Part III Physics

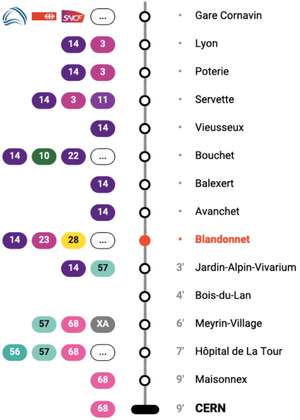
Particle Physics

Dr C.G.Lester

“Les Horribles Cernettes”, CERN’s most famous pop group (and the subjects of the first image to be uploaded to the world wide web) singing “Collider”.

https://www.youtube.com/watch?v=1e1eLe1ihT0

CERN Pizza Recipe:
https://www.hep.phy.cam.ac.uk/~lester/HiggsPizza.pdf

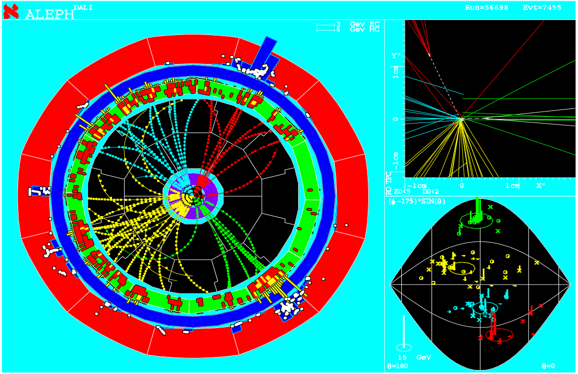


The “Higgs Boson Pizza Day” was held on Monday, 4 July 2016, on the fourth anniversary of the announcement of the discovery of the Higgs boson at CERN. On this occasion, more than 400 pizzas were prepared and served at lunchtime in Restau

Sub-divisions (Handouts)

Particle Physics

Dr Lester



Handout 1 : Introduction

Preliminaries

Web-page

`https://www.hep.phy.cam.ac.uk/~lester/teaching/partIIparticles`

- All course material, old exam questions, corrections, interesting links *etc*.
- Detailed answers will posted after the supervisions.

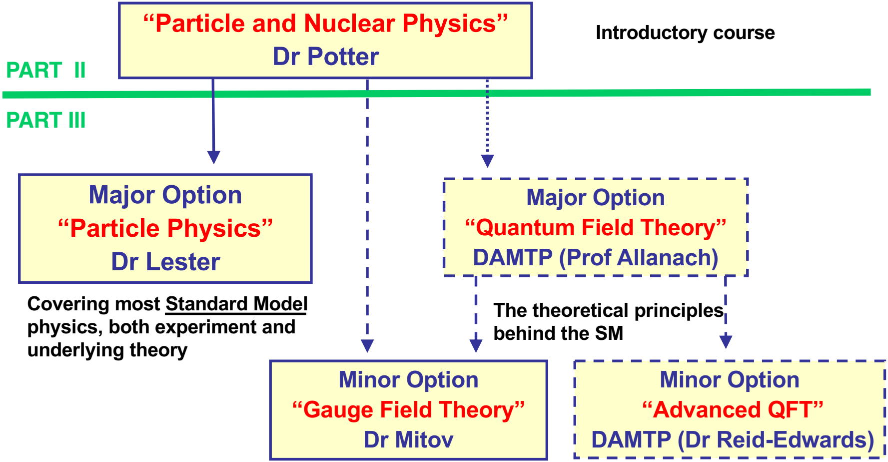
Format

- For historical reasons, the fourteen sections of the course are called 'handouts'.
- Some handouts contain additional theoretical background in non-examinable appendices at their ends.
- Please let me know of any mistakes/corrections: `Lester@hep.phy.cam.ac.uk`

Books

- **"Modern Particle Physics"**, Mark Thomson (Cambridge) BASED ON THIS COURSE!
- **"Particle Physics"**, Martin and Shaw (Wiley): fairly basic but good.
- **"Introductory High Energy Physics"**, Perkins (Cambridge): slightly below level of the course but well written.
- **"Introduction to Elementary Physics"**, Griffiths (Wiley): about right level but doesn't cover the more recent material.
- **"Quarks and Leptons"**, Halzen & Martin (Wiley): good graduate level textbook (slightly above level of this course).

Cambridge Particle Physics Courses



PART II

PART III

"Particle and Nuclear Physics"
Dr Potter

Introductory course

Major Option
"Particle Physics"
Dr Lester

Covering most <u>Standard Model</u> physics, both experiment and underlying theory

Major Option
"Quantum Field Theory"
DAMTP (Prof Allanach)

The theoretical principles behind the SM

Minor Option
"Gauge Field Theory"
Dr Mitov

Minor Option
"Advanced QFT"
DAMTP (Dr Reid-Edwards)

Aims of this course

- The course is intended as an overview-style course. It aims to provide:
  - a context for the other more rigourous courses (QFT, AQFT, Gauge Field Theory),
  - examples of the experiments and types of experimental evidence which have lead to our current underfstanding of The Standard Model, and
  - 'just enough' of the theory to understand how/why the experiments constrain theory.
- Since the QFT, AQFT and Gauge Field Theory courses are either not yet lectured or are lectured in parallel, it is necessary for many results in this course to be presented without proof, or with only plausibility arguments, or with outline theoretical motivations. That will be dissatisfying for some taking the course – but are a necessary evil if this course is to complement those other courses.

Past student advice:

This mini-review was taken from `https://www.reddit.com/r/Physics/comments/iatn6o/an_interesting_question_from_my_2020_particle/`

> *"Technically, this is Part III Physics from the Natural Sciences Tripos. You do get to borrow a QFT course from the Part III Mathematical Tripos though.*
>
> **[redacted]** *the lecturer* **[redacted]** *[likes] to point with a great big stick.*
>
> *This book [Thomson] is based on the course; author is a previous lecturer. Perhaps flicking through the preview might help? It's not a formal QFT course, so there's less maths. It tries to explain both theory and experiment. If you want more theory, I'd recommend the Gauge Field Theory courses or the QFT and AQFT courses from Part III Maths.*
>
> *Pre-req: "Students who are not familiar with the overall structure of The Standard Model, the quark model of the hadrons, scattering processes, and wave equations at some level, have found the course hard in the past." You use quite a lot of Einstein notation / tensors like 4-vectors, Bra-Kets and matrices, so perhaps be comfortable with that (if you aren't already).*
>
> *Have fun in Part III!"*

Non-examinable material

Some parts of the course are marked '**not examinable**' or '**non-examinable**'. What these terms mean is that no student taking the course is expected to revise or learn any material so-labelled for Tripos. In other words: the exam questions should not *require* knowledge of material presented therein.
This does not mean that a Tripos question could never have a domain overlapping with 'non-examinable' material, though. In the rare cases that happens, it simply means that the examiner has judged that material in the overlap can be reasonably deduced from material which was deemed fair game (i.e. which was not labelled 'non-examinable'). Therefore, a more specific (though wordier) name for the material could be 'material-which-does-not-need-to-be-learned-or-revised'.

- Material in these sections is presented purely to provide extra support to other things in the course. Sometimes material from non-examinable sections is discussed in lectures, but most is not. The discussion of such material in lectures does not change its status unless an official announcement to that effect is given.
- Some of the sub-sections of the course ('handouts') are followed by Appendices. All material in appendices is automatically non-examinable, even if not so-labelled.
- In the event that material has been mis-labelled, a correction would be issued to the class by email before the end of Michaelmas Term.).
- If in doubt about the status of any material, ask the lecturer for clarifications before the end of Michaelmas Term.

Lecture Zero

The course proper begins in Monday!
Before then, here are a few things which fit nowhere else:

- Units.
- Assumed knowledge about Dirac δ-Functions.
- Standard Model - review.
- Special Relativity - things you should be familiar with.
- Why Mandelstam variables matter.

Units in Particle Physics

S.I. Units measure:

mass in kg, length in m,
time in s, charge in C.

In principle particle physics 'natural' units measure:[1]

mass in GeV/c^2, length in $\hbar c/GeV$,
time in $\hbar/GeV$, charge in $(\varepsilon_0 \hbar c)^{\frac{1}{2}}$

Heaviside-Lorentz convention:

$c = \hbar = \varepsilon_0 = 1$ (and $\mu_0 = 1$ too since $c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}$)

In practice particle physics units measure:

mass in GeV, length in $1/GeV$,
time in $1/GeV$, and charge is *dimensionless*
on account of using that Heaviside-Lorentz convention!

---

[1] NB: You could change GeV to MeV, TeV or any other eV-based energy unit without upsetting anyone at CERN.

How particle physicists cope with their units:

- Most of the time, they ignore all c and $\hbar$ symbols everywhere.
- They put the back c and $\hbar$ symbols only when they need to talk to 'ordinary' physicists or publish a paper in a journal.
- They remember whether the "GeV"s are on the bottom/top by remembering that they are mostly interested in:
    - **large** energies and **large** momenta: GeV
    - *small* lengthscales and *small* timescales: GeV^{-1}
- To help them rebuild proper units from energies they (mostly) use the following *aides-mémoire*:
    - (to get a mass): $E \sim mc^2$,
    - (to get a momentum): $E \sim (mc)(c) \sim pc$,
    - (to get a time): $\Delta E \Delta t \sim \hbar$
    - (to get a length): $1 = \hbar c \sim 197\ MeV \cdot fm$.
- To get specific S.I. units they may also use:
    - (to get an energy in Joules): $eV \approx 1.60 \times 10^{-19} J$,
    - (to get a length in metres): $1 = \hbar c \approx 197\ MeV \cdot fm$.

Standard results for Dirac δ-Functions:

One variable:

$$\int_X g(x)\delta(u(x))dx = \int g(x(u))\delta(u)\left|\frac{dx}{du}\right| du = \sum_{x\in X \text{ s.t. } u(x)=0} \frac{g(x)}{\left|\frac{du}{dx}\right|}, \tag{1}$$

e.g. $\int_{-\infty}^{\infty} g(x)\delta(x-a)dx = g(a)$ or
$\int_{-\infty}^{\infty} g(x)\delta(x^2-a^2)dx = \sum_{x=\pm a} \frac{g(x)}{|2x|} = \frac{g(a)}{|2(a)|} + \frac{g(-a)}{|2(-a)|} = \frac{1}{2|a|}(g(a)+g(-a)).$

Two variables:

$$\int_X g(x,y)\delta(u(x,y))\delta(v(x,y))dxdy = \int g(x(u,v),y(u,v))\delta(u)\delta(v)\left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| dudv$$
$$= \sum_{(x,y)\in X \text{ s.t. } u(x,y)=v(x,y)=0} \frac{g(x,y)}{\left|\left|\frac{\partial(u,v)}{\partial(x,y)}\right|\right|}.$$

In general:

$$\int_X g(\vec{x})\delta^n(\vec{u}(\vec{x}))d^n x = \int g(\vec{x}(\vec{u}))\delta^n(\vec{u})\left|\left|\frac{\partial(x_1,\ldots,x_n)}{\partial(u_1,\ldots,u_n)}\right|\right| d^n u = \sum_{\vec{x}\in X \text{ s.t. } \vec{u}(\vec{x})=0} \frac{g(\vec{x})}{\left|\left|\frac{\partial(u_1,\ldots,u_n)}{\partial(x_1,\ldots,x_n)}\right|\right|}.$$

Review of The Standard Model

Particle Physics is the study of:

- ★ MATTER: the fundamental constituents of the universe - the elementary particles
- ★ FORCE: the fundamental forces of nature, i.e. the interactions between the elementary particles

Try to categorise the **PARTICLES** and **FORCES** in as simple and fundamental manner possible

★ Current understanding embodied in the STANDARD MODEL:

- **Forces** between particles due to exchange of **particles**
- Consistent with most experimental data !
- Does not account for Dark Matter
- But it is just a "**model**" with many unpredicted parameters, e.g. particle masses.
- As such it is not the ultimate theory (if such a thing exists), there are many mysteries.

Matter in the Standard Model

★ In the Standard Model the fundamental "matter" is described by point-like spin-1/2 fermions

	LEPTONS			QUARKS		
		q	m/GeV		q	m/GeV
First Generation	e^-	–1	0.0005	d	–1/3	0.3
	ν_1	0	≈0	u	+2/3	0.3
Second Generation	μ^-	–1	0.106	s	–1/3	0.5
	ν_2	0	≈0	c	+2/3	1.5
Third Generation	τ^-	–1	1.77	b	–1/3	4.5
	ν_3	0	≈0	t	+2/3	175

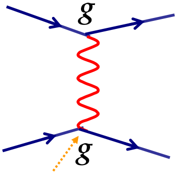
The masses quoted for the quarks are the "constituent masses", i.e. the effective masses for quarks confined in a bound state

- In the SM there are three generations – the particles in each generation are copies of each other differing only in mass. (not understood why three).
- The neutrinos are much lighter than all other particles (e.g. ν_1 has m<3 eV) – we now know that neutrinos have non-zero mass (don't understand why so small)

Forces in the Standard Model

★ Forces mediated by the exchange of spin-1 Gauge Bosons

Force	Boson(s)	J^P	m/GeV
EM (QED)	Photon γ	1^-	0
Weak	$W^\pm$ / Z	1^-	80 / 91
Strong (QCD)	8 Gluons g	1^-	0
Gravity (?)	Graviton?	2^+	0



- Fundamental interaction strength is given by charge g.
- Related to the **dimensionless** coupling "constant" α

e.g. QED $$g_{em} = e = \sqrt{4\pi\alpha\varepsilon_0\hbar c}$$

★ In Natural Units $$g = \sqrt{4\pi\alpha}$$

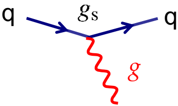
(both g and α are dimensionless, but g contains a "hidden" $\hbar c$)

★ Convenient to express couplings in terms of α which, being genuinely dimensionless does not depend on the system of units (this is not true for the numerical value for e)

Standard Model Vertices

- ★ Interaction of **gauge bosons** with **fermions** described by SM vertices
- ★ Properties of the **gauge bosons** and **nature of the interaction** between the bosons and fermions determine the properties of the interaction

STRONG

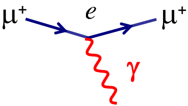


Only quarks

Never changes flavour

$$\alpha_S \sim 1$$

EM



All charged fermions

Never changes flavour

$$\alpha \simeq 1/137$$

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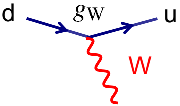

All fermions

Always changes flavour

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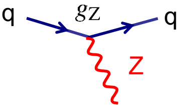


All fermions

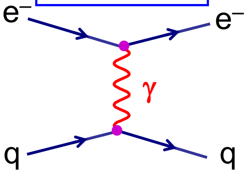
Never changes flavour

$$\alpha_{W/Z} \sim 1/40$$

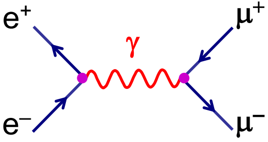
Feynman Diagrams

★ Particle interactions described in terms of Feynman diagrams

e.g. scattering

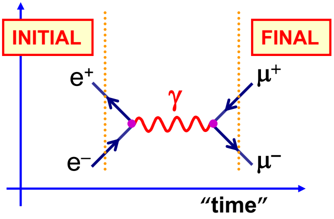


e.g. annihilation



★ **IMPORTANT POINTS TO REMEMBER:**

- "time" runs from left – right, only in sense that:
  - LHS of diagram is initial state
  - RHS of diagram is final state
  - Middle is "how it might have happened"
- anti-particle arrows in –ve "time" direction
- Energy, momentum, angular momentum, etc. conserved at all interaction vertices
- All intermediate particles are "virtual" i.e. $E^2 \neq |\vec{p}|^2 + m^2$ (handout 3)



Special Relativity and 4-Vector Notation

- Will use 4-vector notation with p^0 as the time-like component, e.g.

$$p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\}$$ (contravariant)

$$p_\mu = g_{\mu\nu} p^\nu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\}$$ (covariant)

with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In particle physics, usually deal with relativistic particles. **Require** all calculations to be Lorentz Invariant. L.I. quantities formed from 4-vector scalar products, e.g.

$$p^\mu p_\mu = E^2 - p^2 = m^2$$ **Invariant mass**

$$x^\mu p_\mu = Et - \vec{p}.\vec{r}$$ **Phase**

- A few words on **NOTATION**

  Four vectors written as either: p^μ or p

  Four vector scalar product: $p^\mu q_\mu$ or $p.q$

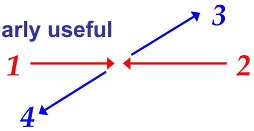
  Three vectors written as: $\vec{p}$

  Quantities evaluated in the **centre of mass frame:** $\vec{p}^*, p^*$ etc.

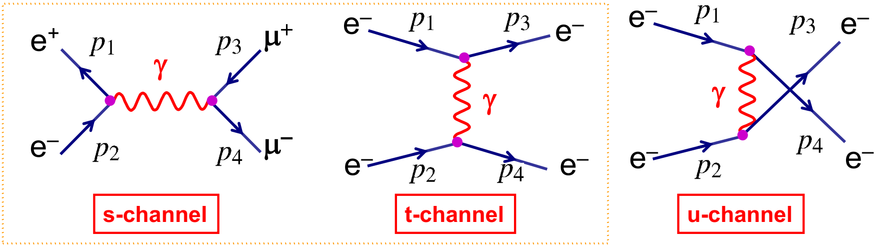
Mandelstam s, t and u

★ In particle scattering/annihilation there are three particularly useful **Lorentz Invariant** quantities: **s**, **t** and **u**

★ Consider the scattering process $1 + 2 \rightarrow 3 + 4$



★ (Simple) Feynman diagrams can be categorised according to the four-momentum of the exchanged particle



•Can define **three** kinematic variables: **s**, **t** and **u** from the following four vector scalar products (squared four-momentum of exchanged particle)

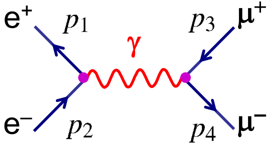
$$\boxed{s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2}$$

Example: Mandelstam s, t and u

$$\boxed{s=(p_1+p_2)^2, \quad t=(p_1-p_3)^2, \quad u=(p_1-p_4)^2}$$

Note: $s+t+u=m_1^2+m_2^2+m_3^2+m_4^2$ **(Question 1)**

★ **e.g. Centre-of-mass energy, *s*:**



$$s=(p_1+p_2)^2=(E_1+E_2)^2-(\vec{p}_1+\vec{p}_2)^2$$

- **This is a scalar product of two four-vectors ➡ Lorentz Invariant**
- **Since this is a L.I. quantity, can evaluate in any frame. Choose the most convenient, i.e. the centre-of-mass frame:**

$$p_1^*=(E_1^*,\vec{p}^*) \quad p_2^*=(E_2^*,-\vec{p}^*)$$

$$\Rightarrow \quad \boxed{s=(E_1^*+E_2^*)^2}$$

★**Hence** $\sqrt{s}$ **is the total energy of collision in the centre-of-mass frame**

From Feynman diagrams to Physics

Particle Physics = Precision Physics

★ Particle physics is about building fundamental theories and testing their predictions against precise experimental data

- Dealing with fundamental particles and can make very precise theoretical predictions – not complicated by dealing with many-body systems
- Many beautiful experimental measurements
  → precise theoretical predictions challenged by precise measurements
- For all its flaws, the Standard Model describes all experimental data ! This is a (the?) remarkable achievement of late 20th century physics.

Requires understanding of theory and experimental data

★ **Part II :** Feynman diagrams mainly used to describe how particles interact

★ **Part III:**
- ♦ will use Feynman diagrams and associated Feynman rules to **perform calculations** for many processes
- ♦ hopefully gain a fairly deep understanding of the Standard Model and how it explains all current data

Before we can start, need calculations for:

- Interaction cross sections;
- Particle decay rates;

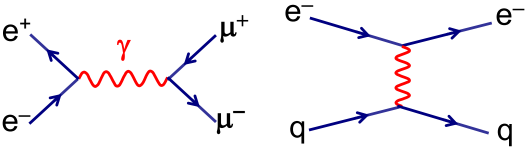
The first five lectures

★ **Aiming towards a proper calculation of decay and scattering processes**

Will concentrate on:

- $e^+e^- \rightarrow \mu^+\mu^-$
- $e^-q \rightarrow e^-q$

($e^-q \rightarrow e^-q$ to probe proton structure)



⅄ **Need relativistic calculations of particle decay rates and cross sections:**

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{phase space})$$

⅄ **Need relativistic treatment of spin-half particles:**

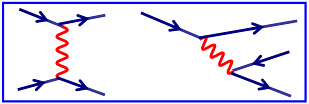
Dirac Equation

⅄ **Need relativistic calculation of interaction Matrix Element:**

Interaction by particle exchange and Feynman rules

\+ **and a few mathematical tricks along, e.g. the Dirac Delta Function**

Cross Sections and Decay Rates



- In particle physics we are mainly concerned with particle <u>interactions</u> and <u>decays</u>, i.e. **transitions between states**
  - these are the experimental observables of particle physics
- Calculate <u>transition rates</u> from **Fermi's Golden Rule**

$$\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)}$$

Form assumes one particle per unit volume and $\int \psi^* \psi \mathrm{dV} = 1$

in a volume V (see next slide)

Γ_{fi} is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$ – **not Lorentz Invariant !**

T_{fi} is Transition **Matrix Element**

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j \neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \dots$$

$\hat{H}$ is the perturbing Hamiltonian

$\rho(E_f)$ is density of final states

★ Rates depend on **MATRIX ELEMENT** and **DENSITY OF STATES**

the ME contains the fundamental particle physics

just kinematics

Non-relativistic Phase Space (revision)

- Apply boundary conditions ($\vec{p}=\hbar\vec{k}$):
- Wave-function vanishing at box boundaries

➡ quantised particle momenta:

$$p_x=\frac{2\pi n_x}{a};\ p_y=\frac{2\pi n_y}{a};\ p_z=\frac{2\pi n_z}{a}$$

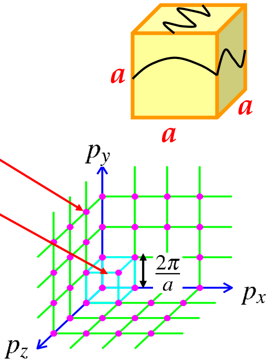
- Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3=\frac{(2\pi)^3}{V}$$

- ~~Normalising to one particle/unit volume gives~~

  number of states in element: $d^3\vec{p}=dp_xdp_ydp_z$ is

$$dn=\frac{d^3\vec{p}}{\frac{(2\pi)^3}{V}}\cancel{\frac{1}{V}}=\frac{d^3\vec{p}}{(2\pi)^3}V$$



- Therefore density of states in Golden rule:

$$\rho(E_f)=\left|\frac{dn}{dE}\right|_{E_f}=\left|\frac{dn}{d|\vec{p}|}\frac{d|\vec{p}|}{dE}\right|_{E_f}$$

with
$p=\beta E$

- Integrating over an elemental shell in momentum-space gives

  $(d^3\vec{p}=4\pi p^2dp)$

$$\rho(E_f)=\frac{4\pi p^2}{(2\pi)^3}V\times\ \beta$$

Intentionally Blank

The Golden Rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- Rewrite the expression for density of states using a delta-function

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_f} = \int \frac{\mathrm{d}n}{\mathrm{d}E}\delta(E-E_i)\mathrm{d}E \qquad \text{since } E_f = E_i$$

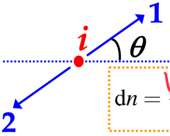
Note : integrating over all final state energies but energy conservation now taken into account explicitly by delta function

- Hence the golden rule becomes: $\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\mathrm{d}n$

the integral is over **all** "allowed" final states of any energy

- For $\mathrm{d}n$ in a two-body decay, only need to consider one particle : mom. conservation fixes the other

$$\frac{1}{V}\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2)\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}$$



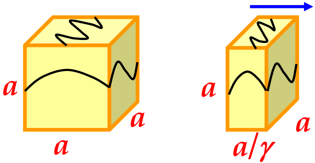
$$\mathrm{d}n = \frac{V\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

- However, can include momentum conservation explicitly by integrating over the momenta of both particles and using another δ-fn

$$\frac{1}{V}\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{Energy cons.}}\underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\text{Mom. cons.}}\underbrace{\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3}}_{\text{Density of states}}$$

Lorentz Invariant Phase Space

- In non-relativistic QM normalise to one particle/unit volume: $\int \psi^* \psi \mathrm{d}V = 1$
- When considering relativistic effects, volume contracts by $\gamma = E/m$



- Particle density therefore increases by $\gamma = E/m$
  - ★ Conclude that a relativistic invariant wave-function normalisation needs to be proportional to E particles per unit volume
- Usual convention: **Normalise to 2E particles/unit volume** $\int \psi'^* \psi' \mathrm{d}V = 2E$
- Previously used ψ normalised to 1 particle per unit volume $\int \psi^* \psi \mathrm{d}V = 1$
- Hence $\psi' = (2E)^{1/2}\psi$ is normalised to $2E$ per unit volume
- **Define** Lorentz Invariant Matrix Element, M_{fi}, in terms of the wave-functions normalised to $2E$ particles per unit volume

$$\boxed{M_{fi} = \langle \psi_1'.\psi_2'...|\hat{H}|...\psi_{n-1}'\psi_n'\rangle = (2E_1.2E_2.2E_3....2E_n)^{1/2} T_{fi}}$$

- For the two body decay

$$i \rightarrow 1 + 2$$

$$\begin{aligned} M_{fi} &= \langle \psi_1' \psi_2' | \hat{H}' | \psi_i' \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} \langle \psi_1 \psi_2 | \hat{H}' | \psi_i \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} T_{fi} \end{aligned}$$

★ Now expressing T_{fi} in terms of M_{fi} gives

$$\boxed{\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}}$$

Note:

- M_{fi} uses relativistically normalised wave-functions. It is Lorentz Invariant
- $\dfrac{d^3\vec{p}}{(2\pi)^3 2E}$ is the Lorentz Invariant Phase Space for each final state particle the factor of $2E$ arises from the wave-function normalisation (prove this in Question 2)
- This form of Γ_{fi} is simply a rearrangement of the original equation **but** the integral is now frame independent (i.e. L.I.)
- Γ_{fi} is inversely proportional to E_i, the energy of the decaying particle. This is exactly what one would expect from time dilation ($E_i = \gamma m$).
- Energy and momentum conservation in the delta functions

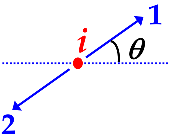
Decay Rate Calculations

$$\frac{1}{V}\Gamma_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2\delta(E_i-E_1-E_2)\delta^3(\vec{p}_i-\vec{p}_1-\vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

★ **Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient**

- **In the C.o.M. frame** $E_i = m_i$ **and** $\vec{p}_i = 0$ $\Rightarrow$

$$\frac{1}{V}\Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2\delta(m_i-E_1-E_2)\delta^3(\vec{p}_1+\vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$$



- **Integrating over** $\vec{p}_2$ **using the δ-function:**

$$\Rightarrow \frac{1}{V}\Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2\delta(m_i-E_1-E_2)\frac{d^3\vec{p}_1}{4E_1E_2}$$

now $E_2^2 = (m_2^2+|\vec{p}_1|^2)$ **since the δ-function imposes** $\vec{p}_2 = -\vec{p}_1$

- **Writing** $d^3\vec{p}_1 = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$

For convenience, here $|\vec{p}_1|$ is written as p_1

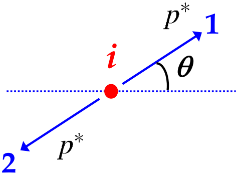
$$\Rightarrow \frac{1}{V}\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2\delta\left(m_i-\sqrt{m_1^2+p_1^2}-\sqrt{m_2^2+p_1^2}\right)\frac{p_1^2 dp_1 d\Omega}{E_1E_2}$$

- Which can be written in the form

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 g(p_1)\delta(f(p_1))\mathrm{d}p_1\mathrm{d}\Omega \qquad (2)$$

where $g(p_1) = p_1^2/(E_1E_2) = p_1^2(m_1^2+p_1^2)^{-1/2}(m_2^2+p_1^2)^{-1/2}$

and $f(p_1) = m_i - (m_1^2+p_1^2)^{1/2} - (m_2^2+p_1^2)^{1/2}$



Note:
- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1) = 0$ determines the C.o.M momenta of the two decay products

  i.e. $f(p_1) = 0$ for $p_1 = p^*$

★ Eq. (2) can be integrated using the property of δ– function derived earlier (eq. (1))

$$\int g(p_1)\delta(f(p_1))\mathrm{d}p_1 = \frac{1}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}\int g(p_1)\delta(p_1-p^*)\mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}$$

where p^* is the value for which $f(p^*) = 0$

- All that remains is to evaluate $\mathrm{d}f/\mathrm{d}p_1$

$$\frac{\mathrm{d}f}{\mathrm{d}p_1} = -\frac{p_1}{(m_1^2+p_1^2)^{1/2}} - \frac{p_1}{(m_2^2+p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1+E_2}{E_1E_2}$$

giving:

$$\frac{1}{V}\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{E_1E_2}{p_1(E_1+E_2)}\,\frac{p_1^2}{E_1E_2}\right|_{p_1=p^*} \mathrm{d}\Omega$$

$$= \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{p_1}{E_1+E_2}\right|_{p_1=p^*} \mathrm{d}\Omega$$

- **But from $f(p_1)=0$, i.e. energy conservation:** $E_1+E_2=m_i$

$$\frac{1}{V}\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i}\int |M_{fi}|^2 \mathrm{d}\Omega$$

In the particle's rest frame $E_i = m_i$

with $\Gamma = \Gamma_{fi}/V$

⇒ $$\boxed{\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 \mathrm{d}\Omega} \qquad \textbf{(3)}$$

$\tau = \frac{1}{\Gamma}$ = mean lifetime of single particle

VALID FOR ALL TWO-BODY DECAYS !

- p^* **can be obtained from** $f(p_1)=0$

$$(m_1^2+p^{*2})^{1/2} + (m_2^2+p^{*2})^{1/2} = m_i$$

(Question 3)

⇒ $$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$

(now try Questions 4 & 5)

Cross section definition

$$\boxed{\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}}$$

Flux = number of incident particles/ unit area/unit time

- The "cross section", σ, can be thought of as the <u>effective</u> cross-sectional area of the target particles for the interaction to occur.
- In general this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption

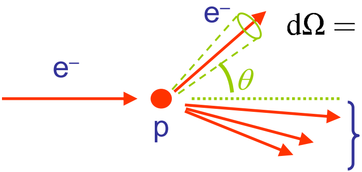
here $\textcircled{\sigma}$ is the projective area of nucleus

Differential Cross section

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}}$$

or generally

$$\frac{d\sigma}{d\ldots}$$



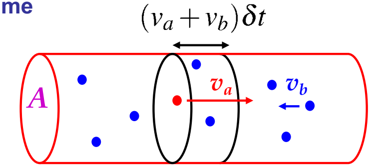
$$d\Omega = d(\cos\theta)d\phi$$

integrate over all other particles

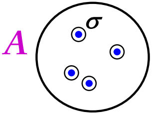
with $$\boxed{\sigma = \int \frac{d\sigma}{d\Omega} d\Omega}$$

example

- Consider a single particle of type a with velocity, v_a, traversing a region of area A containing n_b particles of type b per unit volume



In time δt a particle of type a traverses region containing $n_b(v_a+v_b)A\delta t$ particles of type b



★Interaction probability obtained from effective cross-sectional area occupied by the $n_b(v_a+v_b)A\delta t$ particles of type b

- Interaction Probability = $\dfrac{n_b(v_a+v_b)A\delta t\sigma}{A} = n_b v \delta t \sigma$ $\quad [v = v_a + v_b]$

➡ **Rate per particle of type a = $n_b v \sigma$**

- Consider volume V, total reaction rate = $(n_b v\sigma).(n_a V) = (n_b V)(n_a v)\sigma$

  = $N_b \phi_a \sigma$

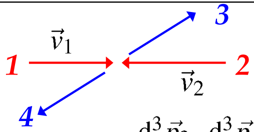
- As anticipated: **Rate = Flux x Number of targets x cross section**

Γ_{fi} = rate in volume V, assuming 1 particle per unit volume of each species of particle.

Cross Section Calculations

- **Consider scattering process**

$$1+2 \rightarrow 3+4$$



Rate per unit volume

- **Start from Fermi's Golden Rule:**

$$\boxed{\frac{1}{V}\Gamma_{fi}} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{d^3\vec{p}_4}{(2\pi)^3}$$

where T_{fi} is the transition matrix for a normalisation of 1/unit volume

- Also:

$$\begin{aligned}\text{Rate/Volume} &= (\text{flux of } 1) \times (\text{number density of } 2) \times \sigma \\ &= n_1(v_1+v_2) \times n_2 \times \sigma\end{aligned}$$

[From last slide]

- **For 1 target particle of each species per unit volume** $\text{Rate/Volume} = (v_1+v_2)\sigma$

[Which is required by our Γ_{fi} given the form of Fermi's Golden Rule we used]

$$\sigma = \frac{\Gamma_{fi}}{(v_1+v_2)}$$

$$\sigma = \underbrace{\frac{(2\pi)^4}{v_1+v_2}} \int \underbrace{|T_{fi}|^2} \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\underbrace{\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{d^3\vec{p}_4}{(2\pi)^3}}$$

the parts are not Lorentz Invariant

- To obtain a Lorentz Invariant form use wave-functions normalised to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2}\psi$$

- Again define L.I. Matrix element $M_{fi} = (2E_1\,2E_2\,2E_3\,2E_4)^{1/2}T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)}\int |M_{fi}|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- The integral is now written in a Lorentz invariant form
- The quantity $F = 2E_1 2E_2(v_1+v_2)$ can be written in terms of a four-vector scalar product and is ~~therefore also~~ Longitudinally Lorentz Invariant **(the Lorentz Inv. Flux)**

$$F = 4\left[(p_1^{\mu}p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2}$$

(see appendix I)

- Consequently cross section is a Longitudinally Lorentz Invariant quantity.

Two special cases of Lorentz Invariant Flux:

- Centre-of-Mass Frame

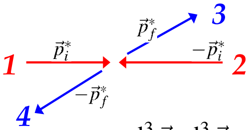
$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1E_2(|\vec{p}^*|/E_1+|\vec{p}^*|/E_2) \\ &= 4|\vec{p}^*|(E_1+E_2) \\ &= 4|\vec{p}^*|\sqrt{s} \end{aligned}$$

- Target (particle 2) at rest

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1m_2v_1 \\ &= 4E_1m_2(|\vec{p}_1|/E_1) \\ &= 4m_2|\vec{p}_1| \end{aligned}$$

2→2 Body Scattering in C.o.M. Frame

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2→2 scattering in C.o.M. frame



- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- Here $\vec{p}_1+\vec{p}_2=0$ and $E_1+E_2=\sqrt{s}$

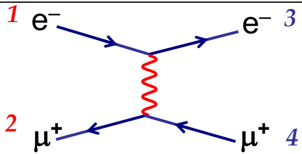
$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s}-E_3-E_4)\delta^3(\vec{p}_3+\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

★The integral is exactly the same integral that appeared in the particle decay calculation but with m_a replaced by $\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}}\frac{|\vec{p}_f^*|}{4\sqrt{s}}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 \mathrm{d}\Omega^*}$$

- In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$



$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$

- For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}|M_{fi}|^2 \mathrm{d}\Omega^*$$

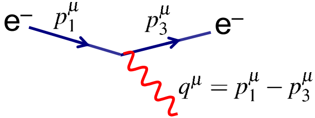
because the angles in $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^*$ refer to the C.o.M frame

- For the last calculation in this section, we need to find a **L.I.** expression for $\mathrm{d}\sigma$

★ Start by expressing $\mathrm{d}\Omega^*$ in terms of Mandelstam t
i.e. the square of the four-momentum transfer

$$\boxed{t = q^2 = (p_1 - p_3)^2}$$

Product of four-vectors therefore L.I.

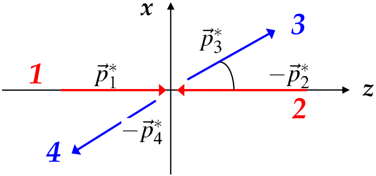


- **Want to express** $\mathrm{d}\Omega^*$ **in terms of Lorentz Invariant** $\mathrm{d}t$

  where $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = m_1^2 + m_3^2 - 2p_1.p_3$

- **In C.o.M. frame:**

$$\begin{aligned} p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|) \\ p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*|\sin\theta^*, 0, |\vec{p}_3^*|\cos\theta^*) \\ p_1^{\mu} p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \\ t &= m_1^2 + m_3^3 - 2E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \end{aligned}$$



giving $$\mathrm{d}t = 2|\vec{p}_1^*||\vec{p}_3^*|\mathrm{d}(\cos\theta^*)$$

therefore $$\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^* = \frac{\mathrm{d}t\,\mathrm{d}\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$$

hence $$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_3^*|}{|\vec{p}_1^*|}|M_{fi}|^2\mathrm{d}\Omega^* = \frac{1}{2\cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2\mathrm{d}\phi^*\mathrm{d}t$$

- **Finally, integrating over** $\mathrm{d}\phi^*$ **(assuming no** ϕ^* **dependence of** $|M_{fi}|^2$ **) gives:**

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}$$

Lorentz Invariant differential cross section

- All quantities in the expression for $\mathrm{d}\sigma/\mathrm{d}t$ are Lorentz Invariant and therefore, it applies to any rest frame. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

- As an example of how to use the invariant expression $\mathrm{d}\sigma/\mathrm{d}t$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$

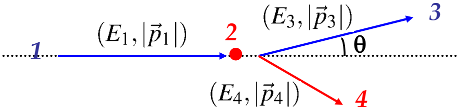
e.g. electron or neutrino scattering

In this limit $$|\vec{p}_i^*|^2 = \frac{(s-m_2^2)^2}{4s}$$

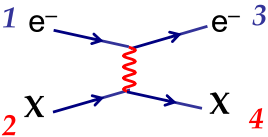
$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi(s-m_2^2)^2}|M_{fi}|^2} \qquad (m_1 = 0)$$

2→2 Body Scattering in Lab. Frame

- A commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0, \quad m_2 = m_4 = M$



e.g.



- Wish to express the cross section in terms of scattering angle of the e⁻

$$d\Omega = 2\pi d(\cos\theta)$$

therefore $$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

Integrating over $d\phi$

- The rest is some rather tedious algebra…. start from four-momenta

$$p_1 = (E_1, 0, 0, E_1), \;\; p_2 = (M, 0, 0, 0), \;\; p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta), \;\; p_4 = (E_4, \vec{p}_4)$$

so here $$t = (p_1 - p_3)^2 = -2p_1.p_3 = -2E_1E_3(1 - \cos\theta)$$

But from (E,p) conservation $p_1 + p_2 = p_3 + p_4$

and, therefore, can also express t in terms of particles 2 and 4

$$\begin{aligned} t &= (p_2 - p_4)^2 = 2M^2 - 2p_2.p_4 = 2M^2 - 2ME_4 \\ &= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3) \end{aligned}$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2M\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}$$

- **Equating the two expressions for t gives** $\boxed{E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}}$

so $$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{E_1^2 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s - M^2)^2}|M_{fi}|^2$$

using $$s = (p_1 + p_2)^2 = M^2 + 2p_1.p_2 = M^2 + 2ME_1$$

Particle 1 massless

➡ $(p_1^2 = 0)$

gives $$(s - M^2) = 2ME_1$$

➡ $$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2}$$

In limit $m_1 \rightarrow 0$

In this equation, E_3 is a function of θ:

$$E_3 \quad = \quad \frac{E_1 M}{M + E_1 - E_1 \cos\theta}$$

giving $$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2} \qquad (m_1 = 0)$$

General form for 2→2 Body Scattering in Lab. Frame

★**The calculation of the differential cross section for the case where m_1 can not be neglected is longer and contains no more "physics" (see appendix II). It gives:**

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{m_2|\vec{p}_1|}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2) - E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}$$

Again there is only one independent variable, θ, which can be seen from conservation of energy

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{\underbrace{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3} + m_4^2}$$

i.e. $\left|\vec{p}_3\right|$ **is a function of** θ

Summary

★ **Used a Lorentz invariant formulation of Fermi′s Golden Rule to derive decay rates and cross-sections in terms of the Lorentz Invariant Matrix Element (wave-functions normalised to 2E/Volume)**

Main Results:

★ **Particle decay:**

$$\boxed{\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 \mathrm{d}\Omega}$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$

★ **Scattering cross section in C.o.M. frame:**

$$\boxed{\sigma \quad = \quad \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 \mathrm{d}\Omega^*}$$

★ **Invariant differential cross section (valid in all frames):**

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

Summary cont.

★ **Differential cross section in the lab. frame ($m_1=0$)**

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2} \Longleftrightarrow \boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2|M_{fi}|^2}$$

★ **Differential cross section in the lab. frame ($m_1\neq 0$)**

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{1}{64\pi^2}\cdot\frac{1}{|m_2|\vec{p}_1|}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2)-E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}$$

with $E_1+m_2=\sqrt{|\vec{p}_3|^2+m_3^2}+\sqrt{|\vec{p}_1|^2+|\vec{p}_3|^2-2|\vec{p}_1||\vec{p}_3|\cos\theta+m_4^2}$

Summary of the summary:

- ★ **Have now dealt with kinematics of particle decays and cross sections**
- ★ **The fundamental particle physics is in the matrix element**
- ★ **The above equations are the basis for all calculations that follow**

Appendix I : Lorentz Invariant Flux

NON-EXAMINABLE

- **Collinear collision:**

$$F = 2E_a 2E_b (v_a + v_b) \;=\; 4E_aE_b\left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b}\right)$$
$$= 4(|\vec{p}_a|E_b + |\vec{p}_b|E_a)$$

To show this is Lorentz invariant, first consider

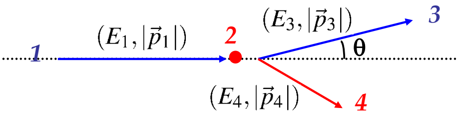
$$p_a.p_b = p_a^\mu p_{b\mu} = E_aE_b - \vec{p}_a.\vec{p}_b = E_aE_b + |\vec{p}_a||\vec{p}_b|$$

Giving

$$\begin{aligned}
F^2/16 - (p_a^\mu p_{b\mu})^2 &= (|\vec{p}_a|E_b + |\vec{p}_b|E_a)^2 - (E_aE_b + |\vec{p}_a||\vec{p}_b|)^2 \\
&= |\vec{p}_a|^2(E_b^2 - |\vec{p}_b|^2) + E_a^2(|\vec{p}_b|^2 - E_b^2) \\
&= |\vec{p}_a|^2 m_b^2 - E_a^2 m_b^2 \\
&= -m_a^2 m_b^2 \\
F &= 4\left[(p_a^\mu p_{b\mu})^2 - m_a^2 m_b^2\right]^{1/2}
\end{aligned}$$

Appendix II : general 2→2 Body Scattering in lab frame

NON-EXAMINABLE



$$p_1=(E_1,0,0,|\vec{p}_1|),\;\; p_2=(M_2,0,0,0),\;\; p_3=(E_3,E_3\sin\theta,0,E_3\cos\theta),\;\; p_4=(E_4,\vec{p}_4)$$

again

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega}=\frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

But now the invariant quantity t:

$$\begin{aligned} t &= (p_2-p_4)^2=m_2^2+m_4^2-2p_2.p_4=m_2^2+m_4^2-2m_2E_4\\ &= m_2^2+m_4^2-2m_2(E_1+m_2-E_3) \end{aligned}$$

$$\Rightarrow\quad \frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}=2m_2\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}$$

Which gives $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_2}{\pi}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

To determine $\mathrm{d}E_3/\mathrm{d}(\cos\theta)$, first differentiate $E_3^2 - |\vec{p}_3|^2 = m_3^2$

$$2E_3\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = 2|\vec{p}_3|\frac{\mathrm{d}|\vec{p}_3|}{\mathrm{d}(\cos\theta)} \qquad \textbf{(AII.1)}$$

Then equate $t = (p_1 - p_3)^2 = (p_4 - p_2)^2$ **to give**

$$m_1^2 + m_3^2 - 2(E_1E_3 - |\vec{p}_1||\vec{p}_3|\cos\theta) = m_4^2 + m_2^2 - 2m_2(E_1 + m_2 - E_3)$$

Differentiate wrt. $\cos\theta$

$$(E_1 + m_2)\frac{\mathrm{d}E_3}{\mathrm{d}\cos\theta} - |\vec{p}_1|\cos\theta\frac{\mathrm{d}|\vec{p}_3|}{\mathrm{d}\cos\theta} = |\vec{p}_1||\vec{p}_3|$$

Using (AII.1) ➡ $$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{|\vec{p}_1||\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \qquad \textbf{(AII.2)}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_2}{\pi}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{m_2}{\pi}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

It is easy to show $|\vec{p}_i^*|\sqrt{s} = m_2|\vec{p}_1|$

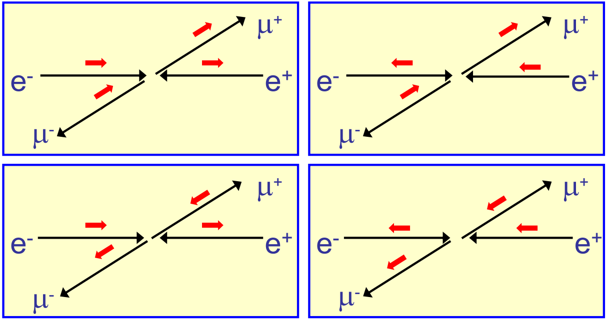
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{m_2}{64\pi^2 m_2^2|\vec{p}_1|^2}|M_{fi}|^2$$

and using (AII.2) obtain

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{m_2|\vec{p}_1|}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2)-E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}$$

Particle Physics

Dr Lester



Handout 2 : The Dirac Equation

Non-Relativistic QM (Revision)

- For particle physics need a relativistic formulation of quantum mechanics. But first take a few moments to review the non-relativistic formulation QM
- Take as the starting point non-relativistic energy:

$$E = T + V = \frac{\vec{p}^2}{2m} + V$$

- In QM we identify the energy and momentum operators:

$$\vec{p} \rightarrow -i\vec{\nabla}, \quad E \rightarrow i\frac{\partial}{\partial t}$$

which gives the time dependent Schrödinger equation (take V=0 for simplicity)

$$-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial \psi}{\partial t}$$

(2)

with plane wave solutions: $\psi = Ne^{i(\vec{p}.\vec{r}-Et)}$ where $\begin{cases} -i\nabla\psi = \vec{p}\psi \\ i\frac{\partial\psi}{\partial t} = E\psi \end{cases}$

- The SE is **first order** in the time derivatives and **second order** in spatial derivatives – and is manifestly not Lorentz invariant.
- In what follows we will use probability density/current extensively. For the non-relativistic case these are derived as follows

(2)* ➡

$$-\frac{1}{2m}\vec{\nabla}^2\psi^* = -i\frac{\partial \psi^*}{\partial t}$$

(3)

$$\begin{aligned}\psi^* \times ((2)) - \psi \times ((3)): \quad -\frac{1}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) &= i\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right)\\ -\frac{1}{2m}\vec{\nabla}.\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) &= i\frac{\partial}{\partial t}(\psi^*\psi)\end{aligned}$$

- **Which by comparison with the continuity equation**

$$\vec{\nabla}.\vec{j} + \frac{\partial\rho}{\partial t} = 0$$

leads to the following expressions for probability density and current:

$$\rho = \psi^*\psi = |\psi|^2 \qquad \vec{j} = \frac{1}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

- **For a plane wave** $\psi = Ne^{i(\vec{p}.\vec{r}-Et)}$

$$\rho = |N|^2 \quad \textbf{and} \quad \vec{j} = |N|^2\frac{\vec{p}}{m} = |N|^2\vec{v}$$

★ **The number of particles per unit volume is** $|N|^2$

★ **For $|N|^2$ particles per unit volume moving at velocity $\vec{v}$, have $|N|^2|\vec{v}|$ passing through a unit area per unit time (particle flux). Therefore $\vec{j}$ is a vector in the particle′s direction with magnitude equal to the flux.**

The Klein-Gordon Equation

- Applying $\vec{p} \to -i\vec{\nabla}, \quad E \to i\partial/\partial t$ to the relativistic equation for energy:

$$E^2 = |\vec{p}|^2 + m^2 \qquad ((4))$$

gives the **Klein-Gordon equation:**

$$\frac{\partial^2 \psi}{\partial t^2} = \vec{\nabla}^2 \psi - m^2 \psi \qquad ((5))$$

- Using $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ ➡ $\partial^\mu \partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

KG can be expressed compactly as

$$\boxed{(\partial^\mu \partial_\mu + m^2)\psi = 0} \qquad ((6))$$

- For plane wave solutions, $\psi = Ne^{i(\vec{p}.\vec{r} - Et)}$ the KG equation gives:

$$-E^2\psi = -|\vec{p}|^2\psi - m^2\psi$$

➡ $$E = \pm\sqrt{|\vec{p}|^2 + m^2}$$

★ Not surprisingly, the KG equation has negative energy solutions – this is just what we started with in eq. ((4))

♦ Historically the **–ve** energy solutions were viewed as problematic. **But** for the KG there is also a problem with the **probability density**…

- Proceeding as before to calculate the probability and current densities:

(KG2)*
$$\frac{\partial^2 \psi^*}{\partial t^2} = \vec{\nabla}^2 \psi^* - m^2 \psi^* \qquad \textbf{(KG4)}$$

$$\psi^* \times (\mathbf{KG2}) - \psi \times (\mathbf{KG4}):$$

$$\begin{aligned} \psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} &= \psi^*(\nabla^2 \psi - m^2 \psi) - \psi(\nabla^2 \psi^* - m^2 \psi^*) \\ \frac{\partial}{\partial t}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) &= \vec{\nabla}.(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) \end{aligned}$$

- Which, again, by comparison with the continuity equation allows us to identify

$$\rho = i\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) \quad \text{and} \quad \vec{j} = i(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$$

- For a plane wave $\psi = N e^{i(\vec{p}.\vec{r} - Et)}$

$$\rho = 2E|N|^2 \quad \textbf{and} \quad \vec{j} = |N|^2 \vec{p}$$

★ Particle densities are proportional to E. We might have anticipated this from the previous discussion of Lorentz invariant phase space (i.e. density of $1/V$ in the particles rest frame will appear as E/V in a frame where the particle has energy E due to length contraction).

The Dirac Equation

- ★ Historically, it was thought that there were **two** main problems with the Klein-Gordon equation:
  - ♦ Negative energy solutions
  - ♦ The negative particle densities associated with these solutions

$$\rho = 2E|N|^2$$

- ★ We now know that in Quantum Field Theory these problems do not arise and the KG equation is used to describe **spin-0** particles (inherently single particle description → multi-particle quantum excitations of a scalar field).

Nevertheless:

- ★ These problems motivated **Dirac (1928)** to search for a different formulation of relativistic quantum mechanics in which all particle densities are positive.
- ★ The resulting wave equation had solutions which not only solved this problem but also fully describe the intrinsic spin and magnetic moment of the electron!

The Dirac Equation :

- **Schrödinger eqn:** $-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t}$ 1st order in $\partial/\partial t$
  2nd order in $\partial/\partial x, \partial/\partial y, \partial/\partial z$
- **Klein-Gordon eqn:** $(\partial^\mu\partial_\mu + m^2)\psi = 0$ 2nd order throughout
- **Dirac looked for an alternative which was 1st order throughout:**

$$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial\psi}{\partial t} \qquad \textbf{(D1)}$$

where $\hat{H}$ is the Hamiltonian operator and, as usual, $\vec{p} = -i\vec{\nabla}$

- **Writing (D1) in full:**

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = \left(i\frac{\partial}{\partial t}\right)\psi$$

"squaring" this equation gives

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = -\frac{\partial^2\psi}{\partial t^2}$$

- **Which can be expanded in gory details as…**

$$\begin{aligned}-\frac{\partial^2\psi}{\partial t^2} \;=\; & -\alpha_x^2\frac{\partial^2\psi}{\partial x^2}-\alpha_y^2\frac{\partial^2\psi}{\partial y^2}-\alpha_z^2\frac{\partial^2\psi}{\partial z^2}+\beta^2 m^2\psi \\ & -(\alpha_x\alpha_y+\alpha_y\alpha_x)\frac{\partial^2\psi}{\partial x\partial y}-(\alpha_y\alpha_z+\alpha_z\alpha_y)\frac{\partial^2\psi}{\partial y\partial z}-(\alpha_z\alpha_x+\alpha_x\alpha_z)\frac{\partial^2\psi}{\partial z\partial x} \\ & -(\alpha_x\beta+\beta\alpha_x)m\frac{\partial\psi}{\partial x}-(\alpha_y\beta+\beta\alpha_y)m\frac{\partial\psi}{\partial y}-(\alpha_z\beta+\beta\alpha_z)m\frac{\partial\psi}{\partial z}\end{aligned}$$

- **For this to be a reasonable formulation of relativistic QM, a free particle must also obey $E^2=\vec{p}^2+m^2$, i.e. it must satisfy the Klein-Gordon equation:**

$$-\frac{\partial^2\psi}{\partial t^2}=-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}+m^2\psi$$

- **Hence for the Dirac Equation to be consistent with the KG equation require:**

$$\alpha_x^2=\alpha_y^2=\alpha_z^2=\beta^2=1 \qquad \textbf{(D2)}$$

$$\alpha_j\beta+\beta\alpha_j=0 \qquad \textbf{(D3)}$$

$$\alpha_j\alpha_k+\alpha_k\alpha_j=0 \quad (j\neq k) \qquad \textbf{(D4)}$$

★ **Immediately we see that the α_j and β cannot be numbers. Require 4 mutually anti-commuting matrices**

★ **Must be (at least) 4x4 matrices (see Appendix I)**

- Consequently the wave-function must be a <u>**four-component**</u> **Dirac Spinor**

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

A consequence of introducing an equation that is 1st order in time/space derivatives is that the wave-function has new degrees of freedom !

- For the Hamiltonian $\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\partial\psi/\partial t$ to be Hermitian requires

$$\boxed{\alpha_x = \alpha_x^\dagger; \quad \alpha_y = \alpha_y^\dagger; \quad \alpha_z = \alpha_z^\dagger; \quad \beta = \beta^\dagger;} \qquad \textbf{(D5)}$$

i.e. the require four anti-commuting Hermitian 4x4 matrices.

