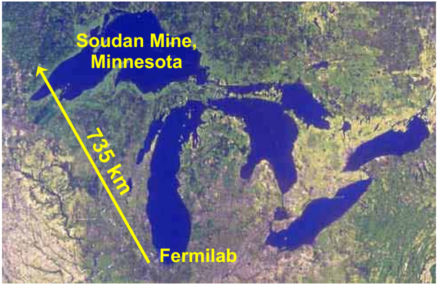
- Create intense neutrino beam.
- Build two "near" and "far" detectors: one close to beam, the other many km away.
- Measure ratio of the neutrino energy spectrum in far detector (oscillated) to that in the near detector (unoscillated)
- Duplication of detectors allows for partial cancellation of systematic biases.



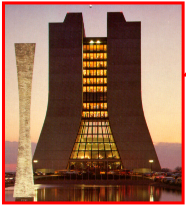
Long baseline example experiment: MINOS (2005-2016)

- 120 GeV proton beam extracted from Fermilab main injector.
- 2.5×10^{13} protons per pulse hit target. Proton beam is 0.3 MW on target!





Two detectors:

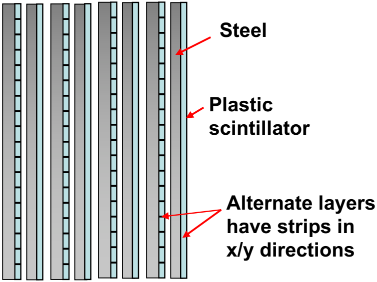


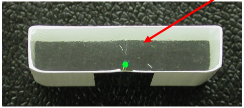
★ 1000 ton, NEAR Detector at Fermilab : 1 km from beam

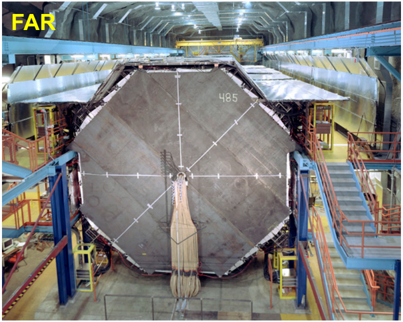
★ 5400 ton FAR Detector, 720m underground in Soudan mine, N. Minnesota: 735 km from beam

The MINOS Detectors:

- Dealing with high energy neutrinos $E_\nu > 1\,\mathrm{GeV}$
- The muons produced by ν_μ interactions travel several metres
- Steel-Scintillator sampling calorimeter
- Each plane: 2.54 cm steel +1 cm scintillator
- A charged particle crossing the scintillator produces light – detect with PMTs

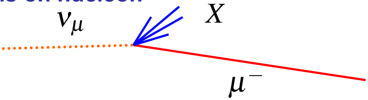




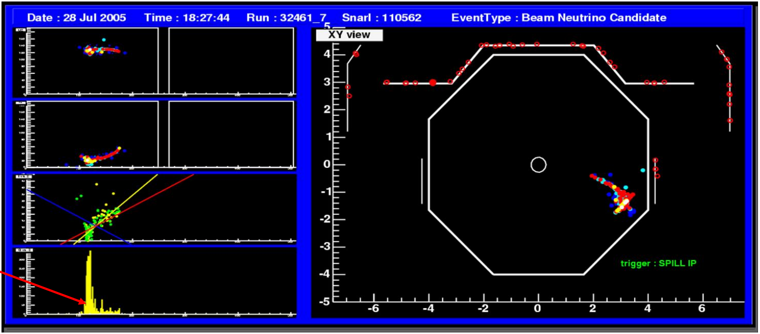


- **Neutrino detection via CC interactions on nucleon**

$$\nu_\mu + N \rightarrow \mu^- + X$$



Example event:



Signal from hadronic shower

- **The main feature of the MINOS detector is the very good neutrino energy resolution**

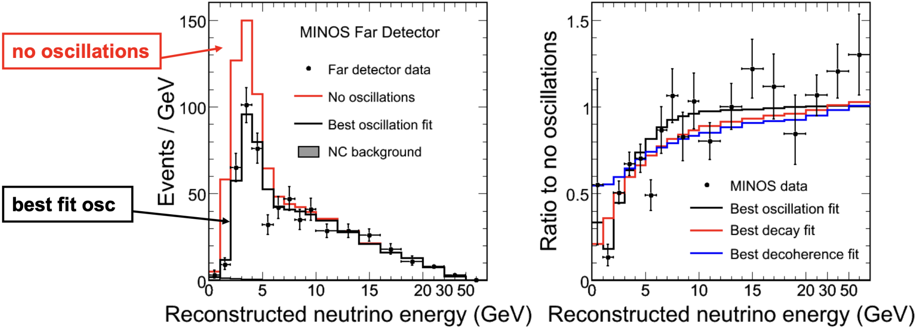
$$\boxed{E_\nu = E_\mu + E_X}$$

- **Muon energy from range/curvature in B-field**
- **Hadronic energy from amount of light observed**