- At this point it is convenient to introduce an explicit representation for $\vec{\alpha}, \beta$ It should be noted that physical results do not depend on the particular representation – everything is in the commutation relations.
- A convenient choice is based on the Pauli spin matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- The matrices are Hermitian and anti-commute with each other

Dirac Equation: Probability Density and Current

- **Now consider probability density/current – this is where the perceived problems with the Klein-Gordon equation arose.**
- **Start with the Dirac equation**

$$-i\alpha_x\frac{\partial\psi}{\partial x}-i\alpha_y\frac{\partial\psi}{\partial y}-i\alpha_z\frac{\partial\psi}{\partial z}+m\beta\psi=i\frac{\partial\psi}{\partial t} \qquad \textbf{(D6)}$$

and its Hermitian conjugate

$$+i\frac{\partial\psi^\dagger}{\partial x}\alpha_x^\dagger+i\frac{\partial\psi^\dagger}{\partial y}\alpha_y^\dagger+i\frac{\partial\psi^\dagger}{\partial z}\alpha_z^\dagger+m\psi^\dagger\beta^\dagger=-i\frac{\partial\psi^\dagger}{\partial t} \qquad \textbf{(D7)}$$

- **Consider** $\psi^\dagger\times(\mathbf{D6})-(\mathbf{D7})\times\psi$ **remembering** α,β **are Hermitian** ➡

$$\psi^\dagger\left(-i\alpha_x\frac{\partial\psi}{\partial x}-i\alpha_y\frac{\partial\psi}{\partial y}-i\alpha_z\frac{\partial\psi}{\partial z}+\beta m\psi\right)-\left(i\frac{\partial\psi^\dagger}{\partial x}\alpha_x+i\frac{\partial\psi^\dagger}{\partial y}\alpha_y+i\frac{\partial\psi^\dagger}{\partial z}\alpha_z+m\psi^\dagger\beta\right)\psi=i\psi^\dagger\frac{\partial\psi}{\partial t}+i\frac{\partial\psi^\dagger}{\partial t}\psi$$

$$\Rightarrow\quad \underbrace{\psi^\dagger\left(\alpha_x\frac{\partial\psi}{\partial x}\right.}_{}+\alpha_y\frac{\partial\psi}{\partial y}+\alpha_z\frac{\partial\psi}{\partial z}\bigg)+\underbrace{\left(\frac{\partial\psi^\dagger}{\partial x}\alpha_x\right.}_{}+\frac{\partial\psi^\dagger}{\partial y}\alpha_y+\frac{\partial\psi^\dagger}{\partial z}\alpha_z\bigg)\psi+\frac{\partial(\psi^\dagger\psi)}{\partial t}=0$$

- **Now using the identity:**

$$\psi^\dagger\alpha_x\frac{\partial\psi}{\partial x}+\frac{\partial\psi^\dagger}{\partial x}\alpha_x\psi\equiv\frac{\partial(\psi^\dagger\alpha_x\psi)}{\partial x}$$

gives the continuity equation $\boxed{\vec{\nabla}.(\psi^\dagger\vec{\alpha}\psi)+\frac{\partial(\psi^\dagger\psi)}{\partial t}=0}$ **(D8)**

where $\psi^\dagger=(\psi_1^*,\psi_2^*,\psi_3^*,\psi_4^*)$

- The probability density and current can be identified as:

$$\boxed{\rho=\psi^\dagger\psi} \quad \text{and} \quad \boxed{\vec{j}=\psi^\dagger\vec{\alpha}\psi}$$

where $\rho=\psi^\dagger\psi=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2+|\psi_4|^2>0$

- Unlike the KG equation, the Dirac equation has probability densities which are always positive.
- In addition, the solutions to the Dirac equation are the four component Dirac Spinors. A great success of the Dirac equation is that these components naturally give rise to the property of intrinsic spin.
- It can be shown that Dirac spinors represent spin-half particles **(appendix II)** with an intrinsic magnetic moment of

$$\boxed{\vec{\mu}=\frac{q}{m}\vec{S}}$$

(appendix III)

Covariant Notation: the Dirac γ Matrices

- The Dirac equation can be written more elegantly by introducing the four Dirac gamma matrices:

$$\gamma^0 \equiv \beta; \;\; \gamma^1 \equiv \beta\alpha_x; \;\; \gamma^2 \equiv \beta\alpha_y; \;\; \gamma^3 \equiv \beta\alpha_z$$

Premultiply the Dirac equation **(D6)** by β

$$i\beta\alpha_x\frac{\partial\psi}{\partial x} + i\beta\alpha_y\frac{\partial\psi}{\partial y} + i\beta\alpha_z\frac{\partial\psi}{\partial z} - \beta^2 m\psi = -i\beta\frac{\partial\psi}{\partial t}$$

➡ $$i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = -i\gamma^0\frac{\partial\psi}{\partial t}$$

using $\partial_\mu = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ this can be written compactly as:

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0} \qquad \textbf{(D9)}$$

★ **NOTE:** it is important to realise that the Dirac gamma matrices are <u>not</u> four-vectors - they are constant matrices which remain invariant under a Lorentz transformation. However it can be shown that the Dirac equation is itself Lorentz covariant ((see page 104))

Properties of the γ matrices

- From the properties of the α and β matrices (D2)-(D4) immediately obtain:

$$(\gamma^0)^2 = \beta^2 = 1 \quad \text{and} \quad (\gamma^1)^2 = \beta\alpha_x\beta\alpha_x = -\alpha_x\beta\beta\alpha_x = -\alpha_x^2 = -1$$

- The full set of relations is

$$\begin{aligned} (\gamma^0)^2 &= 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 &= -1 \\ \gamma^0\gamma^j + \gamma^j\gamma^0 &= 0 \\ \gamma^j\gamma^k + \gamma^k\gamma^j &= 0 \quad (j \neq k) \end{aligned}$$

which can be expressed as:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

(defines the algebra)

- Are the gamma matrices Hermitian?
  - β is Hermitian so γ^0 is Hermitian.
  - The α matrices are also Hermitian, giving

$$\gamma^{1\dagger} = (\beta\alpha_x)^\dagger = \alpha_x^\dagger\beta^\dagger = \alpha_x\beta = -\beta\alpha_x = -\gamma^1$$

  - Hence $\gamma^1, \gamma^2, \gamma^3$ are anti-Hermitian

$$\boxed{\gamma^{0\dagger} = \gamma^0, \ \ \gamma^{1\dagger} = -\gamma^1, \ \ \gamma^{2\dagger} = -\gamma^2, \ \ \gamma^{3\dagger} = -\gamma^3}$$

Pauli-Dirac Representation

- From now on we will use the Pauli-Dirac representation of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$ which when written in full are

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \; \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \; \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \; \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Using the gamma matrices $\rho = \psi^\dagger \psi$ and $\vec{j} = \psi^\dagger \vec{\alpha} \psi$ can be written as:

$$j^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

where j^μ is the four-vector current.

(The proof that j^μ is indeed a four vector is given in page 109)

- In terms of the four-vector current the continuity equation becomes

$$\partial_\mu j^\mu = 0$$

- Finally the expression for the four-vector current

$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

can be simplified by introducing the adjoint spinor

The Adjoint Spinor

- **The adjoint spinor is defined as**

$$\boxed{\overline{\psi} = \psi^{\dagger}\gamma^{0}}$$

i.e. $$\overline{\psi} = \psi^{\dagger}\gamma^{0} \quad = \quad (\psi^{*})^{T}\gamma^{0} = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \text{-}1 & 0 \\ 0 & 0 & 0 & \text{-}1 \end{pmatrix}$$

$$\boxed{\overline{\psi} \quad = \quad (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)}$$

- **In terms the adjoint spinor the four vector current can be written:**

$$\boxed{j^{\mu} = \overline{\psi}\gamma^{\mu}\psi}$$

★ **We will use this expression in deriving the Feynman rules for the Lorentz invariant matrix element for the fundamental interactions.**

★ **That's enough notation, start to investigate the free particle solutions of the Dirac equation...**

Dirac Equation: Free Particle at Rest

- Look for **free particle** solutions to the Dirac equation of form:

$$\boxed{\psi = u(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}}$$

where $u(E,\vec{p})$, which is a constant four-component spinor which must satisfy the Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

- Consider the derivatives of the free particle solution

$$\partial_0\psi = \frac{\partial\psi}{\partial t} = -iE\psi; \quad \partial_1\psi = \frac{\partial\psi}{\partial x} = ip_x\psi, \quad \ldots$$

substituting these into the Dirac equation gives:

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u = 0$$

which can be written: $\boxed{(\gamma^{\mu}p_{\mu} - m)u = 0}$ **(D10)**

- This is the Dirac equation in "momentum" – note it contains no derivatives.
- For a **particle at rest** $\vec{p} = 0$

and $\psi = u(E,0)e^{-iEt}$

eq. (D10) $\longrightarrow$ $E\gamma^0 u - mu = 0$

$$\Rightarrow \qquad E\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}=m\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} \qquad \textbf{(D11)}$$

- **This equation has four orthogonal solutions:**

$$\underbrace{u_1(m,0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\ u_2(m,0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{\textbf{(D11)}\Rightarrow\ \boxed{E=m}};\ \underbrace{u_3(m,0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\ u_4(m,0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{\textbf{(D11)}\Rightarrow\ \boxed{E=-m}}$$

still have NEGATIVE ENERGY SOLUTIONS

(Question 6)

- **Including the time dependence from** $\psi=u(E,0)e^{-iEt}$ **gives**

$$\underbrace{\psi_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt};\quad \psi_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt}}_{\boxed{\text{Two spin states with E>0}}};\quad \underbrace{\psi_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt};\ \text{and}\ \ \psi_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}}_{\boxed{\text{Two spin states with E<0}}}$$

★In QM mechanics can't just discard the E<0 solutions as unphysical as we require a complete set of states - i.e. 4 SOLUTIONS

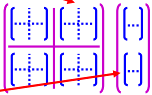
Dirac Equation: Plane Wave Solutions

- Now aim to find general plane wave solutions: $\psi = u(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$
- Start from Dirac equation (D10): $(\gamma^\mu p_\mu - m)u = 0$

and use
$$\begin{aligned}\gamma^\mu p_\mu - m &= E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m \\ &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} .\vec{p} - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -(E+m)I \end{pmatrix}\end{aligned}$$

Note
$\vec{\sigma}.\vec{p} = p_x\sigma_x + p_y\sigma_y + p_z\sigma_z$

Note in the above equation the **4x4 matrix** is written in terms of four **2x2 sub-matrices**



- Writing the four component spinor as $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

$$(\gamma^\mu p_\mu - m)u = 0 \quad \Rightarrow \quad \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & (-E-m)I \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Giving two coupled simultaneous equations for $u_A,\ u_B$

$$\left.\begin{aligned}(\vec{\sigma}.\vec{p})u_B &= (E-m)u_A \\ (\vec{\sigma}.\vec{p})u_A &= (E+m)u_B\end{aligned}\right\} \qquad \text{(D12)}$$

Expanding $$\vec{\sigma}.\vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z$$

$$\vec{\sigma}.\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

- **Therefore (D12)**

$$\left.\begin{array}{rcl} (\vec{\sigma}.\vec{p})u_B & = & (E-m)u_A \\ (\vec{\sigma}.\vec{p})u_A & = & (E+m)u_B \end{array}\right\}$$

gives $$u_B = \frac{\vec{\sigma}.\vec{p}}{E+m} u_A = \frac{1}{E+m} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A$$

- **Solutions can be obtained by making the arbitrary (but simplest) choices for** u_A

i.e. $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **or** $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

giving $$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}; \quad \text{and} \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

where N is the wave-function normalisation

NOTE: **For** $\vec{p} = 0$ **these correspond to the E>0 particle at rest solutions**

★**The choice of** u_A **is arbitrary, but this isn't an issue since we can express any other choice as a linear combination. It is analogous to choosing a basis for spin which could be eigenfunctions of** S_x, S_y **or** S_z

Repeating for $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives the solutions u_3 and u_4

★ The four solutions are: $\psi_i = u_i(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix};\; u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix};\; u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix};\; u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

•If any of these solutions is put back into the Dirac equation, as expected, we obtain

$$E^2 = \vec{p}^2 + m^2$$

which doesn't in itself identify the negative energy solutions.

•One rather subtle point: One could ask the question whether we can interpret **all four** solutions as positive energy solutions. The answer is no. If we take all solutions to have the same value of E, i.e. E = +|E|, only two of the solutions are found to be independent.

•There are only four independent solutions when the two are taken to have E<0.

★To identify which solutions have E<0 energy refer back to particle at rest (eq. D11).

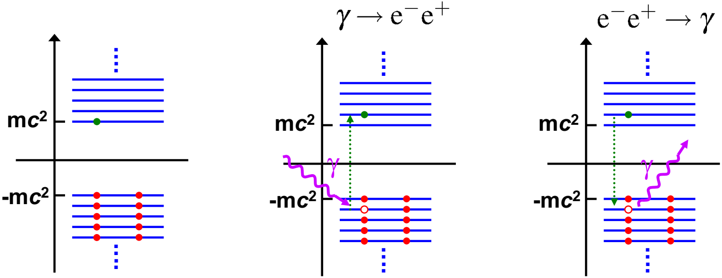
- For $\vec{p} = 0$ u_1, u_2 correspond to the E>0 particle at rest solutions
  u_3, u_4 correspond to the E<0 particle at rest solutions

★ So u_1, u_2 are the +ve energy solutions and u_3, u_4 are the -ve energy solutions

Interpretation of –ve Energy Solutions

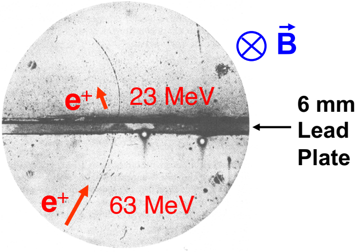
★The Dirac equation has negative energy solutions. Unlike the KG equation these have positive probability densities. But how should –ve energy solutions be interpreted? Why don't all +ve energy electrons fall into to the lower energy –ve energy states?

Dirac Interpretation: the vacuum corresponds to all –ve energy states being full with the Pauli exclusion principle preventing electrons falling into -ve energy states. Holes in the –ve energy states correspond to +ve energy anti-particles with opposite charge. Provides a picture for pair-production and annihilation.

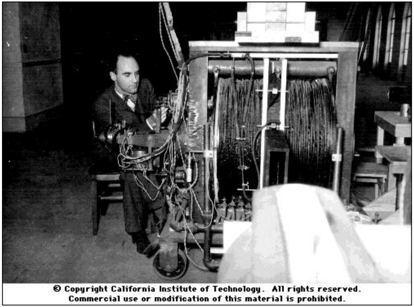


Discovery of the Positron

★**Cosmic ray track in cloud chamber:**



C.D.Anderson, Phys Rev 43 (1933) 491



- e^+ enters at bottom, slows down in the lead plate – know direction
- Curvature in **B-field** shows that it is a **positive particle**
- Can't be a proton as would have stopped in the lead

Provided Verification of Predictions of Dirac Equation

★**Anti-particle solutions exist ! But the picture of the vacuum corresponding to the state where all –ve energy states are occupied is rather unsatisfactory, what about bosons (no exclusion principle),….**

Chronology relating to Negative Energy Solutions

- 1928, Dirac invents his Equation. Probability density is positive, but negative energies are permitted (Proc. Roy. Soc. A117, 610-628) [1].
- 1930, Dirac tries to solve negative energies via the "hole" theory. He relates anti-particles to negative energy eigenstates. (Proc. Cam. Phil. Soc. 26, 376-381) [2].
- 1934, Paulu and Weisskopf present a new interpretation of Klein-Gordon equation: as field equation for a charged spin-0 field. ρ represents the charge density. The energy is given via

  $$\frac{1}{2}\int d^3r\left[|\nabla\psi|^2 + m^2|\psi|^2\right]$$

  and thus positive by definition (Helv. Phys. Acta 7, 709-734) [3].
- 1934, The Dirac equation aquired a field-theoretic interpretation. It no longer represented a probability amplitude. Instead it became the field operator of a spin-$\frac{1}{2}$ field in a QFT. See the QFT and AQFT courses.

Anti-Particle Spinors

Find negative energy plane wave solutions to the Dirac equation of the form: $\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$ **where** $E = |\sqrt{|\vec{p}|^2+m^2}|$

- **Note that although** $E>0$ **these are still negative energy solutions in the sense that**

$$\hat{H}v_1 = i\frac{\partial}{\partial t}v_1 = -Ev_1$$

- **Solving the Dirac equation** $(i\gamma^\mu\partial_\mu - m)\psi = 0$

$$\Rightarrow \quad (-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m)v = 0$$

$$\boxed{(\gamma^\mu p_\mu + m)v = 0} \qquad \textbf{(D13)}$$

✶ **The Dirac equation in terms of momentum for ANTI-PARTICLES** **(c.f. D10)**

- **Proceeding as before:**

$$\left.\begin{aligned}(\vec{\sigma}.\vec{p})v_A &= (E-m)v_B\\ (\vec{\sigma}.\vec{p})v_B &= (E+m)v_A\end{aligned}\right\} \quad \textbf{etc., ...}$$

$$\Rightarrow \quad v_1 = N_1'\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\ \frac{-p_z}{E+m}\\ 0\\ 1\end{pmatrix}; \qquad v_2 = N_2'\begin{pmatrix}\frac{p_z}{E+m}\\ \frac{p_x+ip_y}{E+m}\\ 1\\ 0\end{pmatrix}$$

Particle and anti-particle Spinors

- **Four solutions of form:** $\psi_i = u_i(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$

$$\underbrace{u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix};\; u_2 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}}_{E=+\left|\sqrt{|\vec{p}|^2+m^2}\right|};\; \underbrace{u_3 = N\begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix};\; u_4 = N\begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}}_{E=-\left|\sqrt{|\vec{p}|^2+m^2}\right|}$$

- **Four solutions of form** $\psi_i = v_i(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$

$$\underbrace{v_1 = N\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix};\; v_2 = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}}_{E=+\left|\sqrt{|\vec{p}|^2+m^2}\right|};\; \underbrace{v_3 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \end{pmatrix};\; v_4 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \end{pmatrix}}_{E=-\left|\sqrt{|\vec{p}|^2+m^2}\right|}$$

- **Since we have a four component spinor, only four are linearly independent**
  - **Could choose to work with** $\{u_1,u_2,u_3,u_4\}$ **or** $\{v_1,v_2,v_3,v_4\}$ **or ...**
  - **Natural to use choose +ve energy solutions**

$$\{u_1,u_2,v_1,v_2\}$$

Wave-Function Normalisation

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$$

- From handout 1 want to normalise wave-functions to $2E$ particles per unit volume
- Consider $\psi = u_1 e^{+i(\vec{p}.\vec{r}-Et)}$

  Probability density $\rho = \psi^\dagger \psi = (\psi^*)^T \psi = u_1^\dagger u_1$

$$\begin{aligned} u_1^\dagger u_1 &= |N|^2 \left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2+p_y^2}{(E+m)^2} \right) \\ &= |N|^2 \left(\frac{(E+m)^2 + |\vec{p}|^2}{(E+m)^2} \right) = |N|^2 \left(\frac{(E+m)^2 + E^2 - m^2}{(E+m)^2} \right) \\ &= |N|^2 \frac{2E^2 + 2Em}{(E+m)^2} = |N|^2 \frac{2E}{E+m} \end{aligned}$$

which for the desired $2E$ particles per unit volume, requires that

$$\boxed{N = \sqrt{E+m}}$$

- Obtain same value of ***N*** for $u_1,\ u_2,\ v_1,\ v_2$

Charge Conjugation

- In the part II Relativity and Electrodynamics course it was shown that the motion of a charged particle in an electromagnetic field $A^\mu = (\phi, \vec{A})$ can be obtained by making the *minimal substitution*

$$\vec{p} \to \vec{p} - e\vec{A}; \quad E \to E - e\phi$$

with

$$\vec{p} = -i\vec{\nabla}; \quad E = i\partial/\partial t$$

this can be written

$$\boxed{\partial_\mu \to \partial_\mu + ieA_\mu}$$

and the Dirac equation becomes:

$$\boxed{\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0}$$

- Taking the complex conjugate and pre-multiplying by $-i\gamma^2$

⇨ $$-i\gamma^2\gamma^{\mu*}(\partial_\mu - ieA_\mu)\psi^* - m\gamma^2\psi^* = 0$$

But $\gamma^{0*} = \gamma^0;\ \gamma^{1*} = \gamma^1;\ \gamma^{2*} = -\gamma^2;\ \gamma^{3*} = \gamma^3$ and $\gamma^2\gamma^{\mu*} = -\gamma^\mu\gamma^2$

⇨ $$\gamma^\mu(\partial_\mu - ieA_\mu)\underbrace{i\gamma^2\psi^*}_{} + im\underbrace{i\gamma^2\psi^*}_{} = 0 \qquad \textbf{(D14)}$$

- Define the charge conjugation operator:

$$\boxed{\psi' = \hat{C}\psi = i\gamma^2\psi^*}$$

D14 becomes: $\boxed{\gamma^\mu(\partial_\mu - ieA_\mu)\psi' + im\psi' = 0}$

- **Comparing to the original equation**

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0$$

we see that the spinor ψ' describes a particle of the same mass but with opposite charge, i.e. an anti-particle !

$\hat{C}$ ➡ $\boxed{\textbf{particle spinor} \leftrightarrow \textbf{anti-particle spinor}}$

- **Now consider the action of $\hat{C}$ on the free particle wave-function:**

$$\psi = u_1 e^{i(\vec{p}.\vec{r}-Et)}$$

$$\psi' = \hat{C}\psi = i\gamma^2\psi^* = i\gamma^2 u_1^* e^{-i(\vec{p}.\vec{r}-Et)}$$

$$i\gamma^2 u_1^* = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}^* = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v_1$$

hence $\psi = u_1 e^{i(\vec{p}.\vec{r}-Et)} \xrightarrow{\hat{C}} \psi' = v_1 e^{-i(\vec{p}.\vec{r}-Et)}$

similarly $\psi = u_2 e^{i(\vec{p}.\vec{r}-Et)} \xrightarrow{\hat{C}} \psi' = v_2 e^{-i(\vec{p}.\vec{r}-Et)}$

★ **Under the charge conjugation operator the particle spinors u_1 and u_2 transform to the anti-particle spinors v_1 and v_2**

Using the anti-particle solutions

- There is a subtle but important point about the anti-particle solutions written as

$$\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$$

Applying normal QM operators for momentum and energy $\hat{p} = -i\vec{\nabla}, \quad \hat{H} = i\partial/\partial t$

gives $\hat{H}v_1 = i\partial v_1/\partial t = -Ev_1$ and $\hat{p}v_1 = -i\vec{\nabla}v_1 = -\vec{p}v_1$

★ But have defined solutions to have E>0

★ Hence the quantum mechanical operators giving the physical energy and momenta of the anti-particle solutions are:

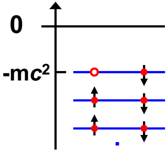
$$\hat{H}^{(v)} = -i\partial/\partial t \quad \text{and} \quad \hat{p}^{(v)} = i\vec{\nabla}$$

- Under the transformation $(E,\vec{p}) \rightarrow (-E,-\vec{p})$: $\vec{L} = \vec{r}\wedge\vec{p} \rightarrow -\vec{L}$

Conservation of total angular momentum $[H,\vec{L}+\vec{S}] = 0$ ➡ $\boxed{\hat{S}^{(v)} \rightarrow -\hat{S}}$

★ The physical spin of the anti-particle solutions is given by $\hat{S}^{(v)} = -\hat{S}$

In the hole picture:



A spin-up hole leaves the negative energy sea in a spin down state

Summary of Solutions to the Dirac Equation

- The normalised free **PARTICLE** solutions to the Dirac equation:

$$\psi = u(E,\vec{p})e^{+i(\vec{p}.\vec{r}-Et)} \quad \text{satisfy} \quad \boxed{(\gamma^\mu p_\mu - m)u = 0}$$

with

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}; \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

- The **ANTI-PARTICLE** solutions in terms of the physical energy and momentum:

$$\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)} \quad \text{satisfy} \quad \boxed{(\gamma^\mu p_\mu + m)v = 0}$$

with

$$v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \qquad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

For these states the spin is given by $\hat{S}^{(v)} = -\hat{S}$

- For both particle and anti-particle solutions: $E = \sqrt{|\vec{p}|^2 + m^2}$

(Now try question 7 – mainly about 4 vector current)

Connection between Dirac Hamiltonian and existence of Intrinsic Spin

- **For a Dirac spinor is orbital angular momentum a good quantum number?** **i.e. does $L=\vec{r}\wedge\vec{p}$ commute with the Hamiltonian?**

$$\begin{aligned}[H,\vec{L}] &= [\vec{\alpha}.\vec{p}+\beta m,\vec{r}\wedge\vec{p}] \\ &= [\vec{\alpha}.\vec{p},\vec{r}\wedge\vec{p}]\end{aligned}$$

Consider the x component of L:

$$\begin{aligned}[H,L_x] &= [\vec{\alpha}.\vec{p},(\vec{r}\wedge\vec{p})_x] \\ &= [\alpha_x p_x+\alpha_y p_y+\alpha_z p_z, yp_z-zp_y]\end{aligned}$$

The only non-zero contributions come from: $[x,p_x]=[y,p_y]=[z,p_z]=i$

$$\begin{aligned}[H,L_x] &= \alpha_y p_z[p_y,y]-\alpha_z p_y[p_z,z] \\ &= -i(\alpha_y p_z-\alpha_z p_y) \\ &= -i(\vec{\alpha}\wedge\vec{p})_x\end{aligned}$$

Therefore $$\boxed{[H,\vec{L}]=-i\vec{\alpha}\wedge\vec{p}} \qquad \text{(A.1)}$$

★**Hence the angular momentum does not commute with the Hamiltonian and is not a constant of motion**

Introduce a new 4x4 operator:

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli spin matrices: i.e.

$$\Sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \Sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad \Sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now consider the commutator

$$[H, \vec{\Sigma}] = [\vec{\alpha}.\vec{p} + \beta m, \vec{\Sigma}]$$

here $$[\beta, \vec{\Sigma}] = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} - \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = 0$$

and hence $$[H, \vec{\Sigma}] = [\vec{\alpha}.\vec{p}, \vec{\Sigma}]$$

Consider the x comp: $$\begin{aligned} [H, \Sigma_x] &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x] \\ &= p_x[\alpha_x, \Sigma_x] + p_y[\alpha_y, \Sigma_x] + p_z[\alpha_z, \Sigma_x] \end{aligned}$$

Taking each of the commutators in turn:

$$
\begin{aligned}
[\alpha_x, \Sigma_x] &= \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = 0 \\
[\alpha_y, \Sigma_x] &= \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sigma_y\sigma_y - \sigma_y\sigma_x \\ \sigma_y\sigma_x - \sigma_x\sigma_y & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z & 0 \end{pmatrix} \\
&= -2i\alpha_z \\
[\alpha_z, \Sigma_x] &= 2i\alpha_y
\end{aligned}
$$

Hence

$$
\begin{aligned}
[H, \Sigma_x] &= p_x[\alpha_x, \Sigma_x] + p_y[\alpha_y, \Sigma_x] + p_z[\alpha_z, \Sigma_x] \\
&= -2ip_y\alpha_x + 2ip_z\alpha_y \\
&= 2i(\vec{\alpha} \wedge \vec{p})_x
\end{aligned}
$$

$$[H, \vec{\Sigma}] = 2i\vec{\alpha} \wedge \vec{p}$$

- Hence the observable corresponding to the operator $\vec{\Sigma}$ is also **not** a constant of motion. However, referring back to **(A.1)**

$$[H,\vec{S}] = \frac{1}{2}[H,\vec{\Sigma}] = i\vec{\alpha}\wedge\vec{p} = -[H,\vec{L}]$$

Therefore: $$\boxed{[H,\vec{L}+\vec{S}] = 0}$$

- Because $$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

the commutation relationships for $\vec{S}$ are the same as for the $\vec{\sigma}$, e.g. $[S_x, S_y] = iS_z$. Furthermore both S^2 and S_z are diagonal

$$S^2 = \tfrac{1}{4}(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2) = \tfrac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad S_z = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Consequently $S^2\psi = S(S+1)\psi = \frac{3}{4}$ and for a particle travelling along the z direction $S_z\psi = \pm\frac{1}{2}\psi$

★ S has all the properties of spin in quantum mechanics and therefore the Dirac equation provides a natural account of the intrinsic angular momentum of fermions

Spin States

- In general the spinors u_1, u_2, v_1, v_2 **are not** Eigenstates of $\hat{S}_z$

$$\hat{S}_z = \tfrac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(Appendix II)

- However particles/anti-particles travelling in the z-direction: $p_z = \pm|\vec{p}|$

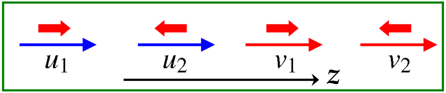
$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{\pm|\vec{p}|}{E+m} \\ 0 \end{pmatrix};\quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp|\vec{p}|}{E+m} \end{pmatrix};\quad v_1 = N\begin{pmatrix} 0 \\ \frac{\mp|\vec{p}|}{E+m} \\ 0 \\ 1 \end{pmatrix};\quad v_2 = N\begin{pmatrix} \frac{\pm|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

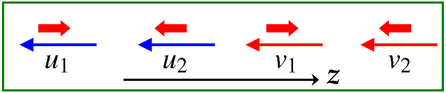
are Eigenstates of $\hat{S}_z$

$$\hat{S}_z u_1 = +\tfrac{1}{2}u_1 \qquad \hat{S}_z^{(v)} v_1 = -\hat{S}_z v_1 = +\tfrac{1}{2}v_1$$

$$\hat{S}_z u_2 = -\tfrac{1}{2}u_2 \qquad \hat{S}_z^{(v)} v_2 = -\hat{S}_z v_2 = -\tfrac{1}{2}v_2$$

Note the change of sign of $\hat{S}$ when dealing with antiparticle spinors





★ Spinors u_1, u_2, v_1, v_2 are only eigenstates of $\hat{S}_z$ for $p_z = \pm|\vec{p}|$

Pause for Breath…

- Have found solutions to the Dirac equation which are also eigenstates $\hat{S}_z$ but only for particles travelling along the z axis.
- Not a particularly useful basis
- More generally, want to label our states in terms of "good quantum numbers", i.e. a set of commuting observables.
- Can't use z component of spin: $[\hat{H}, \hat{S}_z] \neq 0$ (Appendix II)
- Introduce a new concept **"HELICITY"**

Helicity plays an important role in much that follows

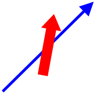
Helicity

★ **The component of a particles spin along its direction of flight is a good quantum number:**

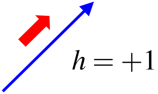
$$[\hat{H},\hat{S}.\hat{p}] = 0$$

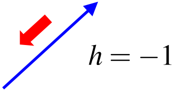
★ **Define the component of a particles spin along its direction of flight as HELICITY:**

$$h \equiv \frac{\vec{S}.\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}.\vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma}.\vec{p}}{|\vec{p}|}$$



- **If we make a measurement of the component of spin of a spin-half particle along any axis it can take two values $\pm 1/2$, consequently the eigenvalues of the helicity operator for a spin-half particle are: ± 1**





Often termed: **“right-handed”** **“left-handed”**

★ **NOTE: these are “RIGHT-HANDED” and LEFT-HANDED HELICITY eigenstates**

★ **In handout 4 we will discuss RH and LH CHIRAL eigenstates. Only in the limit $v \approx c$ are the HELICITY eigenstates the same as the CHIRAL eigenstates**

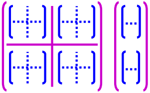
Helicity Eigenstates

★**Wish to find solutions of Dirac equation which are also eigenstates of Helicity:**

$$(\vec{\Sigma}.\hat{p})u_\uparrow = +u_\uparrow \qquad\qquad (\vec{\Sigma}.\hat{p})u_\downarrow = -u_\downarrow$$

where $u_\uparrow$ and $u_\downarrow$ are right and left handed helicity states and here $\hat{p}$ is the unit vector in the direction of the particle.

- **The eigenvalue equation:**

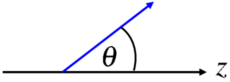


$$\begin{pmatrix} \vec{\sigma}.\hat{p} & 0 \\ 0 & \vec{\sigma}.\hat{p} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

gives the coupled equations:

$$\left.\begin{array}{rcl} (\vec{\sigma}.\hat{p})u_A & = & \pm u_A \\ (\vec{\sigma}.\hat{p})u_B & = & \pm u_B \end{array}\right\} \qquad \textbf{(D15)}$$

- **Consider a particle propagating in (θ,ϕ) direction**



$$\hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{\sigma}.\hat{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$\vec{\sigma}.\hat{p} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

- Writing either $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$ or $u_B = \begin{pmatrix} a \\ b \end{pmatrix}$ then (D15) gives the relation

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

(For helicity ± 1)

So for the components of BOTH u_A and u_B

$$\frac{b}{a} = \frac{\boxed{\pm 1} - \cos\theta}{\sin\theta} e^{i\phi}$$

- For the right-handed helicity state, i.e. helicity +1:

$$\frac{b}{a} = \frac{1-\cos\theta}{\sin\theta} e^{i\phi} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} e^{i\phi} = e^{i\phi}\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

➡ $$u_{A\uparrow} \propto \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_{B\uparrow} \propto \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

- Putting in the constants of proportionality gives:

$$u_\uparrow = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} \kappa_1 \cos\left(\frac{\theta}{2}\right) \\ \kappa_1 e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \kappa_2 \cos\left(\frac{\theta}{2}\right) \\ \kappa_2 e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

- From the Dirac Equation **(D12)** we also have

$$(\vec{\sigma}.\vec{p})u_A \quad = \quad (E+m)u_B$$

$$u_B = \frac{\vec{\sigma}.\vec{p}}{E+m}u_A = \frac{|\vec{p}|}{E+m}(\underbrace{\vec{\sigma}.\hat{p}}_{\boxed{\text{Helicity}}})u_A = \pm\frac{|\vec{p}|}{E+m}u_A \qquad \textbf{(D16)}$$

★**(D15)** determines the relative normalisation of u_A and u_B , i.e. here

$$u_B = +1\frac{|\vec{p}|}{E+m}u_A$$

➡ $$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

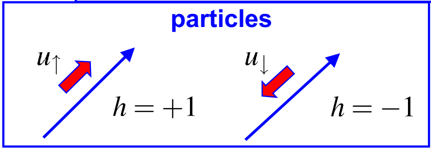
- The **negative helicity particle** state is obtained in the same way.
- The **anti-particle** states can also be obtained in the same manner although it must be remembered that $\hat{S}^{(v)} = -\hat{S}$

i.e. $$\hat{h}^{(v)} = -(\vec{\Sigma}.\hat{p}) \quad \longrightarrow \quad (\vec{\Sigma}.\hat{p})v_\uparrow = -v_\uparrow$$

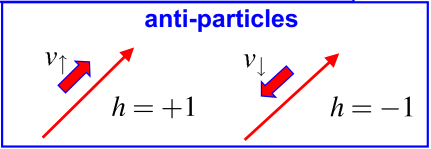
★ **The particle and anti-particle helicity eigenstates states are:**

$$u_{\uparrow}=N\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ e^{i\phi}\sin\left(\frac{\theta}{2}\right)\\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right)\\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right)\end{pmatrix}\qquad u_{\downarrow}=N\begin{pmatrix}-\sin\left(\frac{\theta}{2}\right)\\ e^{i\phi}\cos\left(\frac{\theta}{2}\right)\\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right)\\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right)\end{pmatrix}$$

$$v_{\uparrow}=N\begin{pmatrix}\frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right)\\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right)\\ -\sin\left(\frac{\theta}{2}\right)\\ e^{i\phi}\cos\left(\frac{\theta}{2}\right)\end{pmatrix}\qquad v_{\downarrow}=N\begin{pmatrix}\frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right)\\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right)\\ \cos\left(\frac{\theta}{2}\right)\\ e^{i\phi}\sin\left(\frac{\theta}{2}\right)\end{pmatrix}$$



particles



anti-particles

★ **For all four states, normalising to $2E$ particles/Volume again gives** $\boxed{N=\sqrt{E+m}}$

★ **The helicity eigenstates will be used extensively in the calculations that follow.**

Intrinsic Parity of Dirac Particles

non-examinable

★ **Before leaving the Dirac equation, consider parity**

★ **The parity operation is defined as spatial inversion through the origin:**

$$x' \equiv -x; \quad y' \equiv -y; \quad z' \equiv -z; \quad t' \equiv t$$

•**Consider a Dirac spinor,** $\psi(x,y,z,t)$ **which satisfies the Dirac equation**

$$i\gamma^1\frac{\partial\psi}{\partial x}+i\gamma^2\frac{\partial\psi}{\partial y}+i\gamma^3\frac{\partial\psi}{\partial z}-m\psi=-i\gamma^0\frac{\partial\psi}{\partial t} \qquad \text{(D17)}$$

•**Under the parity transformation:** $\psi'(x',y',z',t')=\hat{P}\psi(x,y,z,t)$

Try $\hat{P}=\gamma^0 \qquad \psi'(x',y',z',t')=\gamma^0\psi(x,y,z,t)$

$(\gamma^0)^2=1$ **so** $\psi(x,y,z,t)=\gamma^0\psi'(x',y',z',t')$

(D17) ➡

$$i\gamma^1\gamma^0\frac{\partial\psi'}{\partial x}+i\gamma^2\gamma^0\frac{\partial\psi'}{\partial y}+i\gamma^3\gamma^0\frac{\partial\psi'}{\partial z}-m\gamma^0\psi'=-i\gamma^0\gamma^0\frac{\partial\psi'}{\partial t}$$

•**Expressing derivatives in terms of the primed system:**

$$-i\gamma^1\gamma^0\frac{\partial\psi'}{\partial x'}-i\gamma^2\gamma^0\frac{\partial\psi'}{\partial y'}-i\gamma^3\gamma^0\frac{\partial\psi'}{\partial z'}-m\gamma^0\psi'=-i\gamma^0\gamma^0\frac{\partial\psi'}{\partial t'}$$

Since γ^0 **anti-commutes with** $\gamma^1,\ \gamma^2,\ \gamma^3$**:**

$$+i\gamma^0\gamma^1\frac{\partial\psi'}{\partial x'}+i\gamma^0\gamma^2\frac{\partial\psi'}{\partial y'}+i\gamma^0\gamma^3\frac{\partial\psi'}{\partial z'}-m\gamma^0\psi'=-i\frac{\partial\psi'}{\partial t'}$$

Pre-multiplying by γ^0 ⇨ $i\gamma^1\frac{\partial\psi'}{\partial x'}+i\gamma^2\frac{\partial\psi'}{\partial y'}+i\gamma^3\frac{\partial\psi'}{\partial z'}-m\psi'=-i\gamma^0\frac{\partial\psi'}{\partial t'}$

- Which is the Dirac equation in the new coordinates.

★ There for under parity transformations the form of the Dirac equation is unchanged **provided** Dirac spinors transform as

$$\boxed{\psi\rightarrow\hat{P}\psi=\pm\gamma^0\psi}$$

(note the above algebra doesn't depend on the choice of $\hat{P}=\pm\gamma^0$)

- For a particle/anti-particle at rest the solutions to the Dirac Equation are:

$$\psi=u_1e^{-imt};\ \psi=u_2e^{-imt};\ \psi=v_1e^{+imt};\psi=v_2e^{+imt}$$

with $$u_1=N\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\ u_2=N\begin{pmatrix}0\\1\\0\\0\end{pmatrix};\ v_1=N\begin{pmatrix}0\\0\\0\\1\end{pmatrix};\ v_2=N\begin{pmatrix}0\\0\\1\\0\end{pmatrix};$$

$$\hat{P}u_1=\pm\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\pm u_1$$ etc. ➡ $$\boxed{\begin{matrix}\hat{P}u_1=\pm u_1 & \hat{P}v_1=\mp v_1\\ \hat{P}u_2=\pm u_2 & \hat{P}v_2=\mp v_2\end{matrix}}$$

★ Hence an anti-particle at rest has opposite intrinsic parity to a particle at rest.

★ Convention: particles are chosen to have +ve parity; corresponds to choosing

$$\boxed{\hat{P}=+\gamma^0}$$

Summary

★ The formulation of relativistic quantum mechanics starting from the linear Dirac equation

$$\hat{H}\psi = (\vec{\alpha}.\vec{p}+\beta m)\psi = i\frac{\partial\psi}{\partial t}$$

➡ New degrees of freedom : found to describe Spin ½ particles

★ In terms of 4x4 gamma matrices the Dirac Equation can be written:

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

★ Introduces the 4-vector current and adjoint spinor:

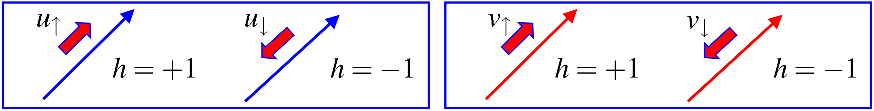
$$j^{\mu} = \psi^{\dagger}\gamma^{0}\gamma^{\mu}\psi = \overline{\psi}\gamma^{\mu}\psi$$

★ With the Dirac equation: forced to have two positive energy and two negative energy solutions

★ Feynman-Stückelberg interpretation: -ve energy particle solutions propagating backwards in time correspond to physical +ve energy anti-particles propagating forwards in time

$$u_1,\ u_2,\ v_1,\ v_2$$

★ **Most useful basis: particle and anti-particle helicity eigenstates**



★ **In terms of 4-component spinors, the charge conjugation and parity operations are:**

$$\boxed{\psi \rightarrow \hat{C}\psi = i\gamma^2\psi^{\dagger}} \qquad \boxed{\psi \rightarrow \hat{P}\psi = \gamma^0\psi}$$

> ★ **Now have all we need to know about a relativistic description of particles… next discuss particle interactions and QED.**

Appendix I : Dimensions of the Dirac Matrices

non-examinable

Starting from $\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t}$

For $\hat{H}$ to be Hermitian for all $\vec{p}$ requires $\alpha_i = \alpha_i^\dagger \quad \beta = \beta^\dagger$

To recover the KG equation: $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$

$$\beta\alpha_j + \alpha_j\beta = 0$$

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 0 \qquad (j \neq k)$$

Consider

with $B^\dagger B = 1$

$$\begin{aligned} Tr(B^\dagger AB) &= B^\dagger_{ij}A_{jk}B_{ki} \\ &= B_{ki}B^\dagger_{ij}A_{jk} \\ &= \delta_{jk}A_{jk} \\ &= Tr(A) \end{aligned}$$

Therefore

$$\begin{aligned} Tr(\alpha) &= Tr(\alpha_j^\dagger\alpha_i\alpha_j) \\ &= -Tr(\alpha_j^\dagger\alpha_j\alpha_i) \\ &= -Tr(\alpha_i) \\ \Rightarrow \quad Tr(\alpha_i) &= 0 \end{aligned}$$

(using commutation relation)

similarly

$$Tr(\beta) = 0$$

We can now show that the matrices are of even dimension by considering the eigenvalue equation, e.g. $\alpha\vec{x} = \lambda\vec{x}$

$$\vec{x}^{\dagger}\vec{x} = \vec{x}\alpha^{\dagger}\alpha\vec{x} = \lambda^{*}\lambda\vec{x}^{\dagger}\vec{x}$$

Eigenvalues of a Hermitian matrix are real so $\lambda^2 = 1 \quad \rightarrow \quad \lambda = \pm 1$

but $$Tr(\alpha) = \sum_i \lambda_i$$

Since the α_i, β are trace zero Hermitian matrices with eigenvalues of ± 1 they must be of even dimension

For **N=2** the 3 Pauli spin matrices satisfy

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 0 \quad (j \neq i)$$

But we require 4 anti-commuting matrices. Consequently the α_i, β of the Dirac equation must be of dimension **4**, **6**, **8**,….. The simplest choice for is to assume that the α_i, β are of dimension **4**.

Appendix III : Magnetic Moment

non-examinable

- In the part II Relativity and Electrodynamics course it was shown that the motion of a charged particle in an electromagnetic field $A^{\mu} = (\phi, \vec{A})$ can be obtained by making the *minimal substitution*

$$\vec{p} \rightarrow \vec{p} - q\vec{A}; \quad E \rightarrow E - q\phi$$

- Applying this to equations (D12)

$$\begin{aligned} (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_B &= (E - m - q\phi)u_A \\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (E + m - q\phi)u_B \end{aligned} \qquad \text{(A.2)}$$

Multiplying (A.2) by $(E + m - q\phi)$

$$\begin{aligned} (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(E + m - q\phi)u_B &= (E - m - q\phi)(E + m - q\phi)u_A \\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (T - q\phi)(T + 2m - q\phi)u_A \end{aligned} \qquad \text{(A.3)}$$

where kinetic energy $T = E - m$

- In the non-relativistic limit $T \ll m$ (A.3) becomes

$$\begin{aligned} (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &\approx 2m(T - q\phi)u_A \\ \left[(\vec{\sigma}.\vec{p})^2 - q(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p}) - q(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{A}) + q^2(\vec{\sigma}.\vec{A})^2\right] u_A &\approx 2m(T - q\phi)u_A \end{aligned} \qquad \text{(A.4)}$$

- **Now** $\vec{\sigma}.\vec{A} = \begin{pmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{pmatrix}$; $\vec{\sigma}.\vec{B} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$;

  which leads to $(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = \vec{A}.\vec{B} + i\vec{\sigma}.(\vec{A}\wedge\vec{B})$

  and $(\vec{\sigma}.\vec{A})^2 = |\vec{A}|^2$

- **The operator on the LHS of (A.4):**

$$\begin{aligned}
&= \vec{p}^2 - q\left[\vec{A}.\vec{p} + i\vec{\sigma}.\vec{A}\wedge\vec{p} + \vec{p}.\vec{A} + i\vec{\sigma}.\vec{p}\wedge\vec{A}\right] + q^2\vec{A}^2 \\
&= (\vec{p} - q\vec{A})^2 - iq\vec{\sigma}.\left[\vec{A}\wedge\vec{p} + \vec{p}\wedge\vec{A}\right] \\
&= (\vec{p} - q\vec{A})^2 - q^2\vec{\sigma}.\left[\vec{A}.\vec{\nabla} + \vec{\nabla}.\vec{A}\right] && \vec{p} = -i\vec{\nabla} \\
&= (\vec{p} - q\vec{A})^2 - q\vec{\sigma}.(\vec{\nabla}\wedge\vec{A}) && (\vec{\nabla}\wedge\vec{A})\psi = \vec{\nabla}\wedge(\vec{A}\psi) + \vec{A}\wedge(\vec{\nabla}\psi) \\
&= (\vec{p} - q\vec{A})^2 - q\vec{\sigma}.\vec{B} && \vec{B} = \vec{\nabla}\wedge\vec{A}
\end{aligned}$$

★**Substituting back into (A.4) gives the Schrödinger-Pauli equation for the motion of a non-relativisitic spin ½ particle in an EM field**

$$\boxed{\left[\frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{\sigma}.\vec{B} + q\phi\right]u_A = Tu_A}$$

$$\boxed{\left[\frac{1}{2m}(\vec{p}-q\vec{A})^2-\frac{q}{2m}\vec{\sigma}.\vec{B}+q\phi\right]u_A=Tu_A}$$

- **Since the energy of a magnetic moment in a field $\vec{B}$ is $-\vec{\mu}.\vec{B}$ we can identify the intrinsic magnetic moment of a spin ½ particle to be:**

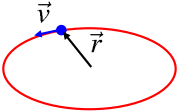
$$\vec{\mu}=\frac{q}{2m}\vec{\sigma}$$

In terms of the spin: $\vec{S}=\frac{1}{2}\vec{\sigma}$

$$\boxed{\vec{\mu}=\frac{q}{m}\vec{S}}$$

- **Classically, for a charged particle current loop**

$$\mu=\frac{q}{2m}\vec{L}$$



- **The intrinsic magnetic moment of a spin half Dirac particle is twice that expected from classical physics. This is often expressed in terms of the gyromagnetic ratio is g=2.**

$$\boxed{\vec{\mu}=g\frac{q}{2m}\vec{S}}$$

Generators of Lorentz Transformations I

It will shortly be seen that the quantities

$$(M^{\alpha\beta})^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta} \tag{7}$$

or the equivalent (but less symmetric) quantities

$$(M^{\alpha\beta})^{\mu}{}_{\nu} = g^{\mu\alpha} \delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu} g^{\mu\beta} \tag{8}$$

are generators of Lorentz Transformations. The indices $\alpha\beta$ choose between generators $M^{\alpha\beta}$, while ${}^{\mu}{}_{\nu}$ in $(M^{\alpha\beta})^{\mu}{}_{\nu}$ are there to act on vector indices. Evident antisymmetry in the $\alpha\beta$ of (7) means that there are only six independent non-zero generators. Suppressing the vector indices (taken to be ${}^{\mu}{}_{\nu}$) and taking $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$ the six independent generators are:

$$K_1 = M^{01} = -M^{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_2 = M^{02} = -M^{20} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_3 = M^{03} = -M^{30} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Generators of Lorentz Transformations II

and

$$J_1 = M^{23} = -M^{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}$$

$$J_2 = M^{31} = -M^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J_3 = M^{12} = -M^{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

or, for short:

$$J_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$$
$$K_i = M^{0i}.$$

[Aside: The generators obey commutation relations

$$[J_i, J_j] = \epsilon_{ijk} J_k, \qquad [J_i, K_j] = \epsilon_{ijk} K_k, \qquad [K_i, K_j] = -\epsilon_{ijk} J_k.$$

Generators of Lorentz Transformations III

The first of these says that the J's generate rotations in three-dimensional space and fixes the overall sign of the *J*s. The second says the *K*s transform as a vector under rotations. End of aside]
With above definition[2] one could represent and arbitrary Lorentz transformation (boost, rotation or both) as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

with

$$\begin{aligned} \Lambda^{\mu}{}_{\nu} &= \left(\exp\left[\frac{1}{2} w_{\alpha\beta} (M^{\alpha\beta})^{\bullet}{}_{\bullet} \right] \right)^{\mu}{}_{\nu} && (9) \\ &= \delta^{\mu}_{\nu} + \frac{1}{2} \omega_{\alpha\beta} (M^{\alpha\beta})^{\mu}{}_{\nu} + O(\omega^2) && (10) \end{aligned}$$

using a set of parameters $w_{\alpha\beta}$ which may as well be antisymmetric in $\alpha\beta$ (since any symmetric part would not participate in (10) on account of the $(\alpha \leftrightarrow \beta)$-antisymmetry of $M^{\alpha\beta}$) and so contain six independent degrees of freedom (controlling three boosts and three rotations) as required. In most of the proofs which follow we use the infinitesimal transformations to first order in ω since if some properties can be proved for infinitesimal transformations then it is always be possible to generalise that result to the exponential form for a finite transformation.

---

[2] Compare to similar but slightly different sign/index conventions in `http://www.phys.ufl.edu/~fry/6607/lorentz.pdf`.

Why do $(M^{\alpha\beta})^{\mu}{}_{\nu}$ generate Lorentz transformations? I

Lorentz transformations should be continuously connected to the identity (which (10) is, when $\omega_{\alpha\beta} = 0$) and should preserve inner products. The transformation in Eq. (10) preserves inner products because:

$$\begin{aligned}
x' \cdot y' &= g_{\mu\nu} x'^{\mu} y'^{\nu} \\
&= g_{\mu\nu} (\Lambda^{\mu}{}_{\sigma} x^{\sigma})(\Lambda^{\nu}{}_{\tau} y^{\tau}) \\
&= g_{\mu\nu} (\delta^{\mu}_{\sigma} + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}{}_{\sigma})(\delta^{\nu}_{\tau} + \frac{1}{2}\omega_{\bar{\alpha}\bar{\beta}}(M^{\bar{\alpha}\bar{\beta}})^{\nu}{}_{\tau}) x^{\sigma} y^{\tau} + O(\omega)^2 \\
&= \left[g_{\sigma\tau} + \frac{1}{2}\left(\omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} + \omega_{\bar{\alpha}\bar{\beta}}(M^{\bar{\alpha}\bar{\beta}})_{\sigma\tau} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \\
&= \left[g_{\sigma\tau} + \frac{1}{2}\left(\omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} + \omega_{\alpha\beta}(M^{\alpha\beta})_{\sigma\tau} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \qquad \text{relabelling} \\
&= \left[g_{\sigma\tau} + \frac{1}{2}\left(\omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} - \omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \qquad \text{antisymmetry of } M \\
&= g_{\sigma\tau} x^{\sigma} y^{\tau} + O(\omega^2) \\
&= x \cdot y + O(\omega^2).
\end{aligned}$$

Why do $(M^{\alpha\beta})^{\mu}{}_{\nu}$ generate Lorentz transformations? II

If the above argument seems too abstract, a more concrete way of checking that we have generators of Lorentz transformations might instead be to compute

$$\exp(\eta K_1) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

as this will be recognised by some as a boost in the positive x-direction with rapidity η (that is with $\cosh\eta = \gamma$ and $\sinh\eta = \beta\gamma$) while

$$\exp(\theta J_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{12}$$

will be recognised by most as a rotation by an angle θ about the x-axis.

Lorentz covariance of the Dirac equation I

If the Dirac Equation:

$$i\gamma^{\mu}\partial_{\mu}\psi = m\psi \tag{13}$$

is to be Lorentz covariant, there would have to exist a matrix $S(\Lambda)$ such that $\psi' = S(\Lambda)\psi$ is the solution of the Lorentz transformed Dirac Equation

$$i\gamma^{\mu}\partial'_{\mu}\psi' = m\psi'. \tag{14}$$

Equation (14) implies

$$i\gamma_{\mu}\partial'^{\mu}\psi' = m\psi' \tag{15}$$

and so

$$i\gamma_{\mu}\Lambda^{\mu}{}_{\nu}\partial^{\nu}S(\Lambda)\psi = mS(\Lambda)\psi \tag{16}$$

and so since $S(\Lambda)$ is independent of position

$$i\gamma_{\mu}S(\Lambda)\Lambda^{\mu}{}_{\nu}\partial^{\nu}\psi = S(\Lambda)m\psi \tag{17}$$

which using (13) becomes

$$i\gamma_{\mu}S(\Lambda)\Lambda^{\mu}{}_{\nu}\partial^{\nu}\psi = S(\Lambda)i\gamma^{\mu}\partial_{\mu}\psi$$

Lorentz covariance of the Dirac equation II

and hence

$$i\gamma^{\mu} S(\Lambda)\Lambda_{\mu}{}^{\nu}\partial_{\nu}\psi = S(\Lambda) i\gamma^{\nu}\partial_{\nu}\psi$$

or

$$i\left[\gamma^{\mu} S(\Lambda)\Lambda_{\mu}{}^{\nu} - S(\Lambda)\gamma^{\nu}\right]\partial_{\nu}\psi = 0. \tag{18}$$

Therefore, if we can show that there exists a matrix $S(\Lambda)$ satisfying

$$\gamma^{\mu} S(\Lambda)\Lambda_{\mu}{}^{\nu} = S(\Lambda)\gamma^{\nu} \tag{19}$$

we will have found a solution to (18) and thus will have found that the Dirac Equation is Lorentz covariant as desired. Thought it would be entirely possible to work directly with (19) it is perhaps nicer to bring both S matrices to the left hand side

$$S^{-1}(\Lambda)\gamma^{\mu} S(\Lambda)\Lambda_{\mu}{}^{\nu} = \gamma^{\nu}$$

and then use the identity

$$\Lambda_{\mu}{}^{\nu}\Lambda^{\sigma}{}_{\nu} \equiv \delta^{\sigma}_{\mu} \tag{20}$$

so that (19) ends up being written in the more common and (perhaps) more suggestive and useful form:

$$S^{-1}(\Lambda)\gamma^{\sigma} S(\Lambda) = \Lambda^{\sigma}{}_{\nu}\gamma^{\nu}. \tag{21}$$

Lorentz covariance of the Dirac equation III

[Aside: Here is (for infinitesimal Lorentz transformations) a proof of the identity (20):

$$\begin{aligned}
\Lambda_\mu{}^\nu \Lambda^\sigma{}_\nu &= \left(g_\mu{}^\nu + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})_\mu{}^\nu\right)\left(g^\sigma{}_\nu + \frac{1}{2}\omega_{\bar\alpha\bar\beta}(M^{\bar\alpha\bar\beta})^\sigma{}_\nu\right) + O(\omega^2)\\
&= \delta^\sigma_\mu + \frac{1}{2}\left[\omega_{\alpha\beta}(M^{\alpha\beta})_\mu{}^\sigma + \omega_{\bar\alpha\bar\beta}(M^{\bar\alpha\bar\beta})^\sigma{}_\mu\right] + O(\omega^2)\\
&= \delta^\sigma_\mu + \frac{1}{2}\left[\omega_{\alpha\beta}(M^{\alpha\beta})_\mu{}^\sigma + \omega_{\alpha\beta}(M^{\alpha\beta})^\sigma{}_\mu\right] + O(\omega^2) \qquad \text{(relabelling)}\\
&= \delta^\sigma_\mu + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})_\mu{}^\sigma + (M^{\alpha\beta})^\sigma{}_\mu\right] + O(\omega^2) \qquad \text{(factorising)}\\
&= \delta^\sigma_\mu + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})^{\tau\sigma} + (M^{\alpha\beta})^{\sigma\tau}\right]g_{\mu\tau} + O(\omega^2) \qquad \text{(tidying)}\\
&= \delta^\sigma_\mu + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})^{\tau\sigma} - (M^{\alpha\beta})^{\tau\sigma}\right]g_{\mu\tau} + O(\omega^2) \qquad \text{(antisymmetry of } M\text{)}\\
&= \delta^\sigma_\mu + O(\omega^2).
\end{aligned}$$

End of aside]

Lemma

A valid choice of $S(\Lambda)$ (for an infinitesimal Lorentz transformation) is given by:

$$S(\Lambda) = 1 + \frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta + O(\omega^2). \qquad (22)$$

Lorentz covariance of the Dirac equation IV

Proof.

$$
\begin{aligned}
S^{-1}(\Lambda)\gamma^\sigma S(\Lambda) &= \left(1 - \frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\right)\gamma^\sigma\left(1 + \frac{1}{4}\omega_{\bar\alpha\bar\beta}\gamma^{\bar\alpha}\gamma^{\bar\beta}\right) + O(\omega^2) \\
&= \gamma^\sigma + \frac{1}{4}\left(\omega_{\bar\alpha\bar\beta}\gamma^\sigma\gamma^{\bar\alpha}\gamma^{\bar\beta} - \omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\gamma^\sigma\right) + O(\omega^2) \\
&= \gamma^\sigma + \frac{1}{4}\omega_{\alpha\beta}\left(\gamma^\sigma\gamma^\alpha\gamma^\beta - \gamma^\alpha\gamma^\beta\gamma^\sigma\right) + O(\omega^2) \\
&= \gamma^\sigma + \frac{1}{4}\omega_{\alpha\beta}\left((\gamma^\sigma\gamma^\alpha + \gamma^\alpha\gamma^\sigma)\gamma^\beta - \gamma^\alpha(\gamma^\sigma\gamma^\beta + \gamma^\beta\gamma^\sigma)\right) + O(\omega^2) \\
&= \gamma^\sigma + \frac{1}{4}\omega_{\alpha\beta}\left(2g^{\sigma\alpha}\gamma^\beta - \gamma^\alpha 2g^{\sigma\beta}\right) + O(\omega^2) \qquad \text{since } \{\gamma^\mu, \gamma^\nu\} \equiv 2g^{\mu\nu} \\
&= \left(\delta^\sigma_\nu + \frac{1}{2}\omega_{\alpha\beta}\left(g^{\sigma\alpha}\delta^\beta_\nu - \delta^\alpha_\nu g^{\sigma\beta}\right)\right)\gamma^\nu + O(\omega^2) \\
&= \left(\delta^\sigma_\nu + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^\sigma{}_\nu\right)\gamma^\nu + O(\omega^2) \qquad \text{using (8)} \\
&= \Lambda^\sigma{}_\nu\gamma^\nu + O(\omega^2) \qquad \text{using (10).}
\end{aligned}
$$

$\square$

Not examinable

Lorentz covariance of the Dirac equation V

[Aside: Since $\gamma^\alpha\gamma^\beta = \frac{1}{2}\{\gamma^\alpha, \gamma^\beta\} + \frac{1}{2}[\gamma^\alpha, \gamma^\beta]$ we can also rewrite (22) in the more frequently seen (conventional) form:

$$S(\Lambda) = 1 + \frac{1}{8}\omega_{\alpha\beta}[\gamma^\alpha, \gamma^\beta] + O(\omega^2). \tag{23}$$

End of aside]

Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. I

Each of the expressions $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$ is of the form $\overline{\phi}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi$. To understand how any of them is affected by a Lorentz transformation it is therefore interesting to consider the following set of manipulations:[3]

$$\begin{aligned}\overline{\phi'}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi' &= \overline{(S(\Lambda)\phi)}[\gamma^\mu\gamma^\nu\cdots\gamma^\tau](S(\Lambda)\psi)\\ &= \phi^\dagger S^\dagger(\Lambda)\gamma^0[\gamma^\mu S(\Lambda)S^{-1}(\Lambda)\gamma^\nu S(\Lambda)\cdots S^{-1}(\Lambda)\gamma^\tau]S(\Lambda)\psi\\ &= \phi^\dagger S^\dagger(\Lambda)\gamma^0 S(\Lambda)(S^{-1}(\Lambda)\gamma^\mu S(\Lambda))(S^{-1}(\Lambda)\gamma^\nu S(\Lambda))\cdots(S^{-1}(\Lambda)\gamma^\tau S(\Lambda))\psi\\ &= \phi^\dagger S^\dagger(\Lambda)\gamma^0 S(\Lambda)(\Lambda^\mu{}_\alpha\gamma^\alpha)(\Lambda^\nu{}_\beta\gamma^\gamma)\cdots(\Lambda^\tau{}_\lambda\gamma^\lambda)\psi \qquad \text{using (21)}\end{aligned}$$

which itself suggests that if we can show that

$$S^\dagger(\Lambda)\gamma^0 S(\Lambda) = \gamma^0 \tag{24}$$

then we will have proved that

$$\overline{\phi'}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi' = \overline{\phi}(\Lambda^\mu{}_\alpha\gamma^\alpha)(\Lambda^\nu{}_\beta\gamma^\gamma)\cdots(\Lambda^\tau{}_\lambda\gamma^\lambda)\psi$$

which will itself have showed that each of the expressions under consideration transforms like a tensor of the appropriate rank.

Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. II

We must therefore prove (24). To do so is a two-stage process. First we compute $S^\dagger(\Lambda)$. Then we combine it with $\gamma^0 S(\Lambda)$. Starting with (22):

$$\begin{aligned}
S^\dagger(\Lambda) &= \left[1 + \frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\right]^\dagger + O(\omega^2) \\
&= 1 + \frac{1}{4}\omega_{\alpha\beta}(\gamma^\alpha\gamma^\beta)^\dagger + O(\omega^2) \qquad (\omega_{\alpha\beta} \text{ are real}) \\
&= 1 + \frac{1}{4}\omega_{\alpha\beta}(\gamma^\beta)^\dagger(\gamma^\alpha)^\dagger + O(\omega^2) \\
&= 1 + \frac{1}{4}\omega_{\alpha\beta}(\gamma^0\gamma^\beta\gamma^0)(\gamma^0\gamma^\alpha\gamma^0) + O(\omega^2) \\
&= 1 + \frac{1}{4}\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0 + O(\omega^2) \qquad (25)
\end{aligned}$$

Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. III

from which we can deduce (using (22)) that

$$\begin{aligned}
S^\dagger(\Lambda)\gamma^0 S(\Lambda) &= \left(1+\frac{1}{4}\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0\right)\gamma^0\left(1+\frac{1}{4}\omega_{\bar{\alpha}\bar{\beta}}\gamma^{\bar{\alpha}}\gamma^{\bar{\beta}}\right)+O(\omega^2)\\
&=\gamma^0+\frac{1}{4}\left(\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0\gamma^0+\omega_{\bar{\alpha}\bar{\beta}}\gamma^0\gamma^{\bar{\alpha}}\gamma^{\bar{\beta}}\right)+O(\omega^2)\\
&=\gamma^0\left[1+\frac{1}{4}\left(\omega_{\alpha\beta}\gamma^\beta\gamma^\alpha+\omega_{\beta\alpha}\gamma^\beta\gamma^\alpha\right)\right]+O(\omega^2) \qquad ((\bar{\alpha},\bar{\beta})\to(\beta,\alpha))\\
&=\gamma^0\left[1+0\right]\psi+O(\omega^2) \qquad (\omega_{\alpha\beta}=-\omega_{\beta\alpha})\\
&=\gamma^0+O(\omega^2)
\end{aligned}$$

verifying (24) as required. This completes our proof that:

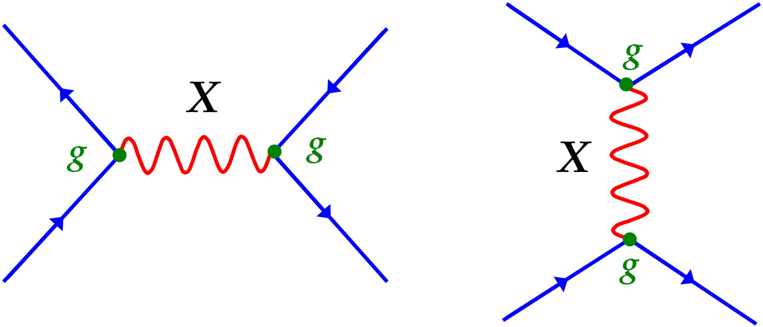
- $\overline{\phi}\psi$ is Lorentz invariant scalar,
- $\overline{\phi}\gamma^\mu\psi$ transforms as a Lorentz vector, and
- $\overline{\phi}\gamma^\mu\gamma^\nu\psi$ transforms as a second-rank tensor, *etc*.

---

[3] These manipulations may look complex but they really only consist of inserting lots of 'ones' in form '$S(\Lambda)S^{-1}(\Lambda)$' at the right places, using $\overline{\phi}\equiv\phi^\dagger\gamma^0$ and using (21) many times.

Particle Physics

Dr Lester



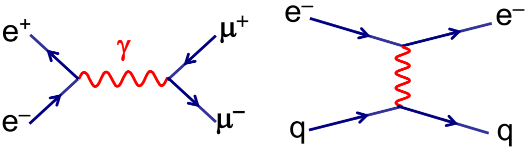
Handout 3 : Interaction by Particle Exchange and QED

Recap

★ **Working towards a proper calculation of decay and scattering processes**

Initially concentrate on:

- $e^+e^- \rightarrow \mu^+\mu^-$
- $e^-q \rightarrow e^-q$



⅄ **In Handout 1 covered the relativistic calculation of particle decay rates and cross sections**

$$\sigma \propto \frac{|M|^2}{\text{flux}} \times (\text{phase space})$$

⅄ **In Handout 2 covered relativistic treatment of spin-half particles**

Dirac Equation

⅄ **This handout concentrate on the Lorentz Invariant Matrix Element**

- **Interaction by particle exchange**
- **Introduction to Feynman diagrams**
- **The Feynman rules for QED**

Interaction by Particle Exchange

- **Calculate transition rates from Fermi′s Golden Rule**

$$\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)}$$

where T_{fi} is perturbation expansion for the Transition Matrix Element

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \ldots$$

- **"Classical picture" – particles act as sources for fields which give rise a potential in which other particles scatter – "action at a distance"**
- **"Quantum Field Theory picture" – forces arise due to the exchange of virtual particles. No action at a distance + forces between particles now due to particles**

We now go to `https://www.hep.phy.cam.ac.uk/~lester/teaching/partIIIparticles/Propagators.pdf` to provide some motivation for why matrix elements of the form

$$M_{fi} = \frac{g^2}{q^2 - m_x^2}$$

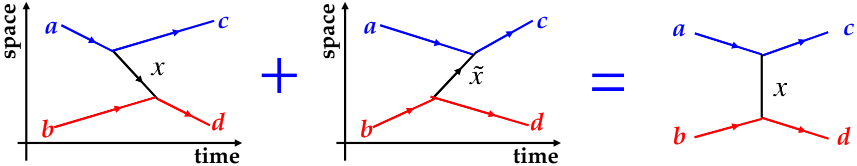
might arise in scattering between two particles when this scattering is caused by the exchange of a *virtual particle* whose non-virtual mass (i.e. if it were it on shell) is m_x. Here q^μ is the four momentum of the virtual particle.

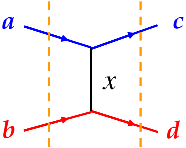
➡ $$\boxed{M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}}$$

- After summing over all possible time orderings, M_{fi} is (as anticipated) **Lorentz invariant**. This is a remarkable result – the sum over all time orderings gives a frame independent matrix element.
- Exactly the same result would have been obtained by considering the annihilation process

Feynman Diagrams

- The sum over all possible time-orderings is represented by a **FEYNMAN diagram**





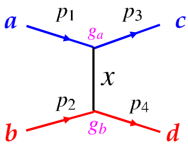
In a Feynman diagram:

- the LHS represents the initial state
- the RHS is the final state
- everything in between is "how the interaction happened"

- It is important to remember that energy and momentum are conserved at each interaction vertex in the diagram.
- The factor $1/(q^2 - m_x^2)$ is the propagator; it arises naturally from the above discussion of interaction by particle exchange

★**The matrix element:** $M_{fi} = \dfrac{g_a g_b}{q^2 - m_x^2}$ **depends on:**

- **The fundamental strength of the interaction at the two vertices** g_a, g_b
- **The four-momentum,** q **, carried by the (virtual) particle which is determined from energy/momentum conservation at the vertices. Note** q^2 **can be either positive or negative.**



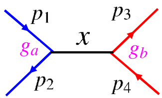
Here $q^2=(p_1-p_3)^2=(p_4-p_2)^2=t$

"t-channel"

For elastic scattering: $p_1 = (E, \vec{p}_1); \quad p_3 = (E, \vec{p}_3)$

$$q^2 = (E-E)^2 - (\vec{p}_1 - \vec{p}_3)^2$$

$\boxed{q^2 < 0}$ **termed "space-like"**



Here $q^2=(p_1+p_2)^2=(p_3+p_4)^2=s$

"s-channel"

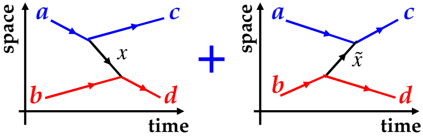
In CoM: $p_1 = (E, \vec{p}); \quad p_2 = (E, -\vec{p})$

$$q^2 = (E+E)^2 - (\vec{p}-\vec{p})^2 = 4E^2$$

$\boxed{q^2 > 0}$ **termed "time-like"**

Virtual Particles

"Time-ordered QM"



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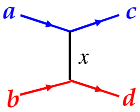
Feynman diagram



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Momentum conserved at vertices
- Energy **not** conserved at vertices
- Exchanged particle **"on mass shell"**

$$E_x^2 - |\vec{p}_x|^2 = m_x^2$$

- Momentum **AND** energy conserved at interaction vertices
- Exchanged particle **"off mass shell"**

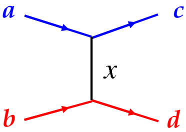
$$E_x^2 - |\vec{p}_x|^2 = q^2 \neq m_x^2$$

VIRTUAL PARTICLE

Aside: V(r) from Particle Exchange

★ **Can view the scattering of an electron by a proton at rest in two ways:**

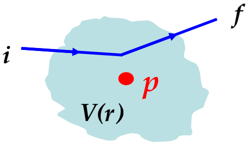
- **Interaction by particle exchange in 2nd order perturbation theory.**



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- **Could also evaluate the same process in first order perturbation theory treating proton as a fixed source of a field which gives rise to a potential V(r)**

$$M = \langle \psi_f | V(r) | \psi_i \rangle$$



Obtain same expression for M_{fi} using

$$V(r) = g_a g_b \frac{e^{-mr}}{r}$$

YUKAWA potential

★ **In this way can relate potential and forces to the particle exchange picture**

★ **However, scattering from a fixed potential $V(r)$ is not a relativistic invariant view**

Quantum Electrodynamics (QED)

★Now consider the interaction of an electron and tau lepton by the exchange of a photon. Although the general ideas we applied previously still hold, we now have to account for the spin of the electron/tau-lepton and also the spin (polarization) of the virtual photon.

(Non-examinable)

- The basic interaction between a photon and a charged particle can be introduced by making the minimal substitution (part II electrodynamics)

$$\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi$$

(here $q =$ charge)

In QM: $$\vec{p} = -i\vec{\nabla}; \quad E = i\partial/\partial t$$

Therefore make substitution: $i\partial_\mu \to i\partial_\mu - qA_\mu$

where $A_\mu = (\phi, -\vec{A}); \quad \partial_\mu = (\partial/\partial t, +\vec{\nabla})$

- The Dirac equation:

$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \quad \Rightarrow \quad \gamma^\mu \partial_\mu \psi + iq\gamma^\mu A_\mu \psi + im\psi = 0$$

$$(\times i) \quad \Rightarrow \quad i\gamma^0 \frac{\partial \psi}{\partial t} + i\vec{\gamma}.\vec{\nabla}\psi - q\gamma^\mu A_\mu \psi - m\psi = 0$$

$$
\begin{aligned}
i\gamma^0\frac{\partial\psi}{\partial t} = \gamma^0\hat{H}\psi &= m\psi - i\vec{\gamma}.\vec{\nabla}\psi + q\gamma^\mu A_\mu\psi \\
\times\gamma^0: \qquad \hat{H}\psi &= \underbrace{(\gamma^0 m - i\gamma^0\vec{\gamma}.\vec{\nabla})}_{\text{Combined rest mass + K.E.}}\psi + \underbrace{q\gamma^0\gamma^\mu A_\mu}_{\text{Potential energy}}\psi
\end{aligned}
$$

- **We can identify the potential energy of a charged spin-half particle in an electromagnetic field as:**

$$\boxed{\hat{V}_D = q\gamma^0\gamma^\mu A_\mu}$$

(note the A_0 term is just: $q\gamma^0\gamma^0 A_0 = q\phi$)

- **The final complication is that we have to account for the photon polarization states.**

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(\vec{p}.\vec{r}-Et)}$$

e.g. for a real photon propagating in the z direction we have two orthogonal transverse polarization states

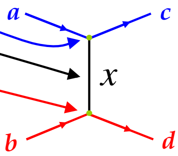
$$\varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Could equally have chosen circularly polarized states

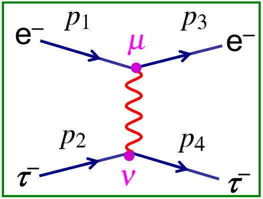
(reasons for choosing these particular polarizations may be found on pages 290-295)

- Previously with the example of a simple spin-less interaction we had:

$$M = \underset{\overset{\shortparallel}{g_a}}{\langle\psi_c|V|\psi_a\rangle}\,\frac{1}{q^2-m_x^2}\,\underset{\overset{\shortparallel}{g_b}}{\langle\psi_d|V|\psi_b\rangle}$$



★ In QED we could again go through the procedure of summing the time-orderings using Dirac spinors and the expression for $\hat{V}_D$. If we were to do this, remembering to sum over all photon polarizations, we would obtain:

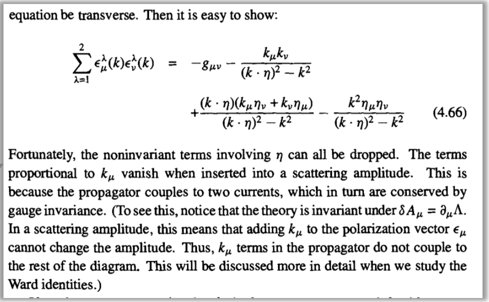


$$M = \underbrace{\left[u_e^\dagger(p_3)q_e\gamma^0\gamma^\mu u_e(p_1)\right]}_{\text{Interaction of } e^- \text{ with photon}}\ \underbrace{\sum_{\lambda=1}^{2}\frac{\varepsilon_\mu^\lambda(\varepsilon_\nu^\lambda)^*}{q^2}}_{\text{Massless photon propagator summing over polarizations}}\ \underbrace{\left[u_\tau^\dagger(p_4)q_\tau\gamma^0\gamma^\nu u_\tau(p_2)\right]}_{\text{Interaction of } \tau^- \text{ with photon}}$$

- All the physics of **QED** is in the above expression !

- The sum $\sum_\lambda \varepsilon^\lambda_\mu (\varepsilon^\lambda_\nu)^*$ over the polarizations of the virtual photon is not $-g_{\mu\nu}$, but in matrix elements it can be replaced by $-g_{\mu\nu}$ in certain circumstances.

(Beyond this course, but see, say, Michio Kaku's "Quantum Field Theory: a modern introduction")



equation be transverse. Then it is easy to show:

$$\sum_{\lambda=1}^{2} \epsilon^\lambda_\mu(k)\epsilon^\lambda_\nu(k) = -g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot\eta)^2 - k^2} + \frac{(k\cdot\eta)(k_\mu\eta_\nu + k_\nu\eta_\mu)}{(k\cdot\eta)^2 - k^2} - \frac{k^2\eta_\mu\eta_\nu}{(k\cdot\eta)^2 - k^2} \qquad (4.66)$$

Fortunately, the noninvariant terms involving η can all be dropped. The terms proportional to k_μ vanish when inserted into a scattering amplitude. This is because the propagator couples to two currents, which in turn are conserved by gauge invariance. (To see this, notice that the theory is invariant under $\delta A_\mu = \partial_\mu \Lambda$. In a scattering amplitude, this means that adding k_μ to the polarization vector ϵ_μ cannot change the amplitude. Thus, k_μ terms in the propagator do not couple to the rest of the diagram. This will be discussed more in detail when we study the Ward identities.)

(end of non-examinable section)

Therefore the invariant matrix element becomes: $M = [u_e^\dagger(p_3) q_e \gamma^0\gamma^\mu u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [u_\tau^\dagger(p_4) q_\tau \gamma^0\gamma^\nu u_\tau(p_2)]$

- Using the definition of the adjoint spinor $\overline{\psi} = \psi^\dagger\gamma^0$

$$\boxed{M = [\overline{u}_e(p_3) q_e \gamma^\mu u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [\overline{u}_\tau(p_4) q_\tau \gamma^\nu u_\tau(p_2)]}$$

★ This is a remarkably simple expression ! It is shown in Appendix V of Handout 2 that $\overline{u}_1\gamma^\mu u_2$ transforms as a four vector. (page 109)

★ Writing $j_e^\mu = \overline{u}_e(p_3)\gamma^\mu u_e(p_1) \qquad j_\tau^\nu = \overline{u}_\tau(p_4)\gamma^\nu u_\tau(p_2)$

we have $M = -q_e q_\tau \frac{j_e \cdot j_\tau}{q^2}$ showing that M is Lorentz Invariant

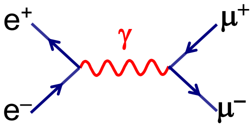
Feynman Rules for QED

- It should be remembered that the expression

$$M = [\overline{u}_e(p_3)q_e\gamma^{\mu}u_e(p_1)]\frac{-g_{\mu\nu}}{q^2}[\overline{u}_\tau(p_4)q_\tau\gamma^{\nu}u_\tau(p_2)]$$

hides a lot of complexity. We have summed over all possible time-orderings and summed over all polarization states of the virtual photon. If we are then presented with a new Feynman diagram we don't want to go through the full calculation again.

Fortunately this isn't necessary – can just write down matrix element using a set of simple rules

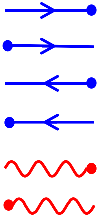


Basic Feynman Rules:

- Propagator factor for each internal line
  (i.e. each internal virtual particle)
- Dirac Spinor for each external line
  (i.e. each real incoming or outgoing particle)
- Vertex factor for each vertex

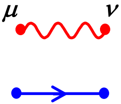
Basic Rules for QED

- **External Lines**



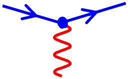
spin 1/2	incoming particle	$u(p)$
	outgoing particle	$\bar{u}(p)$
	incoming antiparticle	$\bar{v}(p)$
	outgoing antiparticle	$v(p)$
spin 1	incoming photon	$\varepsilon^{\mu}(p)$
	outgoing photon	$\varepsilon^{\mu}(p)^{*}$

- **Internal Lines (propagators)**



spin 1	photon	$-\frac{ig_{\mu\nu}}{q^2}$
spin 1/2	fermion	$\frac{i(\gamma^{\mu}q_{\mu}+m)}{q^2-m^2}$

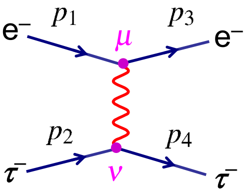
- **Vertex Factors**

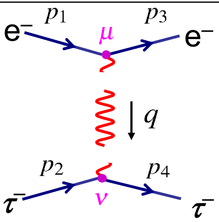


spin 1/2	fermion (charge -|e|)	$ie\gamma^{\mu}$

- **Matrix Element** $-iM$ = product of all factors

e.g.





$$\overline{u}_e(p_3)[ie\gamma^\mu]u_e(p_1)$$

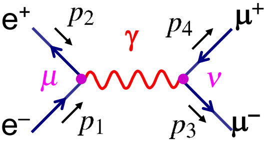
$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\overline{u}_\tau(p_4)[ie\gamma^\nu]u_\tau(p_2)$$

$$-iM = [\overline{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}_\tau(p_4)ie\gamma^\nu u_\tau(p_2)]$$

- **Which is the same expression as we obtained previously**

e.g.



$$-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$$

Note:

- **At each vertex the adjoint spinor is written first**
- **Each vertex has a different index**
- **The $g_{\mu\nu}$ of the propagator connects the indices at the vertices**

Summary

- ★ **Interaction by particle exchange naturally gives rise to Lorentz Invariant Matrix Element of the form**

$$M_{fi} \quad = \quad \frac{g_a g_b}{q^2 - m_x^2}$$

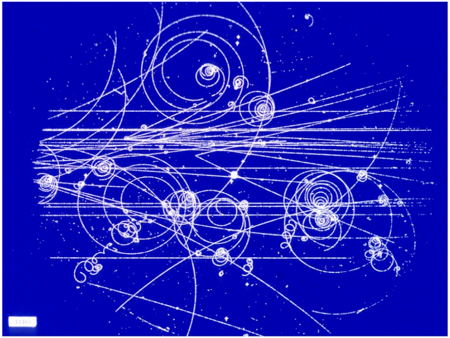
- ★ **Derived the basic interaction in QED taking into account the spins of the fermions and polarization of the virtual photons:**

$$-iM = [\overline{u}(p_3) ie\gamma^{\mu} u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\overline{u}(p_4) ie\gamma^{\nu} u(p_2)]$$

- ★ **We now have all the elements to perform proper calculations in QED !**

Particle Physics

Dr Lester



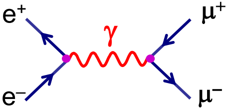
Handout 4 : Electron-Positron Annihilation

QED Calculations

- **How to calculate a cross section using QED (e.g. $e^+e^- \to \mu^+\mu^-$):**

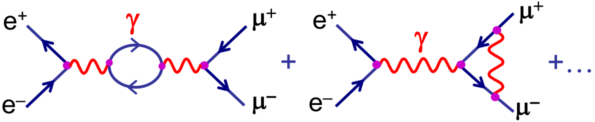
❶ **Draw all possible Feynman Diagrams**

- **For $e^+e^- \to \mu^+\mu^-$ there is just one lowest order diagram**



$$M \propto e^2 \propto \alpha_{em}$$

+ many second order diagrams + …



$$M \propto e^4 \propto \alpha_{em}^2$$

❷ **For each diagram calculate the matrix element using Feynman rules derived in the previous handout.**

❸ **Sum the individual matrix elements (i.e. sum the amplitudes)**

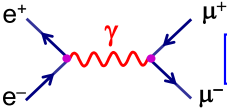
$$M_{fi} = M_1 + M_2 + M_3 + \ldots.$$

- **Note: summing amplitudes therefore different diagrams for the same final state can interfere either positively or negatively!**

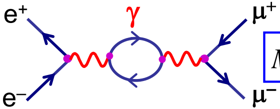
and then square $|M_{fi}|^2 = (M_1 + M_2 + M_3 + ...)(M_1^* + M_2^* + M_3^* + ...)$

➡ this gives the full perturbation expansion in α_{em}

- For QED $\alpha_{em} \sim 1/137$ the lowest order diagram dominates and for most purposes it is sufficient to neglect higher order diagrams.



$$\boxed{M^2 \propto \alpha_{em}^2}$$



$$\boxed{M^2 \propto \alpha_{em}^4}$$

❹ Calculate decay rate/cross section using formulae from handout 1.

- e.g. for a decay

$$\Gamma = \frac{p^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 \mathrm{d}\Omega$$

- For scattering in the centre-of-mass frame

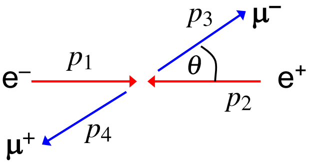
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} |M_{fi}|^2 \qquad (1)$$

- For scattering in lab. frame (neglecting mass of scattered particle)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1} \right)^2 |M_{fi}|^2$$

Electron Positron Annihilation

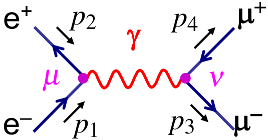
★ **Consider the process:** $e^+e^- \rightarrow \mu^+\mu^-$



- **Work in C.o.M. frame (this is appropriate for most e^+e^- colliders).**

$$p_1 = (E,0,0,p) \qquad p_2 = (E,0,0,-p)$$
$$p_3 = (E,\vec{p}_f) \qquad p_4 = (E,-\vec{p}_f)$$

- **Only consider the lowest order Feynman diagram:**



♦ **Feynman rules give:**

$$-iM = [\overline{v}(p_2)ie\gamma^{\mu}u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^{\nu}v(p_4)]$$

NOTE:
- **Incoming anti-particle** $\overline{v}$
- **Incoming particle** u
- **Adjoint spinor written first**

- **In the C.o.M. frame have**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f|}{|\vec{p}_i|}|M_{fi}|^2 \qquad \text{with} \qquad s = (p_1+p_2)^2 = (E+E)^2 = 4E^2$$

Electron and Muon Currents

- Here $q^2 = (p_1 + p_2)^2 = s$ and matrix element

$$-iM = [\overline{v}(p_2) ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\overline{u}(p_3) ie\gamma^\nu v(p_4)]$$

$$\Rightarrow \quad M = -\frac{e^2}{s} g_{\mu\nu} [\overline{v}(p_2)\gamma^\mu u(p_1)][\overline{u}(p_3)\gamma^\nu v(p_4)]$$

- In handout 2 introduced the four-vector current

$$j^\mu = \overline{\psi}\gamma^\mu \psi$$

which has same form as the two terms in [ ] in the matrix element

- The matrix element can be written in terms of the electron and muon currents

$$(j_e)^\mu = \overline{v}(p_2)\gamma^\mu u(p_1) \qquad \text{and} \qquad (j_\mu)^\nu = \overline{u}(p_3)\gamma^\nu v(p_4)$$

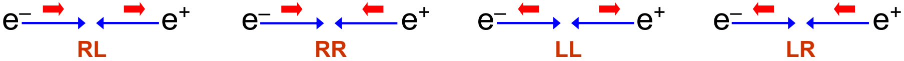
$$\Rightarrow \quad M = -\frac{e^2}{s} g_{\mu\nu} (j_e)^\mu (j_\mu)^\nu$$

$$\boxed{M = -\frac{e^2}{s} j_e . j_\mu}$$

- Matrix element is a four-vector scalar product – confirming it is Lorentz Invariant

Spin in e⁺e⁻ Annihilation

- In general the electron and positron will not be polarized, i.e. there will be equal numbers of positive and negative helicity states
- There are four possible combinations of spins in the initial state !



- Similarly there are four possible helicity combinations in the final state
- In total there are 16 combinations e.g. RL→RR, RL→RL, ….
- To account for these states we need to sum over all 16 possible helicity combinations and then average over the number of initial helicity states:

$$\langle |M|^2\rangle = \frac{1}{4}\sum_{\text{spins}} |M_i|^2 = \frac{1}{4}\left(|M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \ldots\right)$$

★ i.e. need to evaluate:

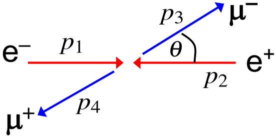
$$M = -\frac{e^2}{s}\, j_e \cdot j_\mu$$

for all 16 helicity combinations !

★ Fortunately, in the limit $E \gg m_\mu$ only 4 helicity combinations give non-zero matrix elements – we will see that this is an important feature of QED/QCD

- In the C.o.M. frame in the limit $E \gg m$

$$p_1 = (E,0,0,E);\;\; p_2 = (E,0,0,-E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta);$$
$$p_4 = (E,\text{-}E\sin\theta, 0, -E\cos\theta)$$



- Left- and right-handed helicity spinors (handout 2) for particles/anti-particles are:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix} \quad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

where $s = \sin\frac{\theta}{2};\quad c = \cos\frac{\theta}{2}$ and $N = \sqrt{E+m}$

- In the limit $E \gg m$ these become:

$$\boxed{u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix};\; u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix};\;\; v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix};\; v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}}$$

- The initial-state electron can either be in a left- or right-handed helicity state

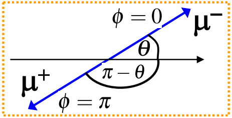
$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix};\; u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix};$$

- For the initial state positron $(\theta = \pi)$ can have either:

$$v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix};\ v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- Similarly for the final state μ^- which has polar angle θ and choosing $\phi = 0$

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix}c\\s\\c\\s\end{pmatrix};\ u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix};$$



- And for the final state μ^+ replacing $\theta \rightarrow \pi - \theta;\quad \phi \rightarrow \pi$ obtain

$$v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix};\ v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix};$$

using

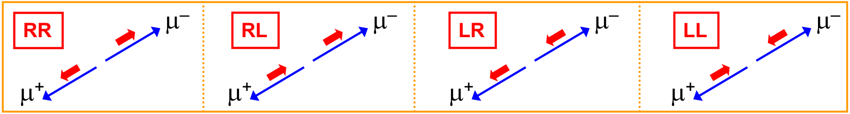
$$\sin\left(\frac{\pi-\theta}{2}\right) = \cos\frac{\theta}{2}$$

$$\cos\left(\frac{\pi-\theta}{2}\right) = \sin\frac{\theta}{2}$$

$$e^{i\pi} = -1$$

- Wish to calculate the matrix element $M = -\frac{e^2}{s} j_e.j_\mu$

★ first consider the muon current j_μ for 4 possible helicity combinations



The Muon Current

- **Want to evaluate** $(j_\mu)^\nu = \overline{u}(p_3)\gamma^\nu v(p_4)$ **for all four helicity combinations**
- **For arbitrary spinors** $\psi,\ \phi$ **with it is straightforward to show that the components of** $\overline{\psi}\gamma^\mu\phi$ **are**

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 && \textbf{(3)}\\
\overline{\psi}\gamma^1\phi &= \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 && \textbf{(4)}\\
\overline{\psi}\gamma^2\phi &= \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) && \textbf{(5)}\\
\overline{\psi}\gamma^3\phi &= \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2 && \textbf{(6)}
\end{aligned}$$

- **Consider the** $\mu_R^-\mu_L^+$ **combination using** $\psi = u_\uparrow \quad \phi = v_\downarrow$

with $v_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix};\ u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix};$

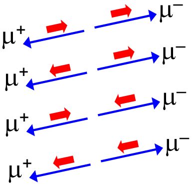
➡

$$\begin{aligned}
\overline{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) &= E(cs - sc + cs - sc) = 0\\
\overline{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) &= E(-c^2 + s^2 - c^2 + s^2) = 2E(s^2 - c^2) = -2E\cos\theta\\
\overline{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) &= -iE(-c^2 - s^2 - c^2 - s^2) = 2iE\\
\overline{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) &= E(cs + sc + cs + sc) = 4Esc = 2E\sin\theta
\end{aligned}$$

- Hence the four-vector muon current for the **RL** combination is

$$\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) \quad = \quad 2E(0,-\cos\theta,i,\sin\theta)$$

- The results for the 4 helicity combinations (obtained in the same manner) are:



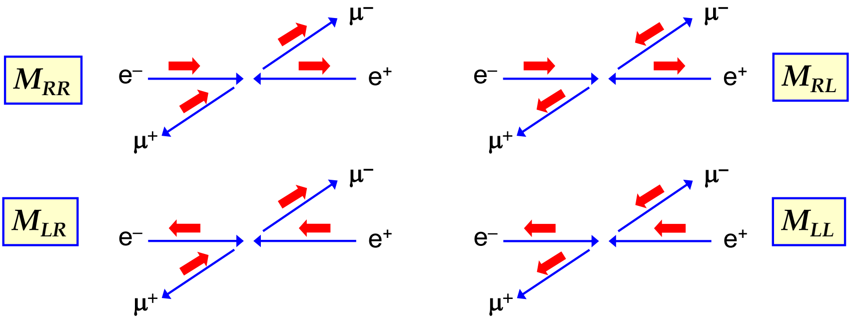
$$\boxed{\begin{array}{rcl} \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) & = & 2E(0,-\cos\theta,i,\sin\theta) \\ \bar{u}_\uparrow(p_3)\gamma^\nu v_\uparrow(p_4) & = & (0,0,0,0) \\ \bar{u}_\downarrow(p_3)\gamma^\nu v_\downarrow(p_4) & = & (0,0,0,0) \\ \bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) & = & 2E(0,-\cos\theta,-i,\sin\theta) \end{array}} \quad \begin{array}{l} \textbf{RL} \\ \textbf{RR} \\ \textbf{LL} \\ \textbf{LR} \end{array}$$

★ **IN THE LIMIT** $E \gg m$ **only two helicity combinations are non-zero !**

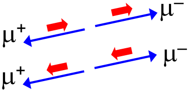
- This is an important feature of QED. It applies equally to QCD.
- In the Weak interaction only one helicity combination contributes.
- The origin of this will be discussed in the last part of this lecture
- But as a consequence of the 16 possible helicity combinations only four given non-zero matrix elements

Electron Positron Annihilation cont.

★ **For** $e^+e^- \to \mu^+\mu^-$ **now only have to consider the 4 matrix elements:**



•**Previously we derived the muon currents for the allowed helicities:**



$$\boxed{\begin{aligned} \mu_R^-\mu_L^+ \quad &: \quad \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) &= \quad 2E(0,-\cos\theta,i,\sin\theta) \\ \mu_L^-\mu_R^+ \quad &: \quad \bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) &= \quad 2E(0,-\cos\theta,-i,\sin\theta) \end{aligned}}$$

•**Now need to consider the electron current**

The Electron Current

- **The incoming electron and positron spinors (L and R helicities) are:**

$$u_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix};\; u_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}; \quad v_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix};\; v_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- **The electron current can either be obtained from equations (3)-(6) as before or it can be obtained directly from the expressions for the muon current.**

$$(j_e)^\mu = \overline{v}(p_2)\gamma^\mu u(p_1) \qquad (j_\mu)^\mu = \overline{u}(p_3)\gamma^\mu v(p_4)$$

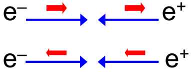
- **Taking the Hermitian conjugate of the muon current gives**

$$\begin{aligned}
[\overline{u}(p_3)\gamma^\mu v(p_4)]^\dagger &= \left[u(p_3)^\dagger\gamma^0\gamma^\mu v(p_4)\right]^\dagger \\
&= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_3) && (AB)^\dagger = B^\dagger A^\dagger \\
&= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^0 u(p_3) && \gamma^{0\dagger} = \gamma^0 \\
&= v(p_4)^\dagger\gamma^0\gamma^\mu u(p_3) && \gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu \\
&= \overline{v}(p_4)\gamma^\mu u(p_3)
\end{aligned}$$

- **Taking the complex conjugate of the muon currents for the two non-zero helicity configurations:**

$$\begin{aligned}\bar{v}_{\downarrow}(p_4)\gamma^{\mu}u_{\uparrow}(p_3) &= \left[\bar{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4)\right]^{*} = 2E(0,-\cos\theta,-i,\sin\theta)\\ \bar{v}_{\uparrow}(p_4)\gamma^{\mu}u_{\downarrow}(p_3) &= \left[\bar{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4)\right]^{*} = 2E(0,-\cos\theta,i,\sin\theta)\end{aligned}$$

To obtain the electron currents we simply need to set $\theta = 0$

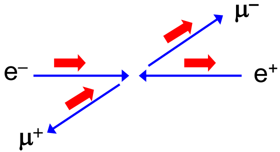


$$\boxed{\begin{array}{llcl} e^-_R e^+_L : & \bar{v}_{\downarrow}(p_2)\gamma^{\nu}u_{\uparrow}(p_1) & = & 2E(0,-1,-i,0)\\ e^-_L e^+_R : & \bar{v}_{\uparrow}(p_2)\gamma^{\nu}u_{\downarrow}(p_1) & = & 2E(0,-1,i,0)\end{array}}$$

Matrix Element Calculation

- We can now calculate $M = -\frac{e^2}{s} j_e . j_\mu$ for the four possible helicity combinations.

e.g. the matrix element for $e^-_R e^+_L \to \mu^-_R \mu^+_L$ which will denote $\boxed{M_{RR}}$



Here the first subscript refers to the helicity of the e⁻ and the second to the helicity of the μ⁻. Don't need to specify other helicities due to "helicity conservation", only certain chiral combinations are non-zero.

★Using:

$$e^-_R e^+_L \ : \qquad (j_e)^\mu = \overline{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1) \quad = \quad 2E(0,-1,-i,0)$$

$$\mu^-_R \mu^+_L \ : \qquad (j_\mu)^\nu = \overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) \quad = \quad 2E(0,-\cos\theta,i,\sin\theta)$$

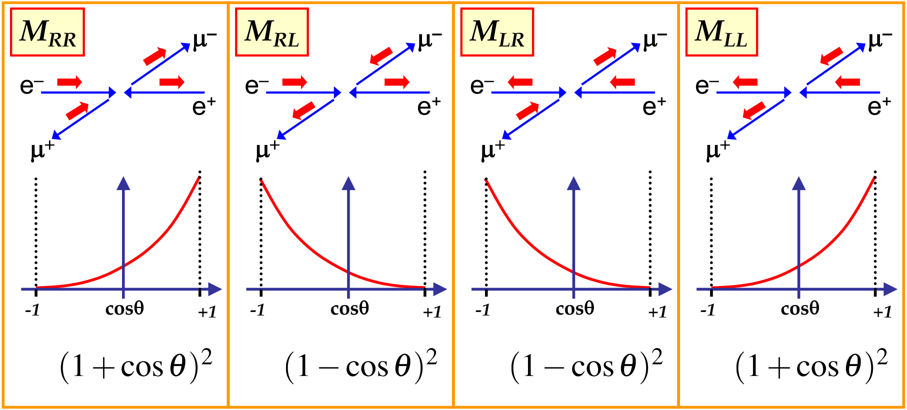
gives

$$\begin{aligned} M_{RR} &= -\frac{e^2}{s}[2E(0,-1,-i,0)].[2E(0,-\cos\theta,i,\sin\theta)] \\ &= e^2(1+\cos\theta) \\ &= 4\pi\alpha(1+\cos\theta) \end{aligned}$$

where $\alpha = e^2/4\pi \approx 1/137$

Similarly

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$$
$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$$



- **Assuming that the incoming electrons and positrons are unpolarized, all 4 possible initial helicity states are equally likely.**

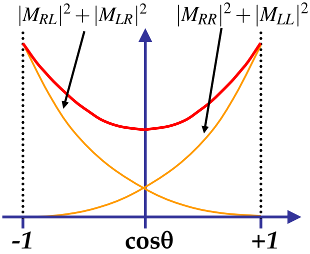
Differential Cross Section

- The cross section is obtained by averaging over the initial spin states and summing over the final spin states:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \times \frac{1}{64\pi^2 s}(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M^2_{LL}|)$$

$$= \frac{(4\pi\alpha)^2}{256\pi^2 s}(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2)$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}$$

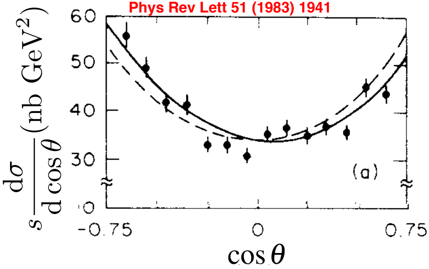


Example:

$e^+e^- \to \mu^+\mu^-$

$\sqrt{s} = 29$ GeV

Mark II Expt., M.E.Levi et al., Phys Rev Lett 51 (1983) 1941



- - - - pure QED, O(α^3)

—— QED plus Z contribution

Angular distribution becomes slightly asymmetric in higher order QED or when Z contribution is included

- The total cross section is obtained by integrating over $\theta,\ \phi$ using

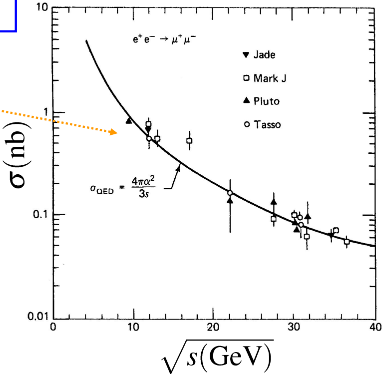
$$\int (1+\cos^2\theta)\mathrm{d}\Omega = 2\pi \int_{-1}^{+1} (1+\cos^2\theta)\mathrm{d}\cos\theta = \frac{16\pi}{3}$$

giving the **QED** total cross-section for the process $e^+e^- \rightarrow \mu^+\mu^-$

$$\boxed{\sigma = \frac{4\pi\alpha^2}{3s}}$$

★ Lowest order cross section calculation provides a good description of the data **!**

This is an impressive result. From first principles we have arrived at an expression for the electron-positron annihilation cross section which is good to **1%**

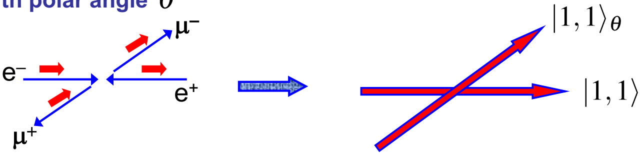


Spin Considerations $(E \gg m)$

★The angular dependence of the QED electron-positron matrix elements can be understood in terms of angular momentum

- Because of the allowed helicity states, the electron and positron interact in a spin state with $S_z = \pm 1$, i.e. in a total spin 1 state aligned along the z axis: $|1, +1\rangle$ or $|1, -1\rangle$
- Similarly the muon and anti-muon are produced in a total spin 1 state aligned along an axis with polar angle θ

e.g. $\boxed{\text{M}_{\text{RR}}}$



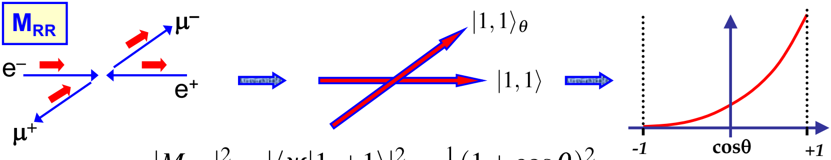
- Hence $M_{\text{RR}} \propto \langle\psi|1,1\rangle$ where ψ corresponds to the spin state, $|1,1\rangle_\theta$, of the muon pair.
- To evaluate this need to express $|1,1\rangle_\theta$ in terms of eigenstates of S_z
- In the appendix (and also in IB QM) it is shown that:

$$\boxed{|1,1\rangle_\theta = \tfrac{1}{2}(1-\cos\theta)|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)|1,+1\rangle}$$

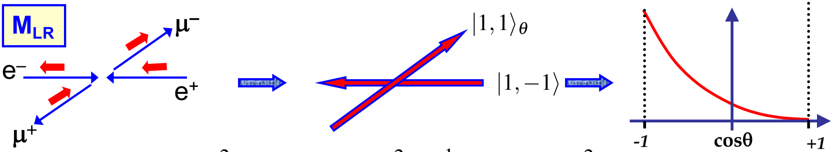
- Using the wave-function for a spin 1 state along an axis at angle θ

$$\psi = |1,1\rangle_\theta = \tfrac{1}{2}(1-\cos\theta)|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)|1,+1\rangle$$

can immediately understand the angular dependence



$$|M_{\rm RR}|^2 \propto |\langle\psi|1,+1\rangle|^2 = \tfrac{1}{4}(1+\cos\theta)^2$$

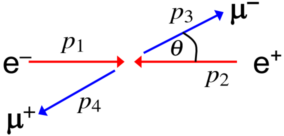


$$|M_{\rm LR}|^2 \propto |\langle\psi|1,-1\rangle|^2 = \tfrac{1}{4}(1-\cos\theta)^2$$

Lorentz Invariant form of Matrix Element

- **Before concluding this discussion, note that the spin-averaged Matrix Element derived above is written in terms of the muon angle in the C.o.M. frame.**

$$\begin{aligned}\langle|M_{fi}|^2\rangle &= \frac{1}{4}\times(|M_{RR}|^2+|M_{RL}|^2+|M_{LR}|^2+|M_{LL}^2|) \\ &= \frac{1}{4}e^4(2(1+\cos\theta)^2+2(1-\cos\theta)^2) \\ &= e^4(1+\cos^2\theta)\end{aligned}$$



- **The matrix element is Lorentz Invariant (scalar product of 4-vector currents) and it is desirable to write it in a frame-independent form, i.e. express in terms of Lorentz Invariant 4-vector scalar products**
- **In the C.o.M.** $p_1 = (E,0,0,E) \quad p_2 = (E,0,0,-E)$

$$p_3 = (E, E\sin\theta, 0, E\cos\theta) \quad p_4 = (E,-E\sin\theta,0,-E\cos\theta)$$

giving: $p_1.p_2 = 2E^2; \quad p_1.p_3 = E^2(1-\cos\theta); \quad p_1.p_4 = E^2(1+\cos\theta)$

- **Hence we can write**

$$\boxed{\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_3)^2+(p_1.p_4)^2}{(p_1.p_2)^2}} \qquad \equiv 2e^4\left(\frac{t^2+u^2}{s^2}\right)$$

★**Valid in any frame !**

CHIRALITY

- The helicity eigenstates for a particle/anti-particle for $E \gg m$ are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix};\ u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix};\ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix};\ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

where $s = \sin\frac{\theta}{2};\quad c = \cos\frac{\theta}{2}$

- Define the matrix

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}$$

- In the limit $E \gg m$ the helicity states are also eigenstates of γ^5

$$\gamma^5 u_\uparrow = +u_\uparrow;\ \gamma^5 u_\downarrow = -u_\downarrow;\ \gamma^5 v_\uparrow = -v_\uparrow;\ \gamma^5 v_\downarrow = +v_\downarrow$$

★ In general, define the eigenstates of γ^5 as LEFT and RIGHT HANDED CHIRAL states $u_R;\quad u_L;\quad v_R;\quad v_L$

i.e. $\gamma^5 u_R = +u_R;\ \gamma^5 u_L = -u_L;\ \gamma^5 v_R = -v_R;\ \gamma^5 v_L = +v_L$

- In the LIMIT $E \gg m$ (and ONLY IN THIS LIMIT):

$$u_R \equiv u_\uparrow;\quad u_L \equiv u_\downarrow;\quad v_R \equiv v_\uparrow;\quad v_L \equiv v_\downarrow$$

- ★ This is a subtle but important point: in general the HELICITY and CHIRAL eigenstates are not the same. It is only in the ultra-relativistic limit that the chiral eigenstates correspond to the helicity eigenstates.
- ★ Chirality is an import concept in the structure of QED, and any interaction of the form $\overline{u}\gamma^{\nu}u$

- In general, the eigenstates of the chirality operator are:

$$\gamma^5 u_R = +u_R; \;\; \gamma^5 u_L = -u_L; \;\; \gamma^5 v_R = -v_R; \;\; \gamma^5 v_L = +v_L$$

- Define the projection operators:

$$\boxed{P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)}$$

- The projection operators, project out the chiral eigenstates

$$\boxed{\begin{array}{llll} P_R u_R = u_R; & P_R u_L = 0; & P_L u_R = 0; & P_L u_L = u_L \\ P_R v_R = 0; & P_R v_L = v_L; & P_L v_R = v_R; & P_L v_L = 0 \end{array}}$$

- Note P_R projects out right-handed particle states and left-handed anti-particle states
- We can then write any spinor in terms of it left and right-handed chiral components:

$$\psi = \psi_R + \psi_L = \tfrac{1}{2}(1+\gamma^5)\psi + \tfrac{1}{2}(1-\gamma^5)\psi$$

Chirality in QED

- In QED the basic interaction between a fermion and photon is:

$$ie\overline{\psi}\gamma^{\mu}\phi$$

- Can decompose the spinors in terms of Left and Right-handed chiral components:

$$\begin{aligned} ie\overline{\psi}\gamma^{\mu}\phi &= ie(\overline{\psi}_L+\overline{\psi}_R)\gamma^{\mu}(\phi_R+\phi_L) \\ &= ie(\overline{\psi}_R\gamma^{\mu}\phi_R+\overline{\psi}_R\gamma^{\mu}\phi_L+\overline{\psi}_L\gamma^{\mu}\phi_R+\overline{\psi}_L\gamma^{\mu}\phi_L) \end{aligned}$$

- Using the properties of γ^5 (Q8 on examples sheet)

$$\boxed{(\gamma^5)^2=1; \quad \gamma^{5\dagger}=\gamma^5; \quad \gamma^5\gamma^{\mu}=-\gamma^{\mu}\gamma^5}$$

it is straightforward to show (Q9 on examples sheet)

$$\overline{\psi}_R\gamma^{\mu}\phi_L=0; \quad \overline{\psi}_L\gamma^{\mu}\phi_R=0$$

★ Hence only certain combinations of chiral eigenstates contribute to the interaction. This statement is **ALWAYS** true.

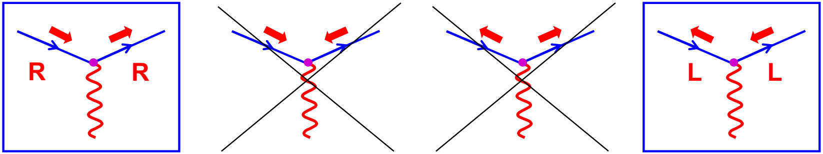
- For $E \gg m$, the chiral and helicity eigenstates are equivalent. This implies that for $E \gg m$ only certain helicity combinations contribute to the QED vertex ! This is why previously we found that for two of the four helicity combinations for the muon current were zero

Allowed QED Helicity Combinations

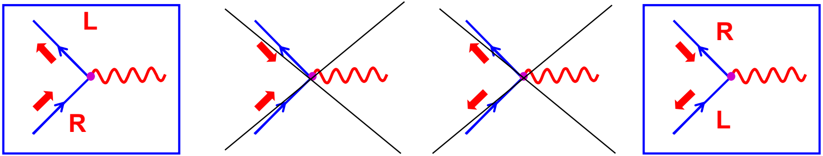
- In the **ultra-relativistic limit** the **helicity eigenstates** ≡ **chiral eigenstates**
- In this limit, the only non-zero helicity combinations in QED are:

Scattering:

"Helicity conservation"



Annihilation:



Summary

★ In the centre-of-mass frame the $e^+e^- \to \mu^+\mu^-$ differential cross-section is

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}$$

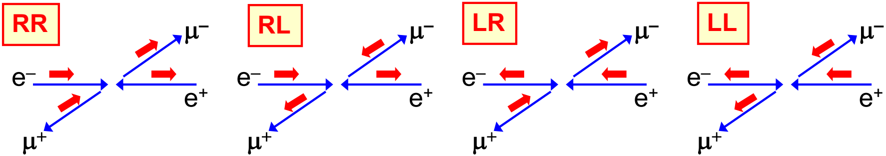
NOTE: neglected masses of the muons, i.e. assumed $E \gg m_\mu$

★ In QED only certain combinations of LEFT- and RIGHT-HANDED CHIRAL states give non-zero matrix elements

★ CHIRAL states defined by chiral projection operators

$$\boxed{P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)}$$

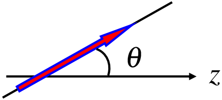
★ In limit $E \gg m$ the chiral eigenstates correspond to the HELICITY eigenstates and only certain HELICITY combinations give non-zero matrix elements



Appendix : Spin 1 Rotation Matrices

- Consider the spin-1 state with spin +1 along the axis defined by unit vector

$$\vec{n} = (\sin\theta, 0, \cos\theta)$$



- Spin state is an eigenstate of $\vec{n}.\vec{S}$ with eigenvalue +1

$$(\vec{n}.\vec{S})|\psi\rangle = +1|\psi\rangle \qquad \textbf{(A1)}$$

- Express in terms of linear combination of spin 1 states which are eigenstates of S_z

$$|\psi\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$$

with
$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

- **(A1)** becomes

$$(\sin\theta S_x + \cos\theta S_z)(\alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle) = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle \qquad \textbf{(A2)}$$

- Write S_x in terms of ladder operators $S_x = \frac{1}{2}(S_+ + S_-)$

where
$$S_+|1,1\rangle = 0 \qquad S_+|1,0\rangle = \sqrt{2}|1,1\rangle \qquad S_+|1,-1\rangle = \sqrt{2}|1,0\rangle$$
$$S_-|1,1\rangle = \sqrt{2}|1,0\rangle \qquad S_-|1,0\rangle = \sqrt{2}|1,-1\rangle \qquad S_-|1,-1\rangle = 0$$

- from which we find
$$S_x|1,1\rangle = \tfrac{1}{\sqrt{2}}|1,0\rangle$$
$$S_x|1,0\rangle = \tfrac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$$
$$S_x|1,-1\rangle = \tfrac{1}{\sqrt{2}}|1,0\rangle$$

- (A2) becomes
$$\sin\theta\left[\frac{\alpha}{\sqrt{2}}|1,0\rangle + \frac{\beta}{\sqrt{2}}|1,-1\rangle + \frac{\beta}{\sqrt{2}}|1,1\rangle + \frac{\gamma}{\sqrt{2}}|1,0\rangle\right] +$$
$$\alpha\cos\theta|1,1\rangle - \gamma\cos\theta|1,-1\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$$

- which gives
$$\left.\begin{aligned} \beta\frac{\sin\theta}{\sqrt{2}} + \alpha\cos\theta &= \alpha \\ (\alpha+\gamma)\frac{\sin\theta}{\sqrt{2}} &= \beta \\ \beta\frac{\sin\theta}{\sqrt{2}} - \gamma\cos\theta &= \gamma \end{aligned}\right\}$$

- using $\alpha^2 + \beta^2 + \gamma^2 = 1$ the above equations yield
$$\alpha = \tfrac{1}{\sqrt{2}}(1+\cos\theta) \qquad \beta = \tfrac{1}{\sqrt{2}}\sin\theta \qquad \gamma = \tfrac{1}{\sqrt{2}}(1-\cos\theta)$$

- hence
$$\psi = \tfrac{1}{2}(1-\cos\theta)|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)|1,+1\rangle$$

- The coefficients α, β, γ are examples of what are known as quantum mechanical **rotation matrices**. The express how angular momentum eigenstate in a particular direction is expressed in terms of the eigenstates defined in a different direction

$$\boxed{d^j_{m',m}(\theta)}$$

- For spin-1 $(j = 1)$ we have just shown that

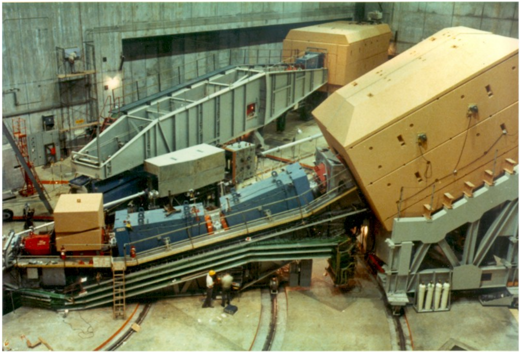
$$d^1_{1,1}(\theta) = \tfrac{1}{2}(1+\cos\theta) \qquad d^1_{0,1}(\theta) = \frac{1}{\sqrt{2}}\sin\theta \qquad d^1_{-1,1}(\theta) = \tfrac{1}{2}(1-\cos\theta)$$

- For spin-1/2 it is straightforward to show

$$d^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta) = \cos\frac{\theta}{2} \qquad d^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\theta) = \sin\frac{\theta}{2}$$

Particle Physics

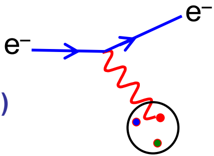
Dr Lester



Handout 5 : Electron-Proton Elastic Scattering

Electron-Proton Scattering

- In this handout aiming towards a study of electron-proton scattering as a probe of the structure of the proton

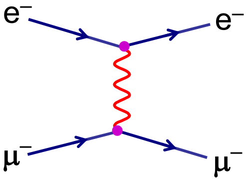


- Two main topics:
  - $e^-p \rightarrow e^-p$ elastic scattering (this handout)
  - $e^-p \rightarrow e^-X$ deep inelastic scattering (handout 6)
- But first consider scattering from a point-like particle e.g.