MINOS Results

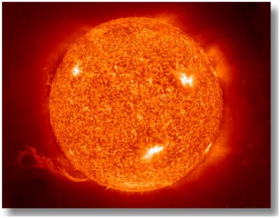
- For the MINOS experiment L is fixed and observe oscillations as function of E_ν.
- For $|\Delta m^2_{32}| \sim 2.5 \times 10^{-3}\ \mathrm{eV}^2$ first oscillation minimum at $E_\nu = 1.5\ \mathrm{GeV}$.
- To a very good approximation can use two flavour formula as oscillations corresponding to $|\Delta m^2_{21}| \sim 8 \times 10^{-5}\ \mathrm{eV}^2$ occur at $E_\nu = 50\ \mathrm{MeV}$, beam contains very few neutrinos at this energy + well below detection threshold.

MINOS Collaboration, Phys. Rev. Lett. 101, 131802, 2008

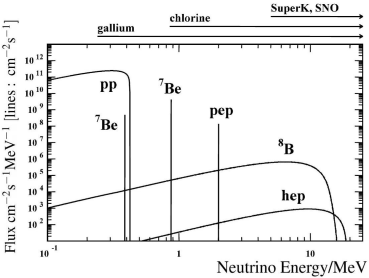


$$\boxed{|\Delta m^2_{32}| = (2.43 \pm 0.12) \times 10^{-3}\ \mathrm{eV}^2}$$

2: Solar Neutrinos



- The Sun is powered by the weak interaction, and produces a very large flux of electron neutrinos: $\sim 2\times10^{38}\ \nu_e\mathrm{s}^{-1}$.
- There are several different nuclear reactions in the sun so neutrino energy spectrum is complex:



$\boxed{p+p\to d+e^{+}+\nu_e}$ $E_\nu < 0.5\ \mathrm{MeV}$

$\boxed{^{8}\mathrm{B}\to {}^{8}\mathrm{Be}^{*}+e^{+}+\nu_e}$ $E_\nu \sim 5\ \mathrm{MeV}$

$\boxed{p+e^{-}+p\to d+\nu_e}$

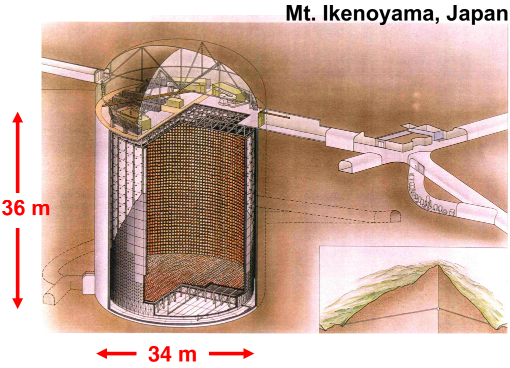
$\boxed{^{7}\mathrm{Be}+e^{-}\to {}^{7}\mathrm{Li}+\nu_e}$

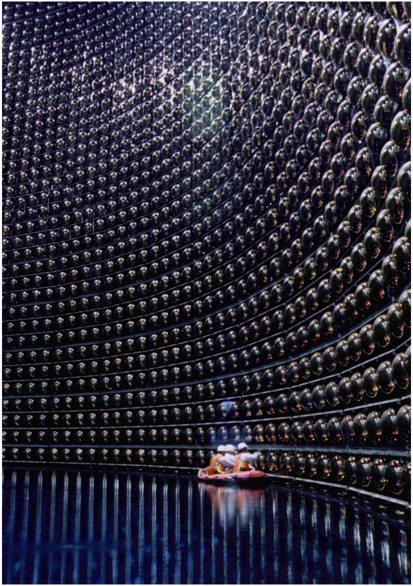
$\boxed{^{3}\mathrm{He}+p\to {}^{4}\mathrm{He}+e^{+}+\nu_e}$

A number of experiments (Homestake Mine, Super Kamiokande, *etc.*) saw a deficit of electron neutrinos compared to experimental prediction.
This was called the SOLAR NEUTRINO PROBLEM!