  $e^-\mu^- \rightarrow e^-\mu^-$

  i.e. the QED part of

  $(e^-q \rightarrow e^-q)$



- Two ways to proceed:
  - perform QED calculation from scratch (Q10 on examples sheet)

$$\langle|M_{fi}|^2\rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1.p_2)(p_3.p_4)+(p_1.p_4)(p_2.p_3)\right] \qquad (1)$$

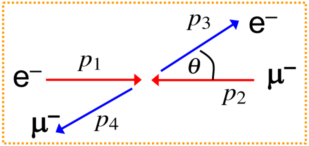
  - take results from $e^+e^- \rightarrow \mu^+\mu^-$ and use **"Crossing Symmetry"** to obtain the matrix element for $e^-\mu^- \rightarrow e^-\mu^-$ (Appendix I)

$$\boxed{\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_4)^2+(p_1.p_2)^2}{(p_1.p_3)^2}} \quad \textbf{(2)}$$

$$\equiv 2e^4\left(\frac{s^2+u^2}{t^2}\right)$$

- **Work in the C.o.M:**

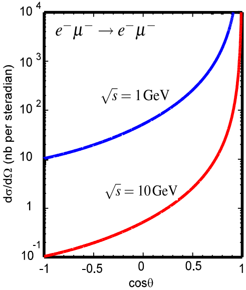
$$p_1 = (E,0,0,E) \qquad p_2 = (E,0,0,-E)$$
$$p_3 = (E,E\sin\theta,0,E\cos\theta)$$
$$p_4 = (E,-E\sin\theta,0,-E\cos\theta)$$



giving $p_1.p_2 = 2E^2;\quad p_1.p_3 = E^2(1-\cos\theta);\quad p_1.p_4 = E^2(1+\cos\theta)$

$$\langle|M_{fi}|^2\rangle = 2e^4\frac{E^4(1+\cos\theta)^2+4E^4}{E^4(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\langle|M_{fi}|^2\rangle = \frac{e^4}{8\pi^2 s}\frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$



- **The <u>denominator</u> arises from the propagator** $-ig_{\mu\nu}/q^2$

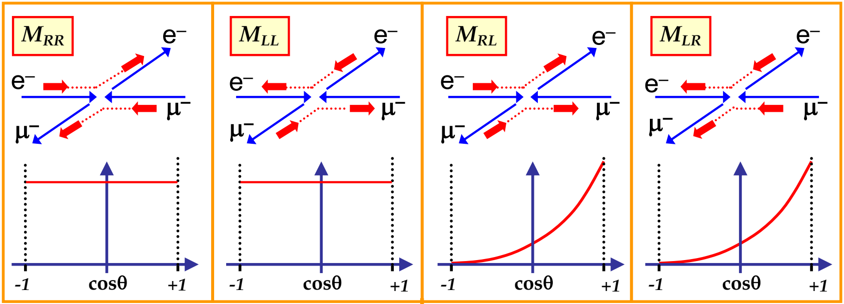
here $q^2 = (p_1-p_3)^2 = -E^2(1-\cos\theta)$

as $q^2 \rightarrow 0$ **the cross section tends to infinity.**

- What about the angular dependence of the numerator ?

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{8\pi^2 s}\frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$

- The factor $1+\frac{1}{4}(1+\cos\theta)^2$ reflects helicity (really chiral) structure of QED
- Of the 16 possible helicity combinations only 4 are non-zero:



$S_z = 0$

$$\rightarrow \frac{d\sigma}{d\Omega} \propto 1$$

i.e. no preferred polar angle

$S_z = +1$ $S_z = -1$

$$\rightarrow \frac{d\sigma}{d\Omega} \propto \frac{1}{4}(1+\cos\theta)^2$$

spin 1 rotation again

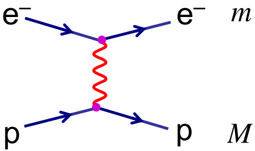
- The cross section calculated above is appropriate for the scattering of two spin half Dirac (i.e. point-like) particles in the ultra-relativistic limit (where both electron and muon masses can be neglected). In this case

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2}$$

- We will use this again in the discussion of "Deep Inelastic Scattering" of electrons from the quarks within a proton (handout 6).

- Before doing so we will consider the scattering of electrons from the composite proton - i.e. how do we know the proton isn't fundamental "point-like" particle ?

- In this discussion we will not be able to use the relativistic limit and require the general expression for the matrix element (derived in the optional part of Q10 in the examples sheet):

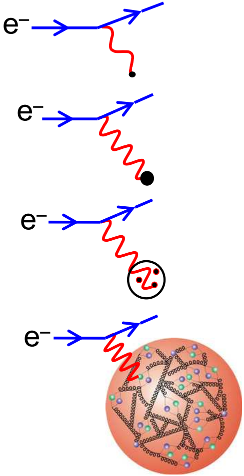


$$\boxed{\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_1.p_4)m^2 + 2m^2M^2\right]} \quad \textbf{(3)}$$

Probing the Structure of the Proton

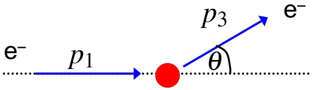
★ In $e^-p \to e^-p$ scattering the nature of the interaction of the virtual photon with the proton depends strongly on wavelength

- At very low electron energies $\lambda \gg r_p$: the scattering is equivalent to that from a "point-like" **spin-less** object
- At low electron energies $\lambda \sim r_p$: the scattering is equivalent to that from a extended charged object
- At high electron energies $\lambda < r_p$: the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks
- At very high electron energies $\lambda \ll r_p$: the proton appears to be a sea of quarks and gluons.



Rutherford Scattering Revisited

★ **Rutherford scattering is the low energy limit where the recoil of the proton can be neglected and the electron is non-relativistic**



(neglect proton recoil)

- **Start from RH and LH Helicity particle spinors**

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad \begin{aligned} N &= \sqrt{E+m}; \\ s &= \sin(\theta/2); \quad c = \cos(\theta/2) \end{aligned}$$

- **Now write in terms of:** $$\alpha = \frac{|\vec{p}|}{E+m_e}$$

Non-relativistic limit: $\alpha \to 0$

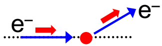
Ultra-relativistic limit: $\alpha \to 1$

$$\Longrightarrow \quad u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \alpha c \\ \alpha e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \alpha s \\ -\alpha e^{i\phi}c \end{pmatrix}$$

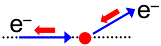
and the possible initial and final state electron spinors are:

$$u_\uparrow(p_1) = N_e\begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = N_e\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix} \quad u_\uparrow(p_3) = N_e\begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix} \quad u_\downarrow(p_3) = N_e\begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}$$

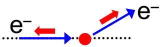
- Consider all four possible electron currents, i.e. Helicities **R→R, L→L, L→R, R→L**



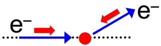
$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad \textbf{(4)}$$



$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad \textbf{(5)}$$



$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right] \quad \textbf{(6)}$$



$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right] \quad \textbf{(7)}$$

- In the relativistic limit ($\alpha = 1$), i.e. $E \gg m$

  (6) and (7) are identically zero; only R→R and L→L combinations non-zero

- In the non-relativistic limit, $|\vec{p}| \ll E$ we have $\alpha = 0$

$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = \overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (2m_e)[c, 0, 0, 0]$$

$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = -\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (2m_e)[s, 0, 0, 0]$$

All four electron helicity combinations have non-zero Matrix Element

i.e. Helicity eigenstates ≠ Chirality eigenstates

- The initial and final state **proton** spinors (assuming no recoil) are:

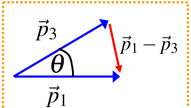
$$u_\uparrow(0) = \sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_\downarrow(0) = \sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Solutions of Dirac equation for a particle at rest

giving the proton currents:

$$j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} = 2M_p(1,0,0,0)$$
$$j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = 0$$

- The spin-averaged ME summing over the 8 allowed helicity states



$$\langle|M_{fi}^2|\rangle = \frac{1}{4}\frac{e^4}{q^4}(16M_p^2m_e^2)(\underline{4c^2+4s^2}) = \frac{16M_p^2m_e^2e^4}{q^4}$$

where $q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -4|\vec{p}|^2\sin^2(\theta/2)$

$$\langle|M_{fi}^2|\rangle = \frac{M_p^2m_e^2e^4}{|\vec{p}|^4\sin^4(\theta/2)}$$

Note: in this limit all angular dependence is in the propagator

- The formula for the differential cross-section in the lab. frame was derived in handout 1:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2 \qquad \textbf{(8)}$$

- Here the electron is non-relativistic so $E \sim m_e \ll M_p$ and we can neglect E_1 in the denominator of equation (8)

$$\rightarrow \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 M_p^2}|M_{fi}|^2 = \frac{m_e^2 e^4}{64\pi^2 |\vec{p}|^4 \sin^4(\theta/2)}$$

- Writing $e^2 = 4\pi\alpha$ and the kinetic energy of the electron as $E_K = p^2/2m_e$

$$\rightarrow \quad \boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Rutherford}} = \frac{\alpha^2}{16E_K^2 \sin^4\theta/2}} \qquad (9)$$

★ This is the normal expression for the Rutherford cross section. It could have been derived by considering the scattering of a non-relativistic particle in the static Coulomb potential of the proton $V(\vec{r})$, **without any consideration of the interaction due to the intrinsic magnetic moments** of the electron or proton. From this we can conclude, that in this non-relativistic limit only the interaction between the electric charges of the particles matters.

The Mott Scattering Cross Section

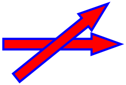
- For Rutherford scattering we are in the limit where the target recoil is neglected and the scattered particle is non-relativistic $E_K \ll m_e$
- The limit where the target recoil is neglected and the scattered particle is **relativistic** (i.e. just neglect the electron mass) is called Mott Scattering
- In this limit the electron currents, equations (4) and (6), become:

$$\overline{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = 2E\,[c, s, -is, c] \qquad \overline{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = E\,[0, 0, 0, 0]$$

Relativistic ➡ Electron "helicity conserved"

- It is then straightforward to obtain the result:

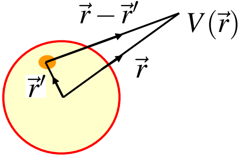
$$\rightarrow \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Mott}} = \underbrace{\frac{\alpha^2}{4E^2\sin^4\theta/2}}_{\text{Rutherford formula with } E_K = E\ (E \gg m_e)} \underbrace{\cos^2\frac{\theta}{2}}_{\text{Overlap between initial/final state electron wave-functions. Just QM of spin ½}} \qquad \textbf{(10)}$$



★ NOTE: we could have derived this expression from scattering of electrons in a static potential from a fixed point in space $V(\vec{r})$.
The interaction is **ELECTRIC** rather than magnetic (spin-spin) in nature.

★ Still haven't taken into account the charge distribution of the proton…..

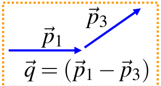
Form Factors



- Consider the scattering of an electron in the static potential due to an extended charge distribution.
- The potential at $\vec{r}$ from the centre is given by:

$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' \qquad \text{with} \quad \int \rho(\vec{r}) \mathrm{d}^3\vec{r} = 1$$

- In first order perturbation theory the matrix element is given by:



$$\begin{aligned} M_{fi} &= \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \int e^{-i\vec{p}_3.\vec{r}} V(\vec{r}) e^{i\vec{p}_1.\vec{r}} \mathrm{d}^3\vec{r} \\ &= \int\int e^{i\vec{q}.\vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}' \mathrm{d}^3\vec{r} = \int\int e^{i\vec{q}.(\vec{r}-\vec{r}')} e^{i\vec{q}.\vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}' \mathrm{d}^3\vec{r} \end{aligned}$$

- Fix $\vec{r}'$ and integrate over $\mathrm{d}^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$

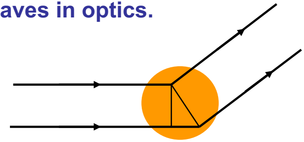
$$M_{fi} = \int e^{i\vec{q}.\vec{R}} \frac{Q}{4\pi|\vec{R}|} \mathrm{d}^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q}.\vec{r}'} \mathrm{d}^3\vec{r}' = (M_{fi})_{point} F(\vec{q}^2)$$

★ The resulting matrix element is equivalent to the matrix element for scattering from a point source multiplied by the form factor

$$\boxed{F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}.\vec{r}} \mathrm{d}^3\vec{r}}$$

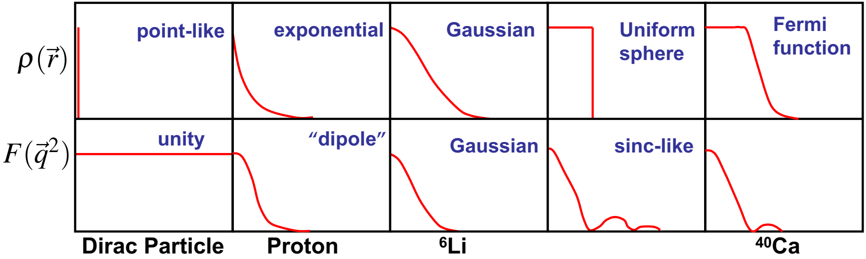
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} \rightarrow \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}|F(\vec{q}^2)|^2$$

- There is nothing mysterious about form factors – similar to diffraction of plane waves in optics.



- The finite size of the scattering centre introduces a phase difference between plane waves "scattered from different points in space". If wavelength is long compared to size all waves in phase and $F(\vec{q}^2) = 1$

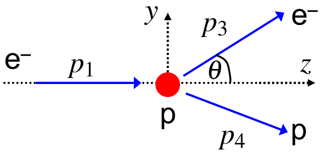
For example:



- NOTE that for a point charge the form factor is unity.

Point-like Electron-Proton Elastic Scattering

- So far have only considered the case we the proton does not recoil... For $E_1 \gg m_e$ the general case is



$$\begin{aligned} p_1 &= (E_1, 0, 0, E_1) \\ p_2 &= (M, 0, 0, 0) \\ \boxed{p_3 = (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)} & \\ p_4 &= (E_4, \vec{p}_4) \end{aligned}$$

- From Eqn. (3) with $m = m_e = 0$ the matrix element for this process is:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 \right] \qquad (11)$$

- Experimentally observe scattered electron so eliminate p_4
- The scalar products not involving p_4 are:

$$p_1.p_2 = E_1 M \qquad p_1.p_3 = E_1 E_3 (1 - \cos\theta) \qquad p_2.p_3 = E_3 M$$

- From momentum conservation can eliminate p_4: $p_4 = p_1 + p_2 - p_3$

$$p_3.p_4 = p_3.p_1 + p_3.p_2 - \cancel{p_3.p_3} = E_1 E_3 (1 - \cos\theta) + E_3 M$$

$$p_1.p_4 = \boxed{\cancel{p_1.p_1}} + p_1.p_2 - p_1.p_3 = E_1 M - E_1 E_3 (1 - \cos\theta)$$

$$\boxed{p_1.p_1 = E_1^2 - |\vec{p}_1|^2 = m_e^2 \approx 0}$$

i.e. neglect m_e

- **Substituting these scalar products in Eqn. (11) gives**

$$\begin{aligned}\langle|M_{fi}|^2\rangle &= \frac{8e^4}{(p_1-p_3)^4}ME_1E_3\left[(E_1-E_3)(1-\cos\theta)+M(1+\cos\theta)\right]\\ &= \frac{8e^4}{(p_1-p_3)^4}2ME_1E_3\left[(E_1-E_3)\sin^2(\theta/2)+M\cos^2(\theta/2)\right] \quad \textbf{(12)}\end{aligned}$$

- **Now obtain expressions for** $q^4=(p_1-p_3)^4$ **and** (E_1-E_3)

$$q^2=(p_1-p_3)^2=p_1^2+p_3^2-2p_1.p_3=-2E_1E_3(1-\cos\theta) \quad \textbf{(13)}$$

$$=-4E_1E_3\sin^2\theta/2 \quad \textbf{(14)}$$

NOTE: $q^2<0$ **Space-like**

- **For** (E_1-E_3) **start from**

$$q.p_2=(p_1-p_3).p_2=M(E_1-E_3)$$

and use $q=(p_1-p_3)=(p_4-p_2)$

$$\begin{aligned}(q+p_2)^2 &= p_4^2\\ q^2+p_2^2+2q.p_2 &= p_4^2\\ q^2+M^2+2q.p_2 &= M^2\\ \rightarrow \quad q.p_2 &= -q^2/2\end{aligned}$$

- **Hence the energy transferred to the proton:**

$$\boxed{E_1 - E_3 = -\frac{q^2}{2M}} \qquad \textbf{(15)}$$

Because q^2 is always negative $E_1 - E_3 > 0$ and the scattered electron is always lower in energy than the incoming electron

- **Combining equations (11), (13) and (14):**

$$\begin{aligned} \langle |M_{fi}|^2 \rangle &= \frac{8e^4}{16E_1^2 E_3^2 \sin^4\theta/2} 2ME_1E_3 \left[M\cos^2\theta/2 - \frac{q^2}{2M}\sin^2\theta/2 \right] \\ &= \frac{M^2e^4}{E_1E_3\sin^4\theta/2}\left[\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right] \end{aligned}$$

- **For $E \gg m_e$ we have (see handout 1)**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2$$

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

➡ $$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)} \qquad \textbf{(16)}$$

Interpretation

- **So far have derived the differential cross-section for** e⁻p → e⁻p **elastic scattering assuming point-like Dirac spin ½ particles. How should we interpret the equation?**

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)$$

- **Compare with** $\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}$

the important thing to note about the Mott cross-section is that it is equivalent to scattering of spin ½ electrons in a fixed electro-static potential. Here the term E_3/E_1 **is due to the proton recoil.**

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \underbrace{\frac{q^2}{2M^2}\sin^2\theta/2}\right)$$

- **the new term:** $\propto \sin^2\frac{\theta}{2}$ ⟷ **Magnetic interaction : due to the spin-spin interaction**

- The above differential cross-section depends on a single parameter. For an electron scattering angle θ, both q^2 and the energy, E_3, are fixed by kinematics

- Equating **(13)** and **(15)**

$$-2M(E_1 - E_3) = -2E_1E_3(1-\cos\theta)$$

$$\Rightarrow \quad \frac{E_3}{E_1} = \frac{M}{M+E_1(1-\cos\theta)}$$

- Substituting back into **(13)**:

$$\Rightarrow \quad q^2 = -\frac{2ME_1^2(1-\cos\theta)}{M+E_1(1-\cos\theta)}$$

- e.g. $e^-p \to e^-p$ at E_{beam}= 529.5 MeV, look at scattered electrons at θ= 75°

For elastic scattering expect:

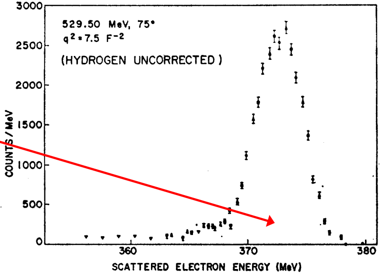
$$E_3 = \frac{ME_1}{M+E_1(1-\cos\theta)}$$

$$E_3 = \frac{938\times 529}{938+529(1-\cos 75^\circ)} = 373\ \text{MeV}$$

The energy identifies the scatter as elastic. Also know squared four-momentum transfer

$$|q^2| = \frac{2\times 938\times 529^2(1-\cos 75^\circ)}{938+529(1-\cos 75^\circ)} = 294\,\text{MeV}^2$$

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



Elastic Scattering from a Finite Size Proton

★**In general** the finite size of the proton can be accounted for by introducing two structure functions. One related to the charge distribution in the proton, $G_E(q^2)$ and the other related to the distribution of the magnetic moment of the proton, $G_M(q^2)$

- It can be shown that equation (16) generalizes to the **ROSENBLUTH FORMULA.**

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2}+2\tau G_M^2\sin^2\frac{\theta}{2}\right)}$$

with the Lorentz Invariant quantity: $\boxed{\tau = -\frac{q^2}{4M^2} > 0}$

- Unlike our previous discussion of form factors, here the form factors are a function of q^2 rather than $\vec{q}^2$ and cannot simply be considered in terms of the FT of the charge and magnetic moment distributions.

But $q^2 = (E_1-E_3)^2 - \vec{q}^2$ and from eq (15) obtain

$$\Rightarrow \quad -\vec{q}^2 = q^2\left[1-\left(\frac{q}{2M}\right)^2\right]$$

So for $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$ and $G(q^2) \approx G(\vec{q}^2)$

- Hence in the limit $q^2/4M^2 \ll 1$ we can interpret the structure functions in terms of the Fourier transforms of the charge and magnetic moment distributions

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}}\rho(\vec{r})\mathrm{d}^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}}\mu(\vec{r})\mathrm{d}^3\vec{r}$$

- Note in deriving the Rosenbluth formula we assumed that the proton was a spin-half Dirac particle, i.e.

$$\vec{\mu} = \frac{e}{M}\vec{S}$$

- However, the experimentally measured value of the proton magnetic moment is larger than expected for a point-like Dirac particle:

$$\vec{\mu} = 2.79\frac{e}{M}\vec{S}$$

So for the proton expect

$$G_E(0) = \int \rho(\vec{r})\mathrm{d}^3\vec{r} = 1 \qquad G_M(0) = \int \mu(\vec{r})\mathrm{d}^3\vec{r} = \mu_p = +2.79$$

- Of course the anomalous magnetic moment of the proton is already evidence that it is not point-like !

Measuring $G_E(q^2)$ and $G_M(q^2)$

- **Express the Rosenbluth formula as:**

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)$$

where $$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}$$

i.e. the Mott cross-section including the proton recoil. It corresponds to scattering from a spin-0 proton.

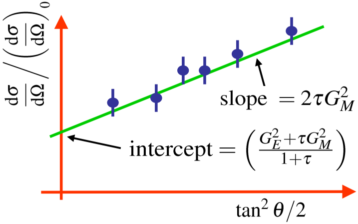
- **At very low q^2:** $\tau = -q^2/4M^2 \approx 0$

$$\frac{d\sigma}{d\Omega}\bigg/\left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2(q^2)$$

- **At high q^2:** $\tau \gg 1$

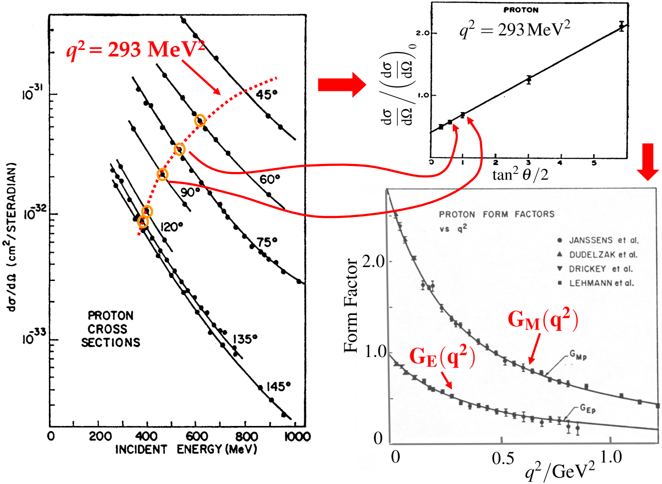
$$\frac{d\sigma}{d\Omega}\bigg/\left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau\tan^2\frac{\theta}{2}\right) G_M^2(q^2)$$

- **In general we are sensitive to both structure functions! These can be resolved from the angular dependence of the cross section at FIXED q^2**



EXAMPLE: $e^-p \rightarrow e^-p$ at E_{beam}= 529.5 MeV

- **Electron beam energies chosen to give certain values of q^2**
- **Cross sections measured to 2-3 %**



E.B.Hughes et al., Phys. Rev. 139 (1965) B458

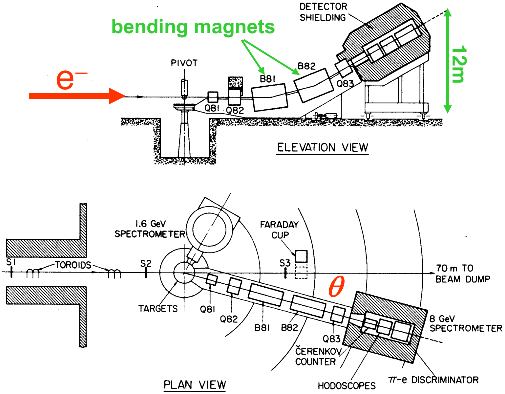
NOTE

Experimentally find $G_M(q^2) = 2.79 G_E(q^2)$, i.e. the electric and and magnetic form factors have same distribution

Higher Energy Electron-Proton Scattering

★ **Use electron beam from SLAC LINAC:** $5 < E_{\text{beam}} < 20$ GeV

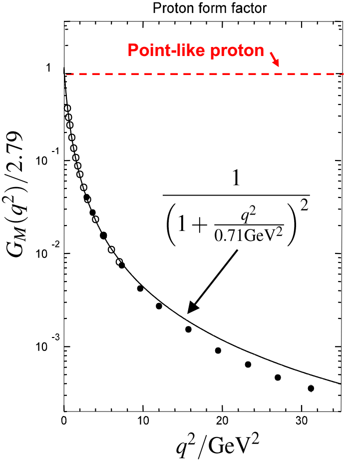
- **Detect scattered electrons using the "8 GeV Spectrometer"**



High q^2 → Measure $G_M(q^2)$

P.N.Kirk et al., Phys Rev D8 (1973) 63

High q^2 Results



R.C.Walker et al., Phys. Rev. D49 (1994) 5671
A.F.Sill et al., Phys. Rev. D48 (1993) 29

★ **Form factor falls rapidly with** q^2

- **Proton is not point-like**
- **Good fit to the data with "dipole form":**

$$G_E^p(q^2) \approx \frac{G_M^p}{2.79} \approx \frac{1}{(1+q^2/0.71\mathrm{GeV}^2)^2}$$

★ **Taking FT find spatial charge and magnetic moment distribution**

$$\boxed{\rho(r) \approx \rho_0 e^{-r/a}}$$

with $a \approx 0.24$ fm

- **Corresponds to a rms charge radius**

$$r_{rms} \approx 0.8 \text{ fm}$$

★ **Although suggestive, does not imply proton is composite !**

★ **Note: so far have only considered ELASTIC scattering; Inelastic scattering is the subject of next handout**

(Try Question 11)

Summary: Elastic Scattering

★ **For elastic scattering of relativistic electrons from a point-like Dirac proton:**

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_1^2\sin^4\theta/2}}_{\textbf{Rutherford}}\,\underbrace{\frac{E_3}{E_1}}_{\textbf{Proton recoil}}\left(\underbrace{\cos^2\frac{\theta}{2}}_{\textbf{Electric/ Magnetic scattering}} - \underbrace{\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}}_{\textbf{Magnetic term due to spin}}\right)$$

★ **For elastic scattering of relativistic electrons from an extended proton:**

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2} + 2\tau G_M^2\sin^2\frac{\theta}{2}\right)}$$

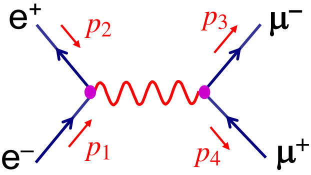
Rosenbluth Formula

★ **Electron elastic scattering from protons demonstrates that the proton is an extended object with rms charge radius of ~0.8 fm**

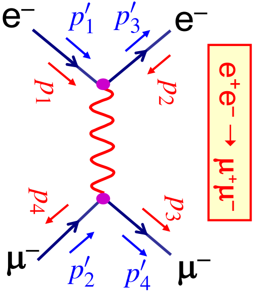
Appendix I : Crossing Symmetry

★ **Having derived the Lorentz invariant matrix element for $e^+e^- \rightarrow \mu^+\mu^-$ "rotate" the diagram to correspond to $e^-\mu^- \rightarrow e^-\mu^-$ and apply the principle of crossing symmetry to write down the matrix element !**

$e^+e^- \rightarrow \mu^+\mu^-$



$e^-\mu^- \rightarrow e^-\mu^-$



★ **The transformation:**

$$\boxed{p_1 \rightarrow p'_1;\; p_2 \rightarrow -p'_3;\; p_3 \rightarrow p'_4;\; p_4 \rightarrow -p'_2}$$

Changes the spin averaged matrix element for

$$\boxed{e^- e^+ \rightarrow \mu^- \mu^+} \quad \boxed{e^- \mu^- \rightarrow e^- \mu^-}$$

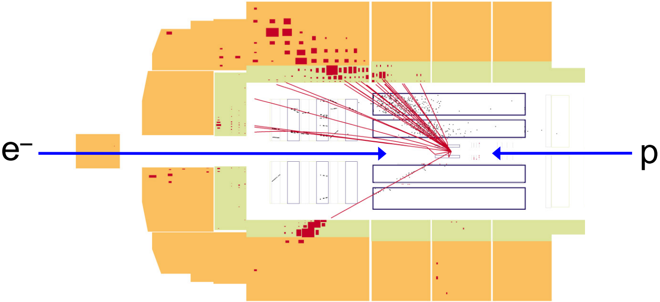
$$p_1\; p_2 \quad p_3\; p_4 \qquad\qquad p'_1\; p'_2 \quad p'_3\; p'_4$$

- Take ME for e$^+$e$^-$ → $\mu^+\mu^-$ (page 143) and apply crossing symmetry:

$$\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_3)^2+(p_1.p_4)^2}{(p_1.p_2)^2} \quad \Rightarrow \quad \boxed{\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1'.p_4')^2+(p_1'.p_2')^2}{(p_1'.p_3')^2}} \qquad (1)$$

Particle Physics

Dr Lester



Handout 6 : Deep Inelastic Scattering

e⁻ p Elastic Scattering at Very High q^2

★At high q^2 the Rosenbluth expression for elastic scattering becomes

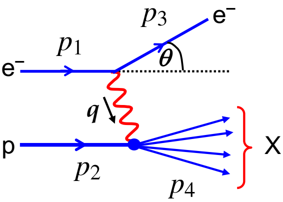
$$\left(\frac{d\sigma}{d\Omega}\right)_{elastic} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{q^2}{2M^2}G_M^2\sin^2\frac{\theta}{2}\right) \qquad \tau = -\frac{q^2}{4M^2} \gg 1$$

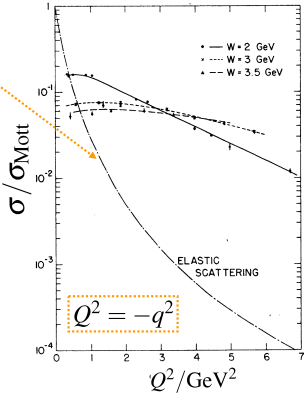
- From **e⁻ p** elastic scattering, the proton magnetic form factor is

$$G_M(q^2) \approx \frac{1}{(1+q^2/0.71\mathrm{GeV}^2)^2} \quad \Rightarrow \quad G_M(q^2) \propto q^{-4} \quad \text{at high } q^2$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{elastic} \propto q^{-6}$$

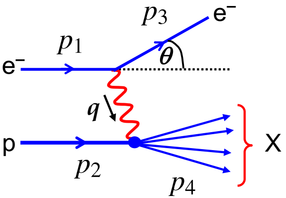
- Due to **the finite proton size**, elastic scattering at high q^2 is unlikely and inelastic reactions where the proton breaks up dominate.





M.Breidenbach et al., Phys. Rev. Lett. 23 (1969) 935

Kinematics of Inelastic Scattering



- For inelastic scattering the mass of the final state hadronic system is no longer the proton mass, M
- The final state hadronic system must contain at least one **baryon** which implies the final state invariant mass $M_X > M$

$$M_X^2 = p_4^2 = (E_4^2 - |\vec{p}_4|^2)$$

★ For inelastic scattering introduce four new kinematic variables: $\boxed{x, y, \nu, Q^2}$

★ **Define:** $\boxed{x \equiv \frac{Q^2}{2p_2.q}}$ **Bjorken x** (Lorentz Invariant)

where $\boxed{Q^2 \equiv -q^2}$ $\quad Q^2 > 0$

- Here $M_X^2 = p_4^2 = (q + p_2)^2 = -Q^2 + 2p_2.q + M^2$

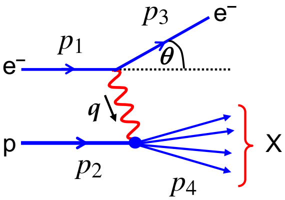
$\Rightarrow$ $Q^2 = 2p_2.q + M^2 - M_X^2$ $\Rightarrow$ $Q^2 \leq 2p_2.q$

Note: in many text books W is often used in place of M_X

hence $\boxed{0 < x < 1 \text{ inelastic}}$ $\quad \boxed{x = 1 \text{ elastic}}$

Proton intact $M_X = M$

★ **Define:** $\boxed{y \equiv \frac{p_2.q}{p_2.p_1}}$ **(Lorentz Invariant)**



- **In the Lab. Frame:**

$$p_1 = (E_1, 0, 0, E_1) \quad p_2 = (M, 0, 0, 0)$$

$$q = (E_1 - E_3, \vec{p}_1 - \vec{p}_3)$$

$$\rightarrow \quad y = \frac{M(E_1 - E_3)}{ME_1} = 1 - \frac{E_3}{E_1}$$

So y is the fractional energy loss of the incoming particle

$$\boxed{0 < y < 1}$$

- **In the C.o.M. Frame (neglecting the electron and proton masses):**

$$p_1 = (E, 0, 0, E); \quad p_2 = (E, 0, 0, -E); \quad p_3 = (E, E\sin\theta^*, 0, E\cos\theta^*)$$

$$\rightarrow \quad y = \tfrac{1}{2}(1 - \cos\theta^*) \qquad \text{for } E \gg M$$

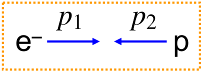
★ **Finally Define:** $\boxed{\nu \equiv \frac{p_2.q}{M}}$ **(Lorentz Invariant)**

- **In the Lab. Frame:** $\nu = E_1 - E_3$

ν **is the energy lost by the incoming particle**

Relationships between Kinematic Variables

- Can rewrite the new kinematic variables in terms of the squared centre-of-mass energy, s, for the electron-proton collision



$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1.p_2 = 2p_1.p_2 + M^2 + \cancel{m_e^2}$$

Neglect mass of electron

$$2p_1.p_2 = s - M^2$$

- For a fixed centre-of-mass energy, it can then be shown that the four kinematic variables

$$Q^2 \equiv -q^2 \qquad x \equiv \frac{Q^2}{2p_2.q} \qquad y \equiv \frac{p_2.q}{p_2.p_1} \qquad \nu \equiv \frac{p_2.q}{M}$$

are not independent.

- i.e. the scaling variables x and y can be expressed as

$$x = \frac{Q^2}{2M\nu} \qquad y = \frac{2M}{s - M^2}\nu$$

Note the simple relationship between y and ν

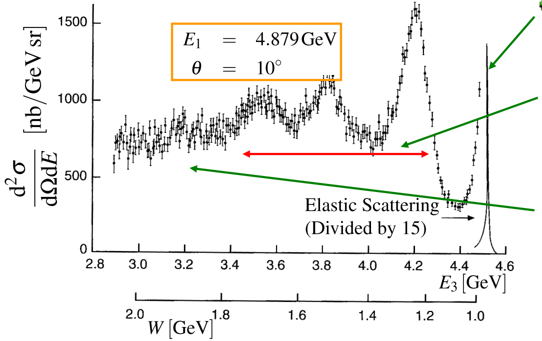
and $$xy = \frac{Q^2}{s - M^2} \quad \Rightarrow \quad Q^2 = (s - M^2)xy$$

- For a fixed centre of mass energy, the interaction kinematics are completely defined by any **two** of the above kinematic variables (except y and ν)
- For elastic scattering $(x = 1)$ there is only one independent variable. As we saw previously if you measure electron scattering angle know everything else.

Inelastic Scattering

Example: Scattering of 4.879 GeV electrons from protons at rest

- Place detector at 10° to beam and measure the **energies** of scattered e^-
- Kinematics fully determined from the **electron energy** and **angle !**
- e.g. for this energy and angle : the invariant mass of the final state hadronic system $W^2 = M_X^2 = 10.06 - 2.03E_3$ (try and show this)

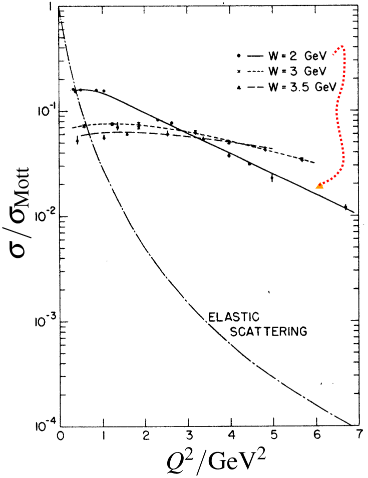


- **Elastic Scattering**
  proton remains intact
  $W = M$
- **Inelastic Scattering**
  produce "excited states" of proton e.g. $\Delta^+(1232)$
  $W = M_\Delta$
- **Deep Inelastic Scattering**
  proton breaks up resulting in a many particle final state

DIS = large W

Inelastic Cross Sections

- Repeat experiments at different angles/beam energies and determine q^2 dependence of elastic and inelastic cross-sections



M.Breidenbach et al.,
Phys. Rev. Lett. 23 (1969) 935

- Elastic scattering falls of rapidly with q^2 due to the proton not being point-like (i.e. form factors)

- Inelastic scattering cross sections only weakly dependent on q^2

- Deep Inelastic scattering cross sections almost independent of q^2 !

i.e. "Form factor" → 1

Scattering from point-like objects within the proton !

Elastic → Inelastic Scattering

★ **Recall:** Elastic scattering (Handout 5)

- Only one independent variable. In Lab. frame express differential cross section in terms of the electron scattering angle (Rosenbluth formula)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2}+2\tau G_M^2\sin^2\frac{\theta}{2}\right) \qquad \tau=\frac{Q^2}{4M^2}$$

Note: here the energy of the scattered electron is determined by the angle.

- In terms of the Lorentz invariant kinematic variables can express this differential cross section in terms of Q^2 (Q13 on examples sheet)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\frac{G_E^2+\tau G_M^2}{(1+\tau)}\left(1-y-\frac{M^2y^2}{Q^2}\right)+\frac{1}{2}y^2G_M^2\right]$$

which can be written as:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[f_2(Q^2)\left(1-y-\frac{M^2y^2}{Q^2}\right)+\frac{1}{2}y^2f_1(Q^2)\right]$$

★ **Inelastic scattering**

- For Deep Inelastic Scattering have two independent variables. Therefore need a double differential cross section

Deep Inelastic Scattering

★ It can be shown that **the most general Lorentz Invariant expression** for $e^-p \to e^-X$ inelastic scattering (via a single exchanged photon is):

$$\boxed{\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1-y-\frac{M^2y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x}+y^2F_1(x,Q^2)\right]} \quad (1)$$

INELASTIC SCATTERING

c.f. $$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1-y-\frac{M^2y^2}{Q^2}\right)f_2(Q^2)+\frac{1}{2}y^2f_1(Q^2)\right]$$

ELASTIC SCATTERING

We will soon see how this connects to the quark model of the proton

- **NOTE:** The form factors have been replaced by the **STRUCTURE FUNCTIONS**

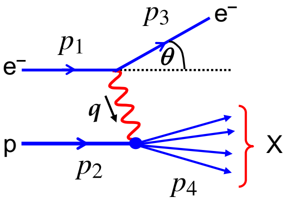
$$F_1(x,Q^2) \quad \text{and} \quad F_2(x,Q^2)$$

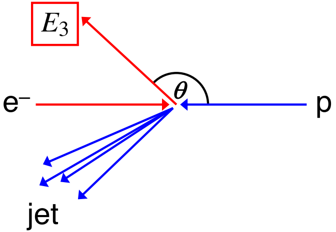
which are a function of x and Q^2: can not be interpreted as the Fourier transforms of the charge and magnetic moment distributions. We shall soon see that they describe the **momentum distribution** of the quarks within the proton

★ In the limit of high energy (or more correctly $Q^2 \gg M^2y^2$) eqn. (1) becomes:

$$\boxed{\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x}+y^2F_1(x,Q^2)\right]} \quad (2)$$

- In the Lab. frame it is convenient to express the cross section in terms of the angle, θ, and energy, E_3, of the scattered electron – **experimentally well measured.**





$$Q^2 = 4E_1E_3\sin^2\theta/2; \quad x = \frac{Q^2}{2M(E_1-E_3)}; \quad y = 1-\frac{E_3}{E_1}; \quad \nu = E_1 - E_3$$

- In the Lab. frame, Equation (2) becomes: (see examples sheet Q13)

$$\boxed{\frac{d^2\sigma}{dE_3\,d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{1}{\nu}F_2(x,Q^2)\cos^2\frac{\theta}{2} + \frac{2}{M}F_1(x,Q^2)\sin^2\frac{\theta}{2}\right]} \qquad (3)$$

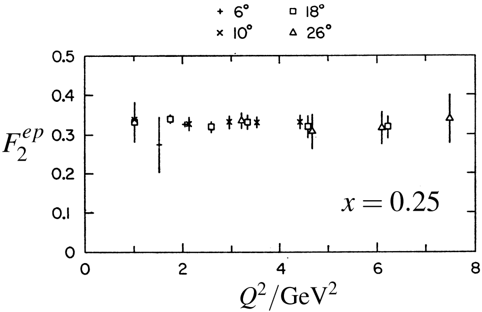
Electromagnetic Structure Function

Pure Magnetic Structure Function

Measuring the Structure Functions

★ To determine $F_1(x,Q^2)$ and $F_2(x,Q^2)$ for a given x and Q^2 need measurements of the differential cross section at several different scattering angles and incoming electron beam energies (see Q13 on examples sheet)

Example: electron-proton scattering F_2 vs. Q^2 at fixed x



J.T.Friedman + H.W.Kendall, Ann. Rev. Nucl. Sci. 22 (1972) 203

- Experimentally it is observed that both F_1 and F_2 are (almost) independent of Q^2

Bjorken Scaling and the Callan-Gross Relation

★The near (see later) independence of the structure functions on Q^2 is known as **Bjorken Scaling**, i.e.

$$\boxed{F_1(x,Q^2) \rightarrow F_1(x) \qquad\qquad F_2(x,Q^2) \rightarrow F_2(x)}$$

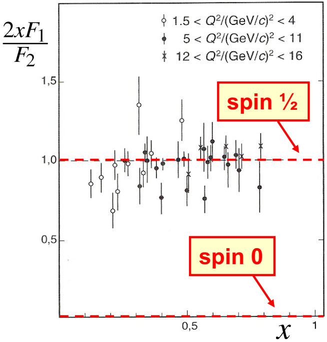
•It is strongly suggestive of scattering from **point-like** constituents within the proton

★It is also observed that $F_1(x)$ and $F_2(x)$ are not independent but satisfy the **Callan-Gross relation**

$$\boxed{F_2(x) = 2xF_1(x)}$$

•As we shall soon see this is exactly what is expected for scattering from **spin-half** quarks.

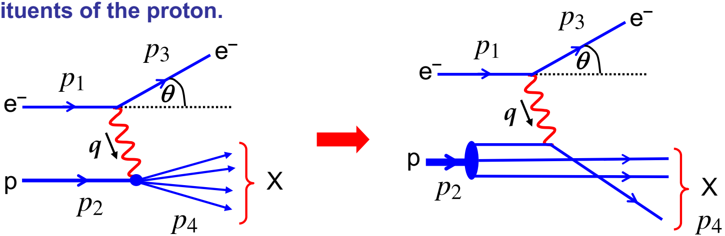
•**Note** if quarks were spin zero particles we would expect the purely magnetic structure function to be zero, i.e. $F_1(x) = 0$



The Quark-Parton Model

- Before quarks and gluons were generally accepted Feynman proposed that the proton was made up of point-like constituents “**partons**”

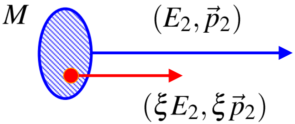
- Both Bjorken Scaling and the Callan-Gross relationship can be explained by assuming that Deep Inelastic Scattering is dominated by the scattering of a single virtual photon from point-like spin-half constituents of the proton.

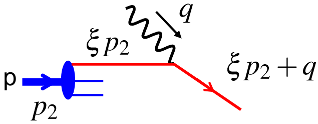


Scattering from a proton with structure functions → **Scattering from a point-like quark within the proton**

★ How do these two pictures of the interaction relate to each other?

- In the parton model the basic interaction is **ELASTIC** scattering from a **"quasi-free"** spin-½ quark in the proton, i.e. treat the quark as a free particle!
- The parton model is most easily formulated in a frame where the proton has very high energy, often referred to as the **"infinite momentum frame"**, where we can neglect the proton mass and $p_2 = (E_2, 0, 0, E_2)$
- In this frame can also neglect the mass of the quark and any momentum transverse to the direction of the proton.
- Let the quark carry a fraction ξ of the proton's four-momentum.





- After the interaction the struck quark's four-momentum is $\xi p_2 + q$

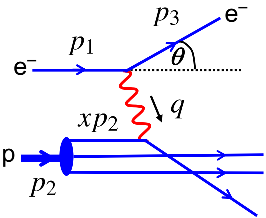
$$(\xi p_2 + q)^2 = m_q^2 \approx 0 \quad \Rightarrow \quad \cancel{\xi^2 p_2^2} + q^2 + 2\xi p_2 . q = 0 \qquad (\xi^2 p_2^2 = m_q^2 \approx 0)$$

$$\Rightarrow \quad \xi = \frac{Q^2}{2 p_2 . q} = x$$

Bjorken x can be identified as the fraction of the proton momentum carried by the struck quark (in a frame where the proton has very high energy)

- In terms of the proton momentum

$$s=(p_1+p_2)^2\simeq 2p_1.p_2 \qquad y=\frac{p_2.q}{p_2.p_1} \qquad x=\frac{Q^2}{2p_2.q}$$



- But for the underlying quark interaction

$$s^q=(p_1+xp_2)^2=2xp_1.p_2=xs$$

$$y_q=\frac{p_q.q}{p_q.p_1}=\frac{xp_2.q}{xp_2.p_1}=y$$

$$x_q=1$$ (elastic, i.e. assume quark does not break up)

- Previously derived the Lorentz Invariant cross section for e⁻μ⁻ → e⁻μ⁻ elastic scattering in the ultra-relativistic limit (handout 4 + Q10 on examples sheet).

  Now apply this to e⁻q → e⁻q

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2}=\frac{2\pi\alpha^2e_q^2}{q^4}\left[1+\left(1+\frac{q^2}{s_q}\right)^2\right]$$

e_q **is quark charge, i.e.** $e_u=+2/3;\quad e_d=-1/3$

- Using $-q^2=Q^2=(s_q-m^2)x_qy_q$ ➡ $\frac{q^2}{s_q}=-y_q=-y$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2}=\frac{2\pi\alpha^2e_q^2}{Q^4}\left[1+(1-y)^2\right]$$

(where the last two expression assume the massless limit m=0)

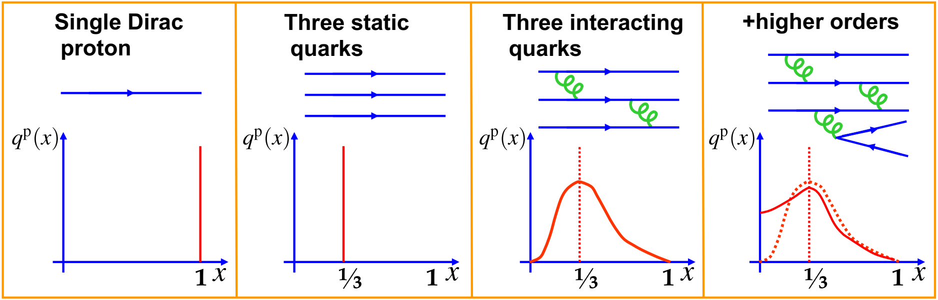
$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2 e_q^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right] \qquad (3)$$

★ This is the expression for the differential cross-section for elastic e⁻q scattering from a quark carrying a fraction x of the proton momentum.

- Now need to account for distribution of quark momenta within proton

★ Introduce parton distribution functions such that $q^{p}(x)dx$ is the number of quarks of type q within a proton with momenta between $x \rightarrow x + dx$

● Expected form of the parton distribution function ?



Single Dirac proton

Three static quarks

Three interacting quarks

+higher orders

- The cross section for scattering from a **particular quark type** within the proton which in the range $x \rightarrow x + \mathrm{d}x$ is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)+\frac{y^2}{2}\right] \times e_q^2 q^{\mathrm{p}}(x)\mathrm{d}x$$

- Summing over all types of quark within the proton gives the expression for the electron-proton scattering cross section

$$\boxed{\frac{\mathrm{d}^2\sigma^{\mathrm{ep}}}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)+\frac{y^2}{2}\right]\sum_q e_q^2 q^{\mathrm{p}}(x)} \qquad (5)$$

- Compare with the electron-proton scattering cross section in terms of structure functions (equation (2)):

$$\boxed{\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right]} \qquad (6)$$

- By comparing (5) and (6) obtain the parton model prediction for the structure functions in the general L.I. form for the differential cross section

$$\boxed{F_2^{\mathrm{p}}(x,Q^2) = 2xF_1^{\mathrm{p}}(x,Q^2) = x\sum_q e_q^2 q^{\mathrm{p}}(x)}$$

⇒ Can relate measured structure functions to the underlying quark distributions

The parton model predicts:

- **Bjorken Scaling** $F_1(x,Q^2) \rightarrow F_1(x) \qquad F_2(x,Q^2) \rightarrow F_2(x)$
  - ✶ **Due to scattering from point-like particles within the proton**
- **Callan-Gross Relation** $F_2(x) = 2xF_1(x)$
  - ✶ **Due to scattering from spin half Dirac particles where the magnetic moment is directly related to the charge; hence the "electro-magnetic" and "pure magnetic" terms are fixed with respect to each other.**

★ **At present parton distributions cannot be calculated from QCD**

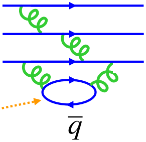
- **Can't use perturbation theory due to large coupling constant**

★ **Measurements of the structure functions enable us to determine the parton distribution functions !**

★ **For electron-proton scattering we have:**

$$F_2^{\mathrm{p}}(x) = x\sum_q e_q^2 q^{\mathrm{p}}(x)$$

- **Due to higher orders, the proton contains not only up and down quarks but also anti-up and anti-down quarks (will neglect the small contributions from heavier quarks)**



- **For electron-proton scattering have:**

$$F_2^{\mathrm{ep}}(x) = x\sum_q e_q^2 q^{\mathrm{p}}(x) = x\left(\frac{4}{9}u^{\mathrm{p}}(x) + \frac{1}{9}d^{\mathrm{p}}(x) + \frac{4}{9}\overline{u}^{\mathrm{p}}(x) + \frac{1}{9}\overline{d}^{\mathrm{p}}(x)\right)$$

- **For electron-neutron scattering have:**

$$F_2^{\mathrm{en}}(x) = x\sum_q e_q^2 q^{\mathrm{n}}(x) = x\left(\frac{4}{9}u^{\mathrm{n}}(x) + \frac{1}{9}d^{\mathrm{n}}(x) + \frac{4}{9}\overline{u}^{\mathrm{n}}(x) + \frac{1}{9}\overline{d}^{\mathrm{n}}(x)\right)$$

★ **Now assume "isospin symmetry", i.e. that the neutron (ddu) is the same as a proton (uud) with up and down quarks interchanged, i.e.**

$$d^{\mathrm{n}}(x) = u^{\mathrm{p}}(x); \qquad u^{\mathrm{n}}(x) = d^{\mathrm{p}}(x)$$

and define the neutron distributions functions in terms of those of the proton

$$u(x) \equiv u^{\mathrm{p}}(x) = d^{\mathrm{n}}(x); \qquad d(x) \equiv d^{\mathrm{p}}(x) = u^{\mathrm{n}}(x)$$

$$\overline{u}(x) \equiv \overline{u}^{\mathrm{p}}(x) = \overline{d}^{\mathrm{n}}(x); \qquad \overline{d}(x) \equiv \overline{d}^{\mathrm{p}}(x) = \overline{u}^{\mathrm{n}}(x)$$

giving:

$$\boxed{F_2^{\mathrm{ep}}(x) = 2xF_1^{\mathrm{ep}}(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\overline{u}(x) + \frac{1}{9}\overline{d}(x)\right)} \quad \textbf{(7)}$$

$$\boxed{F_2^{\mathrm{en}}(x) = 2xF_1^{\mathrm{en}}(x) = x\left(\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\overline{d}(x) + \frac{1}{9}\overline{u}(x)\right)} \quad \textbf{(8)}$$

- **Integrating (7) and (8) :**

$$\int_0^1 F_2^{ep}(x)\mathrm{d}x = \int_0^1 x\left(\frac{4}{9}[u(x)+\overline{u}(x)] + \frac{1}{9}[d(x)+\overline{d}(x)]\right)\mathrm{d}x = \frac{4}{9}f_u + \frac{1}{9}f_d$$

$$\int_0^1 F_2^{en}(x)\mathrm{d}x = \int_0^1 x\left(\frac{4}{9}[d(x)+\overline{d}(x)] + \frac{1}{9}[u(x)+\overline{u}(x)]\right)\mathrm{d}x = \frac{4}{9}f_d + \frac{1}{9}f_u$$

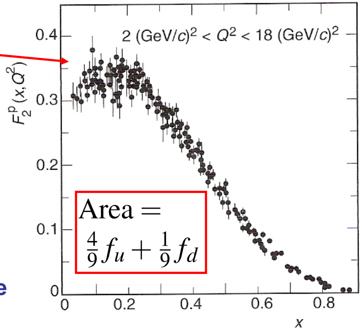
★ $f_u = \int_0^1 [xu(x) + x\overline{u}(x)]\mathrm{d}x$ **is the fraction of the proton momentum carried by the up and anti-up quarks**

Experimentally

$$\int F_2^{ep}(x)\mathrm{d}x \approx 0.18$$

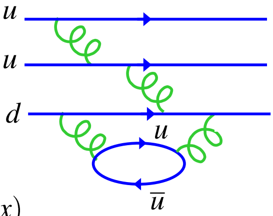
$$\int F_2^{en}(x)\mathrm{d}x \approx 0.12$$

$$\Rightarrow \boxed{f_u \approx 0.36 \qquad f_d \approx 0.18}$$



- ✶ **In the proton, as expected, the up quarks carry twice the momentum of the down quarks**
- ✶ **The quarks carry just over 50 % of the total proton momentum. The rest is carried by gluons (which being neutral doesn't contribute to electron-nucleon scattering).**

Valence and Sea Quarks



- As we are beginning to see the proton is complex…
- The parton distribution function $u^{\mathrm{p}}(x) = u(x)$ includes contributions from the "valence" quarks and the virtual quarks produced by gluons: the "sea"
- Resolving into valence and sea contributions:

$$u(x) = u_{\mathrm{V}}(x) + u_{\mathrm{S}}(x) \qquad d(x) = d_{\mathrm{V}}(x) + d_{\mathrm{S}}(x)$$
$$\overline{u}(x) = \overline{u}_{\mathrm{S}}(x) \qquad \overline{d}(x) = \overline{d}_{\mathrm{S}}(x)$$

- The proton contains two valence up quarks and one valence down quark and would expect: $\int_0^1 u_{\mathrm{V}}(x)\mathrm{d}x = 2 \qquad \int_0^1 d_{\mathrm{V}}(x)\mathrm{d}x = 1$
- But no *a priori* expectation for the total number of sea quarks !
- But sea quarks arise from gluon quark/anti-quark pair production and with $m_u = m_d$ it is reasonable to expect

$$u_{\mathrm{S}}(x) = d_{\mathrm{S}}(x) = \overline{u}_{\mathrm{S}}(x) = \overline{d}_{\mathrm{S}}(x) = S(x)$$

- With these relations (7) and (8) become

$$F_2^{\mathrm{ep}}(x) = x\left(\frac{4}{9}u_{\mathrm{V}}(x) + \frac{1}{9}d_{\mathrm{V}}(x) + \frac{10}{9}S(x)\right) \qquad F_2^{\mathrm{en}}(x) = x\left(\frac{4}{9}d_{\mathrm{V}}(x) + \frac{1}{9}u_{\mathrm{V}}(x) + \frac{10}{9}S(x)\right)$$

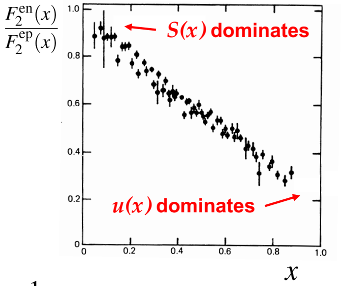
Giving the ratio

$$\frac{F_2^{\mathrm{en}}(x)}{F_2^{\mathrm{ep}}(x)} = \frac{4d_V(x)+u_V(x)+10S(x)}{4u_V(x)+d_V(x)+10S(x)}$$

- The sea component arises from processes such as $g \to \overline{u}u$. Due to the $1/q^2$ dependence of the gluon propagator, much more likely to produce low energy gluons. Expect the sea to comprise of low energy $q/\overline{q}$
- Therefore at low x expect the sea to dominate:

$$\frac{F_2^{\mathrm{en}}(x)}{F_2^{\mathrm{ep}}(x)} \to 1 \quad \text{as} \quad x \to 0$$

Observed experimentally



- At high x expect the sea contribution to be small

$$\frac{F_2^{\mathrm{en}}(x)}{F_2^{\mathrm{ep}}(x)} \to \frac{4d_V(x)+u_V(x)}{4u_V(x)+d_V(x)} \quad \text{as} \quad x \to 1$$

Note: $u_V = 2d_V$ would give ratio 2/3 as $x \to 1$

Experimentally $F_2^{\mathrm{en}}(x)/F_2^{\mathrm{ep}}(x) \to 1/4 \quad \text{as} \quad x \to 1$

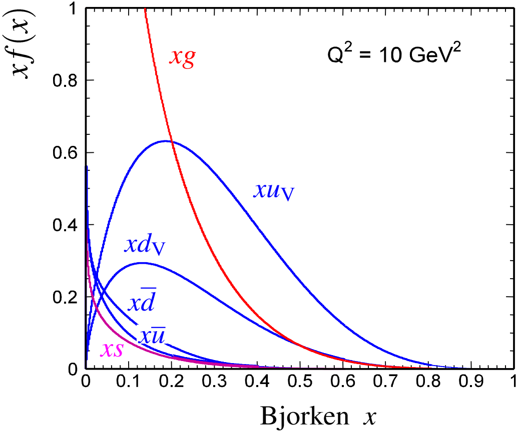
➡ $d(x)/u(x) \to 0 \quad \text{as} \quad x \to 1$

This behaviour is not understood.