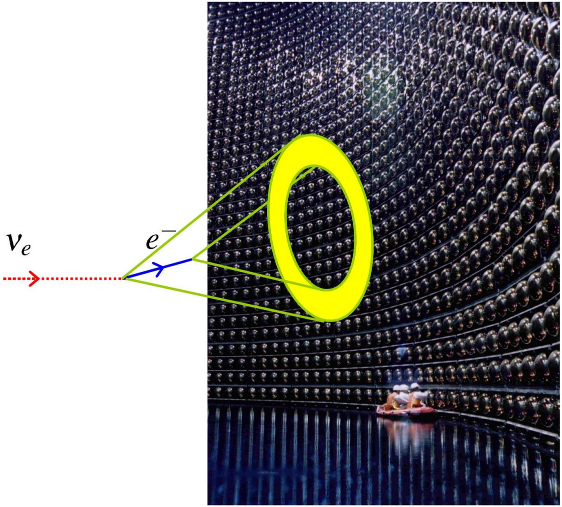
Solar Neutrinos I: Super Kamiokande

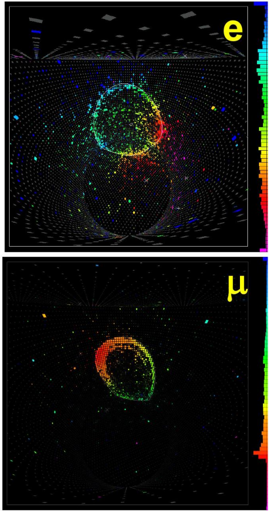
- 50000 ton water Čerenkov detector.
- Water viewed by 11146 photo-multiplier tubes.
- Deep underground to filter out cosmic rays otherwise difficult to detect rare neutrino interactions.





Detect neutrinos by observing Čerenkov radiation from charged particles which travel faster than speed of light in water (which is c/n for refractive index n).

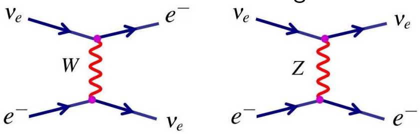




Can distinguish electrons from muons from pattern of light — muons produce clean rings whereas electrons produce more diffuse ‘fuzzy’ rings.

Super K Sensitivity

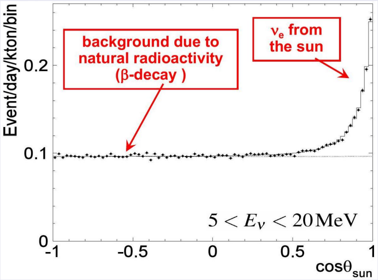
- Detect electron Čerenkov rings from:



- In LAB frame the electron is produced preferentially along the ν_e direction.

- Sensitive to solar neutrinos with $E_\nu > 5\ \mathrm{MeV}$.
- For lower energies too much background from natural radioactivity (β-decays) Hence detect mostly neutrinos from ${}^8B \to {}^8Be^* + e^+ + \nu_e$.

Super K Results



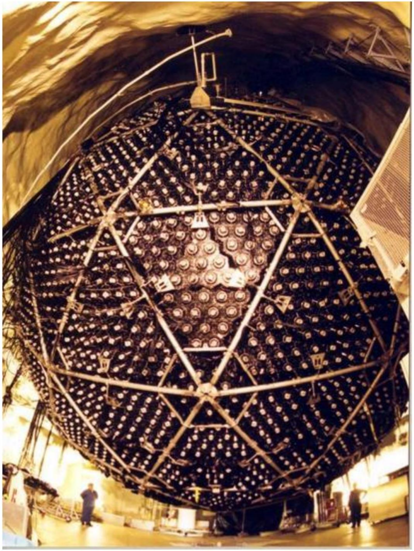
- Plot left from **S.Fukada et al., Phys. Rev. Lett. 86 5651-5655, 2001** [5].
- Clear signal of neutrinos from the sun
- However too few neutrinos:

$$\boxed{\mathrm{DATA/SSM} \;= 0.45 \pm 0.02}.$$

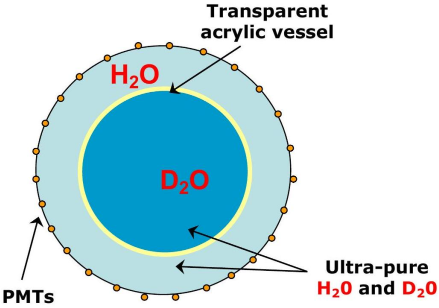
SSM = 'Standard Solar Model' Prediction

Solar Neutrinos II: Sudbury Neutrino Observatory (SNO)

A 1000 ton heavy water (D_2O) Čerenkov detector in a deep mine in Ontario, Canada.



- D_2O inside a 12 m diameter acrylic vessel.
- Surrounded by **3000** tons of normal water.
- Main experimental challenge is the need for very low background from radioactivity.
- Ultra-pure H_2O and D_2O.
- Surrounded by 9546 PMTs.

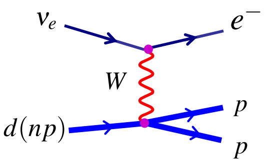


Detect Čerenkov light from three different reactions:

CHARGE CURRENT

- Detect Čerenkov light from electron
- Only sensitive to ν_e (thresholds)
- Gives a measure of ν_e flux

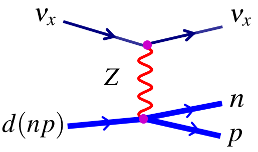
$$\boxed{\text{CC Rate} \propto \phi(\nu_e)}$$