Parton Distribution Functions

- Ultimately the parton distribution functions are obtained from a fit to all experimental data including neutrino scattering (see handout 10)
  - Hadron-hadron collisions give information on gluon pdf $g(x)$

Fit to all data

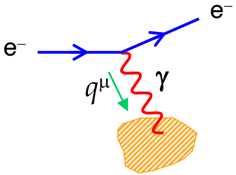


Note:

- Apart from at large x
  $u_V(x) \approx 2d_V(x)$
- For $x < 0.2$
  gluons dominate
- In fits to data assume
  $u_s(x) = \overline{u}(x)$
- $\overline{d}(x) > \overline{u}(x)$
  not understood – exclusion principle?
- Small strange quark component $s(x)$

(Try Question 12)

Scaling Violations

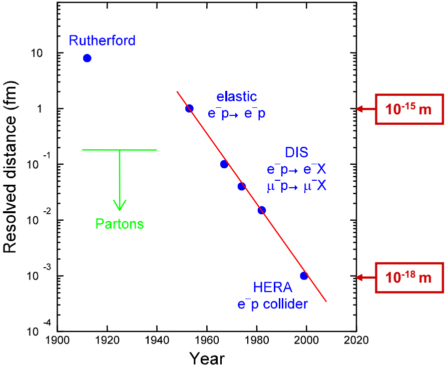


- In last 40 years, experiments have probed the proton with virtual photons of ever increasing energy
- Non-point like nature of the scattering becomes apparent when λ_γ ~ size of scattering centre

$$\lambda_\gamma = \frac{h}{|\vec{q}|} \sim \mathrm{O}\left(\frac{1}{|\vec{q}|\,/\,\mathrm{GeV}}\right)\mathrm{fm}$$

- Scattering from point-like quarks gives rise to **Bjorken scaling**: no q^2 cross section dependence
- IF quarks were not point-like, at high q^2 (when the wavelength of the virtual photon ~ size of quark) would observe rapid decrease in cross section with increasing q^2.
- To search for quark sub-structure want to go to highest q^2

HERA

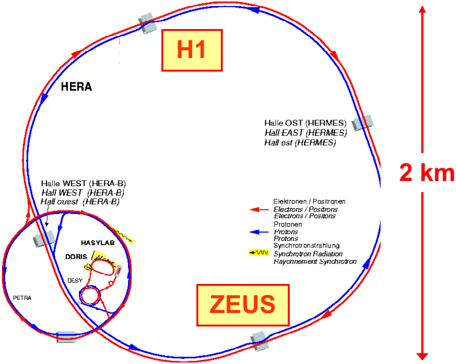


HERA $e^{\pm}p$ Collider : 1991-2007

- ★ **DESY (Deutsches Elektronen-Synchroton) Laboratory, Hamburg, Germany**

$e^{\pm}$ **27.5 GeV** → ← **820 GeV** p

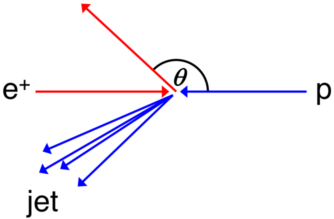
$\sqrt{s}$ **= 300 GeV**



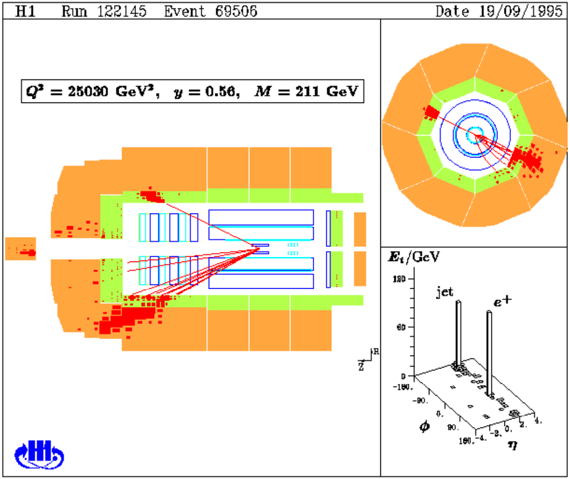
- ★ **Two large experiments : H1 and ZEUS**
- ★ **Probe proton at very high Q^2 and very low x**

Example of a High Q^2 Event in H1

✶ **Event kinematics determined from electron angle and energy**



✶ **Also measure hadronic system (although not as precisely) – gives some redundancy**



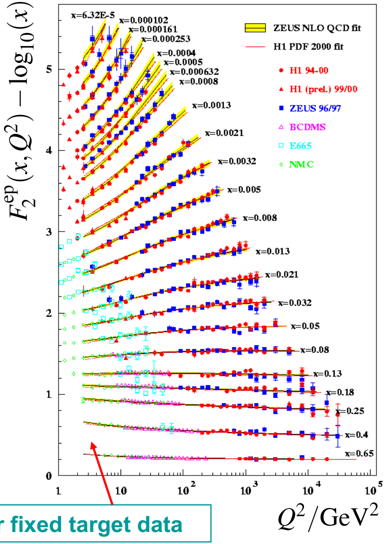
$F_2(x,Q^2)$ Results

- No evidence of rapid decrease of cross section at highest Q^2

$$\Rightarrow \quad \boxed{R_{\text{quark}} < 10^{-18}\,\text{m}}$$

- For $x > 0.05$, only weak dependence of F_2 on Q^2 : consistent with the expectation from the quark-parton model
- But observe clear scaling violations, particularly at low x

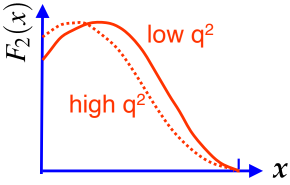
$$F_2(x,Q^2) \neq F_2(x)$$

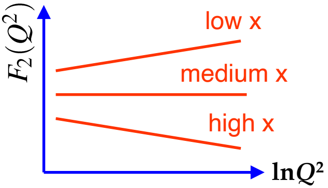


Earlier fixed target data

Origin of Scaling Violations

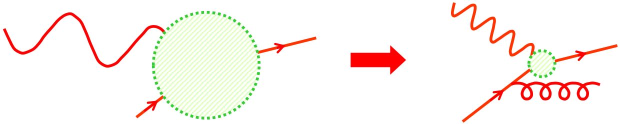
★ Observe “small” deviations from exact Bjorken scaling $F_2(x) \to F_2(x,Q^2)$





★ At high Q^2 observe more low x quarks

★ **“Explanation”:** at high Q^2 (shorter wave-length) resolve finer structure: i.e. reveal quark is sharing momentum with gluons. At higher Q^2 expect to “see” more low x quarks



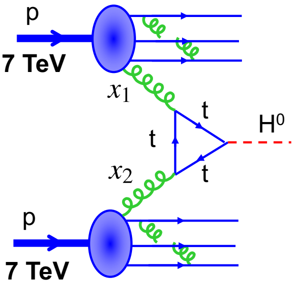
★ QCD cannot predict the x dependence of $F_2(x,Q^2)$

★ But QCD can predict the Q^2 dependence of $F_2(x,Q^2)$

Proton-Proton Collisions at the LHC

★ **Measurements of structure functions not only provide a powerful test of QCD, the parton distribution functions are essential for the calculation of cross sections at pp and p$\bar{\text{p}}$ colliders.**

- **Example: Higgs production at the Large Hadron Collider LHC (2009-)**
  - **The LHC collides up to 7 TeV protons with 7 TeV protons**
  - **However underlying collisions are between partons**
  - **Higgs production the LHC dominated by "gluon-gluon fusion"**



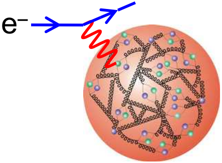
- **Cross section depends on gluon PDFs**

$$\sigma(pp \to HX) \sim \int_0^1 \int_0^1 g(x_1)g(x_2)\sigma(gg \to H)\mathrm{d}x_1\mathrm{d}x_2$$

- **Uncertainty in gluon PDFs lead to a ±5 % uncertainty in Higgs production cross section**

- **Prior to HERA data uncertainty was ±25 %**

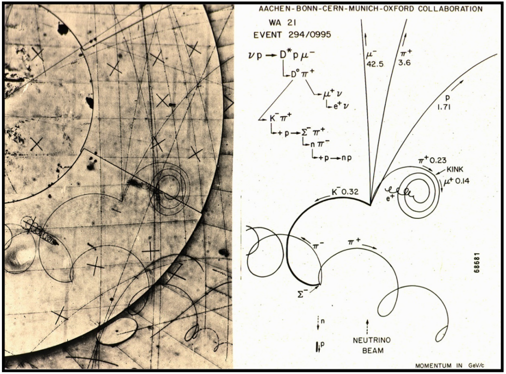
Summary



- At very high electron energies $\lambda \ll r_p$:
  the proton appears to be a sea of
  quarks and gluons.
- Deep Inelastic Scattering = Elastic scattering from the quasi-free constituent quarks
  - ➡ Bjorken Scaling $F_1(x, Q^2) \rightarrow F_1(x)$ **point-like scattering**
  - ➡ Callan-Gross $F_2(x) = 2xF_1(x)$ **Scattering from spin-1/2**
- Describe scattering in terms of parton distribution functions $u(x), d(x), \ldots$ which describe momentum distribution inside a nucleon
- The proton is much more complex than just uud - sea of anti-quarks/gluons
- Quarks carry only 50 % of the protons momentum – the rest is due to low energy gluons
- We will come back to this topic when we discuss neutrino scattering…

Particle Physics

Dr Lester



Handout 7 : Symmetries and the Quark Model

Introduction/Aims

★ Symmetries play a central role in particle physics; one aim of particle physics is to discover the fundamental symmetries of our universe

★ In this handout will apply the idea of symmetry to the quark model with the aim of :

- Deriving **hadron wave-functions**
- Providing an introduction to the more abstract ideas of **colour** and **QCD** (handout 8)
- Ultimately explaining why hadrons only exist as $\bar{q}q$ (mesons) qqq (baryons) or $\bar{q}\bar{q}\bar{q}$ (anti-baryons)

\+ introduce the ideas of the **SU(2)** and **SU(3)** symmetry groups which play a major role in particle physics (see handout 13)

Symmetries and Conservation Laws

★Suppose physics is invariant under the transformation

$$\psi \rightarrow \psi' = \hat{U}\psi$$

e.g. rotation of the coordinate axes

- To conserve probability normalisation require

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle$$

$$\rightarrow \quad \boxed{\hat{U}^\dagger\hat{U} = 1}$$

i.e. $\hat{U}$ has to be unitary

- For physical predictions to be unchanged by the symmetry transformation, also require all QM matrix elements unchanged

$$\boxed{\langle\psi|\hat{H}|\psi\rangle = \langle\psi'|\hat{H}|\psi'\rangle} = \langle\psi|\hat{U}^\dagger\hat{H}\hat{U}|\psi\rangle$$

i.e. require $\quad \hat{U}^\dagger\hat{H}\hat{U} = \hat{H}$

$\times\hat{U} \quad \hat{U}\hat{U}^\dagger\hat{H}\hat{U} = \hat{U}\hat{H} \rightarrow \hat{H}\hat{U} = \hat{U}\hat{H}$

therefore $\quad \boxed{[\hat{H},\hat{U}] = 0}$

$\hat{U}$ commutes with the Hamiltonian

★Now consider the infinitesimal transformation (ε small)

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

($\hat{G}$ is called the generator of the transformation)

- For $\hat{U}$ to be unitary

$$\hat{U}\hat{U}^\dagger = (1 + i\varepsilon\hat{G})(1 - i\varepsilon\hat{G}^\dagger) = 1 + i\varepsilon(\hat{G} - \hat{G}^\dagger) + O(\varepsilon^2)$$

neglecting terms in ε^2 $\quad UU^\dagger = 1$ ➡ $\boxed{\hat{G} = \hat{G}^\dagger}$

i.e. $\hat{G}$ is Hermitian and therefore corresponds to an observable quantity G !

- Furthermore, $\quad [\hat{H}, \hat{U}] = 0 \quad \Rightarrow [\hat{H}, 1 + i\varepsilon\,\hat{G}] = 0 \quad \Rightarrow [\hat{H}, \hat{G}] = 0$

But from QM $\quad \dfrac{\mathrm{d}}{\mathrm{d}t}\langle\hat{G}\rangle = i\langle[\hat{H}, \hat{G}]\rangle = 0$

i.e. G is a conserved quantity.

Symmetry ⟷ Conservation Law

★ **For each symmetry of nature have an observable conserved quantity**

Example: Infinitesimal spatial translation $x \rightarrow x + \varepsilon$

i.e. expect physics to be invariant under $\psi(x) \rightarrow \psi' = \psi(x + \varepsilon)$

$$\psi'(x) = \psi(x + \varepsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\varepsilon = \left(1 + \varepsilon\frac{\partial}{\partial x}\right)\psi(x)$$

but $\quad \hat{p}_x = -i\frac{\partial}{\partial x}$ ➡ $\psi'(x) = (1 + i\varepsilon\hat{p}_x)\psi(x)$

The generator of the symmetry transformation is $\hat{p}_x$ ➡ p_x is conserved

- **Translational invariance** of physics implies **momentum conservation** !

- In general the symmetry operation may depend on more than one parameter

$$\hat{U} = 1 + i\vec{\varepsilon}.\vec{G}$$

For example for an infinitesimal 3D linear translation : $\vec{r} \rightarrow \vec{r} + \vec{\varepsilon}$

$$\longrightarrow \quad \hat{U} = 1 + i\vec{\varepsilon}.\vec{p} \qquad \vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$$

- So far have only considered an infinitesimal transformation, however a finite transformation can be expressed as a series of infinitesimal transformations

$$\hat{U}(\vec{\alpha}) = \lim_{n\rightarrow\infty}\left(1 + i\frac{\vec{\alpha}}{n}.\vec{G}\right)^n = e^{i\vec{\alpha}.\vec{G}}$$

<u>Example:</u> Finite spatial translation in 1D: $x \rightarrow x + x_0$ with $\hat{U}(x_0) = e^{ix_0\hat{p}_x}$

$$\begin{aligned}\psi'(x) = \psi(x+x_0) \quad &= \quad \hat{U}\psi(x) = \exp\left(x_0\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) \qquad \left(p_x = -i\frac{\partial}{\partial x}\right) \\ &= \quad \left(1 + x_0\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x_0^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \ldots\right)\psi(x) \\ &= \quad \psi(x) + x_0\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{x_0^2}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \ldots\end{aligned}$$

i.e. obtain the expected Taylor expansion

Symmetries in Particle Physics : Isospin

- The proton and neutron have very similar masses and the nuclear force is found to be approximately charge-independent, i.e.

$$V_{pp} \approx V_{np} \approx V_{nn}$$

- To reflect this symmetry, Heisenberg (1932) proposed that if you could "switch off" the electric charge of the proton

There would be no way to distinguish between a proton and neutron

- Proposed that the neutron and proton should be considered as two states of a single entity; the **nucleon**

$$p = \begin{pmatrix}1\\0\end{pmatrix} \qquad n = \begin{pmatrix}0\\1\end{pmatrix}$$

★ Analogous to the spin-up/spin-down states of a spin-½ particle

ISOSPIN

★ Expect physics to be invariant under rotations in this space

- The neutron and proton form an isospin doublet with total isospin $I = ½$ and third component $I_3 = \pm ½$

Flavour Symmetry of the Strong Interaction

We can extend this idea to the quarks:

★ Assume the strong interaction treats all quark flavours equally (it does)

- **Because $m_u \approx m_d$:**

  The strong interaction possesses an approximate flavour symmetry
  i.e. from the point of view of the strong interaction nothing changes if all up quarks are replaced by down quarks and *vice versa*.

- Choose the basis $$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Express the invariance of the strong interaction under $u \leftrightarrow d$ as invariance under “rotations” in an abstract isospin space

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

The 2x2 unitary matrix depends on 4 complex numbers, i.e. 8 real parameters
But there are four constraints from $\hat{U}^\dagger \hat{U} = 1$

➡ 8 – 4 = **4 independent matrices**

- In the language of group theory the four matrices form the U(2) group

- One of the matrices corresponds to multiplying by a phase factor

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

not a flavour transformation and of no relevance here.

- The remaining three matrices form an **SU(2)** group **(special unitary)** with $\boxed{\det U = 1}$
- For an infinitesimal transformation, in terms of the **Hermitian** generators $\hat{G}$

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

- $\det U = 1 \quad \Rightarrow \quad Tr(\hat{G}) = 0$
- A linearly independent choice for $\hat{G}$ are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The proposed flavour symmetry of the strong interaction has the same transformation properties as SPIN !
- Define **ISOSPIN**: $\vec{T} = \frac{1}{2}\vec{\sigma} \qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$
- Check this works, for an infinitesimal transformation

$$\hat{U} = 1 + \frac{1}{2}i\vec{\varepsilon}.\vec{\sigma} \quad = \quad 1 + \frac{i}{2}(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3) = \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i(\varepsilon_1 - i\varepsilon_2) \\ \frac{1}{2}i(\varepsilon_1 + i\varepsilon_2) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix}$$

Which is, as required, unitary and has unit determinant

$$U^\dagger U = I + O(\varepsilon^2) \qquad \det U = 1 + O(\varepsilon^2)$$

Properties of Isopin

- Isospin has the exactly the same properties as spin

$$[T_1,T_2] = iT_3 \quad [T_2,T_3] = iT_1 \quad [T_3,T_1] = iT_2$$

$$[T^2,T_3] = 0 \qquad T^2 = T_1^2 + T_2^2 + T_3^2$$

As in the case of spin, have three non-commuting operators, T_1, T_2, T_3 and even though all three correspond to observables, can't know them simultaneously. So label states in terms of total isospin I and the third component of isospin I_3

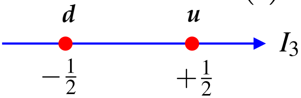
NOTE: isospin has nothing to do with spin – just the same mathematics

- The eigenstates are exact analogues of the eigenstates of ordinary angular momentum $|s,m\rangle \rightarrow |I,I_3\rangle$

  with $T^2|I,I_3\rangle = I(I+1)|I,I_3\rangle \qquad T_3|I,I_3\rangle = I_3|I,I_3\rangle$

- In terms of isospin:

$$u = \begin{pmatrix}1\\0\end{pmatrix} = |\,\tfrac{1}{2},+\tfrac{1}{2}\,\rangle \qquad d = \begin{pmatrix}0\\1\end{pmatrix} = |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$



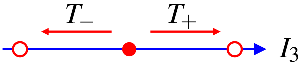
$$\boxed{I = \tfrac{1}{2}, \quad I_3 = \pm\tfrac{1}{2}}$$

- In general $I_3 = \frac{1}{2}(N_u - N_d)$

- Can define isospin ladder operators – analogous to spin ladder operators

$$\boxed{T_- \equiv T_1 - iT_2}$$

u → d



$$\boxed{T_+ \equiv T_1 + iT_2}$$

d → u

$$T_+|I,I_3\rangle = \sqrt{I(I+1) - I_3(I_3+1)}|I,I_3+1\rangle$$

$$T_-|I,I_3\rangle = \sqrt{I(I+1) - I_3(I_3-1)}|I,I_3-1\rangle$$

Step up/down in I_3 until reach end of multiplet $T_+|I,+I\rangle = 0 \quad T_-|I,-I\rangle = 0$

$$\boxed{T_+u = 0 \qquad T_+d = u \qquad T_-u = d \qquad T_-d = 0}$$

- Ladder operators turn $u \longrightarrow d$ and $d \longrightarrow u$

★ Combination of isospin: **e.g.** what is the isospin of a system of two **d** quarks, is exactly analogous to combination of spin (i.e. angular momentum)

$$|I^{(1)}, I_3^{(1)}\rangle |I^{(2)}, I_3^{(2)}\rangle \longrightarrow |I, I_3\rangle$$

- I_3 additive : $I_3 = I_3^{(1)} + I_3^{(2)}$
- I in integer steps from $|I^{(1)} - I^{(2)}|$ to $|I^{(1)} + I^{(2)}|$

★ Assumed symmetry of Strong Interaction under isospin transformations implies the existence of conserved quantites

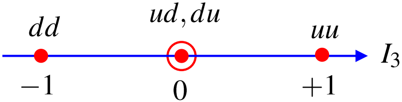
- In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum

Combining Quarks

Goal: derive proton wave-function

- First combine two quarks, then combine the third
- Use requirement that **fermion** wave-functions are **anti-symmetric**

Isospin starts to become useful in defining states of more than one quark.
e.g. two quarks, here we have four possible combinations:



Note: ◉ represents two states with the same value of I_3

- We can immediately identify the extremes (I_3 additive)

$$uu \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |1,+1\rangle \qquad dd \equiv |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |1,-1\rangle$$

To obtain the $|1,0\rangle$ state use ladder operators

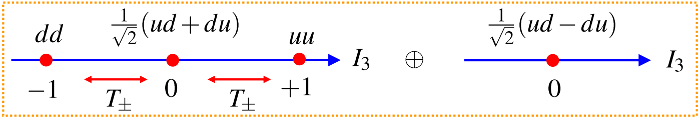
$$T_-|1,+1\rangle = \sqrt{2}|1,0\rangle = T_-(uu) = ud + du$$

$$\Rightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}}(ud+du)$$

The final state, $|0,0\rangle$, can be found from orthogonality with $|1,0\rangle$

$$\Rightarrow \quad |0,0\rangle = \frac{1}{\sqrt{2}}(ud-du)$$

- From four possible combinations of isospin doublets obtain a triplet of **isospin 1** states and a singlet **isospin 0** state $2 \otimes 2 = 3 \oplus 1$

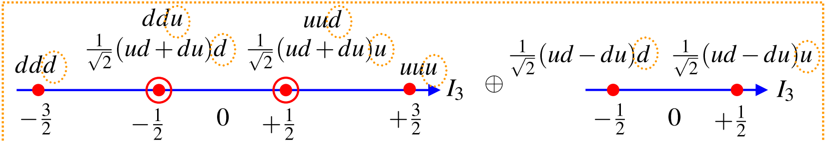


dd $\quad \frac{1}{\sqrt{2}}(ud+du)$ $\quad uu$ $\quad I_3$ $\quad \oplus \quad \frac{1}{\sqrt{2}}(ud-du)$ $\quad I_3$

$-1 \quad T_\pm \quad 0 \quad T_\pm \quad +1 \qquad\qquad 0$

- Can move around **within** multiplets using ladder operators
- note, as anticipated $\quad I_3 = \frac{1}{2}(N_u - N_d)$
- States with different **total isospin** are physically different – the **isospin 1 triplet** is symmetric under interchange of quarks 1 and 2 whereas **singlet** is anti-symmetric

★ Now add an additional up or down quark. From each of the above 4 states get two new isospin states with $I_3' = I_3 \pm \frac{1}{2}$



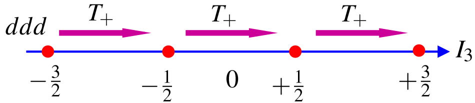
$ddd \quad \frac{1}{\sqrt{2}}(ud+du)d \quad ddu \quad \frac{1}{\sqrt{2}}(ud+du)u \quad uud \quad uuu \quad I_3 \quad \oplus \quad \frac{1}{\sqrt{2}}(ud-du)d \quad \frac{1}{\sqrt{2}}(ud-du)u \quad I_3$

$-\frac{3}{2} \quad -\frac{1}{2} \quad 0 \quad +\frac{1}{2} \quad +\frac{3}{2} \qquad\qquad -\frac{1}{2} \quad 0 \quad +\frac{1}{2}$

6 $\qquad$ **2**

- Use ladder operators and orthogonality to group the 6 states into isospin multiplets, e.g. to obtain the $I = \frac{3}{2}$ states, step up from ddd

★Derive the $I=\frac{3}{2}$ states from $ddd \equiv |\frac{3}{2},-\frac{3}{2}\rangle$



$$\begin{aligned}
T_+|\tfrac{3}{2},-\tfrac{3}{2}\rangle &= T_+(ddd) = (T_+d)dd + d(T_+d)d + dd(T_+)d \\
\sqrt{3}|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= udd + dud + ddu \\
\boxed{|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(udd + dud + ddu)} \\
T_+|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}T_+(udd + dud + ddu) \\
2|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uud + udu + uud + duu + udu + duu) \\
\boxed{|\tfrac{3}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(uud + udu + duu)} \\
T_+|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}T_+(uud + udu + duu) \\
\sqrt{3}|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uuu + uuu + uuu) \\
\boxed{|\tfrac{3}{2},+\tfrac{3}{2}\rangle = uuu}
\end{aligned}$$

★ From the $\boxed{6}$ states on previous page, use orthoganality to find $|\frac{1}{2},\pm\frac{1}{2}\rangle$ states

★ The $\boxed{2}$ states on the previous page give another $|\frac{1}{2},\pm\frac{1}{2}\rangle$ doublet

★The eight states $uuu,\ uud,\ udu,\ udd,\ duu,\ dud,\ ddu,\ ddd$
are grouped into an isospin quadruplet and two isospin doublets

$$2\otimes2\otimes2=2\otimes(3\oplus1)=(2\otimes3)\oplus(2\otimes1)=4\oplus2\oplus2$$

- Different multiplets have different symmetry properties

$$\left.\begin{aligned}
|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= uuu\\
|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uud+udu+duu)\\
|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(ddu+dud+udd)\\
|\tfrac{3}{2},-\tfrac{3}{2}\rangle &= ddd
\end{aligned}\right\}\ \boxed{\mathrm{S}}$$

A quadruplet of states which are symmetric under the interchange of **any** two quarks

$$\left.\begin{aligned}
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= -\tfrac{1}{\sqrt{6}}(2ddu-udd-dud)\\
|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{6}}(2uud-udu-duu)
\end{aligned}\right\}\ \boxed{\mathrm{M_S}}$$

Mixed symmetry.
Symmetric for 1 ⟷ 2

$$\left.\begin{aligned}
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(udd-dud)\\
|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(udu-duu)
\end{aligned}\right\}\ \boxed{\mathrm{M_A}}$$

Mixed symmetry.
Anti-symmetric for 1 ⟷ 2

- Mixed symmetry states have no definite symmetry under interchange of quarks $1\leftrightarrow3$ etc.

Combining Spin

- Can apply exactly the same mathematics to determine the possible spin wave-functions for a combination of 3 spin-half particles

$$\left.\begin{aligned}
|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= \uparrow\uparrow\uparrow \\
|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \\
|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow) \\
|\tfrac{3}{2},-\tfrac{3}{2}\rangle &= \downarrow\downarrow\downarrow
\end{aligned}\right\} \boxed{\mathrm{S}}$$

A quadruplet of states which are symmetric under the interchange of any two quarks

$$\left.\begin{aligned}
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= -\tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow) \\
|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)
\end{aligned}\right\} \boxed{\mathrm{M_S}}$$

Mixed symmetry.
Symmetric for 1 ⟷ 2

$$\left.\begin{aligned}
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow) \\
|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)
\end{aligned}\right\} \boxed{\mathrm{M_A}}$$

Mixed symmetry.
Anti-symmetric for 1 ⟷ 2

- Can now form total wave-functions for combination of three quarks

Baryon Wave-functions (ud)

★ Quarks are fermions so require that the total wave-function is **anti-symmetric** under the interchange of any two quarks

★ the total wave-function can be expressed in terms of:

$$\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\xi_{\text{colour}}\eta_{\text{space}}$$

★ The colour wave-function for all bound qqq states is **anti-symmetric** (see handout 8)

- Here we will only consider the lowest mass, **ground state**, baryons where there is no internal orbital angular momentum.
- For **L=0** the spatial wave-function is **symmetric** $(-1)^L$.

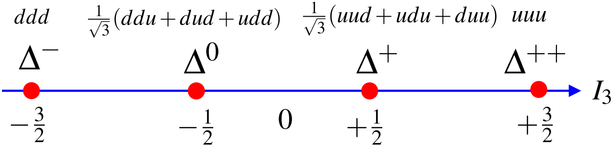
➡ $\xi_{\text{colour}}\eta_{\text{space}}$ **anti-symmetric**

➡ $\phi_{\text{flavour}}\chi_{\text{spin}}$ **symmetric**

Overall anti-symmetric

★ **Two ways** to form a totally symmetric wave-function from spin and isospin states:

❶ combine totally symmetric spin and isospin wave-functions $\phi(S)\chi(S)$



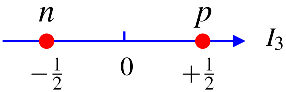
Spin 3/2
Isospin 3/2

❷ combine mixed symmetry spin and mixed symmetry isospin states

- Both $\phi(M_S)\chi(M_S)$ and $\phi(M_A)\chi(M_A)$ are sym. under inter-change of quarks $1 \leftrightarrow 2$
- Not sufficient, these combinations have no definite symmetry under $1 \leftrightarrow 3, \ldots$
- However, it is not difficult to show that the (normalised) linear combination:

$$\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S) + \frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)$$

is **totally symmetric** (i.e. symmetric under $1 \leftrightarrow 2;\ \ 1 \leftrightarrow 3;\ \ 2 \leftrightarrow 3$)



Spin 1/2
Isospin 1/2

- The spin-up proton wave-function is therefore:

$$|p\uparrow\rangle = \frac{1}{6\sqrt{2}}(2uud - udu - duu)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \frac{1}{2\sqrt{2}}(udu - duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

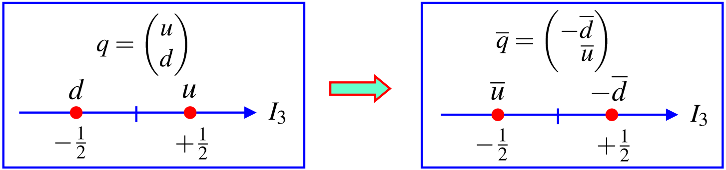
➡

$$\boxed{\begin{aligned} |p\uparrow\rangle = \frac{1}{\sqrt{18}}(\ & 2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + \\ & 2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow + \\ & 2d\downarrow u\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - d\uparrow u\uparrow u\downarrow) \end{aligned}}$$

NOTE: not always necessary to use the fully symmetrised proton wave-function,
e.g. the first 3 terms are sufficient for calculating the proton magnetic moment

Anti-quarks and Mesons (u and d)

★ The **u, d** quarks and **ū, d̄** anti-quarks are represented as isospin doublets



$$q = \begin{pmatrix} u \\ d \end{pmatrix} \qquad \bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$

$$\bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{d} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- **Subtle point:** The ordering and the minus sign in the anti-quark doublet ensures that anti-quarks and quarks transform in the same way (see Appendix I). This is necessary if we want physical predictions to be invariant under $u \leftrightarrow d;\ \bar{u} \leftrightarrow \bar{d}$

- Consider the effect of ladder operators on the anti-quark isospin states

e.g

$$T_+\bar{u} = T_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\bar{d}$$

- The effect of the ladder operators on anti-particle isospin states are:

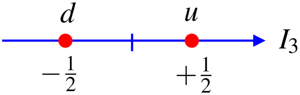
$$\boxed{T_+\bar{u} = -\bar{d} \qquad T_+\bar{d} = 0 \qquad T_-\bar{u} = 0 \qquad T_-\bar{d} = -\bar{u}}$$

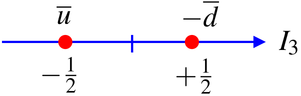
Compare with

$$\boxed{T_+u = 0 \qquad T_+d = u \qquad T_-u = d \qquad T_-d = 0}$$

Light ud Mesons

★ **Can now construct meson states from combinations of up/down quarks**





- **Consider the $q\overline{q}$ combinations in terms of isospin**

$$|1,+1\rangle = |\tfrac{1}{2},+\tfrac{1}{2}\rangle\overline{|\tfrac{1}{2},+\tfrac{1}{2}\rangle} = -u\overline{d}$$

$$|1,-1\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle\overline{|\tfrac{1}{2},-\tfrac{1}{2}\rangle} = d\overline{u}$$

The bar indicates this is the isospin representation of an anti-quark

To obtain the $I_3 = 0$ states use ladder operators and orthogonality

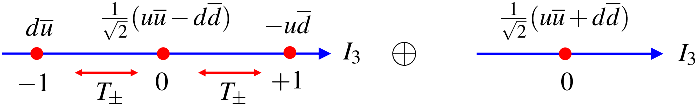
$$\begin{aligned} T_-|1,+1\rangle &= T_-\left[-u\overline{d}\right] \\ \sqrt{2}|1,0\rangle &= -T_-\left[u\right]\overline{d} - uT_-\left[\overline{d}\right] \\ &= -d\overline{d} + u\overline{u} \end{aligned}$$

$$\Rightarrow \qquad |1,0\rangle = \frac{1}{\sqrt{2}}\left(u\overline{u} - d\overline{d}\right)$$

- **Orthogonality gives:** $$|0,0\rangle = \frac{1}{\sqrt{2}}\left(u\overline{u} + d\overline{d}\right)$$

★**To summarise:**

Triplet of $I=1$ states and a singlet $I=0$ state



- You will see this written as $2\otimes\overline{2}=3\oplus 1$

Quark doublet

Anti-quark doublet

- To show the state obtained from orthogonality with $|1,0\rangle$ is a singlet use ladder operators

$$T_+|0,0\rangle = T_+\frac{1}{\sqrt{2}}(u\overline{u}+d\overline{d}) = \frac{1}{\sqrt{2}}\left(-u\overline{d}+u\overline{d}\right)=0$$

similarly $\quad T_-|0,0\rangle = 0$

★ **A singlet state is a "dead-end" from the point of view of ladder operators**

SU(3) Flavour

★ Extend these ideas to include the strange quark. Since $m_s > m_u\,, m_d$ don't have an **exact symmetry**. But m_s not so very different from $m_u\,, m_d$ and can treat the strong interaction (and resulting hadron states) as if it were symmetric under $u \leftrightarrow d \leftrightarrow s$

- **NOTE:** any results obtained from this assumption are only approximate as the symmetry is not exact.
- The assumed uds flavour symmetry can be expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3x3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters
  There are 9 constraints from $\hat{U}^\dagger \hat{U} = 1$

➡ Can form 18 – 9 = 9 linearly independent matrices

These 9 matrices form a U(3) group.

- As before, one matrix is simply the identity multiplied by a complex phase and is of no interest in the context of flavour symmetry
- The remaining 8 matrices have $\det U = 1$ and form an SU(3) group
- The eight matrices (the Hermitian generators) are: $\vec{T} = \frac{1}{2}\vec{\lambda}$ $\qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$

★In SU(3) flavour, the three quark states are represented by:

$$\boxed{u = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad d = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad s = \begin{pmatrix}0\\0\\1\end{pmatrix}}$$

★In SU(3) **uds** flavour symmetry contains SU(2) **ud** flavour symmetry which allows us to write the first three matrices:

$$\lambda_1 = \begin{pmatrix} \sigma_1 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e. **u ↔ d**

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- The third component of isospin is now written $\boxed{I_3 = \frac{1}{2}\lambda_3}$

with $I_3 u = +\frac{1}{2}u \quad I_3 d = -\frac{1}{2}d \quad I_3 s = 0$

- I_3 "counts the number of up quarks – number of down quarks in a state
- As before, ladder operators $T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$ $\quad d \bullet \xleftrightarrow{\;T_\pm\;} \bullet u$

- Now consider the matrices corresponding to the **u ↔ s and d ↔ s**

u ↔ s $\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

d ↔ s $\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

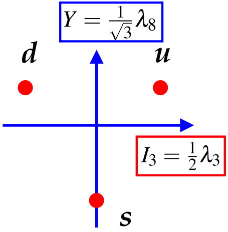
- Hence in addition to $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have two other traceless diagonal matrices

- However the three diagonal matrices are not be independent.
- Define the eighth matrix, λ_8, as the linear combination:

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which specifies the "vertical position" in the 2D plane

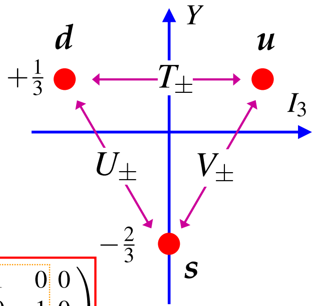
> "Only need two axes (quantum numbers) to specify a state in the 2D plane": **(I_3,Y)**



★The other six matrices form six ladder operators which step between the states

$$\boxed{\begin{aligned} T_\pm &= \tfrac{1}{2}(\lambda_1 \pm i\lambda_2) \\ V_\pm &= \tfrac{1}{2}(\lambda_4 \pm i\lambda_5) \\ U_\pm &= \tfrac{1}{2}(\lambda_6 \pm i\lambda_7) \end{aligned}}$$

with $\boxed{I_3 = \frac{1}{2}\lambda_3}$ $\boxed{Y = \frac{1}{\sqrt{3}}\lambda_8}$



and the eight **Gell-Mann** matrices

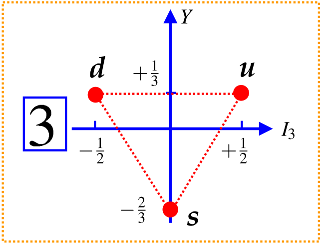
u ↔ d $\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

u ↔ s $\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$

d ↔ s $\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Quarks and anti-quarks in SU(3) Flavour

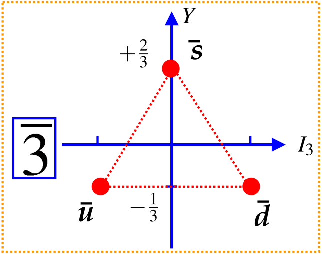


Quarks

$$\boxed{I_3 u = +\tfrac{1}{2}u; \quad I_3 d = -\tfrac{1}{2}d; \quad I_3 s = 0}$$

$$\boxed{Y u = +\tfrac{1}{3}u; \quad Y d = +\tfrac{1}{3}d; \quad Y s = -\tfrac{2}{3}s}$$

- The anti-quarks have opposite **SU(3)** flavour quantum numbers



Anti-Quarks

$$\boxed{I_3 \overline{u} = -\tfrac{1}{2}\overline{u}; \quad I_3 \overline{d} = +\tfrac{1}{2}\overline{d}; \quad I_3 \overline{s} = 0}$$

$$\boxed{Y \overline{u} = -\tfrac{1}{3}\overline{u}; \quad Y \overline{d} = -\tfrac{1}{3}\overline{d}; \quad Y \overline{s} = +\tfrac{2}{3}\overline{s}}$$

SU(3) Ladder Operators

- SU(3) ***uds*** flavour symmetry contains ***ud***, ***us*** and ***ds*** SU(2) symmetries
- Consider the $u \leftrightarrow s$ symmetry “V-spin” which has the associated $s \longrightarrow u$ ladder operator

$$V_+ = \tfrac{1}{2}(\lambda_4 + i\lambda_5) = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \tfrac{i}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $$V_+ s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = +u$$

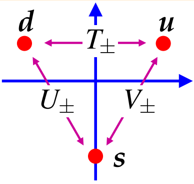
★The effects of the six ladder operators are:

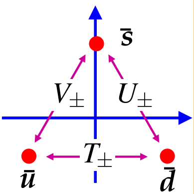
$T_+ d = u; \quad T_- u = d;$	$T_+ \overline{u} = -\overline{d}; \quad T_- \overline{d} = -\overline{u}$
$V_+ s = u; \quad V_- u = s;$	$V_+ \overline{u} = -\overline{s}; \quad V_- \overline{s} = -\overline{u}$
$U_+ s = d; \quad U_- d = s;$	$U_+ \overline{d} = -\overline{s}; \quad U_- \overline{s} = -\overline{d}$

all other combinations give zero

SU(3) LADDER OPERATORS

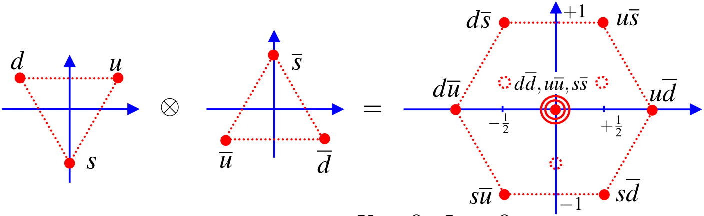
$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$



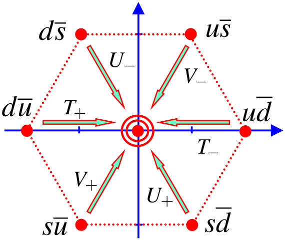


Light (uds) Mesons

- Use ladder operators to construct **uds** mesons from the nine possible $q\bar{q}$ states



- The three central states, all of which have $Y = 0;\ I_3 = 0$ can be obtained using the ladder operators and orthogonality. Starting from the outer states can reach the centre in six ways



$$T_+|d\bar{u}\rangle = |u\bar{u}\rangle - |d\bar{d}\rangle \qquad T_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle$$

$$V_+|s\bar{u}\rangle = |u\bar{u}\rangle - |s\bar{s}\rangle \qquad V_-|u\bar{s}\rangle = |s\bar{s}\rangle - |u\bar{u}\rangle$$

$$U_+|s\bar{d}\rangle = |d\bar{d}\rangle - |s\bar{s}\rangle \qquad U_-|d\bar{s}\rangle = |s\bar{s}\rangle - |d\bar{d}\rangle$$

- Only two of these six states are linearly independent.
- But there are three states with $Y = 0;\ I_3 = 0$
- Therefore one state is not part of the same multiplet, i.e. cannot be reached with ladder ops.

- First form two linearly independent orthogonal states from:

$$\boxed{|u\bar{u}\rangle - |d\bar{d}\rangle} \qquad |u\bar{u}\rangle - |s\bar{s}\rangle \qquad |d\bar{d}\rangle - |s\bar{s}\rangle$$

★ If the SU(3) flavour symmetry were exact, the choice of states wouldn't matter. However, $m_s > m_{u,d}$ and the symmetry is only approximate.

- **Experimentally** observe three light mesons with m~140 MeV: π^+, π^0, π^-
- Identify one state (the π^0) with the isospin triplet (derived previously)

$$\boxed{\psi_1 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})}$$

- The second state can be obtained by taking the linear combination of the other two states which is orthogonal to the π^0

$$\psi_2 = \alpha(|u\bar{u}\rangle - |s\bar{s}\rangle) + \beta(|d\bar{d}\rangle - |s\bar{s}\rangle)$$

with orthonormality: $\langle\psi_1|\psi_2\rangle = 0; \quad \langle\psi_2|\psi_2\rangle = 1$

$$\rightarrow \quad \boxed{\psi_2 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})}$$

- The final state (which is not part of the same multiplet) can be obtained by requiring it to be orthogonal to ψ_1 and ψ_2

$$\rightarrow \quad \boxed{\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})}$$

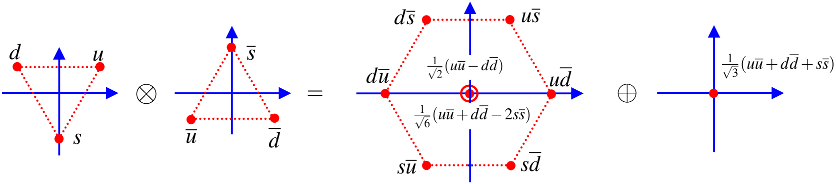
SINGLET

★It is easy to check that ψ_3 is a singlet state using ladder operators

$$T_+\psi_3 = T_-\psi_3 = U_+\psi_3 = U_-\psi_3 = V_+\psi_3 = V_-\psi_3 = 0$$

which confirms that $\psi_3 = \frac{1}{\sqrt{3}}(u\overline{u} + d\overline{d} + s\overline{s})$ is a “flavourless” singlet

- Therefore the combination of a quark and anti-quark yields nine states which breakdown into an OCTET and a SINGLET



- In the language of group theory: $3 \otimes \overline{3} = 8 \oplus 1$

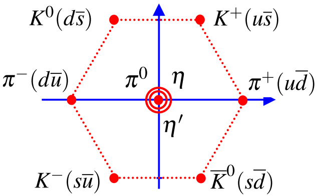
★ Compare with combination of two spin-half particles $2 \otimes 2 = 3 \oplus 1$

TRIPLET of spin-1 states: $|1,-1\rangle,\ |1,0\rangle,\ |1,+1\rangle$

spin-0 SINGLET: $|0,0\rangle$

- These spin triplet states are connected by ladder operators just as the meson *uds* octet states are connected by SU(3) flavour ladder operators
- The singlet state carries no angular momentum – in this sense the SU(3) flavour singlet is “flavourless”

PSEUDOSCALAR MESONS (L=0, S=0, J=0, P= –1)

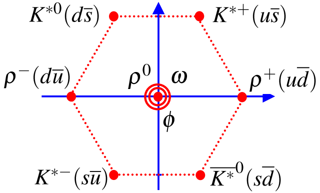


- Because SU(3) flavour is only approximate the physical states with $I_3 = 0,\ Y = 0$ can be mixtures of the octet and singlet states. **Empirically find:**

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\eta \approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\eta' \approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

singlet

VECTOR MESONS (L=0, S=1, J=1, P= –1)



- For the vector mesons the physical states are found to be approximately **"ideally mixed"**:

$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\omega \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$$\phi \approx s\bar{s}$$

MASSES

$\pi^\pm : 140\,\mathrm{MeV}$ $\quad\pi^0 : 135\,\mathrm{MeV}$

$K^\pm : 494\,\mathrm{MeV}$ $\quad K^0/\bar{K}^0 : 498\,\mathrm{MeV}$

$\eta : 549\,\mathrm{MeV}$ $\quad\eta' : 958\,\mathrm{MeV}$

$\rho^\pm : 770\,\mathrm{MeV}$ $\quad\rho^0 : 770\,\mathrm{MeV}$

$K^{*\pm} : 892\,\mathrm{MeV}$ $\quad K^{*0}/\bar{K}^{*0} : 896\,\mathrm{MeV}$

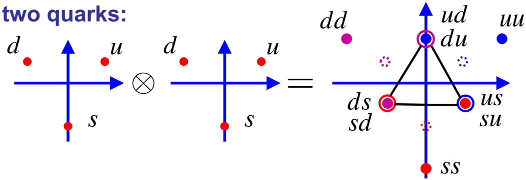
$\omega : 782\,\mathrm{MeV}$ $\quad\phi : 1020\,\mathrm{MeV}$

Combining uds Quarks to form Baryons

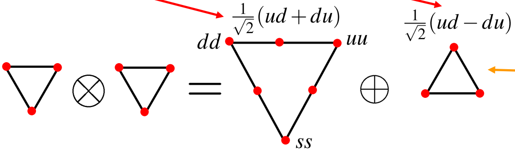
★ **Have already seen that constructing Baryon states is a fairly tedious process when we derived the proton wave-function. Concentrate on multiplet structure rather than deriving all the wave-functions.**

★ **Everything we do here is relevant to the treatment of colour**

- **First combine two quarks:**



★**Yields a symmetric sextet and anti-symmetric triplet:** $3 \otimes 3 = 6 \oplus \overline{3}$

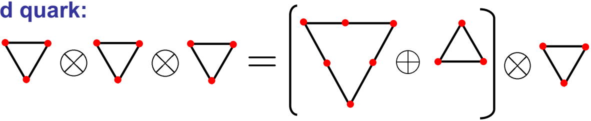


SYMMETRIC

ANTI-SYMMETRIC

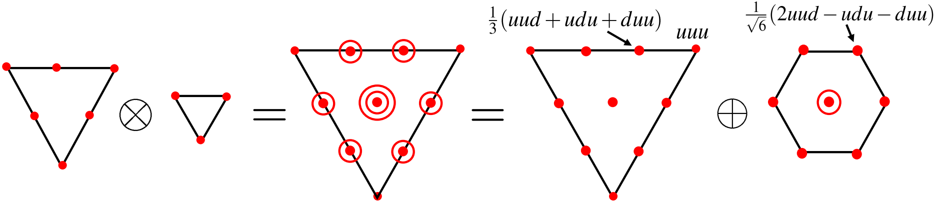
Same "pattern" as the anti-quark representation

•**Now add the third quark:**



•**Best considered in two parts, building on the sextet and triplet. Again concentrate on the multiplet structure (for the wave-functions refer to the discussion of proton wave-function).**

❶ **Building on the sextet:** $3\otimes 6 = 10 \oplus 8$

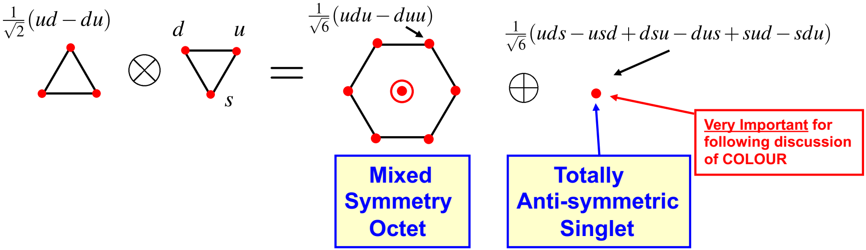


Symmetric Decuplet

Mixed Symmetry Octet

❷ Building on the triplet:

- Just as in the case of uds mesons we are combining $\overline{3} \times 3$ and again obtain an octet and a singlet



$\frac{1}{\sqrt{2}}(ud - du)$ $\otimes$ $=$ $\frac{1}{\sqrt{6}}(udu - duu)$ $\oplus$ $\frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$

Mixed Symmetry Octet

Totally Anti-symmetric Singlet

Very Important for following discussion of COLOUR

- Can verify the wave-function $\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$ is a singlet by using ladder operators, e.g.

$$T_+ \psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uus - usu + usu - uus + suu - suu) = 0$$

★ In summary, the combination of three ***uds*** quarks decomposes into

$$3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \overline{3}) = 10 \oplus 8 \oplus 8 \oplus 1$$

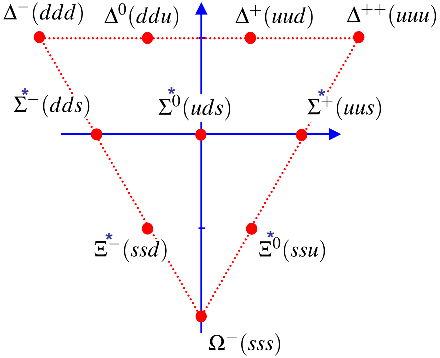
Baryon Decuplet

★ The baryon states (L=0) are:

- the **spin 3/2 decuplet** of symmetric flavour and symmetric spin wave-functions $\phi(S)\chi(S)$

<u>BARYON DECUPLET</u> (L=0, S=3/2, J=3/2, P= +1)

Mass in MeV



Δ(1232)

Σ(1318)

Ξ(1384)

Ω(1672)

★ If SU(3) flavour were an exact symmetry all masses would be the same (broken symmetry)

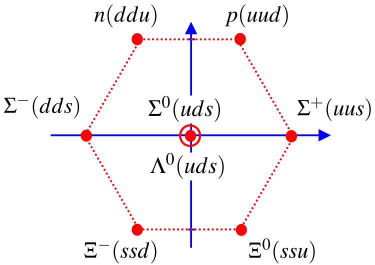
Baryon Octet

★ The **spin 1/2 octet** is formed from mixed symmetry flavour and mixed symmetry spin wave-functions

$$\alpha\phi(M_S)\chi(M_S)+\beta\phi(M_A)\chi(M_A)$$

See previous discussion proton for how to obtain wave-functions

BARYON OCTET (L=0, S=1/2, J=1/2, P= +1)



Mass in MeV

939

Σ(1193)
Λ(1116)

Ξ(1318)

★ **NOTE:** Cannot form a totally symmetric wave-function based on the anti-symmetric flavour singlet as there no totally anti-symmetric spin wave-function for 3 quarks

Summary

- ★ Considered SU(2) **ud** and SU(3) **uds** flavour symmetries
- ★ Although these flavour symmetries are only approximate can still be used to explain observed multiplet structure for mesons/baryons
- ★ In case of SU(3) flavour symmetry results, e.g. predicted wave-functions should be treated with a pinch of salt as $m_s \neq m_{u/d}$
- ★ Introduced idea of singlet states being "spinless" or "flavourless"
- ★ In the next handout apply these ideas to colour and QCD…

Appendix: the SU(2) anti-quark representation

Non-examinable

- **Define anti-quark doublet** $\overline{q} = \begin{pmatrix} -\overline{d} \\ \overline{u} \end{pmatrix} = \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$

- **The quark doublet** $q = \begin{pmatrix} u \\ d \end{pmatrix}$ **transforms as** $q' = Uq$

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix} \xrightarrow[\text{conjugate}]{\text{Complex}} \begin{pmatrix} u'^* \\ d'^* \end{pmatrix} = U^* \begin{pmatrix} u^* \\ d^* \end{pmatrix}$$

- **Express in terms of anti-quark doublet**

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q}' = U^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q}$$

- **Hence** $\overline{q}$ **transforms as**

$$\overline{q}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q}$$

- A special 2x2 unitary matrix can always be written in the form

$$U = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix}$$

… provided that $|c_{11}|^2+|c_{12}|^2=1$. This gives:

$$\begin{aligned} \overline{q}' &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_{11}^* & c_{12}^* \\ -c_{12} & c_{11} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q} \\ &= \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix} \\ &= U\overline{q} \end{aligned}$$

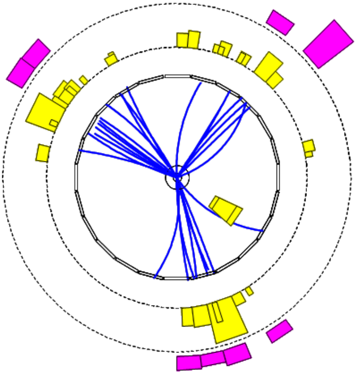
- Therefore the anti-quark doublet $\overline{q} = \begin{pmatrix} -\overline{d} \\ \overline{u} \end{pmatrix}$

transforms in the same way as the quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$

★ **NOTE:** this is a special property of SU(2) and for SU(3) there is no analogous representation of the anti-quarks

Particle Physics

Dr Lester



Handout 8 : Quantum Chromodynamics

The Local Gauge Principle

(see the Appendices A, B and C for more details)

★ All the interactions between fermions and spin-1 bosons in the SM are specified by the principle of **LOCAL GAUGE INVARIANCE**

★ To arrive at **QED**, require physics to be invariant under the **local phase transformation** of particle wave-functions

$$\psi \rightarrow \psi' = \psi e^{iq\chi(x)}$$

★ Note that the change of phase depends on the space-time coordinate: $\chi(t, \vec{x})$

- Under this transformation the Dirac Equation transforms as

$$\boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0} \Rightarrow \boxed{i\gamma^\mu(\partial_\mu + iq\partial_\mu\chi)\psi - m\psi = 0}$$

- To make "physics", i.e. the Dirac equation, invariant under this local phase transformation **FORCED** to introduce a **massless gauge boson**, A_μ .

+ The Dirac equation has to be modified to include this new field:

$$\boxed{i\gamma^\mu(\partial_\mu + iqA_\mu)\psi - m\psi = 0}$$

- The modified Dirac equation is invariant under local phase transformations if:

$$\boxed{A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu\chi}$$

Gauge Invariance

★ **For physics to remain unchanged – must have GAUGE INVARIANCE of the new field, i.e. physical predictions unchanged for** $A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$

★ **Hence the principle of invariance under local phase transformations completely specifies the interaction between a fermion and the gauge boson (i.e. photon):**

$$i\gamma^\mu(\partial_\mu + \boxed{iqA_\mu})\psi - m\psi = 0$$

⇨ **interaction vertex:** $i\gamma^\mu q A_\mu$

(see pages 121 and 299-301)

⇨ **QED !**

★ **The local phase transformation of QED is a unitary U(1) transformation**

$$\psi \to \psi' = \hat{U}\psi \quad \text{i.e.} \quad \psi \to \psi' = \psi e^{iq\chi(x)} \quad \text{with} \quad U^\dagger U = 1$$

Now extend this idea…

From QED to QCD

★ Suppose there is another fundamental symmetry of the universe, say

"invariance under SU(3) local phase transformations"

- i.e. require invariance under $\psi \to \psi' = \psi e^{ig\vec{\lambda}.\vec{\theta}(x)}$ where

  $\vec{\lambda}$ are the eight 3x3 Gell-Mann matrices introduced in handout 7

  $\vec{\theta}(x)$ are 8 functions taking different values at each point in space-time

  ⇨ 8 spin-1 gauge bosons

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

wave function is now a vector in COLOUR SPACE

⇨ QCD !

★ QCD is fully specified by require invariance under SU(3) **local** phase transformations

Corresponds to rotating states in colour space about an axis whose direction is different at every space-time point

⇨ interaction vertex: $-\frac{1}{2} i g_s \lambda^a \gamma^\mu$

★ Predicts 8 massless gauge bosons – the gluons (one for each λ)

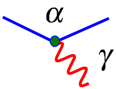
★ Also predicts exact form for interactions between gluons, i.e. the 3 and 4 gluon vertices – the details are beyond the level of this course

Colour in QCD

★The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved "colour" charges

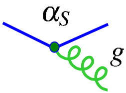
In QED:

- the electron carries one unit of charge $-e$
- the anti-electron carries one unit of anti-charge $+e$
- the force is mediated by a massless "gauge boson" – the photon



In QCD:

- quarks carry colour charge: r, g, b
- anti-quarks carry anti-charge: $\bar{r}, \bar{g}, \bar{b}$
- The force is mediated by massless gluons



★ In QCD, the strong interaction is invariant under rotations in colour space

$$r \leftrightarrow b;\ r \leftrightarrow g;\ b \leftrightarrow g$$

i.e. the same for all three colours

⇨ **SU(3) colour symmetry**

•This is an **exact** symmetry, unlike the approximate uds flavour symmetry discussed previously.

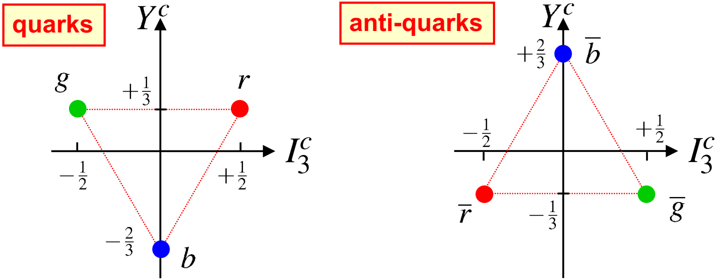
- ★ Represent r, g, b **SU(3)** colour states by:

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- ★ Colour states can be labelled by two quantum numbers:
  - ♦ I_3^c colour isospin
  - ♦ Y^c colour hypercharge

  Exactly analogous to labelling u,d,s flavour states by I_3 and Y

- ★ Each quark (anti-quark) can have the following colour quantum numbers:



Colour Confinement

- ★ It is believed (although not yet proven) that all observed free particles are "colourless"
  - •i.e. never observe a free quark (which would carry colour charge)
  - •consequently quarks are always found in bound states colourless hadrons

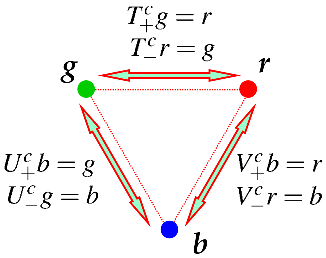
★ **Colour Confinement Hypothesis:**

> only colour singlet states can exist as free particles

- ★ All hadrons must be "colourless" i.e. colour singlets
- ★ To construct colour wave-functions for hadrons can apply results for SU(3) flavour symmetry to SU(3) colour with replacement

$$\boxed{\begin{array}{c} u \longrightarrow r \\ d \longrightarrow g \\ s \longrightarrow b \end{array}}$$

- ★ just as for uds flavour symmetry can define colour ladder operators



Colour Singlets

★ It is important to understand what is meant by a singlet state

★ Consider spin states obtained from two spin 1/2 particles.

- Four spin combinations: $\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$
- Gives four eigenstates of $\hat{S}^2,\ \hat{S}_z$ $\qquad (2 \otimes 2 = 3 \oplus 1)$

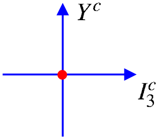
$$\begin{aligned} |1,+1\rangle &= \uparrow\uparrow \\ |1,0\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1,-1\rangle &= \downarrow\downarrow \end{aligned} \quad \boxed{\text{spin-1 triplet}} \quad \oplus \quad |0,0\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \quad \boxed{\text{spin-0 singlet}}$$

★ The singlet state is "spinless": it has zero angular momentum, is invariant under SU(2) spin transformations and spin ladder operators yield zero

$$S_{\pm}|0,0\rangle = 0$$

★ In the same way **COLOUR SINGLETS** are "colourless" combinations:

- they have zero colour quantum numbers $I_3^c = 0,\ Y^c = 0$
- invariant under SU(3) colour transformations
- ladder operators $T_{\pm},\ U_{\pm},\ V_{\pm}$ all yield zero

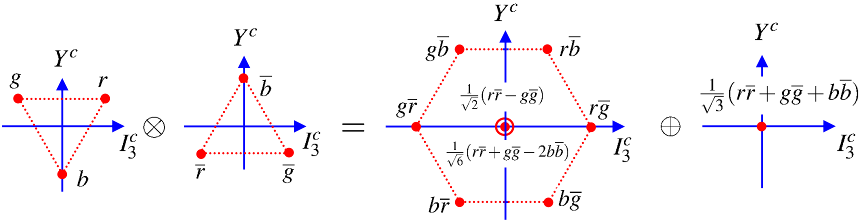


★ NOT sufficient to have $I_3^c = 0,\ Y^c = 0$: does not mean that state is a singlet

Meson Colour Wave-function

★ Consider colour wave-functions for $q\overline{q}$

★ The combination of colour with anti-colour is mathematically identical to construction of meson wave-functions with uds flavour symmetry



➡ **Coloured octet and a colourless singlet**

•Colour confinement implies that hadrons only exist in colour singlet states so the colour wave-function for mesons is:

$$\boxed{\psi_c^{q\overline{q}} = \frac{1}{\sqrt{3}}(r\overline{r}+g\overline{g}+b\overline{b})}$$

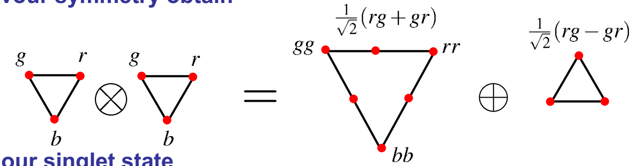
★ Can we have a $qq\overline{q}$ state ? i.e. by adding a quark to the above octet can we form a state with $Y^c = 0;\ I_3^c = 0$. The answer is clear no.

➡ $qq\overline{q}$ bound states do not exist in nature.

Baryon Colour Wave-function

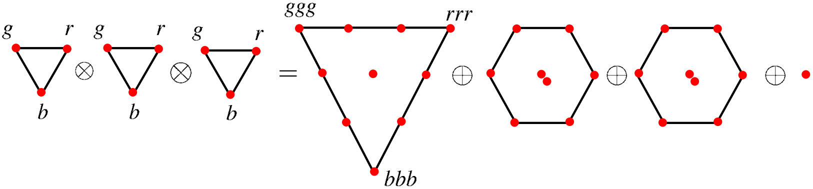
★ Do **qq** bound states exist ? This is equivalent to asking whether it possible to form a colour singlet from two colour triplets ?

- Following the discussion of construction of baryon wave-functions in SU(3) flavour symmetry obtain



- No **qq** colour singlet state
- Colour confinement ➡bound states of **qq** do not exist

BUT combination of three quarks (three colour triplets) gives a colour singlet state (p(pages 244-246)7)



★ **The singlet colour wave-function is:**

$$\boxed{\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)}$$

Check this is a colour singlet…

- It has $I_3^c = 0, \ Y^c = 0$: a necessary but not sufficient condition
- Apply ladder operators, e.g. T_+ (recall $T_+ g = r$)

$$T_+ \psi_c^{qqq} = \frac{1}{\sqrt{6}}(rrb - rbr + rbr - rrb + brr - brr) = 0$$

- Similarly $T_- \psi_c^{qqq} = 0; \quad V_\pm \psi_c^{qqq} = 0; \quad U_\pm \psi_c^{qqq} = 0;$

★ **Colourless singlet - therefore qqq bound states exist !**

⇒ **Anti-symmetric colour wave-function**

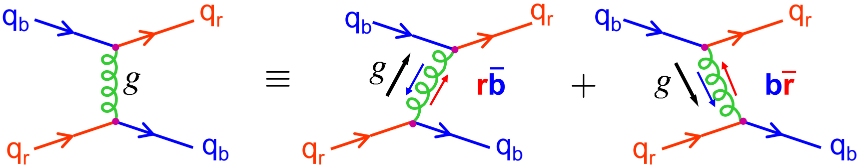
Allowed Hadrons i.e. the possible colour singlet states

- ● $q\bar{q}, \ qqq$ **Mesons and Baryons**
- ● $q\bar{q}q\bar{q}, \ qqqq\bar{q}$ **Exotic states, e.g. pentaquarks**

To date all confirmed hadrons are either mesons or baryons. However, some recent (but not entirely convincing) "evidence" for pentaquark states

Gluons

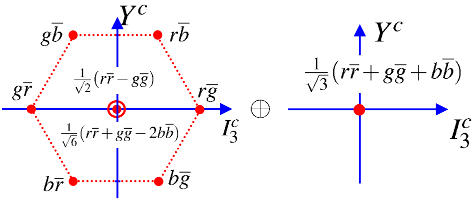
★ In QCD quarks interact by exchanging virtual massless gluons, e.g.



★ Gluons carry colour and anti-colour, e.g.

★ Gluon colour wave-functions (colour + anti-colour) are the same as those obtained for mesons (also colour + anti-colour)

⇨ OCTET +
“COLOURLESS” SINGLET



★ So we might expect 9 physical gluons:

OCTET: $r\overline{g},\ r\overline{b},\ g\overline{r},\ g\overline{b},\ b\overline{r},\ b\overline{g},\ \frac{1}{\sqrt{2}}(r\overline{r}-g\overline{g}),\ \frac{1}{\sqrt{6}}(r\overline{r}+g\overline{g}-2b\overline{b})$

SINGLET: $\frac{1}{\sqrt{3}}(r\overline{r}+g\overline{g}+b\overline{b})$

★ **BUT**, colour confinement hypothesis:

only colour singlet states can exist as free particles ⇒

Colour singlet gluon would be unconfined. It would behave like a strongly interacting photon ➡ infinite range Strong force.

★ **Empirically**, the strong force is short range and therefore know that the physical gluons are confined. The colour singlet state does not exist in nature !

NOTE: this is not entirely ad hoc. In the context of gauge field theory (see minor option) the strong interaction arises from a fundamental SU(3) symmetry. The gluons arise from the generators of the symmetry group (the Gell-Mann λ matrices). There are 8 such matrices ➡ 8 gluons. Had nature "chosen" a U(3) symmetry, would have 9 gluons, the additional gluon would be the colour singlet state and QCD would be an unconfined long-range force.

NOTE: the "gauge symmetry" determines the exact nature of the interaction ➡ FEYNMAN RULES

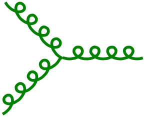
Gluon-Gluon Interactions

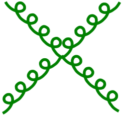
- ★ In **QED** the **photon** does not carry the charge of the EM interaction (photons are electrically neutral)
- ★ In contrast, in **QCD** the **gluons** do carry **colour charge**

⇒ **Gluon Self-Interactions**

- ★ Two new vertices (no QED analogues)

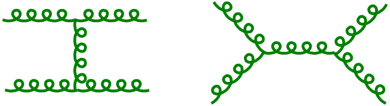
triple-gluon vertex



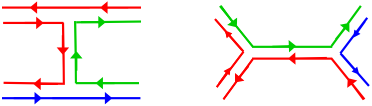


quartic-gluon vertex

- ★ In addition to quark-quark scattering, therefore can have gluon-gluon scattering

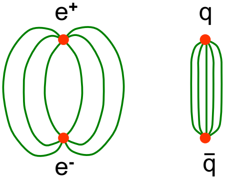


e.g. possible way of arranging the colour flow

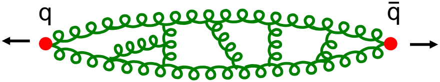


Gluon self-Interactions and Confinement

- ★ Gluon self-interactions are believed to give rise to colour confinement
- ★ Qualitative picture:
  - •Compare QED with QCD
  - •In QCD "gluon self-interactions squeeze lines of force into a flux tube"



★ What happens when try to separate two coloured objects e.g. $q\bar{q}$



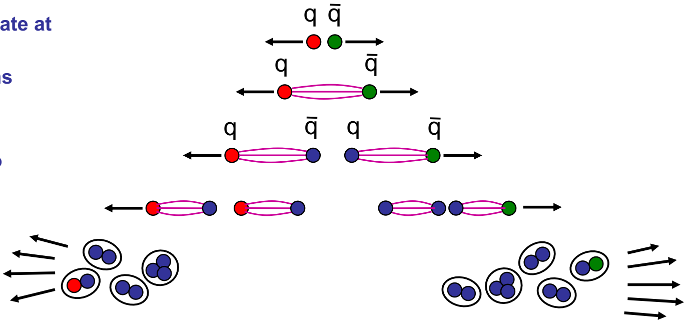
•Form a flux tube of interacting gluons of approximately constant energy density $\sim 1\,\mathrm{GeV/fm}$

$$\Rightarrow \quad V(r) \sim \lambda r$$

- •Require infinite energy to separate coloured objects to infinity
- •Coloured quarks and gluons are always confined within colourless states
- •In this way QCD provides a plausible explanation of confinement – but not yet proven (although there has been recent progress with Lattice QCD)

Hadronisation and Jets

★ **Consider a quark and anti-quark produced in electron positron annihilation**



i) **Initially Quarks separate at high velocity**

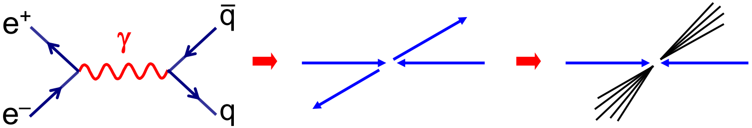
ii) **Colour flux tube forms between quarks**

iii) **Energy stored in the flux tube sufficient to produce $q\bar{q}$ pairs**

iv) **Process continues until quarks pair up into jets of colourless hadrons**

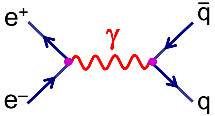
★ **This process is called hadronisation. It is not (yet) calculable.**

★ **The main consequence is that at collider experiments quarks and gluons observed as jets of particles**



QCD and Colour in e^+e^- Collisions

★ e^+e^- **colliders are an excellent place to study QCD**

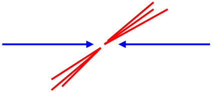


★ **Well defined production of quarks**
- **QED process well-understood**
- **no need to know parton structure functions**
- **+ experimentally very clean – no proton remnants**

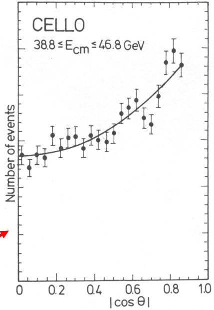
★ **In handout 5 obtained expressions for the** $e^+e^- \to \mu^+\mu^-$ **cross-section**

$$\sigma = \frac{4\pi\alpha^2}{3s} \qquad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

- **In e^+e^- collisions produce all quark flavours for which** $\sqrt{s} > 2m_q$
- **In general, i.e. unless producing a** $q\overline{q}$ **bound state, produce jets of hadrons**
- **Usually can't tell which jet came from the quark and came from anti-quark**



★ **Angular distribution of jets** $\propto (1+\cos^2\theta)$



H.J.Behrend et al., Phys Lett 183B (1987) 400

➡ **Quarks are spin ½**

- ★ Colour is conserved and quarks are produced as $r\bar{r},\ g\bar{g},\ b\bar{b}$
- ★ For a **single quark flavour and single colour**

$$\sigma(e^+e^- \to q_i\bar{q}_i) = \frac{4\pi\alpha^2}{3s}Q_q^2$$

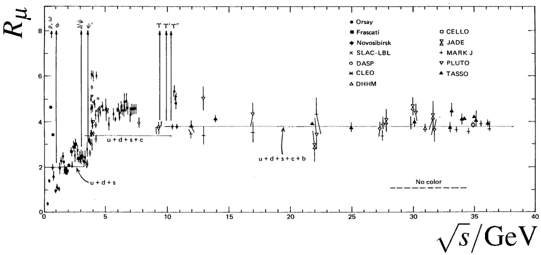
- Experimentally observe jets of hadrons:

$$\sigma(e^+e^- \to \text{hadrons}) = 3\sum_{u,d,s,..}\frac{4\pi\alpha^2}{3s}Q_q^2$$

Factor 3 comes from colours

- Usual to express as ratio compared to $\sigma(e^+e^- \to \mu^+\mu^-)$

$$R_\mu = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{u,d,s,..}Q_q^2$$



u,d,s: $R_\mu = 3\times(\frac{1}{9}+\frac{4}{9}+\frac{1}{9}) = 2$

u,d,s,c: $R_\mu = \frac{10}{3}$

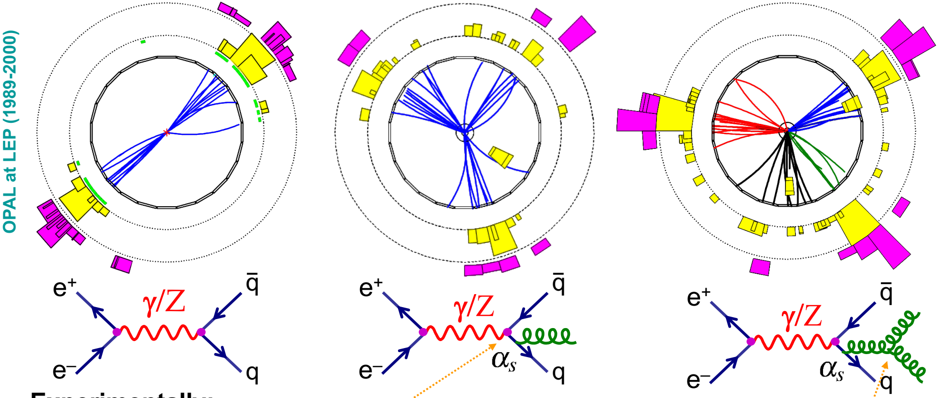
u,d,s,c,b: $R_\mu = \frac{11}{3}$

★ Data consistent with expectation **with factor 3 from colour**

Jet production in e+e- Collisions

★ e^+e^- **colliders are also a good place to study gluons**

$e^+e^- \to q\overline{q} \to 2\text{jets}$ $e^+e^- \to q\overline{q}g \to 3\text{jets}$ $e^+e^- \to q\overline{q}gg \to 4\text{jets}$



Experimentally:

- **Three jet rate ➡ measurement of** α_s
- **Angular distributions ➡ gluons are spin-1**
- **Four-jet rate and distributions ➡ QCD has an underlying SU(3) symmetry**

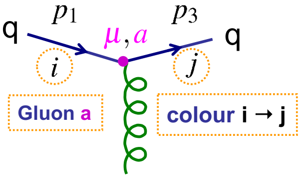
The Quark – Gluon Interaction

- Representing the colour part of the fermion wave-functions by:

$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$
- The QCD qqg vertex is written:

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



- Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices
- Isolating the colour part:

$$c_j^\dagger\lambda^a c_i = c_j^\dagger\begin{pmatrix}\lambda^a_{1i}\\\lambda^a_{2i}\\\lambda^a_{3i}\end{pmatrix} = \lambda^a_{ji}$$

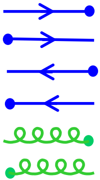
- Hence the fundamental quark - gluon QCD interaction can be written

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Feynman Rules for QCD

- **External Lines**

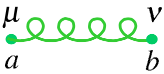
spin 1/2	incoming quark	$u(p)$	
	outgoing quark	$\overline{u}(p)$	
	incoming anti-quark	$\overline{v}(p)$	
	outgoing anti-quark	$v(p)$	
spin 1	incoming gluon	$\varepsilon^{\mu}(p)$	
	outgoing gluon	$\varepsilon^{\mu}(p)^{*}$	



- **Internal Lines (propagators)**

spin 1 gluon

$$\boxed{\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}}$$

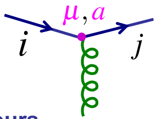


a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark

$$\boxed{-ig_s\tfrac{1}{2}\lambda^{a}_{ji}\gamma^{\mu}}$$



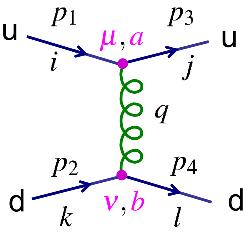
i, j = 1,2,3 are quark colours,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element** $-iM$ **= product of all factors**

Matrix Element for quark-quark scattering

★ **Consider QCD scattering of an up and a down quark**



- The incoming and out-going quark colours are labelled by $i, j, k, l = \{1, 2, 3\}$ (or $\{r, g, b\}$)
- In terms of colour this scattering is $$ik \to jl$$
- The 8 different gluons are accounted for by the colour indices $a, b = 1, 2, \ldots, 8$
- NOTE: the δ-function in the propagator ensures $a = b$, i.e. the gluon "emitted" at a is the same as that "absorbed" at b

★ **Applying the Feynman rules:**

$$-iM = \left[\overline{u}_u(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u_u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\overline{u}_d(p_4)\{-\tfrac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\}u_d(p_2)\right]$$

where summation over a and b (and μ and ν) is implied.

★ **Summing over a and b using the δ-function gives:**

$$\boxed{M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]}$$

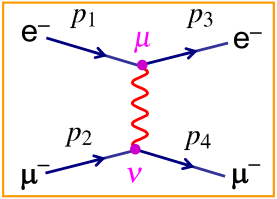
Sum over all 8 gluons (repeated indices)

QCD vs QED

QED

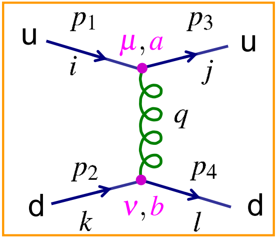
$$-iM = [\overline{u}(p_3)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^\nu u(p_2)]$$

$$\boxed{M = -e^2\frac{1}{q^2}g_{\mu\nu}[\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\gamma^\nu u(p_2)]}$$



QCD

$$\boxed{M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]}$$



★ **QCD Matrix Element = QED Matrix Element with:**

- $\boxed{e^2 \rightarrow g_s^2}$ **or equivalently** $\boxed{\alpha = \frac{e^2}{4\pi} \rightarrow \alpha_s = \frac{g_s^2}{4\pi}}$

+ QCD Matrix Element includes an additional "colour factor"

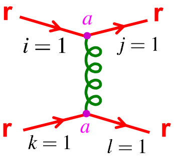
$$\boxed{C(ik \rightarrow jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}}$$

Evaluation of QCD Colour Factors

- QCD colour factors reflect the gluon states that are involved

$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\bar{g}\,,\, g\bar{r}$	$r\bar{b}\,,\, b\bar{r}$	$g\bar{b}\,,\, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

❶ Configurations involving a single colour



- Only matrices with non-zero entries in **11** position are involved

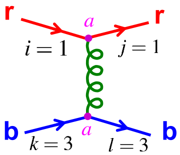
$$C(rr \to rr) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{11}\lambda^a_{11} = \frac{1}{4}(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11})$$

$$= \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3}$$

Similarly find

$$\boxed{C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \frac{1}{3}}$$

❷ Other configurations where quarks don’t change colour e.g. $rb \longrightarrow rb$



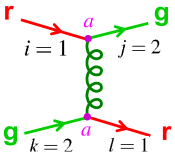
- Only matrices with non-zero entries in **11** and **33** position are involved

$$C(rb \to rb) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{11}\lambda^a_{33} = \frac{1}{4}(\lambda^8_{11}\lambda^8_{33})$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}.\frac{-2}{\sqrt{3}}\right) = -\frac{1}{6}$$

Similarly $C(rb\to rb) = C(rg\to rg) = C(gr\to gr) = C(gb\to gb) = C(br\to br) = C(bg\to bg) = -\frac{1}{6}$

❸ Configurations where quarks swap colours e.g. $rg \longrightarrow gr$



- Only matrices with non-zero entries in **12** and **21** position are involved

Gluons $r\bar{g}$, $g\bar{r}$

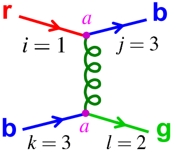
$$C(rg \to gr) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{21}\lambda^a_{12} = \frac{1}{4}(\lambda^1_{21}\lambda^1_{12} + \lambda^2_{21}\lambda^2_{12})$$

$$\hat{T}_+^{(ij)}\hat{T}_-^{(kl)}$$

$$= \frac{1}{4}(i(-i)+1) = \frac{1}{2}$$

$C(rb\to br) = C(rg\to gr) = C(gr\to rg) = C(gb\to bg) = C(br\to rb) = C(bg\to gb) = \frac{1}{2}$

❹ Configurations involving 3 colours e.g. $rb \longrightarrow bg$



- Only matrices with non-zero entries in the **13** and **32** position
- But none of the λ matrices have non-zero entries in the **13** and **32** positions. Hence the colour factor is zero

★ colour is conserved

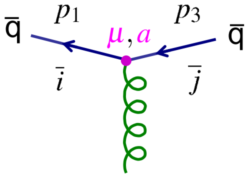
Colour Factors : Quarks vs Anti-Quarks

- **Recall the colour part of wave-function:**

$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

- **The QCD** qqg **vertex was written:**

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



★ **Now consider the anti-quark vertex**

- **The QCD** q̄q̄g **vertex is:**

$$\overline{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3)$$

Note that the incoming anti-particle now enters on the LHS of the expression

- **For which the colour part is**

$$c_i^\dagger\lambda^a c_j = c_i^\dagger\begin{pmatrix}\lambda^a_{1j}\\ \lambda^a_{2j}\\ \lambda^a_{3j}\end{pmatrix} = \lambda^a_{ij}$$

i.e indices *ij* are swapped with respect to the quark case

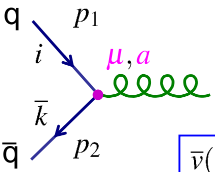
- **Hence**

$$\boxed{\overline{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3) \equiv \overline{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda^a_{ij}\gamma^\mu\}v(p_3)}$$

- **c.f. the quark - gluon QCD interaction**

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

★**Finally we can consider the quark – anti-quark annihilation**



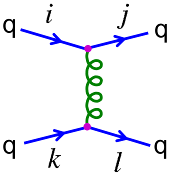
QCD vertex: $\overline{v}(p_2)c_k^\dagger\{-\frac{1}{2}ig_s\lambda^a\gamma^\mu\}c_iu(p_1)$

with $c_k^\dagger\lambda^a c_i = \lambda^a_{ki}$

$$\boxed{\overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_iu(p_1) \equiv \overline{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda^a_{ki}\gamma^\mu\}u(p_1)}$$

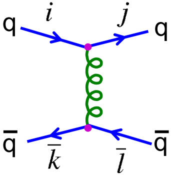
- **Consequently the colour factors for the different diagrams are:**

e.g.



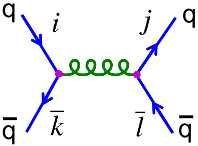
$$C(ik \rightarrow jl) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}$$

$$C(rr \rightarrow rr) = \tfrac{1}{3}$$
$$C(rg \rightarrow rg) = -\tfrac{1}{6}$$
$$C(rg \rightarrow gr) = \tfrac{1}{2}$$



$$C(i\bar{k} \rightarrow j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{kl}$$

$$C(r\bar{r} \rightarrow r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \rightarrow r\bar{g}) = -\tfrac{1}{6}$$
$$C(r\bar{r} \rightarrow g\bar{g}) = \tfrac{1}{2}$$

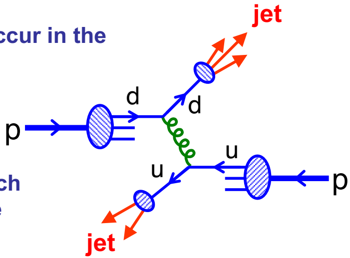


$$C(i\bar{k} \rightarrow j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ki}\lambda^a_{jl}$$

$$C(r\bar{r} \rightarrow r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \rightarrow r\bar{g}) = \tfrac{1}{2}$$
$$C(r\bar{r} \rightarrow g\bar{g}) = -\tfrac{1}{6}$$

Colour index of adjoint spinor comes first

Quark-Quark Scattering



- Consider the process $u+d\rightarrow u+d$ which can occur in the high energy proton-proton scattering
- There are nine possible colour configurations of the colliding quarks which are all equally likely.
- Need to determine the average matrix element which is the sum over all possible colours divided by the number of possible initial colour states

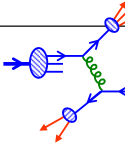
$$\langle|M_{fi}|^2\rangle=\frac{1}{3}\cdot\frac{1}{3}\sum_{i,j,k,l=1}^{3}|M_{fi}(ij\rightarrow kl)|^2$$

- The colour average matrix element contains the average colour factor

$$\langle|C|^2\rangle=\frac{1}{9}\sum_{i,j,k,l=1}^{3}|C(ij\rightarrow kl)|^2$$

- For $qq\rightarrow qq$ $\boxed{rr\rightarrow rr,..}$ $\boxed{rb\rightarrow rb,..}$ $\boxed{rb\rightarrow br,..}$

$$\langle|C|^2\rangle=\frac{1}{9}\left[3\times\left(\frac{1}{3}\right)^2+6\times\left(-\frac{1}{6}\right)^2+6\times\left(\frac{1}{2}\right)^2\right]=\frac{2}{9}$$



- Previously derived the Lorentz Invariant cross section for $e^-\mu^- \rightarrow e^-\mu^-$ elastic scattering in the ultra-relativistic limit (handout 6).

QED

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s}\right)^2\right]$$

- For ud → ud in QCD replace $\alpha \longrightarrow \alpha_s$ and multiply by $\langle |C|^2 \rangle$

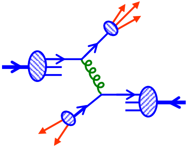
QCD

$$\frac{d\sigma}{dq^2} = \frac{2}{9}\frac{2\pi\alpha_S^2}{q^4}\left[1 + \left(1 + \frac{q^2}{\hat{s}}\right)^2\right]$$

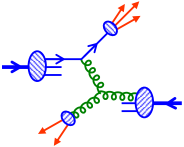
Never see colour, but enters through colour factors. Can tell QCD is SU(3)

- Here $\hat{s}$ is the centre-of-mass energy of the quark-quark collision
- The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions

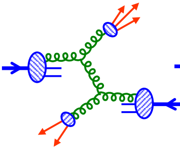
e.g. two jet production in proton-antiproton collisions



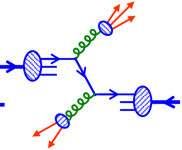
$q\overline{q} \rightarrow q\overline{q}$



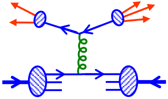
$qg \rightarrow qg$



$gg \rightarrow gg$



$q\overline{q} \rightarrow gg$



$q\overline{q} \rightarrow q\overline{q}$

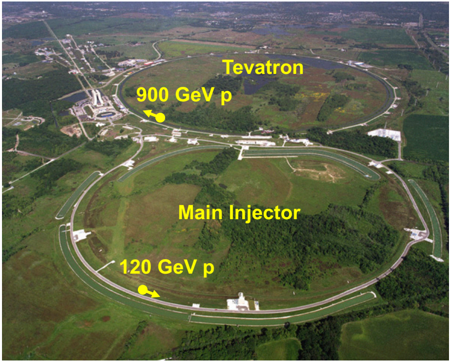
e.g. $p\bar{p}$ collisions at the Tevatron

★ Tevatron collider at **Fermi National Laboratory** (FNAL)

- located ~40 miles from Chigaco, US
- started operation in **1987** (ran until 2010)

★ $p\bar{p}$ collisions at $\sqrt{s}$ = 1.8 TeV

c.f. 14 TeV at the LHC



Two main accelerators:

★ Main Injector

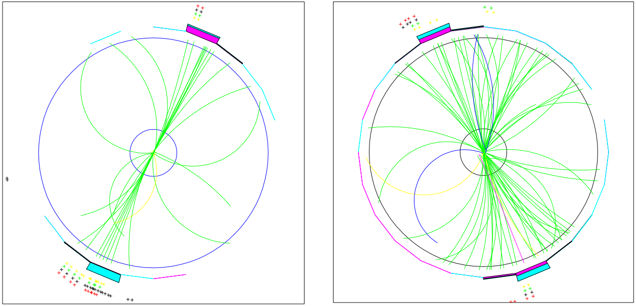
- Accelerated 8 GeV p to 120 GeV
- also $\bar{p}$ to 120 GeV
- Protons sent to Tevatron & MINOS
- $\bar{p}$ all went to Tevatron

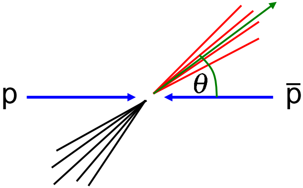
★ Tevatron

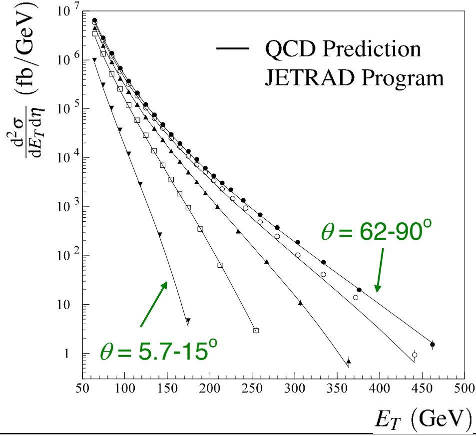
- 4 mile circumference
- accelerated $p/\bar{p}$ from 120 GeV to 900 GeV

★ **Test QCD predictions by looking at production of pairs of high energy jets**

$p\bar{p} \rightarrow \text{jet jet} + X$







D0 Collaboration, Phys. Rev. Lett. 86 (2001)

★ **Measure cross-section in terms of**

- **"transverse energy"** $E_T = E_{\text{jet}} \sin\theta$
- **"pseudorapidity"** $\eta = \ln\left[\cot\left(\frac{\theta}{2}\right)\right]$

...don't worry too much about the details here, what matters is that...

★ **QCD predictions provide an excellent description of the data**

★ **NOTE:**

- **at low E_T cross-section is dominated by low x partons**

  i.e. gluon-gluon scattering

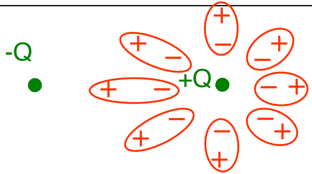
- **at high E_T cross-section is dominated by high x partons**

  i.e. quark-antiquark scattering

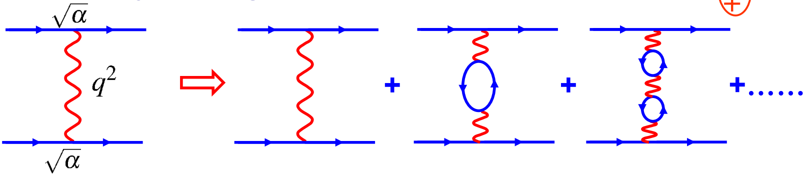
Running Coupling Constants

QED

- "bare" charge of electron screened by virtual e$^+$e$^-$ pairs
- behaves like a polarizable dielectric

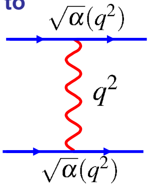


★ In terms of Feynman diagrams:



★ Same final state so add matrix element **amplitudes:** $M = M_1 + M_2 + M_3 + \ldots$

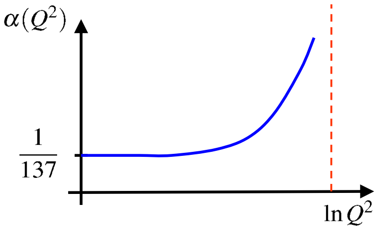
★ Giving an infinite series which can be summed and is equivalent to a single diagram with "running" coupling constant



$$\alpha(Q^2) = \alpha(Q_0^2) \Bigg/ \left[1 - \frac{\alpha(Q_0^2)}{3\pi} \ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

$$Q^2 \gg Q_0^2$$

Note sign



- ★ Might worry that coupling becomes infinite at

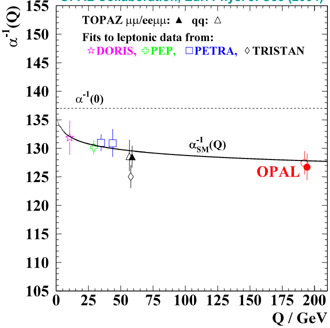
$$\ln\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$$

i.e. at

$$Q \sim 10^{26}\,\mathrm{GeV}$$

  - But quantum gravity effects would come in way below this energy and it is highly unlikely that QED "as is" would be valid in this regime

OPAL Collaboration, Eur. Phys. J. C33 (2004)



- ★ In QED, running coupling **increases** very slowly
  - Atomic physics: $Q^2 \sim 0$

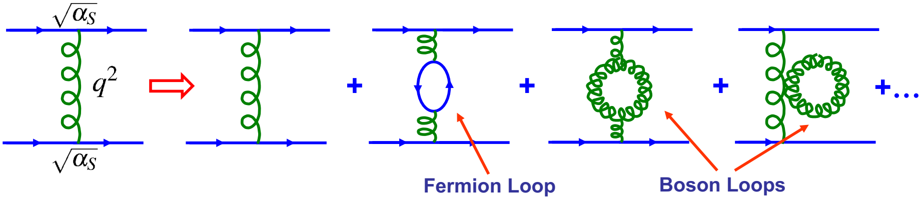
$$1/\alpha = 137.03599976(50)$$

  - High energy physics:

$$1/\alpha(193\,\mathrm{GeV}) = 127.4 \pm 2.1$$

Running of α_s

QCD Similar to QED but also have gluon loops



- ★ Remembering **adding amplitudes**, so can get negative interference and the sum can be smaller than the original diagram alone
- ★ Bosonic loops "interfere negatively"

$$\alpha_S(Q^2) = \alpha_S(Q_0^2) \Big/ \left[1 + B\alpha_S(Q_0^2) \ln\left(\frac{Q^2}{Q_0^2}\right) \right]$$

with $$B = \frac{11N_c - 2N_f}{12\pi}$$

N_c = no. of colours
N_f = no. of quark flavours

$N_c = 3;\ N_f = 6$ ➡ $B > 0$

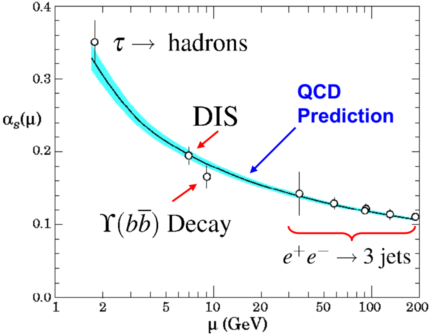
➡ α_S **decreases** with Q^2

Nobel Prize for Physics, 2004
(Gross, Politzer, Wilczek)

★ **Measure α_s in many ways:**

- jet rates
- DIS
- tau decays
- bottomonium decays
- +…

✶ **As predicted by QCD, α_s decreases with Q^2**



$\mu \approx Q$

★ **At low Q^2: α_s is large, e.g. at $Q^2 = 1\,\mathrm{GeV}^2$ find α_s ~ 1**

•Can't use perturbation theory ! This is the reason why QCD calculations at low energies are so difficult, e.g. properties hadrons, hadronisation of quarks to jets,…

★ **At high Q^2 : α_s is rather small, e.g. at $Q^2 = M_Z^2$ find α_s ~ 0.12**

➡ **Asymptotic Freedom**

•Can use perturbation theory and this is the reason that in DIS at high Q^2 **quarks behave as if they are quasi-free** (i.e. only weakly bound within hadrons)

Summary

★ Superficially QCD very similar to QED

★ But gluon self-interactions are believed to result in colour confinement

★ All hadrons are colour singlets which explains why only observe

Mesons **Baryons**

★ A low energies $\alpha_S \sim 1$

➡ Can't use perturbation theory !

Non-Perturbative regime

★ Coupling constant runs, smaller coupling at higher energy scales

$$\alpha_S(100\,\mathrm{GeV}) \sim 0.1$$

➡ Can use perturbation theory

Asymptotic Freedom

★ Where calculations can be performed, QCD provides a good description of relevant experimental data

Appendix A1 : Electromagnetism

(Non-examinable)

★ In Heaviside-Lorentz units $\varepsilon_0 = \mu_0 = c = 1$ Maxwell's equations in the vacuum become

$$\vec{\nabla}.\vec{E} = \rho; \quad \vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}; \quad \vec{\nabla} \cdot \vec{B} = 0; \quad \vec{\nabla} \wedge \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}$$

★ The electric and magnetic fields can be expressed in terms of scalar and vector potentials

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi; \quad \vec{B} = \vec{\nabla} \wedge \vec{A} \qquad (26)$$

★ In terms of the 4-vector potential $A^\mu = (\phi, \vec{A})$ and the 4-vector current $j^\mu = (\rho, \vec{J})$ Maxwell's equations can be expressed in the covariant form:

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad (27)$$

where $F^{\mu\nu}$ is the anti-symmetric field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad (28)$$

•Combining (27) and (28)

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu \qquad (29)$$

which can be written $\Box^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu$ (30)

where the D′Alembertian operator

$$\Box^2 = \partial_\nu \partial^\nu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

- Acting on equation (30)) with ∂_ν gives

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial_\nu \partial^\nu A^\mu = 0$$

⇨ $$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$ **Conservation of Electric Charge**

- Conservation laws are associated with symmetries. Here the symmetry is the **GAUGE INVARIANCE** of electro-magnetism

Appendix A2 : Gauge Invariance **(Non-examinable)**

★ The electric and magnetic fields are unchanged for the **gauge transformation**:

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\chi; \qquad \phi \rightarrow \phi' = \phi - \frac{\partial \chi}{\partial t}$$

where $\chi = \chi(t, \vec{x})$ is any finite differentiable function of position and time

★ In 4-vector notation the **gauge transformation** can be expressed as:

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \chi$$

★ Using the fact that the physical fields are gauge invariant, choose χ to be a solution of

$$\Box^2\chi = -\partial_\mu A^\mu$$

★ In this case we have

$$\partial^\mu A'_\mu = \partial^\mu(A_\mu + \partial_\mu\chi) = \partial^\mu A_\mu + \Box^2\chi = 0$$

★ Dropping the prime we have a chosen a gauge in which

$$\boxed{\partial_\mu A^\mu = 0} \qquad \boxed{\text{The Lorentz Condition}} \qquad (31)$$

★ With the Lorentz condition, equation (30)) becomes:

$$\boxed{\Box^2 A^\mu = j^\mu} \qquad (32)$$

★ Having imposed the Lorentz condition we still have freedom to make a further gauge transformation, i.e.

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu\Lambda$$

where $\Lambda(t,\vec{x})$ is any function that satisfies

$$\Box^2\Lambda = 0 \qquad (33)$$

★ Clearly (32)) remains unchanged, in addition the Lorentz condition still holds:

$$\partial^\mu A'_\mu = \partial^\mu(A_\mu + \partial_\mu\Lambda) = \partial^\mu A_\mu + \Box^2\Lambda = \partial^\mu A_\mu = 0$$

Appendix A3 : Photon Polarization

(Non-examinable)

- For a free photon (i.e. $j^\mu = 0$) equation (32) becomes

$$\boxed{\Box^2 A^\mu = 0} \qquad (34)$$

(note have chosen a gauge where the Lorentz condition is satisfied)

★ Equation (33) has solutions (i.e. the wave-function for a free photon)

$$A^\mu = \varepsilon^\mu(q) e^{-iq.x}$$

where ε^μ is the four-component polarization vector and q is the photon four-momentum

$$0 = \Box^2 A^\mu = -q^2 \varepsilon^\mu e^{-iq.x}$$

$$\Rightarrow \quad q^2 = 0$$

★ Hence equation (34) describes a massless particle.

★ But the solution has four components – might ask how it can describe a spin-1 particle which has three polarization states?

★ But for (A8) to hold we must satisfy the Lorentz condition:

$$0 = \partial_\mu A^\mu = \partial_\mu(\varepsilon^\mu e^{-iq.x}) = \varepsilon^\mu \partial_\nu(e^{-iq.x}) = -i\varepsilon^\mu q_\mu e^{-iq.x}$$

Hence the Lorentz condition gives $\boxed{q_\mu \varepsilon^\mu = 0}$ (35)

i.e. only 3 independent components.

- ★ However, in addition to the Lorentz condition still have the addional gauge freedom of $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda$ with (33) $\Box^2 \Lambda = 0$

- •Choosing $\Lambda = iae^{-iq.x}$ which has $\Box^2 \Lambda = q^2 \Lambda = 0$

$$\begin{aligned} A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda &= \varepsilon_\mu e^{-iq.x} + ia\partial_\mu e^{-iq.x} \\ &= \varepsilon_\mu e^{-iq.x} + ia(-iq_\mu)e^{-iq.x} \\ &= (\varepsilon_\mu + aq_\mu)e^{-iq.x} \end{aligned}$$

- ★ Hence the electromagnetic field is left unchanged by

$$\varepsilon_\mu \rightarrow \varepsilon'_\mu = \varepsilon_\mu + aq_\mu$$

- ★ Hence the two polarization vectors which differ by a mulitple of the photon four-momentum describe the same photon. Choose a such that the time-like component of ε_μ is zero, i.e. $\varepsilon_0 \equiv 0$

- ★ With this choice of gauge, which is known as the **COULOMB GAUGE**, the Lorentz condition (35) gives

$$\boxed{\vec{\varepsilon} \cdot \vec{q} = 0} \qquad (36)$$

i.e. only 2 independent components, both transverse to the photons momentum

- ★ **A massless photon has two transverse polarisation states. For a photon travelling in the z direction these can be expressed as the transversly polarized states:**

$$\varepsilon_1^\mu = (0,1,0,0); \qquad \varepsilon_2^\mu = (0,0,1,0)$$

- ★ **Alternatively take linear combinations to get the circularly polarized states**

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \qquad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

- ★ **It can be shown that the ε_+ state corresponds the state in which the photon spin is directed in the +z direction, i.e. $S_z = +1$**

These are used on page 122

Appendix A4 : Massive Spin-1 particles

(Non-examinable)

- **For a massless photon we had (before imposing the Lorentz condition) we had from equation (30))**

$$\Box^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu$$

★ **The Klein-Gordon equation for a spin-0 particle of mass m is**

$$(\Box^2 + m^2)\phi = 0$$

suggestive that the appropriate equations for a massive spin-1 particle can be obtained by replacing $\Box^2 \longrightarrow \Box^2 + m^2$

★ **This is indeed the case, and from QFT it can be shown that for a massive spin 1 particle equation (30)) becomes**

$$(\Box^2 + m^2)B^\mu - \partial^\mu(\partial_\nu B^\nu) = j^\mu$$

★ **Therefore a free particle must satisfy**

$$(\Box^2 + m^2)B^\mu - \partial^\mu(\partial_\nu B^\nu) = 0 \qquad (37)$$

- Acting on equation (37) with ∂_ν gives

$$\begin{aligned}(\Box^2+m^2)\partial_\mu B^\mu-\partial_\mu\partial^\mu(\partial_\nu B^\nu) &= 0\\ (\Box^2+m^2)\partial_\mu B^\mu-\Box^2(\partial_\nu B^\nu) &= 0\\ m^2\partial_\mu B^\mu &= 0 \qquad (38)\end{aligned}$$

★ Hence, for a massive spin-1 particle, unavoidably have $\partial_\mu B^\mu=0$; note this is not a relation that reflects to choice of gauge.

- Equation (37) becomes

$$\boxed{(\Box^2+m^2)B^\mu=0} \qquad (39)$$

★ For a free spin-1 particle with 4-momentum, p^μ , equation (39) admits solutions

$$B_\mu=\varepsilon_\mu e^{-ip.x}$$

★ Substituting into equation (38) gives

$$\boxed{p_\mu\varepsilon^\mu=0}$$

★The four degrees of freedom in ε^μ are reduced to three, but for a massive particle, equation (39) does <u>not</u> allow a choice of gauge and we can not reduce the number of degrees of freedom any further.

★ **Hence we need to find three orthogonal polarisation states satisfying**

$$\boxed{p_\mu \varepsilon^\mu = 0} \tag{40}$$

★ **For a particle travelling in the z direction, can still admit the circularly polarized states.**

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \qquad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

★ **Writing the third state as**

These are used on page 478

$$\varepsilon_L^\mu = \frac{1}{\sqrt{\alpha^2+\beta^2}}(\alpha,0,0,\beta)$$

equation (40) gives $\alpha E - \beta p_z = 0$

$$\Rightarrow \qquad \varepsilon_L^\mu = \frac{1}{m}(p_z,0,0,E)$$

★ **This <u>longitudinal</u> polarisation state is only present for massive spin-1 particles, i.e. there is no analogous state for a free on-shell photon.**

Appendix B : Local Gauge Invariance

(Non-examinable)

★ The Dirac equation for a charged particle in an electro-magnetic field can be obtained from the free particle wave-equation by making the minimal substitution

$$\vec{p} \rightarrow \vec{p} - q\vec{A}; \quad E \rightarrow E - q\phi \qquad (q = \text{charge})$$

(see page 121)

In QM: $i\partial_\mu \rightarrow i\partial_\mu - qA_\mu$ and the Dirac equation becomes

$$\boxed{\gamma^\mu (i\partial_\mu - qA_\mu)\psi - m\psi = 0} \qquad (41)$$

★ In Appendix A2 : saw that the physical EM fields where invariant under the gauge transformation

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \chi \qquad \Box^2 \chi = 0$$

★ Under this transformation the Dirac equation becomes

$$\gamma^\mu (i\partial_\mu - qA_\mu + q\partial_\mu \chi)\psi - m\psi = 0$$

which is not the same as the original equation. If we require that the Dirac equation is invariant under the Gauge transformation then under the gauge transformation we need to modify the wave-functions

$$\psi \rightarrow \psi' = \psi e^{iq\chi}$$

A Local Phase Transformation

★**To prove this, applying the gauge transformation :**

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \chi \qquad \psi \rightarrow \psi' = \psi e^{iq\chi}$$

to the original Dirac equation gives

$$\gamma^\mu (i\partial_\mu - qA_\mu + q\partial_\mu \chi)\psi e^{iq\chi} - m\psi e^{iq\chi} = 0 \qquad (42)$$

★ **But** $$i\partial_\mu(\psi e^{iq\chi}) = ie^{iq\chi}\partial_\mu \psi - q(\partial_\mu \chi)e^{iq\chi}\psi$$

★ **Equation (42) becomes**

$$\gamma^\mu e^{iq\chi}(i\partial_\mu - qA_\mu + q\partial_\mu \chi - q\partial_\mu \chi)\psi - m\psi e^{iq\chi} = 0$$

$$\Rightarrow \qquad \gamma^\mu e^{iq\chi}(i\partial_\mu - qA_\mu)\psi - m\psi e^{iq\chi} = 0$$

$$\Rightarrow \qquad \gamma^\mu (i\partial_\mu - qA_\mu)\psi - m\psi = 0$$

which is the original form of the Dirac equation

Appendix C : Local Gauge Invariance 2

(Non-examinable)

- ★ Reverse the argument of Appendix B. Suppose there is a fundamental symmetry of the universe under local phase transformations

$$\psi(x) \rightarrow \psi'(x) = \psi(x)e^{iq\chi(x)}$$

- ★ Note that the local nature of these transformations: the phase transformation depends on the space-time coordinate $x = (t, \vec{x})$

- ★ Under this transformation the free particle Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

becomes

$$i\gamma^\mu \partial_\mu (\psi e^{iq\chi}) - m\psi e^{iq\chi} = 0$$

$$ie^{iq\chi}\gamma^\mu(\partial_\mu \psi + iq\psi\partial_\mu \chi) - m\psi e^{iq\chi} = 0$$

$$i\gamma^\mu(\partial_\mu + iq\partial_\mu \chi)\psi - m\psi = 0$$

Local phase invariance is not possible for a free theory, i.e. one without interactions

- ★ To restore invariance under local phase transformations have to introduce a massless "gauge boson" A^μ which transforms as

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \chi$$

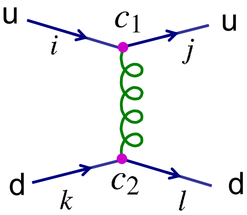
and make the substitution

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + iqA_\mu$$

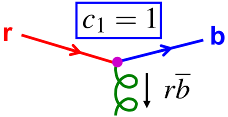
Appendix D: Alternative evaluation of colour factors

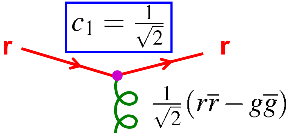
“Non-examinable” but can be used to derive colour factors.