NEUTRAL CURRENT

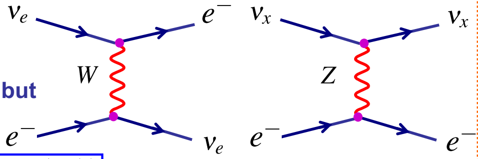
- Neutron capture on a deuteron gives **6.25 MeV**
- Detect Čerenkov light from electrons scattered by γ
- Measures total neutrino flux

$$\boxed{\text{NC Rate} \propto \phi(\nu_e) + \phi(\nu_\mu) + \phi(\nu_\tau)}$$



ELASTIC SCATTERING

- Detect Čerenkov light from electron
- Sensitive to all neutrinos (NC part) – but larger cross section for ν_e

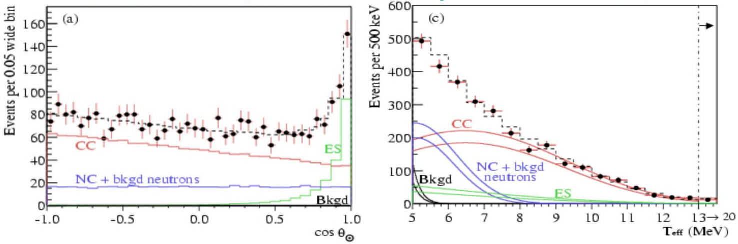


$$\boxed{\text{ES Rate} \propto \phi(\nu_e) + 0.154(\phi(\nu_\mu) + \phi(\nu_\tau))}$$

Experimentally can determine rates for different interactions from:

- Electrons from elastic scattering (ES) point back to sun.
- Energy: NC events have lower energy — 6.25 MeV photon from neutron capture.
- Radius from centre of detector gives a measure of background from neutrons.

SNO Collaboration, Q.R. Ahmad et al., Phys. Rev. Lett. 89:011301, 2002



Using different distributions obtain a measure of numbers of events of each type:

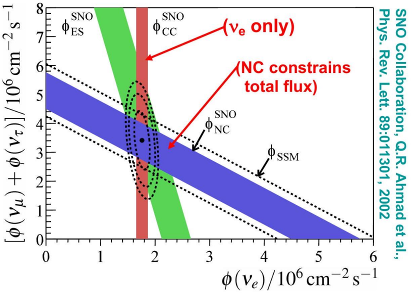
CC : 1968 ± 61	$\propto \phi(\nu_e)$
ES : 264 ± 26	$\propto \phi(\nu_e) + 0.154\,[\phi(\nu_\mu) + \phi(\nu_\tau)]$
NC: 576 ± 50	$\propto \phi(\nu_e) + \phi(\nu_\mu) + \phi(\nu_\tau)$

Uniquely:

SNO is able to masure of electron neutrino flux **and total flux**!

SNO Results

- Using known cross sections can convert observed numbers of events into fluxes
- The different processes impose different constraints
- Where constraints meet gives separate measurements of ν_e and $\nu_\mu + \nu_\tau$ fluxes



SNO Results:
$$\left\{ \begin{array}{l} \phi(\nu_e) = (1.8 \pm 0.1) \times 10^{-6}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1} \\ \phi(\nu_\mu) + \phi(\nu_\tau) = (3.4 \pm 0.6) \times 10^{-6}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1} \end{array} \right\}.$$

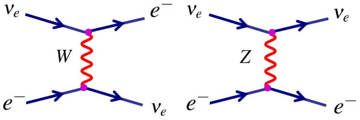
SSM Prediction: $\phi(\nu_e) = 5.1 \times 10^{-6}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}$.

- Clear evidence for a flux of ν_μ and/or ν_τ from the sun.
- Total neutrino flux is consistent with expectation from SSM.
- Clear evidence of $\nu_e \to \nu_\mu$ and/or $\nu_e \to \nu_\tau$ neutrino transitions.

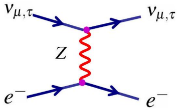
Complication for interpretation of Solar Neutrino Data

The interpretation of the solar neutrino data is complicated by **matter effects**:

- A quantitative treatment is non-trivial and is not given here.
- Basic idea is that as a neutrino leaves the sun it crosses a region of high electron density.
- The coherent forward scattering process ($\nu_e \to \nu_e$) for an electron neutrino



  is different to that for a muon or tau neutrino:



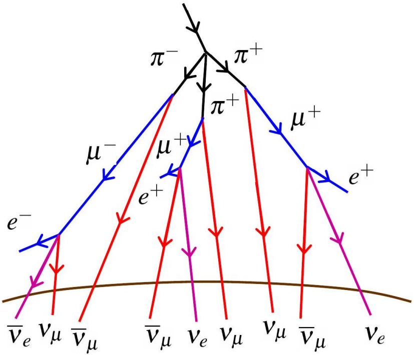
- Effect can enhance oscillations — is called "MSW effect".

A combined analysis of all solar neutrino data gives:

$$\Delta m^2_{\text{solar}} \approx 8 \times 10^{-5}\ \mathrm{eV}^2, \qquad \sin^2 2\theta_{\text{solar}} \approx 0.85.$$

1b : Atmospheric Neutrinos

- High energy cosmic rays (up to 10^{20} eV) interact in the upper part of the Earth's atmosphere.
- The cosmic rays (26% protons, 11% He Nuclei, $\sim$ 1% heavier nuclei, 2% electrons) mostly interact hadronically giving showers of mesons (mainly pions)



- Neutrinos produced by:

$$\pi^+ \to \mu^+ \nu_\mu \to e^+ \nu_e \bar{\nu}_\mu \nu_\mu$$

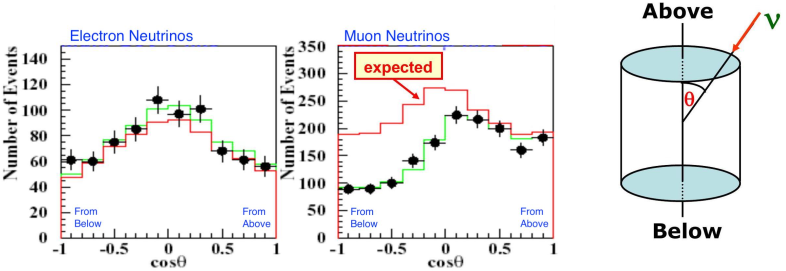
$$\pi^- \to \mu^- \bar{\nu}_\mu \to e^- \bar{\nu}_e \nu_\mu \bar{\nu}_\mu$$

therefore expect $\dfrac{N(\nu_\mu + \bar{\nu}_\mu)}{N(\nu_e + \bar{\nu}_e)} \approx 2.$

- Neutrino flux $\sim 1\ \mathrm{cm}^{-2}\mathrm{Sr}^{-1}\ \mathrm{s}^{-1}$.
- Typical energy : $\mathrm{E}_\nu \sim 1\ \mathrm{GeV}$.
- Observe a lower ratio with deficit of $\nu_\mu/\bar{\nu}_\mu$ coming from below the horizon, i.e. large distance from production point on other side of the Earth. (See evidence on next slide.)

Super Kamiokande Atmospheric Results

- Typical energy: $\mathrm{E}_\nu \sim 1\ \mathrm{GeV}$ (much greater than solar neutrinos — no confusion).
- Identify ν_e and ν_μ interactions from nature of Čerenkov rings.
- Measure rate as a function of angle with respect to local vertical.
- Neutrinos coming from above travel 20 km.
- Neutrinos coming from below (i.e. other side of the Earth) travel $\sim$ 12800 km.



⋆ Prediction for ν_e rate agrees with data.

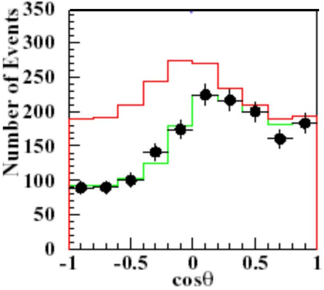
⋆ Strong evidence for disappearance of ν_μ for large distances.

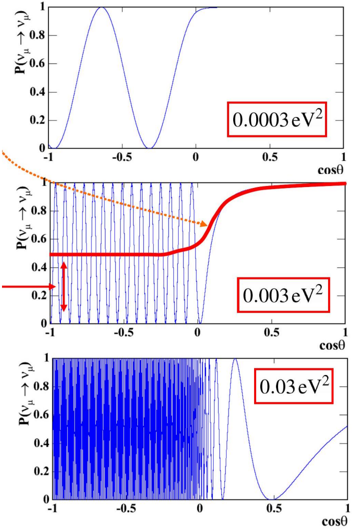
⋆ Consistent with $\nu_\mu \to \nu_\tau$ oscillations.

⋆ Don't detect the oscillated ν_τ, as typically below interaction threshold of 3.5 GeV.

Interpretation of Atmospheric Neutrino Data

- Measure muon direction and energy not neutrino direction/energy.
- Don't have E or θ resolution to see oscillations.
- Oscillations 'smeared' out in data.
- Compare data to predictions for $\left|\Delta m^2\right|$.





- Data consistent with: $\left|\Delta m^2_{\text{atmos}}\right| \approx 0.0025\ \mathrm{eV}^2$ and $\sin^2 2\theta_{\text{atmos}} \approx 1$.