★The colour factors can be obtained (more intuitively) as follows :



- Write $C(ik \to jl) = \frac{1}{2}c_1 c_2$
- Where the colour coefficients at the two vertices depend on the quark and gluon colours

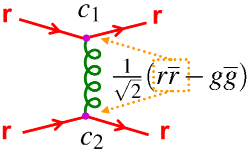


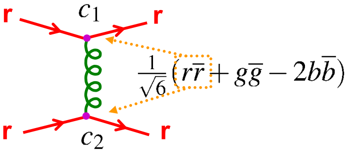


- Sum over all possible exchanged gluons conserving colour at both vertices

① Configurations involving a single colour

e.g. $rr \rightarrow rr$ **: two possible exchanged gluons**



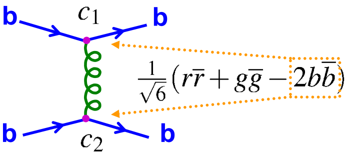


$$c_1 = c_2 = \frac{1}{\sqrt{2}}$$

$$c_1 = c_2 = \frac{1}{\sqrt{6}}$$

$$C(rr \rightarrow rr) = \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{6}\right) = \tfrac{1}{3}$$

e.g. $bb \rightarrow bb$ **: only one possible exchanged gluon**

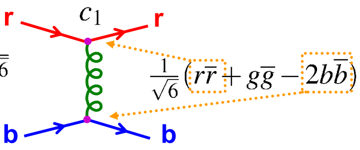


$$c_1 = c_2 = -\frac{2}{\sqrt{6}}$$

➡ $$C(bb \rightarrow bb) = \tfrac{1}{2}\left(\frac{2}{\sqrt{6}}\frac{2}{\sqrt{6}}\right) = \tfrac{1}{3}$$

② Other configurations where quarks don't change colour

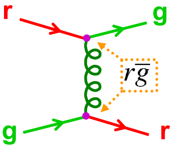
$$c_1 = \frac{1}{\sqrt{6}}$$



$$c_2 = -\frac{2}{\sqrt{6}}$$

$$C(rb \rightarrow rb) = \frac{1}{2}\left(-\frac{1}{\sqrt{6}}\frac{2}{\sqrt{6}}\right) = -\frac{1}{6}$$

③ Configurations where quarks swap colours



$$c_1 = c_2 = 1$$

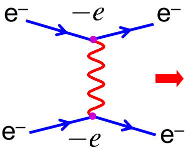
$$C(rg \rightarrow gr) = \frac{1}{2}$$

Appendix E: Colour Potentials

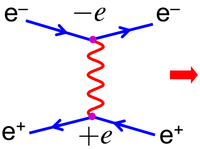
Non-examinable

- Previously argued that gluon self-interactions lead to a $+\lambda r$ long-range potential and that this is likely to explain colour confinement
- Have yet to consider the short range potential – i.e. for quarks in mesons and baryons does QCD lead to an attractive potential?
- Analogy with QED: (NOTE this is very far from a formal proof)

QED



$$V(r) = +\frac{\alpha}{r}$$



$$V(r) = -\frac{\alpha}{r}$$

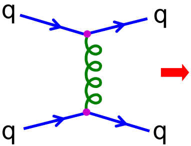
Repulsive Potential

Static

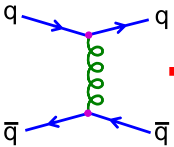
Attractive Potential

QCD

★ by analogy with QED expect potentials of form



$$V(r) = +C\frac{\alpha_S}{r}$$



$$V(r) = -C\frac{\alpha_S}{r}$$

★ Whether it is a attractive or repulsive potential depends on sign of colour factor

★ **Consider the colour factor for a q$\bar{q}$ system in the colour singlet state:**

$$\psi = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

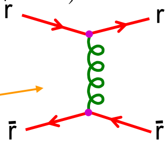
with colour potential $\langle V_{q\bar{q}}\rangle = \langle\psi|V_{\rm QCD}|\psi\rangle$

➡ $$\langle V_{q\bar{q}}\rangle = \frac{1}{3}\left(\langle r\bar{r}|V_{\rm QCD}|r\bar{r}\rangle + \ldots + \langle r\bar{r}|V_{\rm QCD}|b\bar{b}\rangle + \ldots\right)$$

- **Following the QED analogy:**

$$\langle r\bar{r}|V_{\rm QCD}|r\bar{r}\rangle = -C(r\bar{r}\to r\bar{r})\frac{\alpha_S}{r}$$

which is the term arising from $r\bar{r}\to r\bar{r}$



- **Have 3 terms like** $r\bar{r}\to r\bar{r}, b\bar{b}\to b\bar{b},\ldots$ **and 6 like** $r\bar{r}\to g\bar{g}, r\bar{r}\to b\bar{b},\ldots$

$$\langle V_{q\bar{q}}\rangle = -\frac{1}{3}\frac{\alpha_S}{r}[3\times C(r\bar{r}\to r\bar{r}) + 6\times C(r\bar{r}\to g\bar{g})] = -\frac{1}{3}\frac{\alpha_S}{r}\left[3\times\frac{1}{3} + 6\times\frac{1}{2}\right]$$

➡ $$\boxed{\langle V_{q\bar{q}}\rangle = -\frac{4}{3}\frac{\alpha_S}{r}}$$

NEGATIVE ➡ATTRACTIVE

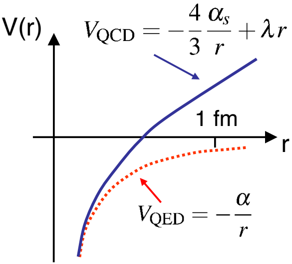
- **The same calculation for a q$\bar{q}$ colour octet state, e.g.** $r\bar{g}$ **gives a positive repulsive potential:** $C(r\bar{g}\to r\bar{g}) = -\frac{1}{6}$

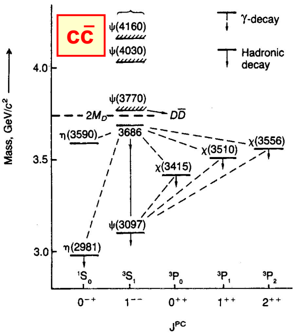
★ **Whilst not a formal proof, it is comforting to see that in the colour singlet** $q\bar{q}$ **state the QCD potential is indeed attractive.** **(question 15)**

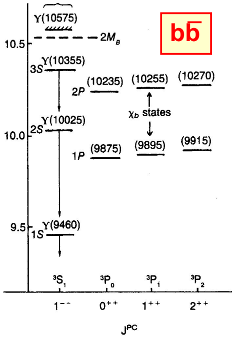
- ★ **Combining the short-range QCD potential with the linear long-range term discussed previously:**

$$\boxed{V_{\rm QCD} = -\frac{4}{3}\frac{\alpha_s}{r} + \lambda r}$$

- ★ **This potential is found to give a good description of the observed charmonium (c$\bar{\text{c}}$) and bottomonium (b$\bar{\text{b}}$) bound states.**







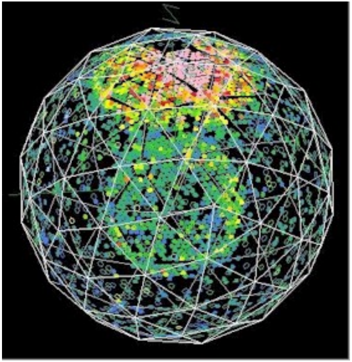
NOTE:

- **c, b are heavy quarks**
- **approx. non-relativistic**
- **orbit close together**
- **probe 1/r part of $V_{\rm QCD}$**

Agreement of data with prediction provides strong evidence that $V_{\rm QCD}$ has the Expected form

Particle Physics

Dr Lester



Handout 9 : The Weak Interaction and V-A

Parity

★The parity operator performs spatial inversion through the origin:

$$\psi'(\vec{x},t)=\hat{P}\psi(\vec{x},t)=\psi(-\vec{x},t)$$

•applying $\hat{P}$ twice: $\hat{P}\hat{P}\psi(\vec{x},t)=\hat{P}\psi(-\vec{x},t)=\psi(\vec{x},t)$

so $\hat{P}\hat{P}=I$ ➡ $\hat{P}^{-1}=\hat{P}$

•To preserve the normalisation of the wave-function

$$\langle\psi|\psi\rangle=\langle\psi'|\psi'\rangle=\langle\psi|\hat{P}^{\dagger}\hat{P}|\psi\rangle$$

$\hat{P}^{\dagger}\hat{P}=I$ ➡ $\hat{P}$ Unitary

• But since $\hat{P}\hat{P}=I$ $\hat{P}=\hat{P}^{\dagger}$ ➡ $\hat{P}$ Hermitian

which implies Parity is an observable quantity. If the interaction Hamiltonian commutes with $\hat{P}$, parity is an observable conserved quantity

• If $\psi(\vec{x},t)$ is an eigenfunction of the parity operator with eigenvalue P

$\hat{P}\psi(\vec{x},t)=P\psi(\vec{x},t)$ ➡ $\hat{P}\hat{P}\psi(\vec{x},t)=P\hat{P}\psi(\vec{x},t)=P^2\psi(\vec{x},t)$

since $\hat{P}\hat{P}=I$ $P^2=1$

➡ Parity has eigenvalues $P=\pm1$

★ QED and QCD are invariant under parity

★ **Experimentally** observe that Weak Interactions do not conserve parity

Intrinsic Parities of fundamental particles:

Spin-1 Bosons

- From Gauge Field Theory can show that the gauge bosons have $P = -1$

$$P_\gamma = P_g = P_{W^+} = P_{W^-} = P_Z = -1$$

Spin-½ Fermions

- From the Dirac equation showed (handout 2):

  Spin ½ particles have opposite parity to spin ½ anti-particles

- Conventional choice: spin ½ particles have $P = +1$

$$P_{e^-} = P_{\mu^-} = P_{\tau^-} = P_\nu = P_q = +1$$

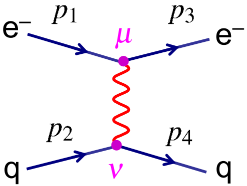
and anti-particles have opposite parity, i.e.

$$P_{e^+} = P_{\mu^+} = P_{\tau^+} = P_{\overline{\nu}} = P_{\overline{q}} = -1$$

★ For Dirac spinors it was shown (handout 2) that the parity operator is:

$$\boxed{\hat{P} = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}$$

Parity Conservation in QED and QCD



- Consider the QED process $e^-q \to e^-q$
- The Feynman rules for QED give:

$$-iM \propto [\overline{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}_q(p_4)ie\gamma^\nu u_q(p_2)]$$

- Which can be expressed in terms of the electron and quark 4-vector currents:

$$M \propto -\frac{e^2}{q^2}g_{\mu\nu}j_e^\mu j_q^\nu = -\frac{e^2}{q^2}j_e.j_q$$

with $j_e = \overline{u}_e(p_3)\gamma^\mu u_e(p_1)$ and $j_q = \overline{u}_q(p_4)\gamma^\mu u_q(p_2)$

★ Consider the what happen to the matrix element under the parity transformation

- ♦ Spinors transform as $\boxed{u \xrightarrow{\hat{P}} \hat{P}u = \gamma^0 u}$
- ♦ Adjoint spinors transform as

$$\overline{u} = u^\dagger\gamma^0 \xrightarrow{\hat{P}} (\hat{P}u)^\dagger\gamma^0 = u^\dagger\gamma^{0\dagger}\gamma^0 = u^\dagger\gamma^0\gamma^0 = \overline{u}\gamma^0$$

$$\boxed{\overline{u} \xrightarrow{\hat{P}} \overline{u}\gamma^0}$$

- ♦ Hence $j_e = \overline{u}_e(p_3)\gamma^\mu u_e(p_1) \xrightarrow{\hat{P}} \overline{u}_e(p_3)\gamma^0\gamma^\mu\gamma^0 u_e(p_1)$

★ **Consider the components of the four-vector current**

0: $j_e^0 \xrightarrow{\hat{P}} \overline{u}\gamma^0\gamma^0\gamma^0 u = \overline{u}\gamma^0 u = j_e^0$ **since** $\gamma^0\gamma^0 = 1$

k=1,2,3: $j_e^k \xrightarrow{\hat{P}} \overline{u}\gamma^0\gamma^k\gamma^0 u = -\overline{u}\gamma^k\gamma^0\gamma^0 u = -\overline{u}\gamma^k u = -j_e^k$ **since** $\gamma^0\gamma^k = -\gamma^k\gamma^0$

- **The time-like component remains unchanged and the space-like components change sign**
- **Similarly** $j_q^0 \xrightarrow{\hat{P}} j_q^0 \qquad j_q^k \xrightarrow{\hat{P}} -j_q^k \quad k = 1,2,3$

★ **Consequently the four-vector scalar product**

$$j_e.j_q = j_e^0 j_q^0 - j_e^k j_q^k \xrightarrow{\hat{P}} j_e^0 j_q^0 - (-j_e^k)(-j_q^k) = j_e.j_q \quad k = 1,3$$

or $j^\mu \xrightarrow{\hat{P}} j_\mu$

$$j^\mu.j^\nu \xrightarrow{\hat{P}} j_\mu.j_\nu \xrightarrow{\hat{P}} j^\mu.j^\nu$$

QED Matrix Elements are Parity Invariant

Parity Conserved in QED

★ **The QCD vertex has the same form and, thus,**

Parity Conserved in QCD

Parity Violation in β-Decay

★The parity operator $\hat{P}$ corresponds to a discrete transformation $x \longrightarrow -x, etc.$

★Under the parity transformation:

Vectors
change sign

$$\left\{\begin{array}{ll} \vec{r} \xrightarrow{\hat{P}} -\vec{r} & \\ \vec{p} \xrightarrow{\hat{P}} -\vec{p} & \left(p_x = \frac{\partial}{\partial x},\ etc.\right) \end{array}\right.$$

Axial-Vectors
unchanged

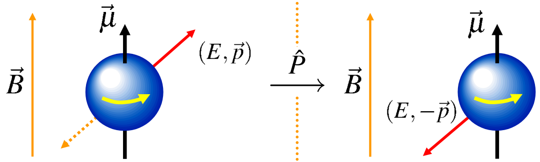
$$\left\{\begin{array}{ll} \vec{L} \xrightarrow{\hat{P}} \vec{L} & (\vec{L} = \vec{r} \wedge \vec{p}) \\ \vec{\mu} \xrightarrow{\hat{P}} \vec{\mu} & (\vec{\mu} \propto \vec{L}) \end{array}\right.$$

Note B is an axial vector
$\mathrm{d}\vec{B} \propto \vec{J} \wedge \vec{r}\, \mathrm{d}^3\vec{r}$

★**1957:** C.S.Wu et al. studied beta decay of polarized cobalt-60 nuclei:

$${}^{60}\mathrm{Co} \longrightarrow {}^{60}Ni^{*} + e^{-} + \overline{\nu}_e$$

★Observed electrons emitted preferentially in direction opposite to applied field



more e- in ↙ c.f. ↗

If parity were conserved: expect equal rate for producing e⁻ in directions along and opposite to the nuclear spin.

★Conclude parity is violated in WEAK INTERACTION
➡ that the WEAK interaction vertex is NOT of the form $\overline{u}_e \gamma^{\mu} u_{\nu}$

Bilinear Covariants

- ★The requirement of Lorentz invariance of the matrix element severely restricts the form of the interaction vertex. QED and QCD are "VECTOR" interactions:

$$j^\mu = \overline{\psi}\gamma^\mu\phi$$

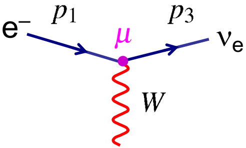
- ★This combination transforms as a 4-vector (Handout 2 appendix V)
- ★ In general, there are only 5 possible combinations of two spinors and the gamma matrices that form Lorentz covariant currents, called "bilinear covariants":

Type	Form	Components	"Boson Spin"
♦ SCALAR	$\overline{\psi}\phi$	1	0
♦ PSEUDOSCALAR	$\overline{\psi}\gamma^5\phi$	1	0
♦ VECTOR	$\overline{\psi}\gamma^\mu\phi$	4	1
♦ AXIAL VECTOR	$\overline{\psi}\gamma^\mu\gamma^5\phi$	4	1
♦ TENSOR	$\overline{\psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\phi$	6	2

- ★ Note that in total the sixteen components correspond to the 16 elements of a general 4x4 matrix: "decomposition into Lorentz covariant combinations"
- ★ In QED the factor $g_{\mu\nu}$ arose from the sum over polarization states of the virtual photon (2 transverse + 1 longitudinal, 1 scalar) = (2J+1) + 1
- ★ Associate SCALAR and PSEUDOSCALAR interactions with the exchange of a SPIN-0 boson, etc. – no spin degrees of freedom

V-A Structure of the Weak Interaction

- ★ The most general form for the interaction between a fermion and a boson is a linear combination of bilinear covariants
- ★ For an interaction corresponding to the **exchange of a spin-1** particle the most general form is a linear combination of **VECTOR** and **AXIAL-VECTOR**
- ★ The form for WEAK interaction is <u>determined from experiment</u> to be **VECTOR – AXIAL-VECTOR** **(V – A)**



$$j^\mu \propto \overline{u}_{\nu_e}(\gamma^\mu - \gamma^\mu\gamma^5)u_e$$

V – A

- ★ Can this account for parity violation?
- ★ First consider parity transformation of a **pure AXIAL-VECTOR** current

$$j_A = \overline{\psi}\gamma^\mu\gamma^5\phi \qquad \text{with} \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3; \quad \gamma^5\gamma^0 = -\gamma^0\gamma^5$$

$$j_A = \overline{\psi}\gamma^\mu\gamma^5\phi \xrightarrow{\hat{P}} \overline{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\phi = -\overline{\psi}\gamma^0\gamma^\mu\gamma^0\gamma^5\phi$$

$$j_A^0 = \xrightarrow{\hat{P}} -\overline{\psi}\gamma^0\gamma^0\gamma^0\gamma^5\phi = -\overline{\psi}\gamma^0\gamma^5\phi = -j_A^0$$

$$j_A^k = \xrightarrow{\hat{P}} -\overline{\psi}\gamma^0\gamma^k\gamma^0\gamma^5\phi = +\overline{\psi}\gamma^k\gamma^5\phi = +j_A^k \qquad k = 1,2,3$$

or $j_A^\mu \xrightarrow{\hat{P}} -j_{A\mu}$

- **The space-like components remain unchanged and the time-like components change sign (the opposite to the parity properties of a vector-current)**

$$\boxed{j_A^0 \xrightarrow{\hat{P}} -j_A^0; \quad j_A^k \xrightarrow{\hat{P}} +j_A^k;} \boxed{j_V^0 \xrightarrow{\hat{P}} +j_V^0; \quad j_V^k \xrightarrow{\hat{P}} -j_V^k}$$

- **Now consider the matrix elements**

$$M \propto g_{\mu\nu} j_1^\mu j_2^\nu = j_1^0 j_2^0 - \sum_{k=1,3} j_1^k j_2^k$$

- **For the combination of a two axial-vector currents**

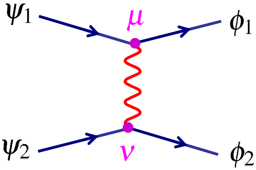
$$j_{A1} . j_{A2} \xrightarrow{\hat{P}} (-j_1^0)(-j_2^0) - \sum_{k=1,3} (j_1^k)(j_2^k) = j_{A1} . j_{A2}$$

- **Consequently parity is conserved for both a pure vector and pure axial-vector interactions**
- **However the combination of a vector current and an axial vector current**

$$j_{V1} . j_{A2} \xrightarrow{\hat{P}} (j_1^0)(-j_2^0) - \sum_{k=1,3} (-j_1^k)(j_2^k) = -j_{V1} . j_{A2}$$

changes sign under parity – can give parity violation !

★ **Now consider a general linear combination of VECTOR and AXIAL-VECTOR (note this is relevant for the Z-boson vertex)**



$$\begin{cases} j_1 = \overline{\phi}_1(g_V\gamma^\mu + g_A\gamma^\mu\gamma^5)\psi_1 = g_V j_1^V + g_A j_1^A \\ \dfrac{g_{\mu\nu}}{q^2-m^2} \\ j_2 = \overline{\phi}_2(g_V\gamma^\mu + g_A\gamma^\mu\gamma^5)\psi_2 = g_V j_2^V + g_A j_2^A \end{cases}$$

$$M_{fi} \propto j_1.j_2 = g_V^2 j_1^V.j_2^V + g_A^2 j_1^A.j_2^A + g_V g_A(j_1^V.j_2^A + j_1^A.j_2^V)$$

- **Consider the parity transformation of this scalar product**

$$j_1.j_2 \xrightarrow{\hat{P}} g_V^2 j_1^V.j_2^V + g_A^2 j_1^A.j_2^A - g_V g_A(j_1^V.j_2^A + j_1^A.j_2^V)$$

- **If either g_A or g_V is zero, Parity is conserved, i.e. parity conserved in a pure VECTOR or pure AXIAL-VECTOR interaction**

- **Relative strength of parity violating part** $\propto \dfrac{g_V g_A}{g_V^2 + g_A^2}$

Maximal Parity Violation for V-A (or V+A)

Chiral Structure of QED (Reminder)

★ **Recall (Handout 4) introduced CHIRAL projections operators**

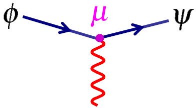
$$P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)$$

project out chiral right- and left- handed states

★ **In the ultra-relativistic limit, chiral states correspond to helicity states**

★ **Any spinor can be expressed as:**

$$\psi = \tfrac{1}{2}(1+\gamma^5)\psi + \tfrac{1}{2}(1-\gamma^5)\psi = P_R\psi + P_L\psi = \psi_R + \psi_L$$



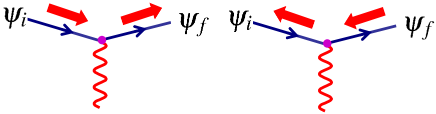
•**The QED vertex** $\overline{\psi}\gamma^\mu\phi$ **in terms of chiral states:**

$$\overline{\psi}\gamma^\mu\phi = \overline{\psi}_R\gamma^\mu\phi_R + \overline{\psi}_R\gamma^\mu\phi_L + \overline{\psi}_L\gamma^\mu\phi_R + \overline{\psi}_L\gamma^\mu\phi_L$$

conserves chirality, e.g.

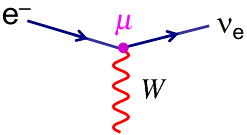
$$\begin{aligned}\overline{\psi}_R\gamma^\mu\phi_L &= \tfrac{1}{2}\psi^\dagger(1+\gamma^5)\gamma^0\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\phi \\ &= \tfrac{1}{4}\psi^\dagger\gamma^0(1-\gamma^5)\gamma^\mu(1-\gamma^5)\phi \\ &= \tfrac{1}{4}\overline{\psi}\gamma^\mu(1+\gamma^5)(1-\gamma^5)\phi = 0\end{aligned}$$

★ **In the ultra-relativistic limit only two helicity combinations are non-zero**



Chiral and Helicity Structure of the Weak Interaction

★**The charged current (W^±) weak vertex is:**



$$\boxed{\frac{-ig_w}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5)}$$

★**Since** $\frac{1}{2}(1-\gamma^5)$ **projects out left-handed chiral particle states:**

$$\overline{\psi}\tfrac{1}{2}\gamma^\mu(1-\gamma^5)\phi = \overline{\psi}\gamma^\mu\phi_L$$

(question 16)

★**Writing** $\overline{\psi} = \overline{\psi}_R + \overline{\psi}_L$ **and from discussion of QED,** $\overline{\psi}_R\gamma^\mu\phi_L = 0$ **gives**

$$\overline{\psi}\tfrac{1}{2}\gamma^\mu(1-\gamma^5)\phi = \overline{\psi}_L\gamma^\mu\phi_L$$

⇒ **Only the left-handed chiral components of particle spinors and right-handed chiral components of anti-particle spinors participate in charged current weak interactions**

★**At very high energy** $(E \gg m)$ **, the left-handed chiral components are helicity eigenstates :**

$\frac{1}{2}(1-\gamma^5)u \Rightarrow$

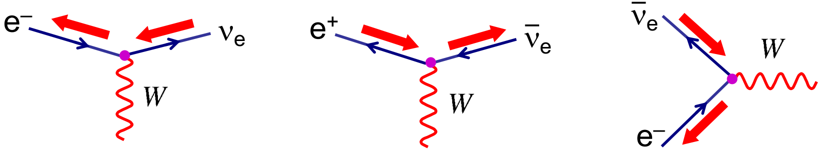
LEFT-HANDED PARTICLES
Helicity = -1

$\frac{1}{2}(1-\gamma^5)v \Rightarrow$

RIGHT-HANDED ANTI-PARTICLES
Helicity = +1

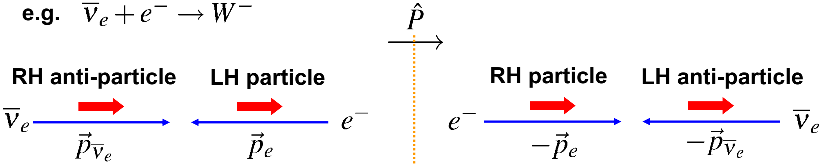
In the ultra-relativistic limit only left-handed particles and right-handed antiparticles participate in charged current weak interactions

e.g. In the relativistic limit, the only possible electron – neutrino interactions are:



★ The helicity dependence of the weak interaction ⟷ parity violation

e.g. $\overline{\nu}_e + e^- \rightarrow W^-$


$\hat{P}$

RH anti-particle **LH particle**

$\overline{\nu}_e$ $\vec{p}_{\overline{\nu}_e}$ $\vec{p}_e$ e^-

RH particle **LH anti-particle**

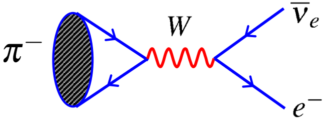
e^- $-\vec{p}_e$ $-\vec{p}_{\overline{\nu}_e}$ $\overline{\nu}_e$

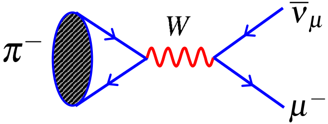
Valid weak interaction

Does not occur

Helicity in Pion Decay

★The decays of charged pions provide a good demonstration of the role of helicity in the weak interaction





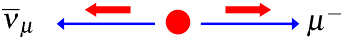
EXPERIMENTALLY:

$$\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} = 1.23 \times 10^{-4}$$

- Might expect the decay to electrons to dominate – due to increased phase space…. The opposite happens, the electron decay is helicity suppressed

★Consider decay in pion rest frame.

- Pion is spin zero: so the spins of the $\bar{\nu}$ and μ are opposite
- Weak interaction only couples to RH chiral anti-particle states. Since neutrinos are (almost) massless, must be in RH Helicity state
- Therefore, to conserve angular mom. muon is emitted in a RH **HELICITY** state



- But only left-handed **CHIRAL** particle states participate in weak interaction

★The general **right-handed helicity** solution to the Dirac equation is

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad \text{with} \quad c=\cos\frac{\theta}{2} \quad \text{and} \quad s=\sin\frac{\theta}{2}$$

- project out the **left-handed** **chiral** part of the wave-function using

$$P_L = \tfrac{1}{2}(1-\gamma^5) = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

giving

$$P_L u_\uparrow = \tfrac{1}{2}N\left(1-\frac{|\vec{p}|}{E+m}\right)\begin{pmatrix} c \\ e^{i\phi}s \\ -c \\ -e^{i\phi}s \end{pmatrix} = \tfrac{1}{2}\left(1-\frac{|\vec{p}|}{E+m}\right)u_L$$

In the limit $m \ll E$ this tends to zero

- similarly

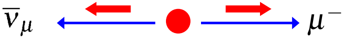
$$P_R u_\uparrow = \tfrac{1}{2}N\left(1+\frac{|\vec{p}|}{E+m}\right)\begin{pmatrix} c \\ e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix} = \tfrac{1}{2}\left(1+\frac{|\vec{p}|}{E+m}\right)u_R$$

In the limit $m \ll E$, $P_R u_\uparrow \rightarrow u_R$

★ **Hence**

$$\underset{\text{RH Helicity}}{\boxed{u_\uparrow}} = P_R u_\uparrow + P_L u_\uparrow = \frac{1}{2}\left(1+\frac{|\vec{p}|}{E+m}\right)\underset{\text{RH Chiral}}{\boxed{u_R}} + \frac{1}{2}\left(1-\frac{|\vec{p}|}{E+m}\right)\underset{\text{LH Chiral}}{\boxed{u_L}}$$

- In the limit $E \gg m$, as expected, the RH chiral and helicity states are identical
- Although only LH chiral particles participate in the weak interaction the contribution from RH **Helicity** states is not necessarily zero !



$m_\nu \approx 0$: RH Helicity ≡ RH Chiral

$m_\mu \neq 0$: RH Helicity has LH Chiral Component

★ Expect matrix element to be proportional to **LH chiral component of RH Helicity** electron/muon **spinor**

$$M_{fi} \propto \frac{1}{2}\left(1-\frac{|\vec{p}|}{E+m}\right) = \boxed{\frac{m_\mu}{m_\pi + m_\mu}}$$

from the kinematics of pion decay at rest

★ Hence because the electron mass is much smaller than the pion mass the decay $\pi^- \to e^- \overline{\nu}_e$ **is heavily suppressed.**

Evidence for V-A

★**The V-A nature of the charged current weak interaction vertex fits with experiment**

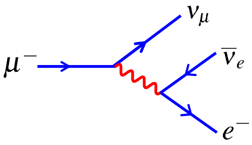
EXAMPLE charged pion decay (question 17)

- **Experimentally measure:** $\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} = (1.230 \pm 0.004) \times 10^{-4}$
- **Theoretical predictions** (depend on Lorentz Structure of the interaction)

V-A $(\overline{\psi}\gamma^\mu(1-\gamma^5)\phi)$ or **V+A** $(\overline{\psi}\gamma^\mu(1+\gamma^5)\phi)$ ➡ $\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} \approx 1.3 \times 10^{-4}$

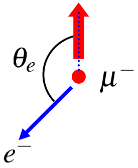
Scalar $(\overline{\psi}\phi)$ or **Pseudo-Scalar** $(\overline{\psi}\gamma^5\phi)$ ➡ $\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} = 5.5$

EXAMPLE muon decay



e.g. TWIST expt: 6x10⁹ μ decays
Phys. Rev. Lett. 95 (2005) 101805

Measure electron energy and angular distributions relative to muon spin direction. Results expressed in terms of general S+P+V+A+T form in "**Michel Parameters**"



$$\rho = 0.75080 \pm 0.00105$$

V-A Prediction: $\rho = 0.75$

Weak Charged Current Propagator

★The charged-current Weak interaction is different from QED and QCD in that it is mediated by massive W-bosons (80.3 GeV)

★This results in a more complicated form for the propagator:

- in handout 4 showed that for the exchange of a massive particle:

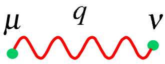
$$\overset{\text{massless}}{\frac{1}{q^2}} \longrightarrow \overset{\text{massive}}{\frac{1}{q^2-m^2}}$$

- In addition the sum over W boson polarization states modifies the numerator

W-boson propagator

spin 1 $W^{\pm}$

$$\frac{-i\left[g_{\mu\nu}-q_{\mu}q_{\nu}/m_W^2\right]}{q^2-m_W^2}$$



★ However in the limit where q^2 is small compared with $m_W = 80.3\,\mathrm{GeV}$ the interaction takes a simpler form.

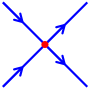
W-boson propagator ($q^2 \ll m_W^2$)

$$\frac{ig_{\mu\nu}}{m_W^2}$$

- The interaction appears point-like (i.e no q^2 dependence)

Connection to Fermi Theory

★ **In 1934, before the discovery of parity violation, Fermi proposed, in analogy with QED, that the invariant matrix element for β-decay was of the form:**



$$M_{fi} = G_{\mathrm{F}}\, g_{\mu\nu}[\overline{\psi}\gamma^{\mu}\psi][\overline{\psi}\gamma^{\nu}\psi]$$

where $G_{\mathrm{F}} = 1.166\times10^{-5}\,\mathrm{GeV}^{-2}$

- **Note the absence of a propagator : i.e. this represents an interaction at a point**

★ **After the discovery of parity violation in 1957 this was modified to**

$$M_{fi} = \frac{G_{\mathrm{F}}}{\sqrt{2}}\, g_{\mu\nu}[\overline{\psi}\gamma^{\mu}(1-\gamma^{5})\psi][\overline{\psi}\gamma^{\nu}(1-\gamma^{5})\psi]$$

(the factor of √2 was included so the numerical value of G_{F} did not need to be changed)

★ **Compare to the prediction for W-boson exchange**

$$M_{fi} = [\tfrac{g_W}{\sqrt{2}}\overline{\psi}\tfrac{1}{2}\gamma^{\mu}(1-\gamma^{5})\psi]\frac{g_{\mu\nu}-q_{\mu}q_{\nu}/m_W^2}{q^2-m_W^2}[\tfrac{g_W}{\sqrt{2}}\overline{\psi}\tfrac{1}{2}\gamma^{\nu}(1-\gamma^{5})\psi]$$

which for $q^2 \ll m_W^2$ **becomes:**

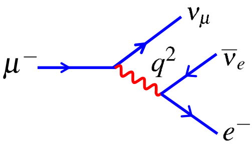
$$M_{fi} = \frac{g_W^2}{8m_W^2}\, g_{\mu\nu}[\overline{\psi}\gamma^{\mu}(1-\gamma^{5})\psi][\overline{\psi}\gamma^{\nu}(1-\gamma^{5})\psi]$$

➡ $$\boxed{\frac{G_{\mathrm{F}}}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}}$$

Still usually use G_{F} to express strength of weak interaction as the is the quantity that is precisely determined in muon decay

Strength of Weak Interaction

★ **Strength of weak interaction most precisely measured in muon decay**



- **Here** $q^2 < m_\mu \; (0.106\,\mathrm{GeV})$
- **To a very good approximation the W-boson propagator can be written**

$$\frac{-i\left[g_{\mu\nu} - q_\mu q_\nu / m_W^2\right]}{q^2 - m_W^2} \approx \frac{i g_{\mu\nu}}{m_W^2}$$

- **In muon decay measure** g_W^2 / m_W^2
- **Muon decay** ➡ $G_F = 1.16639(1) \times 10^{-5}\,\mathrm{GeV}^{-2}$

$$\boxed{\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8 m_W^2}}$$

★ **To obtain the intrinsic strength of weak interaction need to know mass of W-boson:** $m_W = 80.403 \pm 0.029\,\mathrm{GeV}$ **(see handout 14)**

$$\Rightarrow \quad \alpha_W = \frac{g_W^2}{4\pi} = \frac{8 m_W^2 G_F}{4\sqrt{2}\pi} = \frac{1}{30}$$

★ **The intrinsic strength of the weak interaction is similar to, but greater than, the EM interaction ! It is the massive W-boson in the propagator which makes it appear weak. For** $q^2 \gg m_W^2$ **weak interactions are more likely than EM.**

Summary

- ★ Weak interaction is of form **Vector – Axial-vector (V-A)**

$$\boxed{\frac{-ig_w}{\sqrt{2}}\frac{1}{2}\gamma^{\mu}(1-\gamma^5)}$$

- ★ Consequently only **left-handed chiral particle states** and **right-handed chiral anti-particle states** participate in the weak interaction

➡ **MAXIMAL PARITY VIOLATION**

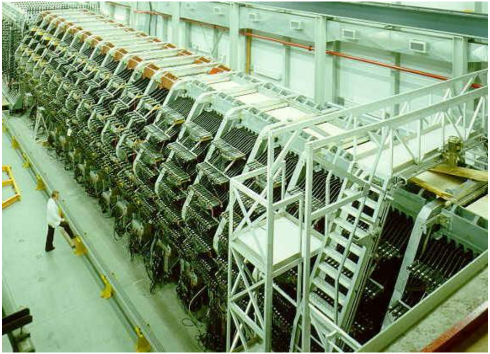
- ★ Weak interaction also violates Charge Conjugation symmetry
- ★ At low q^2 weak interaction is only weak because of the large W-boson mass

$$\boxed{\frac{G_F}{\sqrt{2}}=\frac{g_W^2}{8m_W^2}}$$

- ★ Intrinsic strength of weak interaction is similar to that of QED

Particle Physics

Dr Lester

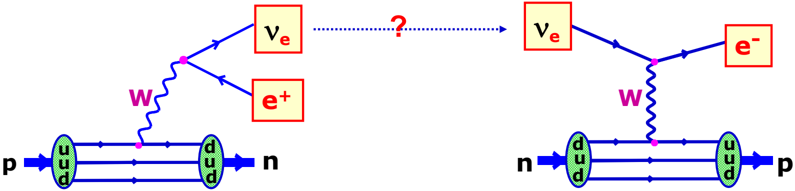


Handout 10 : Leptonic Weak Interactions and Neutrino Deep Inelastic Scattering

Aside : Neutrino Flavours

- ★ **Recent experiments (see Handout 11) ➡ neutrinos have mass (albeit very small)**
- ★ **The textbook neutrino states, ν_e, ν_μ, ν_τ, are not the fundamental particles; these are ν_1, ν_2, ν_3**
- ★ **Concepts like "electron number" conservation are now known not to hold.**
- ★ **So what are ν_e, ν_μ, ν_τ ?**
- ★ **Never directly observe neutrinos – can only detect them by their weak interactions. Hence by definition ν_e is the neutrino state produced along with an electron. Similarly, charged current weak interactions of the state ν_e produce an electron**

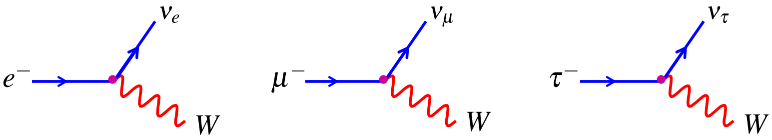
➡ $\boxed{\nu_e, \nu_\mu, \nu_\tau \text{ = weak eigenstates}}$



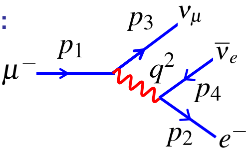
- ★ **Unless dealing with very large distances: the neutrino produced with a positron will interact to produce an electron. For the discussion of the weak interaction continue to use ν_e, ν_μ, ν_τ as if they were the fundamental particle states.**

Muon Decay and Lepton Universality

★The leptonic **charged current** (W±) interaction vertices are:



★Consider muon decay:



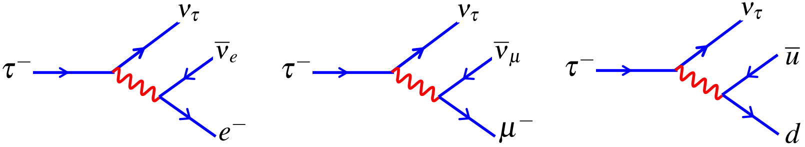
- It is straight-forward to write down the matrix element

$$M_{fi} = \frac{g_W^{(e)} g_W^{(\mu)}}{8m_W^2}[\overline{u}(p_3)\gamma^\mu(1-\gamma^5)u(p_1)]g_{\mu\nu}[\overline{u}(p_2)\gamma^\nu(1-\gamma^5)v(p_4)]$$

Note: for lepton decay $q^2 \ll m_W^2$ so propagator is a constant $1/m_W^2$
i.e. in limit of Fermi theory

- Its evaluation and subsequent treatment of a three-body decay is rather tricky (and not particularly interesting). Here will simply quote the result

- The muon to electron rate $\Gamma(\mu \to e\nu\nu) = \dfrac{G_F^e G_F^\mu m_\mu^5}{192\pi^3} = \dfrac{1}{\tau_\mu}$ with $G_F = \frac{g_W^2}{4\sqrt{2}m_W^2}$
- Similarly for tau to electron $\Gamma(\tau \to e\nu\nu) = \dfrac{G_F^e G_F^\tau m_\tau^5}{192\pi^3}$
- However, the tau can decay to a number of final states:



$Br(\tau \to e\nu\nu) = 0.1784(5)$ $Br(\tau \to \mu\nu\nu) = 0.1736(5)$

- Recall total width (total transition rate) is the sum of the partial widths

$$\Gamma = \sum_i \Gamma_i = \frac{1}{\tau}$$

- Can relate partial decay width to total decay width and therefore lifetime:

$$\Gamma(\tau \to e\nu\nu) = \Gamma_\tau Br(\tau \to e\nu\nu) = Br(\tau \to e\nu\nu)/\tau_\tau$$

- Therefore predict

$$\tau_\mu = \frac{192\pi^3}{G_F^e G_F^\mu m_\mu^5} \qquad \tau_\tau = \frac{192\pi^3}{G_F^e G_F^\tau m_\tau^5} Br(\tau \to e\nu\nu)$$

- **All these quantities are precisely measured:**

$$m_\mu = 0.1056583692(94)\,\mathrm{GeV} \qquad \tau_\mu = 2.19703(4)\times 10^{-6}\,\mathrm{s}$$

$$m_\tau = 1.77699(28)\,\mathrm{GeV} \qquad \tau_\tau = 0.2906(10)\times 10^{-12}\,\mathrm{s}$$

$$Br(\tau\to e\nu\nu) = 0.1784(5)$$

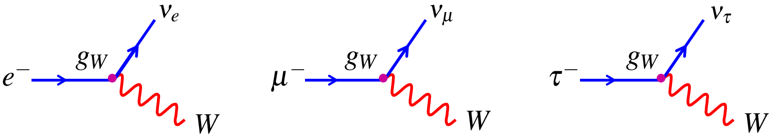
➡

$$\boxed{\frac{G_F^\tau}{G_F^\mu} = \frac{m_\mu^5\tau_\mu}{m_\tau^5\tau_\tau}Br(\tau\to e\nu\nu) = 1.0024\pm 0.0033}$$

- **Similarly by comparing** $Br(\tau\to e\nu\nu)$ **and** $Br(\tau\to\mu\nu\nu)$

$$\boxed{\frac{G_F^e}{G_F^\mu} = 1.000\pm 0.004}$$

★ **Demonstrates the weak charged current is the same for all leptonic vertices**



➡ **Charged Current Lepton Universality**

Neutrino Scattering

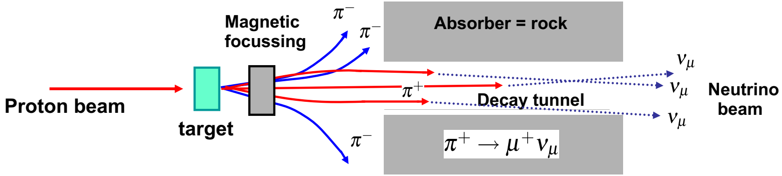
- In handout 6 considered electron-proton **Deep Inelastic Scattering** where a virtual photon is used to probe nucleon structure
- Can also consider the weak interaction equivalent: **Neutrino Deep Inelastic Scattering** where a virtual W-boson probes the structure of nucleons

➡ additional information about parton structure functions

\+ provides a good example of calculations of weak interaction cross sections.

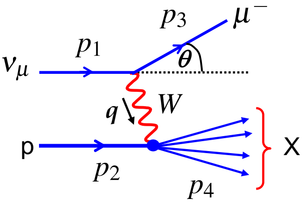
★ **Neutrino Beams:**

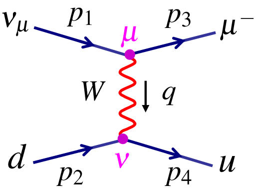
- Smash high energy protons into a fixed target ➡ hadrons
- Focus positive pions/kaons
- Allow them to decay $\pi^+ \to \mu^+ \nu_\mu$ + $K^+ \to \mu^+ \nu_\mu \; (BR \approx 64\%)$
- Gives a beam of "collimated" ν_μ
- Focus negative pions/kaons to give beam of $\overline{\nu}_\mu$



Neutrino-Quark Scattering

★For ν_μ-proton Deep Inelastic Scattering the underlying process is $\nu_\mu d \rightarrow \mu^- u$





★In the limit $q^2 \ll m_W^2$ the W-boson propagator is $\approx i g_{\mu\nu}/m_W^2$

- The Feynman rules give:

$$-iM_{fi} = \left[-i\frac{g_W}{\sqrt{2}}\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_1)\right]\frac{ig_{\mu\nu}}{m_W^2}\left[-i\frac{g_W}{\sqrt{2}}\overline{u}(p_4)\tfrac{1}{2}\gamma^\nu(1-\gamma^5)u(p_2)\right]$$

$$\boxed{M_{fi} = \frac{g_W^2}{2m_W^2}g_{\mu\nu}\left[\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_1)\right]\left[\overline{u}(p_4)\tfrac{1}{2}\gamma^\nu(1-\gamma^5)u(p_2)\right]}$$

- Evaluate the matrix element in the extreme relativistic limit where the muon and quark masses can be neglected

- **In this limit the helicity states are equivalent to the chiral states and**

$$\tfrac{1}{2}(1-\gamma^5)u_\uparrow(p_1)=0 \qquad \tfrac{1}{2}(1-\gamma^5)u_\downarrow(p_1)=u_\downarrow(p_1)$$

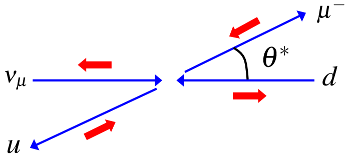
➡ $M_{fi}=0$ **for** $u_\uparrow(p_1)$ **and** $u_\uparrow(p_2)$

- **Since the weak interaction "conserves the helicity", the only helicity combination where the matrix element is non-zero is**

$$M_{fi}=\frac{g_W^2}{2m_W^2}g_{\mu\nu}\left[\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)\right]\left[\overline{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)\right]$$

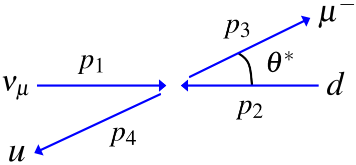
NOTE: **we could have written this down straight away as in the ultra-relativistic limit only LH helicity particle states participate in the weak interaction.**

★**Consider the scattering in the C.o.M frame**



Evaluation of Neutrino-Quark Scattering ME

- Go through the calculation in gory detail (fortunately only one helicity combination)
- In the $\nu_\mu d$ CMS frame, neglecting particle masses:



$$p_1 = (E, 0, 0, E),$$
$$p_2 = (E, 0, 0, -E)$$
$$p_3 = (E, E\sin\theta^*, 0, E\cos\theta^*)$$
$$p_4 = (E, -E\sin\theta^*, 0, -E\cos\theta^*)$$

- Dealing with LH helicity particle spinors. From handout 3 ((p.89)), for a massless particle travelling in direction (θ, ϕ):

$$u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix} \qquad c = \cos\frac{\theta}{2}; \quad s = \sin\frac{\theta}{2}$$

- Here $(\theta_1, \phi_1) = (0, 0)$; $(\theta_2, \phi_2) = (\pi, 0)$; $(\theta_3, \phi_3) = (\theta^*, 0)$; $(\theta_4, \phi_4) = (\pi - \theta^*, \pi)$

giving:

$$u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}; \; u_\downarrow(p_2) = \sqrt{E}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \; u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}; \; u_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} -c \\ -s \\ c \\ s \end{pmatrix}$$

- **To calculate**

$$M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\nu} \left[\overline{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1)\right] \left[\overline{u}_{\downarrow}(p_4)\gamma^{\nu}u_{\downarrow}(p_2)\right]$$

need to evaluate two terms of form

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi^{\dagger}\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\
\overline{\psi}\gamma^1\phi &= \psi^{\dagger}\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\overline{\psi}\gamma^2\phi &= \psi^{\dagger}\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\overline{\psi}\gamma^3\phi &= \psi^{\dagger}\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2
\end{aligned}$$

- **Using**

$$u_{\downarrow}(p_1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix};\; u_{\downarrow}(p_2) = \sqrt{E}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix};\; u_{\downarrow}(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix};\; u_{\downarrow}(p_4) = \sqrt{E}\begin{pmatrix}-c\\-s\\c\\s\end{pmatrix}$$

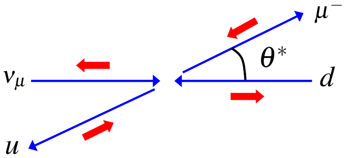
➡ $$\boxed{\begin{aligned}\overline{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) &= 2E(c, s, -is, c) \\ \overline{u}_{\downarrow}(p_4)\gamma^{\nu}u_{\downarrow}(p_2) &= 2E(c, -s, -is, -c)\end{aligned}}$$

➡ $$M_{fi} = \frac{g_W^2}{2m_W^2} 4E^2(c^2 + s^2 + s^2 + c^2) = \frac{g_W^2\hat{s}}{m_W^2}$$

$$\hat{s} = (2E)^2$$

★ **Note the Matrix Element is isotropic**

$$\boxed{M_{fi} = \frac{g_W^2}{m_W^2}\hat{s}}$$



we could have anticipated this since the helicity combination (spins anti-parallel) has $S_z = 0$ ➡ no preferred polar angle

★ As before need to sum over all possible spin states and average over all possible initial state spin states. Here only one possible spin combination (LL→LL) and only 2 possible initial state combinations (the neutrino is always produced in a LH helicity state)

$$\langle |M_{fi}|^2 \rangle = \frac{1}{2} \cdot \left| \frac{g_W^2}{m_W^2}\hat{s} \right|^2$$

The factor of a half arises because half of the time the quark will be in a RH states and won't participate in the charged current Weak interaction

★ From handout 1, in the extreme relativistic limit, the cross section for any 2→2 body scattering process is

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 \hat{s}} \langle |M_{fi}|^2 \rangle$$

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2\hat{s}}\langle|M_{fi}|^2\rangle = \frac{1}{64\pi\hat{s}}\frac{1}{2}\left(\frac{g_W^2\hat{s}}{m_W^2}\right)^2 = \left(\frac{g_W^2}{8\sqrt{2}\pi m_W^2}\right)^2\hat{s}$$

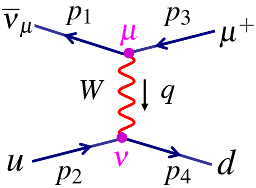
using $\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}$ ➡ $\boxed{\frac{d\sigma}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}}$

★**Integrating this isotropic distribution over** $d\Omega^*$

➡ $\boxed{\sigma_{\nu q} = \frac{G_F^2\hat{s}}{\pi}}$ **(1)**

•**cross section is a Lorentz invariant quantity so this is valid in any frame**

Antineutrino-Quark Scattering



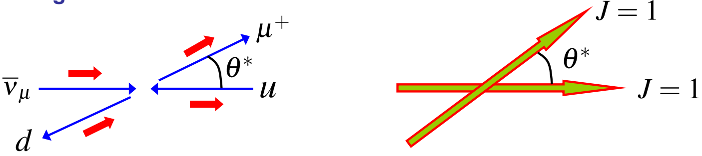
- In the ultra-relativistic limit, the charged-current interaction matrix element is:

$$M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\nu} \left[\overline{v}(p_1)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_3)\right] \left[\overline{u}(p_4)\gamma^\nu \tfrac{1}{2}(1-\gamma^5)u(p_2)\right]$$

★ In the extreme relativistic limit only **LH Helicity particles** and **RH Helicity anti-particles** participate in the charged current weak interaction:

$$\Rightarrow \quad M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\nu} \left[\overline{v}_\uparrow(p_1)\gamma^\mu v_\uparrow(p_3)\right] \left[\overline{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)\right]$$

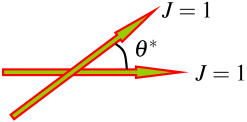
★ In terms of the particle spins it can be seen that the interaction occurs in a total angular momentum 1 state



★In a similar manner to the neutrino-quark scattering calculation obtain:

$$\frac{d\sigma_{\overline{\nu}q}}{d\Omega^*} = \frac{d\sigma_{\nu q}}{d\Omega^*}\frac{1}{4}(1+\cos\theta^*)^2$$

- The factor $\frac{1}{4}(1+\cos\theta^*)^2$ can be understood in terms of the overlap of the initial and final angular momentum wave-functions



★Similarly to the neutrino-quark scattering calculation obtain:

$$\frac{d\sigma_{\overline{\nu}q}}{d\Omega^*} = \frac{G_F^2}{16\pi^2}(1+\cos\theta^*)^2\hat{s}$$

★Integrating over solid angle:

$$d\Omega = d\phi\sin\theta d\theta \longrightarrow d\phi d(\cos\theta)$$

$$\int(1+\cos\theta^*)^2 d\Omega^* = \int(1+\cos\theta^*)^2 d(\cos\theta^*)d\phi = 2\pi\int_{-1}^{+1}(1+\cos\theta^*)^2 d(\cos\theta^*) = \frac{16\pi}{3}$$

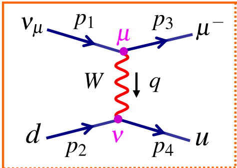
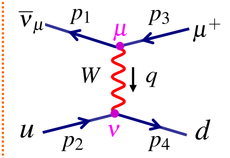
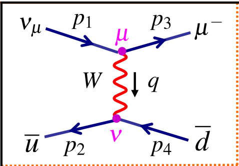
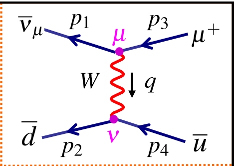
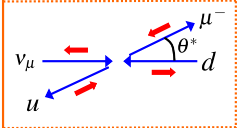
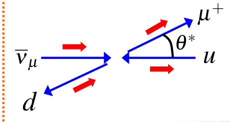
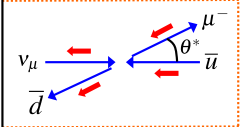
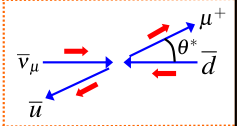
$$\Rightarrow \boxed{\sigma_{\overline{\nu}q} = \frac{G_F^2\hat{s}}{3\pi}}$$

★This is a factor three smaller than the neutrino quark cross-section

$$\boxed{\frac{\sigma_{\overline{\nu}q}}{\sigma_{\nu q}} = \frac{1}{3}}$$

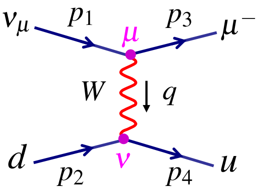
(Anti)neutrino-(Anti)quark Scattering

- Non-zero anti-quark component to the nucleon ⇒ also consider scattering from $\overline{q}$
- Cross-sections can be obtained immediately by comparing with quark scattering and remembering to only include **LH particles and RH anti-particles**

			
			
$S_z = 0$	$S_z = +1$	$S_z = -1$	$S_z = 0$
$\frac{d\sigma_{\nu q}}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}$	$\frac{d\sigma_{\overline{\nu} q}}{d\Omega^*} = \frac{G_F^2}{16\pi^2}(1+\cos\theta^*)^2\hat{s}$	$\frac{d\sigma_{\nu \overline{q}}}{d\Omega^*} = \frac{G_F^2}{16\pi^2}(1+\cos\theta^*)^2\hat{s}$	$\frac{d\sigma_{\overline{\nu}\overline{q}}}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}$
$\sigma_{\nu q} = \frac{G_F^2\hat{s}}{\pi}$	$\sigma_{\overline{\nu} q} = \frac{G_F^2\hat{s}}{3\pi}$	$\sigma_{\nu \overline{q}} = \frac{G_F^2\hat{s}}{3\pi}$	$\sigma_{\overline{\nu}\overline{q}} = \frac{G_F^2\hat{s}}{\pi}$

Differential Cross Section dσ/dy

★ Derived differential neutrino scattering cross sections in C.o.M frame, can convert to Lorentz invariant form



- As for DIS use Lorentz invariant $y \equiv \frac{p_2.q}{p_2.p_1}$
- In relativistic limit y can be expressed in terms of the C.o.M. scattering angle

$$y = \tfrac{1}{2}(1 - \cos\theta^*)$$

- In lab. frame $y = 1 - \frac{E_3}{E_1}$

★ Convert from $\frac{d\sigma}{d\Omega^*} \to \frac{d\sigma}{dy}$ using

$$\frac{d\sigma}{dy} = \left|\frac{d\cos\theta^*}{dy}\right| \frac{d\sigma}{d\cos\theta^*} = \left|\frac{d\cos\theta^*}{dy}\right| 2\pi \frac{d\sigma}{d\Omega^*} = 4\pi \frac{d\sigma}{d\Omega^*}$$

- Already calculated (1)

$$\frac{d\sigma}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}$$

- Hence:

$$\boxed{\frac{d\sigma_{\nu q}}{dy} = \frac{d\sigma_{\overline{\nu}\overline{q}}}{dy} = \frac{G_F^2}{\pi}\hat{s}}$$

and

$$\frac{d\sigma_{\overline{\nu}q}}{d\Omega^*} = \frac{d\sigma_{\nu\overline{q}}}{d\Omega^*} = \frac{G_F^2}{16\pi^2}(1+\cos\theta^*)^2\hat{s}$$

becomes

$$\frac{d\sigma_{\overline{\nu}q}}{dy} = \frac{d\sigma_{\nu\overline{q}}}{dy} = \frac{G_F^2}{4\pi}(1+\cos\theta^*)^2\hat{s}$$

from $y = \frac{1}{2}(1-\cos\theta^*) \rightarrow 1-y = \frac{1}{2}(1+\cos\theta^*)$

and hence

$$\boxed{\frac{d\sigma_{\overline{\nu}q}}{dy} = \frac{d\sigma_{\nu\overline{q}}}{dy} = \frac{G_F^2}{\pi}(1-y)^2\hat{s}}$$

★**For comparison, the electro-magnetic** $e^{\pm}q \rightarrow e^{\pm}q$ **cross section is:**

QED

$$\frac{d\sigma_{e^{\pm}q}}{dy} = \underbrace{\frac{2\pi\alpha^2}{q^4}e_q^2}\,\underbrace{\left[1+(1-y)^2\right]}\,\hat{s}$$

DIFFERENCES:

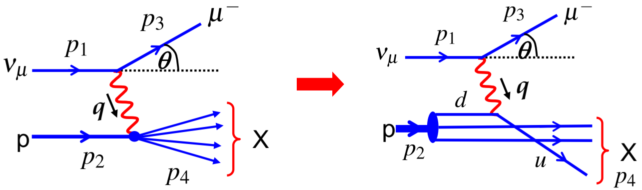
Interaction +propagator

Helicity Structure

WEAK

$$\frac{d\sigma_{\overline{\nu}q}}{dy} = \frac{d\sigma_{\nu\overline{q}}}{dy} = \overbrace{\frac{G_F^2}{\pi}}\,\overbrace{(1-y)^2}\,\hat{s}$$

Parton Model For Neutrino Deep Inelastic Scattering



Scattering from a proton with structure functions

→

Scattering from a point-like quark within the proton

★ **Neutrino-proton scattering can occur via scattering from a <u>down-quark</u> or from an <u>anti-up quark</u>**

- **In the parton model, number of down quarks within the proton in the momentum fraction range $x \rightarrow x + \mathrm{d}x$ is $d^p(x)\mathrm{d}x$. Their contribution to the neutrino scattering cross-section is obtained by multiplying by the $\nu_\mu d \rightarrow \mu^- u$ cross-section derived previously**

$$\frac{\mathrm{d}\sigma^{\nu p}}{\mathrm{d}y} = \frac{G_\mathrm{F}^2}{\pi}\hat{s}d^p(x)\mathrm{d}x$$

where $\hat{s}$ is the centre-of-mass energy of the $\nu_\mu d$

- Similarly for the $\overline{u}$ contribution

$$\frac{\mathrm{d}\sigma^{\nu p}}{\mathrm{d}y} = \frac{G_\mathrm{F}^2}{\pi}\hat{s}(1-y)^2\overline{u}^p(x)\mathrm{d}x$$

★ Summing the two contributions and using $\hat{s} = xs$

$$\Rightarrow \quad \frac{\mathrm{d}^2\sigma^{\nu p}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2}{\pi}sx\left[d^p(x) + (1-y)^2\overline{u}^p(x)\right]$$

★ The anti-neutrino proton differential cross section can be obtained in the same manner:

$$\frac{\mathrm{d}^2\sigma^{\overline{\nu} p}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2}{\pi}sx\left[(1-y)^2u^p(x) + \overline{d}^p(x)\right]$$

★ For (anti)neutrino – neutron scattering:

$$\frac{\mathrm{d}^2\sigma^{\nu n}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2}{\pi}sx\left[d^n(x) + (1-y)^2\overline{u}^n(x)\right]$$

$$\frac{\mathrm{d}^2\sigma^{\overline{\nu} n}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2}{\pi}sx\left[(1-y)^2u^n(x) + \overline{d}^n(x)\right]$$

- As before, define neutron distributions functions in terms of those of the proton

$$u(x) \equiv u^{\mathrm{p}}(x) = d^{\mathrm{n}}(x); \qquad d(x) \equiv d^{\mathrm{p}}(x) = u^{\mathrm{n}}(x)$$

$$\overline{u}(x) \equiv \overline{u}^{\mathrm{p}}(x) = \overline{d}^{\mathrm{n}}(x); \qquad \overline{d}(x) \equiv \overline{d}^{\mathrm{p}}(x) = \overline{u}^{\mathrm{n}}(x)$$

$$\boxed{\begin{aligned} \frac{\mathrm{d}^2\sigma^{\nu p}}{\mathrm{d}x\mathrm{d}y} &= \frac{G_{\mathrm{F}}^2}{\pi} sx\left[d(x) + (1-y)^2\overline{u}(x)\right] \qquad (2) \\ \frac{\mathrm{d}^2\sigma^{\overline{\nu} p}}{\mathrm{d}x\mathrm{d}y} &= \frac{G_{\mathrm{F}}^2}{\pi} sx\left[(1-y)^2 u(x) + \overline{d}(x)\right] \qquad (3) \end{aligned}}$$

$$\frac{\mathrm{d}^2\sigma^{\nu n}}{\mathrm{d}x\mathrm{d}y} = \frac{G_{\mathrm{F}}^2}{\pi} sx\left[u(x) + (1-y)^2\overline{d}(x)\right] \qquad (4)$$

$$\frac{\mathrm{d}^2\sigma^{\overline{\nu} n}}{\mathrm{d}x\mathrm{d}y} = \frac{G_{\mathrm{F}}^2}{\pi} sx\left[(1-y)^2 d(x) + \overline{u}(x)\right] \qquad (5)$$

★ **Because neutrino cross sections are very small, need massive detectors. These are usually made of Iron, hence, experimentally measure a combination of proton/neutron scattering cross sections**

★ **For an isoscalar target (i.e. equal numbers of protons and neutrons), the mean cross section per nucleon:**

$$\frac{d^2\sigma^{\nu N}}{dxdy} = \frac{1}{2}\left(\frac{d^2\sigma^{\nu p}}{dxdy} + \frac{d^2\sigma^{\nu n}}{dxdy}\right)$$

$$\Rightarrow \quad \boxed{\frac{d^2\sigma^{\nu N}}{dxdy} = \frac{G_F^2}{2\pi} sx\left[u(x) + d(x) + (1-y)^2(\overline{u}(x) + \overline{d}(x))\right]}$$

- **Integrate over momentum fraction x**

$$\boxed{\frac{d\sigma^{\nu N}}{dy} = \frac{G_F^2}{2\pi} s\left[f_q + (1-y)^2 f_{\overline{q}})\right]} \qquad \textbf{(6)}$$

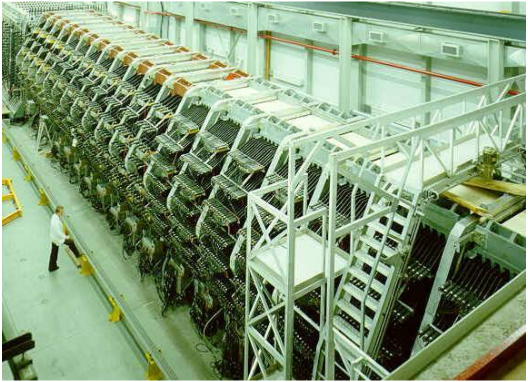
where f_q and $f_{\overline{q}}$ are the total momentum fractions carried by the quarks and by the anti-quarks within a nucleon

$$f_q \equiv f_d + f_u = \int_0^1 x\left[u(x) + d(x)\right] dx; \quad f_{\overline{q}} \equiv f_{\overline{d}} + f_{\overline{u}} = \int_0^1 x\left[\overline{u}(x) + \overline{d}(x)\right] dx$$

- **Similarly**

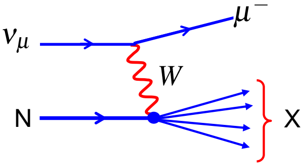
$$\boxed{\frac{d\sigma^{\overline{\nu} N}}{dy} = \frac{G_F^2}{2\pi} s\left[(1-y)^2 f_q + f_{\overline{q}})\right]} \qquad \textbf{(7)}$$

e.g. CDHS Experiment (CERN 1976-1984)