3: Reactor Experiments

- To explain reactor neutrino experiments we need the full three neutrino expression for the electron neutrino survival probability (157) which depends on U_{e1}, U_{e2}, U_{e3}.
- Substituting these PMNS matrix elements in (157):

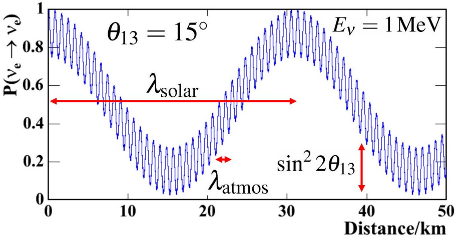
$$
\begin{aligned}
P(\nu_e \to \nu_e) &\approx 1 - 4U_{e1}^2 U_{e2}^2 \sin^2 \Delta_{21} - 4\left(1 - U_{e3}^2\right) U_{e3}^2 \sin^2 \Delta_{32} \\
&= 1 - 4(c_{12}c_{13})^2 (s_{12}c_{13})^2 \sin^2 \Delta_{21} - 4\left(1 - s_{13}^2\right) s_{13}^2 \sin^2 \Delta_{32} \\
&= 1 - c_{13}^4 (2s_{12}c_{12})^2 \sin^2 \Delta_{21} - (2c_{13}s_{13})^2 \sin^2 \Delta_{32} \\
&= 1 - \cos^4 \theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21} - \sin^2 2\theta_{13} \sin^2 \Delta_{32}
\end{aligned}
$$

- Contributions with short wavelength (atmospheric) and long wavelength (solar).

- For a 1 MeV neutrino:

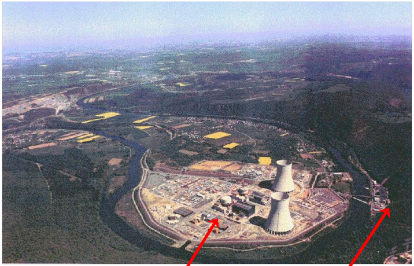
$$
\begin{aligned}
\lambda_{\mathrm{osc}}(\mathrm{km}) &= 2.47 \frac{E(\mathrm{GeV})}{\Delta m^2\,(\mathrm{eV}^2)} \\
\Rightarrow \lambda_{21} &= 30.0\ \mathrm{km} \\
\lambda_{32} &= 0.8\ \mathrm{km}.
\end{aligned}
$$

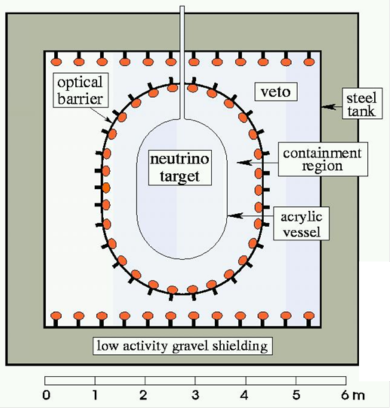
- Amplitude of short wavelength oscillations given by: $\sin^2 2\theta_{13}$.



Reactor Experiments I : CHOOZ (France) near two 4.2 GW nuclear reactors

- Place detector 1 km from reactor cores.
- Reactors produce intense flux of $\bar{\nu}_e$.



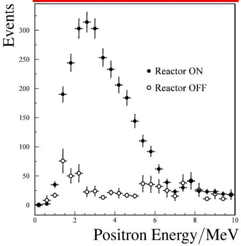


- Anti-neutrinos interact via inverse beta decay $\bar{\nu}_e + p \to e^+ + n$.
- Detector is liquid scintillator loaded with Gadolinium (large n capture cross section).
- Detect photons from positron annihilation and a delayed signal from photons from neutron capture on Gadolinium:

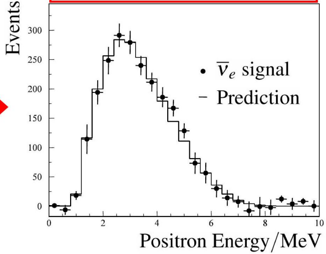
$$e^+ + e^- \to \gamma + \gamma$$
$$n + \mathrm{Gd} \to \mathrm{Gd}^* \to \mathrm{Gd} + \gamma + \gamma + \ldots$$

At $1\ \mathrm{km}$ and energies $> 1\ \mathrm{MeV}$, only the short wavelength component matters
$P(\nu_e \to \nu_e) = 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \sin^2 2\theta_{13}\sin^2\Delta_{32} \approx 1 - \sin^2 2\theta_{13}\sin^2\Delta_{32}$.

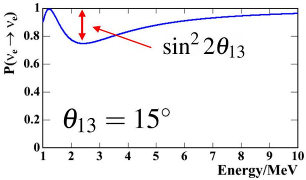
CHOOZ Raw Data



Background subtracted



Compare to effect of oscillations



- Data agree with unoscillated prediction both in terms of rate and energy spectrum

$$\text{Data in [6]} \implies N_{\text{data}}/N_{\text{expect}} = 1.01 \pm 0.04.$$

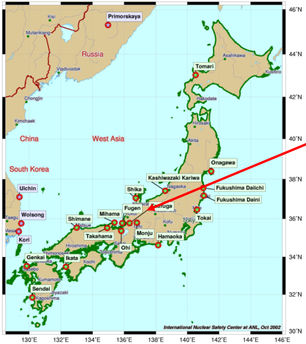
- Hence $\sin^2 2\theta_{13}$ must be small !

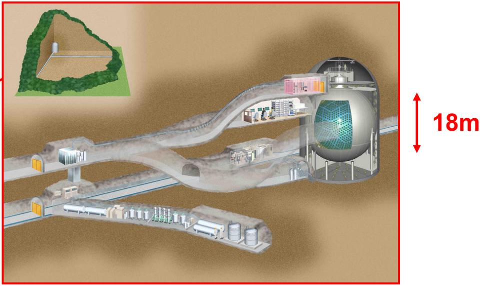
$$\Rightarrow \sin^2 2\theta_{13} < 0.12 - 0.2.$$

- Atmospheric neutrino results (elsewhere) can exclude $\theta_{13} \sim \frac{\pi}{2}$.
- Hence the CHOOZ limit: $\sin^2 2\theta_{13} < 0.2$ can be interpreted as $\sin^2\theta_{13} < 0.05$.

Reactor Experiments II : KamLAND

- Detector located in same mine as Super Kamiokande





- 70 GW from nuclear power (7% of World total) from reactors within $130 - 240\ \mathrm{km}$
- Liquid scintillator detector, 1789 PMTs
- Detection via inverse beta decay: $\bar{\nu}_e + p \to e^+ + n$ followed promptly by

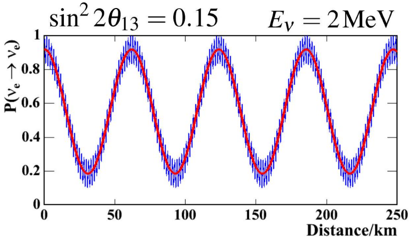
$$e^+ + e^- \to \gamma + \gamma$$

and then, after a (very helpful) characteristic delay by

$$n + p \to d + \gamma_{(2.2\ \mathrm{MeV})}.$$

- For MeV neutrinos at a distance of 130-240 km oscillations due to Δm^2_{32} are very rapid.
- Experimentally, only see average effect

$$\left\langle \sin^2 \Delta_{32} \right\rangle = 0.5.$$



- Here:

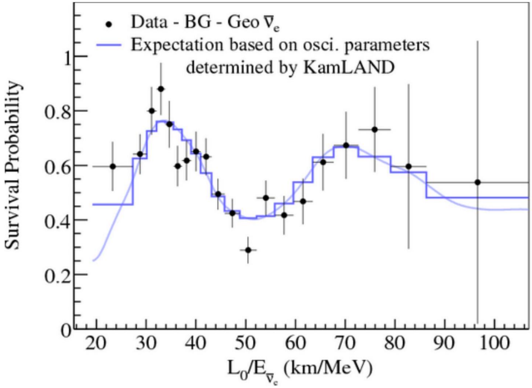
$$\begin{aligned}
P(\nu_e \to \nu_e) &= 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \sin^2 2\theta_{13}\sin^2\Delta_{32} \\
&\approx 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \frac{1}{2}\sin^2 2\theta_{13} \qquad \text{(averaging over rapid oscillations)} \\
&= \cos^4\theta_{13} + \sin^4\theta_{13} - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} \\
&\approx \cos^4\theta_{13}\left(1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}\right) \qquad (\text{neglect } \sin^4\theta_{13})
\end{aligned}$$

- Obtain two-flavour oscillation formula multiplied by $\cos^4\theta_{13}$.
- From CHOOZ $\quad \cos^4\theta_{13} > 0.9$.

KamLAND Results

Observe: 1609 events.
Expect: 2179 ± 89 events (if no oscillations).

KamLAND Collaboration, Phys. Rev. Lett., 221803, 2008

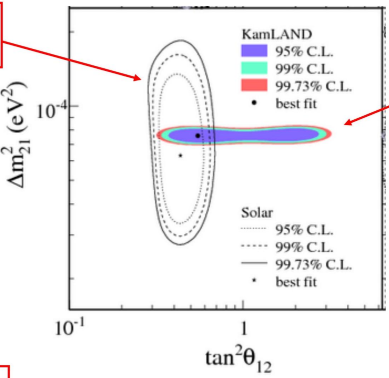


- Clear evidence of electron anti-neutrino oscillations consistent with the results from solar neutrinos.
- Oscillatory structure clearly visible.
- Compare data with expectations for different osc. parameters and perform χ^2 fit to extract measurment.

Combined Solar Neutrino and KamLAND Results

- KamLAND data provides strong constraints on $|\Delta m^2_{21}|$.
- Solar neutrino data (especially SNO) provides a strong constraint on θ_{12}.

Solar Neutrinos



KamLAND

Combined

$$|\Delta m^2_{21}| = (7.59 \pm 0.21) \times 10^{-5}\,\mathrm{eV}^2$$

$$\tan^2\theta_{12} = 0.47^{+0.06}_{-0.05}$$

Other more recent work ...

$\nu_\mu \to \nu_e$ appearance:

- T2K: $\nu_\mu \to \nu_e$ appearance (3.1σ) [arXiv:1304.0841].
- MINOS: $\nu_\mu \to v_e$ appearance (2σ).

$\bar{\nu}_e$ disappearance:

- Double-CHOOZ (2012): $\bar{\nu}_e$ disappearance (2σ) (doi = 10.1103/PhysRevLett.108.131801)

$$\sin^2 2\theta_{13} \approx 0.086 \pm 0.06.$$

- RENO (2012): 4.9σ evidence for $\bar{\nu}_e$ disappearance [arXiv:1204.0626], and

$$\sin^2 2\theta_{13} \approx 0.113 \pm 0.003$$

Throughout lifetime of course, evidence for non-zero value of θ_{13} has grown from 'none' to 'very clear':

- In Daya Bay experiment (see Question 21) measured:

$$\sin^2 2\theta_{13} = 0.090 \pm 0.009 \quad \text{[arXiv:1310.6732] (2013), and then}$$

$$\sin^2 2\theta_{13} = 0.0851 \pm 0.0024 \quad \text{[arXiv:2211.14988] (2022).}$$

Summary of Current Knowledge (from Review of Particle Physics 2020)

Param	bfp $\pm 1\sigma$	3σ range
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.10^{+0.13}_{-0.12}$	$2.75 \to 3.50$
$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.58^{+0.20}_{-0.33}$	$4.27 \to 6.09$
$\theta_{23}/^\circ$	$48.3^{+1.2}_{-1.9}$	$40.8 \to 51.3$
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.241^{+0.066}_{-0.065}$	$2.046 \to 2.440$
$\theta_{13}/^\circ$	$8.61^{+0.13}_{-0.13}$	$8.22 \to 8.99$
$\delta_{CP}/^\circ$	222^{+38}_{-28}	$141 \to 370$
$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$
$\frac{\Delta m^2_{32}}{10^{-3}\ \mathrm{eV}^2}$	$2.449^{+0.032}_{-0.030}$	$2.358 \to 2.544$

Experiment	Dominant	Important
Solar Experiments	θ_{12}	Δm^2_{21} , θ_{13}
Reactor LBL (KamLAND)	Δm^2_{21}	θ_{12} , θ_{13}
Reactor MBL (Daya-Bay, Reno, D-Chooz)	θ_{13}, $\lvert\Delta m^2_{31,32}\rvert$	
Atmospheric Experiments (SK, IC-DC)		θ_{23},$\lvert\Delta m^2_{31,32}\rvert$, θ_{13},δ_{CP}
Accel LBL ν_μ,$\bar{\nu}_\mu$, Disapp (K2K, MINOS, T2K, NOνA)	$\lvert\Delta m^2_{31,32}\rvert$, θ_{23}	
Accel LBL ν_e,$\bar{\nu}_e$ App (MINOS, T2K, NOνA)	δ_{CP}	θ_{13} , θ_{23}

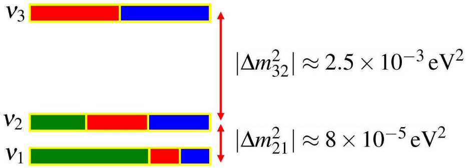
Approximate interpretation of neutrino mixing parameters:

- For the central values of the mixing angles on the previous page obtain:

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \approx \begin{pmatrix} 0.82 & 0.55 & 0.15e^{-i\delta} \\ -0.38 & 0.55 & 0.74 \\ 0.42 & -0.62 & 0.66 \end{pmatrix}.$$

- i.e. approximate expressions for mass eigenstates in terms of weak eigenstates are:

$$|\nu_3\rangle \approx \frac{1}{\sqrt{2}}\left(|\nu_\mu\rangle + |\nu_\tau\rangle\right),$$
$$|\nu_2\rangle \approx 0.56\left(|\nu_e\rangle + |\nu_\mu\rangle - |\nu_\tau\rangle\right),$$
$$|\nu_1\rangle \approx 0.82\,|\nu_e\rangle - 0.4\left(|\nu_\mu\rangle - |\nu_\tau\rangle\right).$$



Final words on Neutrino Masses

- Neutrino oscillations require non-zero neutrino masses.
- Oscillation measuremments only determine mass-squared differences — not the masses themselves.
- We have no direct measure of neutrino mass — only mass limits:

$$m_\nu(e) < 2 \text{ eV}; \quad m_\nu(\mu) < 0.17 \text{ MeV}; \quad m_\nu(\tau) < 18.2 \text{ MeV}.$$

  Note the e, μ, τ refer to charged lepton flavour in the experiment, e.g. $m_\nu(e) < 2$ eV refers to the limit from tritium beta-decay.

- Also from cosmological evolution infer that the sum

$$\sum_i m_{\nu_i} < \text{ few eV } .$$

- Before $\sim$2008 neutrinos were assumed to be massless, but there were hints that neutrinos might oscillate.
- Now, know a great deal about massive neutrinos, but there are still many unknowns. including: δ, the mass hierarchy, and the absolute values of neutrino masses.
- Measurements of these SM parameters is the focus of the next generation of neutrino expts (e.g. DUNE).

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