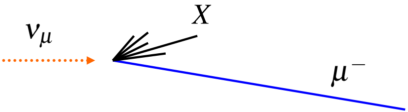


- **1250 tons**
- **Magnetized iron modules**
- **Separated by drift chambers**

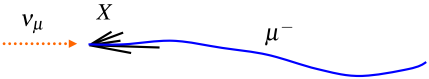
Study Neutrino Deep Inelastic Scattering

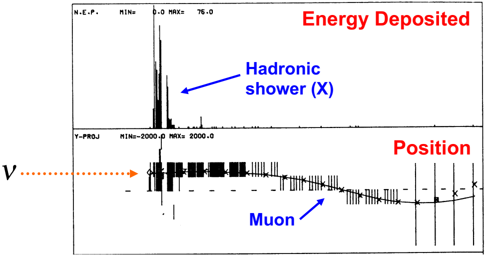


Experimental Signature:



Example Event:





- Measure energy of X

$$E_X$$

- Measure muon momentum from curvature in B-field

$$E_\mu$$

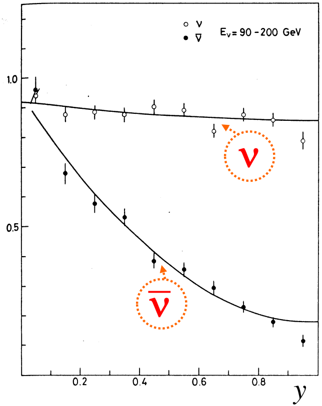
★ **For each event can determine neutrino energy and y !**

$$E_\nu = E_X + E_\mu$$

$$E_\mu = (1-y)E_\nu \quad \longrightarrow \quad y = \left(1 - \frac{E_\mu}{E_\nu}\right)$$

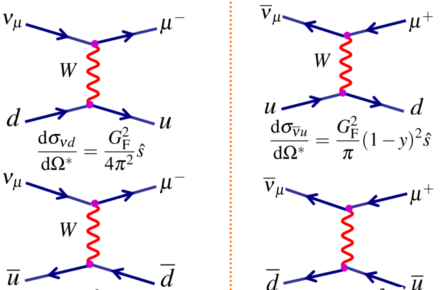
Measured y Distribution

- CDHS measured y distribution



J. de Groot et al., Z.Phys. C1 (1979) 143

- Shapes can be understood in terms of (anti)neutrino – (anti)quark scattering



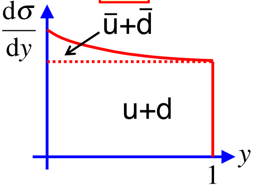
$$\frac{d\sigma_{\nu d}}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}$$

$$\frac{d\sigma_{\overline{\nu} u}}{d\Omega^*} = \frac{G_F^2}{\pi}(1-y)^2\hat{s}$$

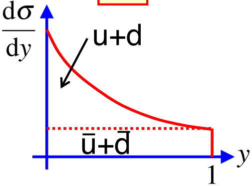
$$\frac{d\sigma_{\nu \overline{u}}}{d\Omega^*} = \frac{G_F^2}{\pi}(1-y)^2\hat{s}$$

$$\frac{d\sigma_{\overline{\nu}\overline{d}}}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}$$

νN



ν̄N



Measured Total Cross Sections

★ **Integrating the expressions for $\frac{d\sigma}{dy}$ (equations (6) and (7))**

$$\boxed{\sigma^{\nu N} = \frac{G_F^2 s}{2\pi}\left[f_q + \frac{1}{3}f_{\overline{q}}\right]} \qquad \boxed{\sigma^{\overline{\nu} N} = \frac{G_F^2 s}{2\pi}\left[\frac{1}{3}f_q + f_{\overline{q}}\right]}$$

$$\underset{(E_\nu, 0, 0, +E_\nu)}{\nu} \longrightarrow \underset{(m_p, 0, 0, 0)}{\bullet\, p} \qquad s = (E_\nu + m_p)^2 - E_\nu^2 = 2E_\nu m_p + m_p^2 \approx 2E_\nu m_p$$

➡ **DIS cross section $\propto$ lab. frame neutrino energy**

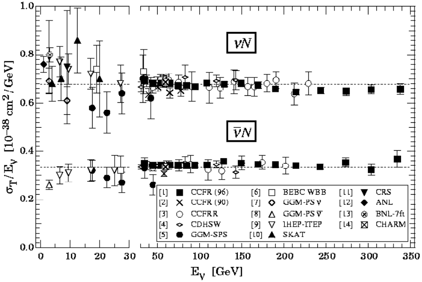
★ **Measure cross sections can be used to determine fraction of protons momentum carried by quarks, f_q , and fraction carried by anti-quarks, $f_{\overline{q}}$**

- **Find:** $f_q \approx 0.41; \quad f_{\overline{q}} \approx 0.08$
- **~50% of momentum carried by gluons (which don't interact with virtual W boson)**
- **If no anti-quarks in nucleons expect**

$$\frac{\sigma^{\nu N}}{\sigma^{\overline{\nu} N}} = 3$$

- **Including anti-quarks**

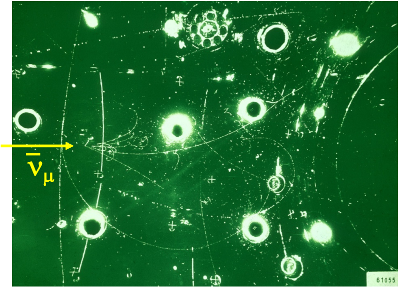
$$\boxed{\frac{\sigma^{\nu N}}{\sigma^{\overline{\nu} N}} \approx 2}$$



Weak Neutral Current

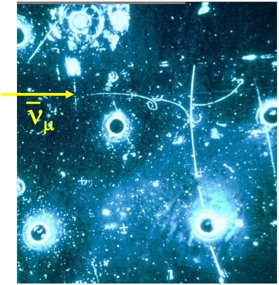
★ **Neutrinos also interact via the Neutral Current. First observed in the Gargamelle bubble chamber in 1973. Interaction of muon neutrinos produce a final state muon**

$$\overline{\nu}_\mu + N \to \overline{\nu}_\mu + \text{hadrons}$$



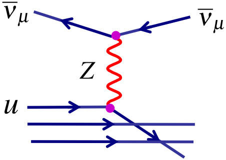
F.J. Hasert et al., Phys. Lett. 46B (1973) 138

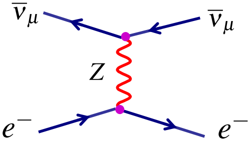
$$\overline{\nu}_\mu + e^- \to \overline{\nu}_\mu + e^-$$



F.J. Hasert et al., Phys. Lett. 46B (1973) 121

★ **Cannot be due to W exchange - first evidence for Z boson**





Summary

- ★ Derived neutrino/anti-neutrino – quark/anti-quark weak charged current (CC) interaction cross sections

- ★ Neutrino – nucleon scattering yields extra information about parton distributions functions:
  - ν couples to d and $\overline{u}$; $\overline{\nu}$ couples to u and $\overline{d}$

    ➡ investigate flavour content of nucleon
  - can measure anti-quark content of nucleon

    $\nu\overline{q}$ suppressed by factor $(1-y)^2$ compared with νq

    $\overline{\nu}q$ suppressed by factor $(1-y)^2$ compared with $\overline{\nu}\overline{q}$

- ★ Further aspects of neutrino deep-inelastic scattering (expressed in general structure functions) are covered in **Appendix II**

- ★ Finally observe that neutrinos interact via weak neutral currents (NC)

Appendix I

- For the adjoint spinors $\overline{u} = u^{\dagger}\gamma^{0}$ consider

$$\overline{\tfrac{1}{2}(1-\gamma^5)u} = [\tfrac{1}{2}(1-\gamma^5)u]^{\dagger}\gamma^0 = u^{\dagger}\tfrac{1}{2}(1-\gamma^5)\gamma^0 = u^{\dagger}\gamma^0\tfrac{1}{2}(1+\gamma^5) = \overline{u}\tfrac{1}{2}(1+\gamma^5)$$

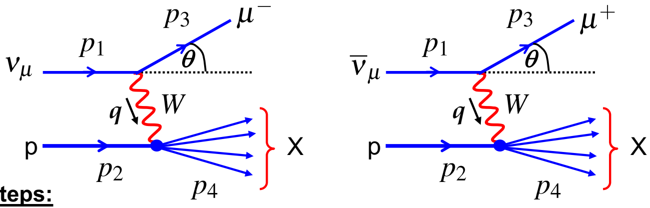
$$\tfrac{1}{2}(1-\gamma^5)u_{\uparrow} = 0 \quad \Rightarrow \quad \overline{u}\tfrac{1}{2}(1+\gamma^5) = 0$$

Using the fact that γ^5 and γ^{μ} anti-commute can rewrite ME:

$$M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\nu}\left[\overline{u}(p_3)\tfrac{1}{2}(1+\gamma^5)\gamma^{\mu}u(p_1)\right]\left[\overline{u}(p_4)\tfrac{1}{2}(1+\gamma^5)\gamma^{\nu}u(p_2)\right]$$

$\Rightarrow$ $M_{fi} = 0$ for $\overline{u}_{\uparrow}(p_3)$ and $\overline{u}_{\uparrow}(p_4)$

Appendix II: Deep-Inelastic Neutrino Scattering



Two steps:

- First write down most general cross section in terms of structure functions
- Then evaluate expressions in the quark-parton model

QED Revisited

★ In the limit $s \gg M^2$ the most general electro-magnetic deep-inelastic cross section (from single photon exchange) can be written (Eq. 2 of handout 6)

$$\frac{\mathrm{d}^2\sigma_{e^\pm p}}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2F_1(x,Q^2)\right]$$

- For neutrino scattering typically measure the energy of the produced muon $E_\mu = E_\nu(1-y)$ and differential cross-sections expressed in terms of $\mathrm{d}x\mathrm{d}y$

- Using $Q^2 = (s-M^2)xy \approx sxy$ $\Longrightarrow$ $\dfrac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y} = \left|\dfrac{\mathrm{d}Q^2}{\mathrm{d}y}\right| \dfrac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = sx\dfrac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}$

- In the limit $s \gg M^2$ the general Electro-magnetic DIS cross section can be written

$$\boxed{\frac{\mathrm{d}^2\sigma^{e^\pm p}}{\mathrm{d}x\mathrm{d}y} = \frac{4\pi\alpha^2 s}{Q^4}\left[(1-y)F_2(x,Q^2)+y^2xF_1(x,Q^2)\right]}$$

- **NOTE:** This is the most general **Lorentz Invariant** parity conserving expression

★ For neutrino DIS parity is violated and the general expression includes an additional term to allow for parity violation. New structure function $F_3(x,Q^2)$

$$\boxed{\nu_\mu p \to \mu^- X} \quad \boxed{\frac{\mathrm{d}^2\sigma^{\nu p}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2 s}{2\pi}\left[(1-y)F_2^{\nu p}(x,Q^2)+y^2xF_1^{\nu p}(x,Q^2)+y\left(1-\frac{y}{2}\right)xF_3^{\nu p}(x,Q^2)\right]}$$

- For anti-neutrino scattering new structure function enters with opposite sign

$$\boxed{\overline{\nu}_\mu p \to \mu^+ X} \quad \boxed{\frac{\mathrm{d}^2\sigma^{\overline{\nu} p}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2 s}{2\pi}\left[(1-y)F_2^{\overline{\nu} p}(x,Q^2)+y^2xF_1^{\overline{\nu} p}(x,Q^2)-y\left(1-\frac{y}{2}\right)xF_3^{\overline{\nu} p}(x,Q^2)\right]}$$

- Similarly for neutrino-neutron scattering

$$\boxed{\nu_\mu n \to \mu^- X} \quad \boxed{\frac{\mathrm{d}^2\sigma^{\nu n}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2 s}{2\pi}\left[(1-y)F_2^{\nu n}(x,Q^2)+y^2xF_1^{\nu n}(x,Q^2)+y\left(1-\frac{y}{2}\right)xF_3^{\nu n}(x,Q^2)\right]}$$

$$\boxed{\overline{\nu}_\mu n \to \mu^+ X} \quad \boxed{\frac{\mathrm{d}^2\sigma^{\overline{\nu} n}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2 s}{2\pi}\left[(1-y)F_2^{\overline{\nu} n}(x,Q^2)+y^2xF_1^{\overline{\nu} n}(x,Q^2)-y\left(1-\frac{y}{2}\right)xF_3^{\overline{\nu} n}(x,Q^2)\right]}$$

Neutrino Interaction Structure Functions

★In terms of the parton distribution functions we found (2) :

$$\frac{\mathrm{d}^2\sigma^{\nu p}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2}{\pi} sx\left[d(x) + (1-y)^2\overline{u}(x)\right]$$

- Compare coefficients of y with the general Lorentz Invariant form ((p.358)) and assume Bjorken scaling, i.e. $F(x,Q^2) \to F(x)$

$$\frac{\mathrm{d}^2\sigma^{\nu p}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2 s}{2\pi}\left[(1-y)F_2^{\nu p}(x) + y^2 x F_1^{\nu p}(x) + y\left(1-\frac{y}{2}\right)xF_3^{\nu p}(x)\right]$$

- Re-writing (2) $$\frac{\mathrm{d}^2\sigma^{\nu p}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2}{2\pi} s\left[2xd(x) + 2x\overline{u}(x) - 4xy\overline{u}(x) + 2xy^2\overline{u}(x)\right]$$

and equating powers of **y**

$$\left.\begin{array}{rcl} 2xd + 2x\overline{u} & = & F_2 \\ -4x\overline{u} & = & -F_2 + xF_3 \\ 2\overline{u} & = & F_1 - xF_3/2 \end{array}\right\}$$

gives:

$$\boxed{\begin{array}{rl} F_2^{\nu p} = 2xF_1^{\nu p} & = 2x[d(x) + \overline{u}(x)] \\ xF_3^{\nu p} & = 2x[d(x) - \overline{u}(x)] \end{array}}$$

NOTE: again we get the **Callan-Gross** relation $F_2 = 2xF_1$

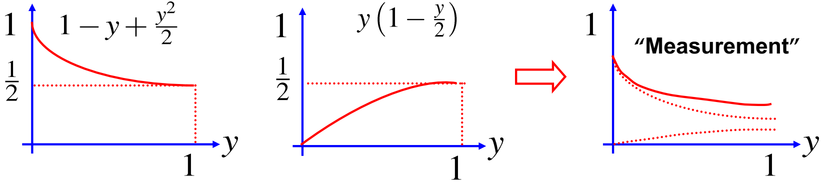
No surprise, underlying process is scattering from point-like spin-1/2 quarks

★ Substituting back in to expression for differential cross section:

$$\frac{\mathrm{d}^2\sigma^{\nu p}}{\mathrm{d}x\mathrm{d}y} = \frac{G_F^2 s}{2\pi}\left[\left(1-y+\frac{y^2}{2}\right)F_2^{\nu p}(x) + y\left(1-\frac{y}{2}\right)xF_3^{\nu p}(x)\right]$$

★ Experimentally measure F_2 and F_3 from y distributions at fixed x

- Different **y** dependencies (from different rest frame angular distributions) allow contributions from the two structure functions to be measured



★ Then use $F_2^{\nu p} = 2x[d(x)+\overline{u}(x)]$ and $F_3^{\nu p} = 2[d(x)-\overline{u}(x)]$

➡ Determine $d(x)$ and $\overline{u}(x)$ separately

★**Neutrino experiments require large detectors (often iron) i.e. isoscalar target**

$$\boxed{\begin{aligned} F_2^{\nu N} = 2xF_1^{\nu N} = \tfrac{1}{2}\left(F_2^{\nu p} + F_2^{\nu n}\right) &= x[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)] \\ xF_3^{\nu N} = \tfrac{1}{2}\left(xF_3^{\nu p} + xF_3^{\nu n}\right) &= x[u(x) + d(x) - \overline{u}(x) - \overline{d}(x)] \end{aligned}}$$

★**For electron – nucleon scattering:**

$$F_2^{ep} = 2xF_1^{ep} = x[\tfrac{4}{9}u(x) + \tfrac{1}{9}d(x) + \tfrac{4}{9}\overline{u}(x) + \tfrac{1}{9}\overline{d}(x)]$$

$$F_2^{en} = 2xF_1^{en} = x[\tfrac{4}{9}d(x) + \tfrac{1}{9}u(x) + \tfrac{4}{9}\overline{d}(x) + \tfrac{1}{9}\overline{u}(x)]$$

- **For an isoscalar target**

$$\boxed{F_2^{eN} = \tfrac{1}{2}\left(F_2^{ep} + F_2^{en}\right) = \tfrac{5}{18}x[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)]}$$

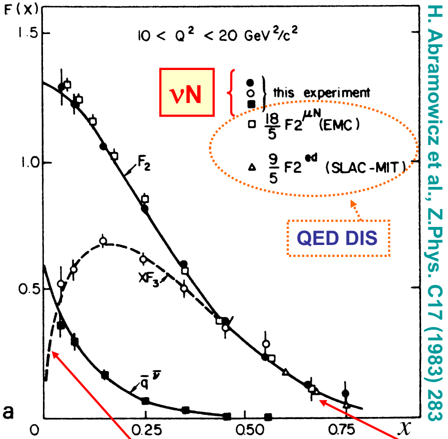
$$\Rightarrow \quad \boxed{F_2^{\nu N} = \tfrac{18}{5}F_2^{eN}}$$

- **Note that the factor** $\frac{5}{18} = \frac{1}{2}\left(q_u^2 + q_d^2\right)$ **and by comparing neutrino to electron scattering structure functions measure the sum of quark charges**

Experiment: 0.29 ± 0.02

Measurements of $F_2(x)$ and $F_3(x)$

- **CDHS Experiment** $\nu_\mu + \mathrm{Fe} \rightarrow \mu^- + X$



H. Abramowicz et al., Z.Phys. C17 (1983) 283

$$F_2^{\nu N} = x[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)]$$
$$xF_3^{\nu N} = x[u(x) + d(x) - \overline{u}(x) - \overline{d}(x)]$$

$$\Rightarrow F_2^{\nu N} - xF_3^{\nu N} = 2x[\overline{u} + \overline{d}]$$

✶ **Difference in neutrino structure functions measures anti-quark (sea) parton distribution functions**

Sea dominates so expect xF_3 to go to zero as q(x) = $\overline{\text{q}}$(x)

Sea contribution goes to zero

Valence Contribution

★ **Separate parton density functions into sea and valence components**

$$u(x) = u_V(x) + u_S(x) = u_V(x) + S(x)$$
$$d(x) = d_V(x) + d_S(x) = d_V(x) + S(x)$$
$$\overline{u}(x) = \overline{u}_S(x) = S(x)$$
$$\overline{d}(x) = \overline{d}_S(x) = S(x)$$

➡ $$\boxed{F_3^{\nu N} = [u(x) + d(x) - \overline{u}(x) - \overline{d}(x)] = u_V(x) + d_V(x)}$$

➡ $$\int_0^1 F_3^{\nu N}(x)\mathrm{d}x = \int_0^1 (u_V(x) + d_V(x))\,\mathrm{d}x = N_u^V + N_d^V$$

★ **Area under measured function $F_3^{\nu N}(x)$ gives a measurement of the total number of valence quarks in a nucleon !**

expect $$\int_0^1 F_3^{\nu N}(x)\mathrm{d}x = 3$$

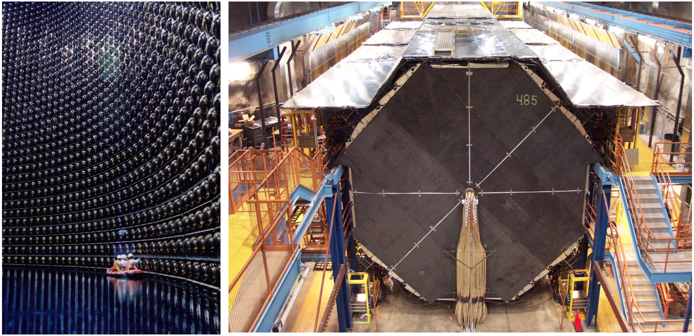
"Gross – Llewellyn-Smith sum rule"

Experiment: **3.0±0.2**

•**Note:** $F_2^{\overline{\nu}p} = F_2^{\nu n}$; $F_2^{\overline{\nu}n} = F_2^{\nu p}$; $F_3^{\overline{\nu}p} = F_3^{\nu n}$; $F_3^{\overline{\nu}n} = F_3^{\nu p}$ **and anti-neutrino structure functions contain same pdf information**

Particle Physics

Dr Lester



Handout 11 : Neutrino Oscillations

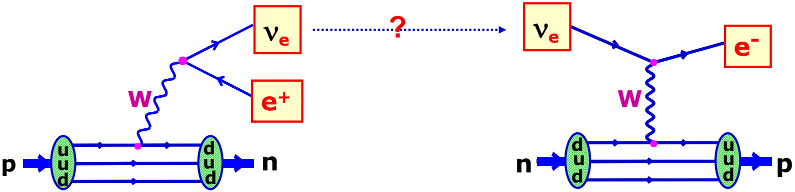
Neutrino Flavours Revisited

★ **Never directly observe neutrinos – can only detect them by their weak interactions.**
Hence by definition ν_e is the neutrino state produced along with an electron.
Similarly, charged current weak interactions of the state ν_e produce an electron

$$\boxed{\nu_e, \nu_\mu, \nu_\tau \quad \text{= weak eigenstates}}$$

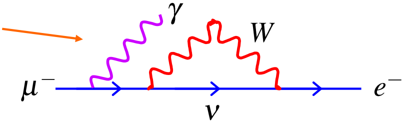
★ **For many years, assumed that ν_e, ν_μ, ν_τ were massless fundamental particles**

- **Experimental evidence:** neutrinos produced along with an electron always produced an electron in CC Weak interactions, etc.



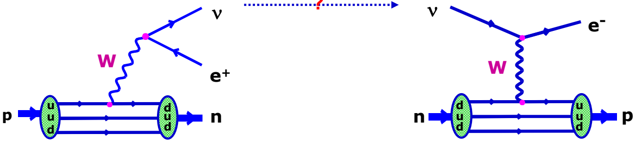
- **Experimental evidence:** absence $\mu^- \to e^-\gamma$ $\quad \mathrm{BR}(\mu^- \to e^-\gamma) < 10^{-11}$

Suggests that ν_e and ν_μ are distinct particles otherwise decay could go via:



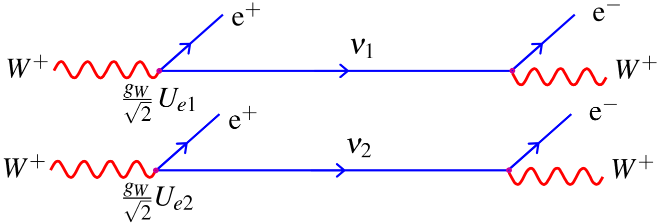
Mass Eigenstates and Weak Eigenstates

- ★The essential feature in understanding the physics of neutrino oscillations is to understand what is meant by weak eigenstates and mass eigenstates ν_1, ν_2
- ★Suppose the process below proceeds via two fundamental particle states



i.e.

and



- ★ Can't know which **mass eigenstate** (fundamental particle ν_1, ν_2) was involved
- ★ In Quantum mechanics treat as a coherent state $\psi = \nu_e = U_{e1}\nu_1 + U_{e2}\nu_2$
- ★ ν_e represents the wave-function of the coherent state produced along with an electron in the weak interaction, i.e. the weak eigenstate

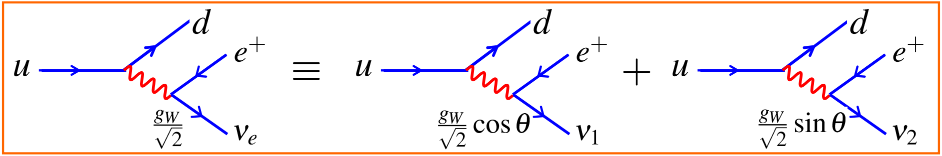
Neutrino Oscillations for Two Flavours

- ★ Neutrinos are produced and interact as weak eigenstates, ν_e, ν_μ
- ★ The weak eigenstates as coherent linear combinations of the fundamental "mass eigenstates" ν_1, ν_2
- ★ The mass eigenstates are the free particle solutions to the wave-equation and will be taken to propagate as plane waves

$$|\nu_1(t)\rangle = |\nu_1\rangle e^{i\vec{p}_1.\vec{x}-iE_1t} \qquad |\nu_2(t)\rangle = |\nu_2\rangle e^{i\vec{p}_2.\vec{x}-iE_2t}$$

- ★ The weak and mass eigenstates are related by the unitary 2x2 matrix

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \qquad (1)$$



- ★ Equation (1) can be inverted to give

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \qquad (2)$$

- Suppose at time $t = 0$ a neutrino is produced in a pure ν_e state, e.g. in a decay $u \rightarrow de^+\nu_e$

$$|\psi(0)\rangle = |\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$$

- Take the z-axis to be along the neutrino direction
- The wave-function evolves according to the time-evolution of the mass eigenstates (free particle solutions to the wave equation)

$$|\psi(t)\rangle = \cos\theta|\nu_1\rangle e^{-ip_1.x} + \sin\theta|\nu_2\rangle e^{-ip_2.x}$$

where $p_i.x = E_i t - \vec{p}_i.\vec{x} = E_i t - |\vec{p}_i|z$

- Suppose the neutrino interacts in a detector at a distance **L** and at a time **T**

$$\phi_i = p_i.x = E_i T - |\vec{p}_i|L$$

gives $$\boxed{|\psi(L,T)\rangle = \cos\theta|\nu_1\rangle e^{-i\phi_1} + \sin\theta|\nu_2\rangle e^{-i\phi_2}}$$

★ Expressing the mass eigenstates, $|\nu_1\rangle$, $|\nu_2\rangle$, in terms of weak eigenstates (eq 2)

$$|\psi(L,T)\rangle = \cos\theta(\cos\theta|\nu_e\rangle - \sin\theta|\nu_\mu\rangle)e^{-i\phi_1} + \sin\theta(\sin\theta|\nu_e\rangle + \cos\theta|\nu_\mu\rangle)e^{-i\phi_2}$$

$$\boxed{|\psi(L,T)\rangle = |\nu_e\rangle(\cos^2\theta e^{-i\phi_1} + \sin^2\theta e^{-i\phi_2}) + |\nu_\mu\rangle\sin\theta\cos\theta(-e^{-i\phi_1} + e^{-i\phi_2})}$$

- If the masses of $|\nu_1\rangle, |\nu_2\rangle$ **are the same**, the mass eigenstates remain in phase, $\phi_1 = \phi_2$, and the state remains the linear combination corresponding to $|\nu_e\rangle$ and in a weak interaction will produce an electron
- If the masses are different, the wave-function no longer remains a pure $|\nu_e\rangle$

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= |\langle \nu_\mu | \psi(L,T)\rangle|^2 \\ &= \cos^2\theta \sin^2\theta(-e^{-i\phi_1} + e^{-i\phi_2})(-e^{+i\phi_1} + e^{+i\phi_2}) \\ &= \frac{1}{4}\sin^2 2\theta(2 - 2\cos(\phi_1 - \phi_2)) \\ &= \sin^2 2\theta \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right) \end{aligned}$$

- The treatment of the phase difference

$$\Delta\phi_{12} = \phi_1 - \phi_2 = (E_1 - E_2)T - (|p_1| - |p_2|)L$$

in most text books is dubious. Here we will be more careful…

- One could assume $|p_1| = |p_2| = \mathrm{p}$ in which case

$$\Delta\phi_{12} = (E_1 - E_2)T = \left[(\mathrm{p}^2 + m_1^2)^{1/2} - (\mathrm{p}^2 + m_2^2)^{1/2}\right]L \qquad L \approx (c)T$$

$$\Delta\phi_{12} = \mathrm{p}\left[\left(1+\frac{m_1^2}{\mathrm{p}^2}\right)^{1/2} - \left(1+\frac{m_2^2}{\mathrm{p}^2}\right)^{1/2}\right] L \approx \frac{m_1^2-m_2^2}{2\mathrm{p}} L$$

- **However we have neglected that fact that for the same momentum, different mass eigenstates will propagate at different velocities and be observed at different times**
- **The full derivation requires a wave-packet treatment and gives the same result**
- **Nevertheless it is worth noting that the phase difference can be written**

$$\Delta\phi_{12} = (E_1-E_2)T - \left(\frac{|p_1|^2-|p_2|^2}{|p_1|+|p_2|}\right) L$$

$$\Delta\phi_{12} = (E_1-E_2)\left[T - \left(\frac{E_1+E_2}{|p_1|+|p_2|}\right) L\right] + \left(\frac{m_1^2-m_2^2}{|p_1|+|p_2|}\right) L$$

- **The first term on the RHS vanishes if we assume** $E_1 = E_2$ **or** $\beta_1 = \beta_2$

in all cases

$$\boxed{\Delta\phi_{12} = \frac{m_1^2-m_2^2}{2\mathrm{p}} L = \frac{\Delta m^2}{2E} \mathbf{L}}$$

- ★ **Hence the two-flavour oscillation probability is:**

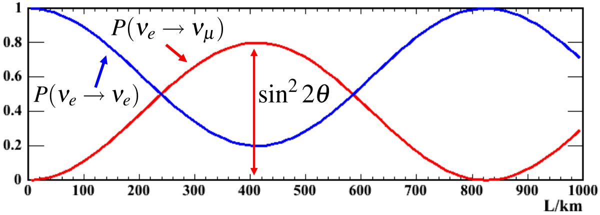
$$\boxed{P(\nu_e \rightarrow \nu_\mu) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right)}$$

with $\boxed{\Delta m^2_{21} = m^2_2 - m^2_1}$

- ★ **The corresponding two-flavour survival probability is:**

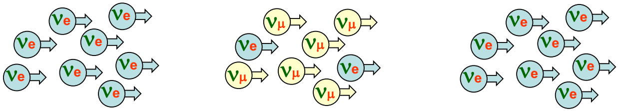
$$\boxed{P(\nu_e \rightarrow \nu_e) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right)}$$

- **e.g.** $\Delta m^2 = 0.003\,\mathrm{eV}^2, \qquad \sin^2 2\theta = 0.8, \qquad E_\nu = 1\,\mathrm{GeV}$



- **wavelength**

$$\boxed{\lambda_{\rm osc} = \frac{4\pi E}{\Delta m^2}}$$

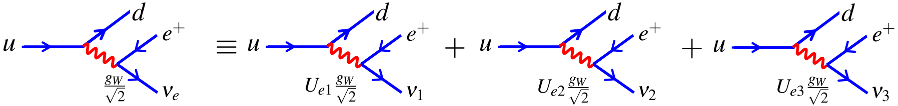


Neutrino Oscillations for Three Flavours

★ It is simple to extend this treatment to three generations of neutrinos.

★ In this case we have:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$



★ The 3x3 Unitary matrix U is known as the Pontecorvo-Maki-Nakagawa-Sakata matrix, usually abbreviated **PMNS**

★ Note : has to be unitary to conserve probability

•Using $U^\dagger U = I \quad \Rightarrow \quad U^{-1} = U^\dagger = (U^*)^T$

gives

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U_{e1}^* & U_{\mu 1}^* & U_{\tau 1}^* \\ U_{e2}^* & U_{\mu 2}^* & U_{\tau 2}^* \\ U_{e3}^* & U_{\mu 3}^* & U_{\tau 3}^* \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

Unitarity Relations

★**The Unitarity of the PMNS matrix gives several useful relations:** $UU^{\dagger} = I \quad \Rightarrow$

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1} \\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2} \\ U^*_{e3} & U^*_{\mu 3} & U^*_{\tau 3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

gives:

$$\begin{aligned} U_{e1}U^*_{e1} + U_{e2}U^*_{e2} + U_{e3}U^*_{e3} &= 1 & \textbf{(U1)} \\ U_{\mu 1}U^*_{\mu 1} + U_{\mu 2}U^*_{\mu 2} + U_{\mu 3}U^*_{\mu 3} &= 1 & \textbf{(U2)} \\ U_{\tau 1}U^*_{\tau 1} + U_{\tau 2}U^*_{\tau 2} + U_{\tau 3}U^*_{\tau 3} &= 1 & \textbf{(U3)} \\ U_{e1}U^*_{\mu 1} + U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3} &= 0 & \textbf{(U4)} \\ U_{e1}U^*_{\tau 1} + U_{e2}U^*_{\tau 2} + U_{e3}U^*_{\tau 3} &= 0 & \textbf{(U5)} \\ U_{\mu 1}U^*_{\tau 1} + U_{\mu 2}U^*_{\tau 2} + U_{\mu 3}U^*_{\tau 3} &= 0 & \textbf{(U6)} \end{aligned}$$

★**To calculate the oscillation probability proceed as before…**

•**Consider a state which is produced at** $t = 0$ **as a** $|\nu_e\rangle$ **(i.e. with an electron)**

$$|\psi(t=0)\rangle = |\nu_e\rangle = U_{e1}|\nu_1\rangle + U_{e2}|\nu_2\rangle + U_{e3}|\nu_3\rangle$$

- The wave-function evolves as:

$$|\psi(t)\rangle = U_{e1}|\nu_1\rangle e^{-ip_1.x} + U_{e2}|\nu_2\rangle e^{-ip_2.x} + U_{e3}|\nu_3\rangle e^{-ip_3.x}$$

where $p_i.x = E_i t - \vec{p}_i.\vec{x} = E_i t - |\vec{p}|z$

z axis in direction of propagation

- After a travelling a distance L

$$|\psi(L)\rangle = U_{e1}|\nu_1\rangle e^{-i\phi_1} + U_{e2}|\nu_2\rangle e^{-i\phi_2} + U_{e3}|\nu_3\rangle e^{-i\phi_3}$$

where $\phi_i = p_i.x = E_i t - |\vec{p}|L = (E_i - |\vec{p}_i|)L$

- As before we can approximate $\phi_i \approx \frac{m_i^2}{2E_i}L$

- Expressing the mass eigenstates in terms of the weak eigenstates

$$\begin{aligned}|\psi(L)\rangle &= U_{e1}(U_{e1}^*|\nu_e\rangle + U_{\mu 1}^*|\nu_\mu\rangle + U_{\tau 1}^*|\nu_\tau\rangle)e^{-i\phi_1}\\ &+ U_{e2}(U_{e2}^*|\nu_e\rangle + U_{\mu 2}^*|\nu_\mu\rangle + U_{\tau 2}^*|\nu_\tau\rangle)e^{-i\phi_2}\\ &+ U_{e3}(U_{e3}^*|\nu_e\rangle + U_{\mu 3}^*|\nu_\mu\rangle + U_{\tau 3}^*|\nu_\tau\rangle)e^{-i\phi_3}\end{aligned}$$

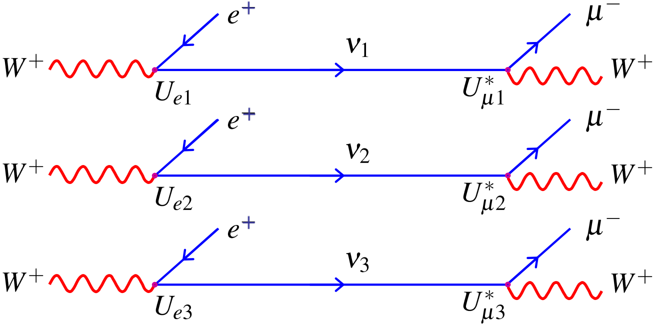
- Which can be rearranged to give

$$\begin{aligned}|\psi(L)\rangle &= (U_{e1}U_{e1}^*e^{-i\phi_1} + U_{e2}U_{e2}^*e^{-i\phi_2} + U_{e3}U_{e3}^*e^{-i\phi_3})|\nu_e\rangle\\ &+ (U_{e1}U_{\mu 1}^*e^{-i\phi_1} + U_{e2}U_{\mu 2}^*e^{-i\phi_2} + U_{e3}U_{\mu 3}^*e^{-i\phi_3})|\nu_\mu\rangle\\ &+ (U_{e1}U_{\tau 1}^*e^{-i\phi_1} + U_{e2}U_{\tau 2}^*e^{-i\phi_2} + U_{e3}U_{\tau 3}^*e^{-i\phi_3})|\nu_\tau\rangle\end{aligned} \qquad (3)$$

- **From which**

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= |\langle \nu_\mu | \psi(L) \rangle|^2 \\ &= |U_{e1}U^*_{\mu 1}e^{-i\phi_1} + U_{e2}U^*_{\mu 2}e^{-i\phi_2} + U_{e3}U^*_{\mu 3}e^{-i\phi_3}|^2 \end{aligned}$$

- **The terms in this expression can be represented as:**



- **Because of the unitarity of the PMNS matrix we have (U4):**

$$U_{e1}U^*_{\mu 1} + U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3} = 0$$

and, consequently, unless the phases of the different components are different, the sum of these three diagrams is zero, i.e., require different neutrino masses for osc.

- **Evaluate**

$$P(\nu_e \rightarrow \nu_\mu) \quad = \quad |U_{e1}U^*_{\mu 1}e^{-i\phi_1} + U_{e2}U^*_{\mu 2}e^{-i\phi_2} + U_{e3}U^*_{\mu 3}e^{-i\phi_3}|^2$$

using $|z_1 + z_2 + z_3|^2 \equiv |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\Re(z_1 z_2^* + z_1 z_3^* + z_2 z_3^*)$ **(4)**

which gives:

$$P(\nu_e \rightarrow \nu_\mu) \quad = \quad |U_{e1}U^*_{\mu 1}|^2 + |U_{e2}U^*_{\mu 2}|^2 + |U_{e3}U^*_{\mu 3}|^2 + \qquad \textbf{(5)}$$

$$2\Re(U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2}e^{-i(\phi_1-\phi_2)} + U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3}e^{-i(\phi_1-\phi_3)} + U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3}e^{-i(\phi_2-\phi_3)})$$

- **This can be simplified by applying identity (4) to |(U4)|^2**

$$|U_{e1}U^*_{\mu 1} + U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3}|^2 = 0$$

➡

$$|U_{e1}U^*_{\mu 1}|^2 + |U_{e2}U^*_{\mu 2}|^2 + |U_{e3}U^*_{\mu 3}|^2 =$$

$$-2\Re(U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2} + U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3} + U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3})$$

- **Substituting into equation (5) gives**

$$\boxed{\begin{aligned} P(\nu_e \rightarrow \nu_\mu) \quad &= \quad 2\Re\{U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2}[e^{-i(\phi_1-\phi_2)} - 1]\} \\ &+ \quad 2\Re\{U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3}[e^{-i(\phi_1-\phi_3)} - 1]\} \\ &+ \quad 2\Re\{U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3}[e^{-i(\phi_2-\phi_3)} - 1]\} \end{aligned}} \qquad \textbf{(6)}$$

★ **This expression for the electron survival probability is obtained from the coefficient for $|\nu_e\rangle$ in eqn. (3):**

$$
\begin{aligned}
P(\nu_e \to \nu_e) &= |\langle \nu_e|\psi(L)\rangle|^2 \\
&= |U_{e1}U_{e1}^* e^{-i\phi_1} + U_{e2}U_{e2}^* e^{-i\phi_2} + U_{e3}U_{e3}^* e^{-i\phi_3}|^2
\end{aligned}
$$

which using the unitarity relation (U1)

$$|U_{e1}U_{e1}^* + U_{e2}U_{e2}^* + U_{e3}U_{e3}^*|^2 = 1$$

can be written

$$
\boxed{
\begin{aligned}
P(\nu_e \to \nu_e) = 1 \quad &+ \quad 2|U_{e1}|^2|U_{e2}|^2\Re\{[e^{-i(\phi_1-\phi_2)} - 1]\} \\
&+ \quad 2|U_{e1}|^2|U_{e3}|^2\Re\{[e^{-i(\phi_1-\phi_3)} - 1]\} \\
&+ \quad 2|U_{e2}|^2|U_{e3}|^2\Re\{[e^{-i(\phi_2-\phi_3)} - 1]\}
\end{aligned}
}
\tag{7}
$$

★ **This expression can simplified using**

$$
\begin{aligned}
\Re\{e^{-i(\phi_1-\phi_2)} - 1\} &= \cos(\phi_2 - \phi_1) - 1 \\
&= -2\sin^2\left(\frac{\phi_2 - \phi_1}{2}\right) \\
&= -2\sin^2\left(\frac{(m_2^2 - m_1^2)L}{4E}\right)
\end{aligned}
$$

with $\phi_i \approx \frac{m_i^2}{2E}L$

Phase of mass eigenstate i at $z = L$

- **Define:** $\boxed{\Delta_{21} = \frac{(m_2^2 - m_1^2)L}{4E} = \frac{\Delta m_{21}^2 L}{4E}}$ **with** $\boxed{\Delta m_{21}^2 = m_2^2 - m_1^2}$

NOTE: $\Delta_{21} = (\phi_2 - \phi_1)/2$ **is a phase difference (i.e. dimensionless)**

- **Which gives the electron neutrino survival probability**

$$\boxed{P(\nu_e \to \nu_e) = 1 - 4|U_{e1}|^2|U_{e2}|^2 \sin^2\Delta_{21} - 4|U_{e1}|^2|U_{e3}|^2 \sin^2\Delta_{31} - 4|U_{e2}|^2|U_{e3}|^2 \sin^2\Delta_{32}}$$

- **Similar expressions can be obtained for the muon and tau neutrino survival probabilities for muon and tau neutrinos.**

★ **Note that since we only have three neutrino generations there are only two independent mass-squared differences, i.e.**

$$m_3^2 - m_1^2 = (m_3^2 - m_2^2) + (m_2^2 - m_1^2)$$

and in the above equation only two of the Δ_{ij} **are independent**

★ **All expressions are in Natural Units, conversion to more useful units here gives:**

$$\boxed{\Delta_{21} = 1.27\frac{\Delta m_{21}^2(\mathrm{eV}^2)L(\mathrm{km})}{E(\mathrm{GeV})}} \quad \textbf{and} \quad \boxed{\lambda_{\mathrm{osc}}(\mathrm{km}) = 2.47\frac{E(\mathrm{GeV})}{\Delta m^2(\mathrm{eV}^2)}}$$

CP and CPT in the Weak Interaction

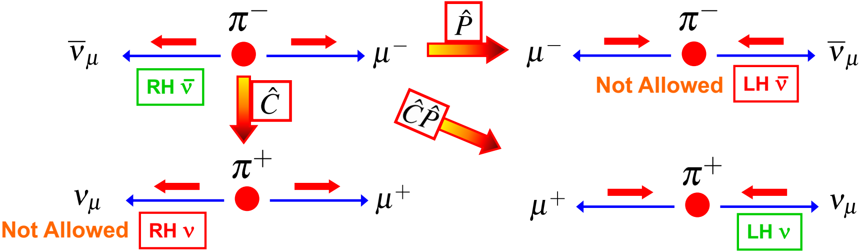
- ★ In addition to parity there are two other important discrete symmetries:

Parity $\hat{P}:$ $\vec{r} \longrightarrow -\vec{r}$

Time Reversal $\hat{T}:$ $t \longrightarrow -t$

Charge Conjugation $\hat{C}:$ Particle ⟷ Anti-particle

- ★ The weak interaction violates parity conservation, but what about C ? Consider pion decay remembering that the neutrino is ultra-relativistic and only **left-handed neutrinos** and **right-handed anti-neutrinos** participate in WI



- ★ Hence weak interaction also violates charge conjugation symmetry but appears to be invariant under combined effect of C and P

CP transforms:

RH Particles ⇔ LH Anti-particles

LH Particles ⇔ RH Anti-particles

★ If the weak interaction were invariant under CP expect

$$\boxed{\Gamma(\pi^+ \rightarrow \mu^+ \nu_\mu) = \Gamma(\pi^- \rightarrow \mu^- \overline{\nu}_\mu)}$$

★ All Lorentz invariant Quantum Field Theories can be shown to be invariant under CPT (charge conjugation + parity + time reversal)

➡ Particles/anti-particles have identical mass, lifetime, magnetic moments,…

Best current experimental test: $m_{K^0} - m_{\overline{K}^0} < 6 \times 10^{-19} m_{K^0}$

★ Believe CPT has to hold:

if CP invariance holds ➡ time reversal symmetry

if CP is violated ➡ time reversal symmetry violated

★ To account for the small excess of matter over anti-matter that must have existed early in the universe require CP violation in particle physics !

★ CP violation can arise in the weak interaction (see also handout 12).

CP and T Violation in Neutrino Oscillations

- Previously derived the oscillation probability for $\nu_e \rightarrow \nu_\mu$

$$\begin{aligned} P(\nu_e \rightarrow \nu_\mu) &= 2\Re\{U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2}[e^{-\imath(\varphi_1-\varphi_2)}-1]\} \\ &+ 2\Re\{U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3}[e^{-i(\phi_1-\phi_3)}-1]\} \\ &+ 2\Re\{U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3}[e^{-i(\phi_2-\phi_3)}-1]\} \end{aligned}$$

- The oscillation probability for $\nu_\mu \rightarrow \nu_e$ can be obtained in the same manner or by simply exchanging the labels $(e) \leftrightarrow (\mu)$

$$\begin{aligned} P(\nu_\mu \rightarrow \nu_e) &= 2\Re\{U_{\mu 1}U^*_{e1}U^*_{\mu 2}U_{e2}[e^{-i(\phi_1-\phi_2)}-1]\} \\ &+ 2\Re\{U_{\mu 1}U^*_{e1}U^*_{\mu 3}U_{e3}[e^{-i(\phi_1-\phi_3)}-1]\} \\ &+ 2\Re\{U_{\mu 2}U^*_{e2}U^*_{\mu 3}U_{e3}[e^{-i(\phi_2-\phi_3)}-1]\} \end{aligned} \qquad (8)$$

★ Unless the elements of the PMNS matrix are real (see note below)

$$P(\nu_e \rightarrow \nu_\mu) \neq P(\nu_\mu \rightarrow \nu_e) \qquad (9)$$

- If any of the elements of the PMNS matrix are complex, neutrino oscillations are not invariant under time reversal $t \rightarrow -t$

NOTE: can multiply entire PMNS matrix by a complex phase without changing the oscillation prob. T is violated if one of the elements has a different complex phase than the others

- Consider the effects of T, CP and CPT on neutrino oscillations

T $$\nu_e \to \nu_\mu \xrightarrow{\hat{T}} \nu_\mu \to \nu_e$$

CP $$\nu_e \to \nu_\mu \xrightarrow{\hat{C}\hat{P}} \overline{\nu}_e \to \overline{\nu}_\mu$$

CPT $$\nu_e \to \nu_\mu \xrightarrow{\hat{C}\hat{P}\hat{T}} \overline{\nu}_\mu \to \overline{\nu}_e$$

Note C alone is not sufficient as it transforms **LH neutrinos** into **LH anti-neutrinos** (not involved in Weak Interaction)

- If the weak interactions is invariant under **CPT**

$$P(\nu_e \to \nu_\mu) = P(\overline{\nu}_\mu \to \overline{\nu}_e)$$

and similarly $$P(\nu_\mu \to \nu_e) = P(\overline{\nu}_e \to \overline{\nu}_\mu) \qquad (10)$$

- If the PMNS matrix is not purely real, then (9)

$$P(\nu_e \to \nu_\mu) \neq P(\nu_\mu \to \nu_e)$$

and from (10) $$\boxed{P(\nu_e \to \nu_\mu) \neq P(\overline{\nu}_e \to \overline{\nu}_\mu)}$$

★**Hence unless the PMNS matrix is real, CP is violated in neutrino oscillations!**

Future experiments, e.g. “a neutrino factory”, are being considered as a way to investigate CP violation in neutrino oscillations. However, CP violating effects are well below the current experimental sensitivity. In the following discussion we will take the PMNS matrix to be real. (question 22)

Neutrino Mass Hierarchy

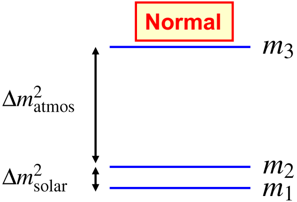
★ **To date, results on neutrino oscillations only determine**

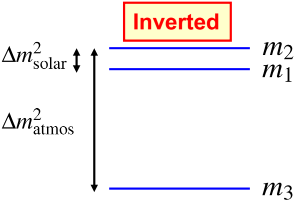
$$|\Delta m^2_{ji}| = |m^2_j - m^2_i|$$

★ **Two distinct and very different mass scales:**

- **Atmospheric neutrino oscillations :** $|\Delta m^2|_{\text{atmos}} \sim 2.5 \times 10^{-3}\,\text{eV}^2$
- **Solar neutrino oscillations:** $|\Delta m^2|_{\text{solar}} \sim 8 \times 10^{-5}\,\text{eV}^2$

•**Two possible assignments of mass hierarchy:**





•**In both cases:** $\Delta m^2_{21} \sim 8 \times 10^{-5}\,\text{eV}^2$ **(solar)**

$|\Delta m^2_{31}| \approx |\Delta m^2_{32}| \sim 2.5 \times 10^{-3}\,\text{eV}^2$ **(atmospheric)**

•**Hence we can approximate** $\Delta m^2_{31} \approx \Delta m^2_{32}$

Three Flavour Oscillations Neglecting CP Violation

- **Neglecting CP violation considerably simplifies the algebra of three flavour neutrino oscillations. Taking the PMNS matrix to be real, equation (6) becomes:**

$$P(\nu_e \rightarrow \nu_\mu) = -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} - 4U_{e1}U_{\mu 1}U_{e3}U_{\mu 3}\sin^2\Delta_{31} - 4U_{e2}U_{\mu 2}U_{e3}U_{\mu 3}\sin^2\Delta_{32}$$

with $\Delta_{ji} = \frac{(m_j^2 - m_i^2)L}{4E} = \frac{\Delta m_{ji}^2 L}{4E}$

- **Using:** $\Delta_{31} \approx \Delta_{32}$ **(see p.383)**

$$P(\nu_e \rightarrow \nu_\mu) \approx -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} - 4(U_{e1}U_{\mu 1} + U_{e2}U_{\mu 2})U_{e3}U_{\mu 3}\sin^2\Delta_{32}$$

- **Which can be simplified using (U4)** $U_{e1}U_{\mu 1}^* + U_{e2}U_{\mu 2}^* + U_{e3}U_{\mu 3}^* = 0$

➡ $$\boxed{P(\nu_e \rightarrow \nu_\mu) \approx -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} + 4U_{e3}^2U_{\mu 3}^2\sin^2\Delta_{32}}$$

- **Can apply** $\Delta_{31} \approx \Delta_{32}$ **to the expression for electron neutrino survival probability**

$$P(\nu_e \rightarrow \nu_e) = 1 - 4U_{e1}^2U_{e2}^2\sin^2\Delta_{21} - 4U_{e1}^2U_{e3}^2\sin^2\Delta_{31} - 4U_{e2}^2U_{e3}^2\sin^2\Delta_{32}$$

$$\approx 1 - 4U_{e1}^2U_{e2}^2\sin^2\Delta_{21} - 4(U_{e1}^2 + U_{e2}^2)U_{e3}^2\sin^2\Delta_{32}$$

- **Which can be simplified using (U1)** $U_{e1}^2 + U_{e2}^2 + U_{e3}^2 = 1$

➡ $$\boxed{P(\nu_e \rightarrow \nu_e) \approx 1 - 4U_{e1}^2U_{e2}^2\sin^2\Delta_{21} - 4(1 - U_{e3}^2)U_{e3}^2\sin^2\Delta_{32}}$$

★ **Neglecting CP violation (i.e. taking the PMNS matrix to be real) and making the approximation that $|\Delta m^2_{31}| \approx |\Delta m^2_{32}|$ obtain the following expressions for neutrino oscillation probabilities:**

$$\boxed{P(\nu_e \to \nu_e) \approx 1 - 4U^2_{e1}U^2_{e2}\sin^2\Delta_{21} - 4(1-U^2_{e3})U^2_{e3}\sin^2\Delta_{32}} \tag{11}$$

$$\boxed{P(\nu_\mu \to \nu_\mu) \approx 1 - 4U^2_{\mu1}U^2_{\mu2}\sin^2\Delta_{21} - 4(1-U^2_{\mu3})U^2_{\mu3}\sin^2\Delta_{32}} \tag{12}$$

$$\boxed{P(\nu_\tau \to \nu_\tau) \approx 1 - 4U^2_{\tau1}U^2_{\tau2}\sin^2\Delta_{21} - 4(1-U^2_{\tau3})U^2_{\tau3}\sin^2\Delta_{32}} \tag{13}$$

$$\boxed{P(\nu_e \to \nu_\mu) = P(\nu_\mu \to \nu_e) \approx -4U_{e1}U_{\mu1}U_{e2}U_{\mu2}\sin^2\Delta_{21} + 4U^2_{e3}U^2_{\mu3}\sin^2\Delta_{32}} \tag{14}$$

$$\boxed{P(\nu_e \to \nu_\tau) = P(\nu_\tau \to \nu_e) \approx -4U_{e1}U_{\tau1}U_{e2}U_{\tau2}\sin^2\Delta_{21} + 4U^2_{e3}U^2_{\tau3}\sin^2\Delta_{32}} \tag{15}$$

$$\boxed{P(\nu_\mu \to \nu_\tau) = P(\nu_\tau \to \nu_\mu) \approx -4U_{\mu1}U_{\tau1}U_{\mu2}U_{\tau2}\sin^2\Delta_{21} + 4U^2_{\mu3}U^2_{\tau3}\sin^2\Delta_{32}} \tag{16}$$

★ **The wavelengths associated with $\sin^2\Delta_{21}$ and $\sin^2\Delta_{32}$ are:**

"SOLAR"

$$\lambda_{21} = \frac{4\pi E}{\Delta m^2_{21}}$$

"Long"-Wavelength

and

$$\lambda_{32} = \frac{4\pi E}{\Delta m^2_{32}}$$

"ATMOSPHERIC"

"Short"-Wavelength

PMNS Matrix

★ The PMNS matrix is usually expressed in terms of 3 rotation angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a complex phase δ, using the notation $s_{ij} = \sin\theta_{ij}, \quad c_{ij} = \cos\theta_{ij}$

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{“Atmospheric”}} \times \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{+i\delta} & 0 & c_{13} \end{pmatrix} \times \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{“Solar”}}$$

Dominates:

- Writing this out in full:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

★There are six **SM parameters** that can be measured in ν oscillation experiments

$\lvert\Delta m_{21}\rvert^2 = \lvert m_2^2 - m_1^2\rvert$	θ_{12}	Solar and reactor neutrino experiments
$\lvert\Delta m_{32}\rvert^2 = \lvert m_3^2 - m_2^2\rvert$	θ_{23}	Atmospheric and beam neutrino experiments
	θ_{13}	Reactor neutrino experiments + future beam
	δ	Future beam experiments

Neutrino Experiments

- Before discussing current experimental data, need to consider how neutrinos interact in matter (i.e. our detectors)

Two processes:

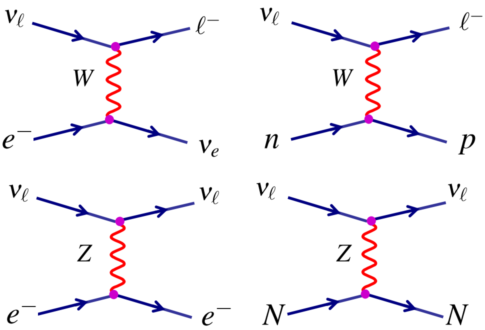
- Charged current (CC) interactions (via a W-boson) ⇨ charged lepton
- Neutral current (NC) interactions (via a Z-boson)

Two possible "targets": can have neutrino interactions with

- atomic electrons
- nucleons within the nucleus

CHARGED CURRENT

NEUTRAL CURRENT



Neutrino Interaction Thresholds

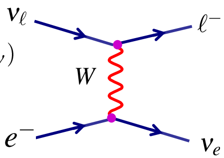
★ **Neutrino detection method depends on the neutrino energy and (weak) flavour**

- **Neutrinos from the sun and nuclear reactions have** $E_\nu \sim 1\,\mathrm{MeV}$
- **Atmospheric neutrinos have** $E_\nu \sim 1\,\mathrm{GeV}$

★ **These energies are relatively low and not all interactions are kinematically allowed, i.e. there is a threshold energy before an interaction can occur. Require sufficient energy in the centre-of-mass frame to produce the final state particles**

➊ **Charged current interactions on atomic electrons (in laboratory frame)**

$$p_\nu = (E_\nu, 0, 0, E_\nu)$$
$$p_e = (m_e, 0, 0, 0)$$



$$s = (p_\nu + p_e)^2 = (E_\nu + m_e)^2 - E_\nu^2$$

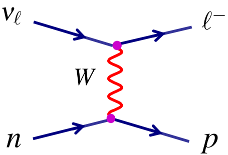
Require: $s > m_\ell^2$

$$\Rightarrow \boxed{E_\nu > \left[\left(\frac{m_\ell}{m_e}\right)^2 - 1\right]\frac{m_e}{2}}$$

- **Putting in the numbers, for CC interactions with atomic electrons require**

$$E_{\nu_e} > 0 \qquad \underbrace{E_{\nu_\mu} > 11\,\mathrm{GeV} \qquad E_{\nu_\tau} > 3090\,\mathrm{GeV}}_{\text{High energy thresholds compared to typical energies considered here}}$$

❷ charged current interactions on nucleons (in lab. frame)



$$s=(p_\nu+p_n)^2=(E_\nu+m_n)^2-E_\nu^2$$

Require: $s>(m_\ell+m_p)^2$

➡ $$\boxed{E_\nu>\frac{(m_p^2-m_n^2)+m_\ell^2+2m_pm_\ell}{2m_n}}$$

•For CC interactions from neutrons require

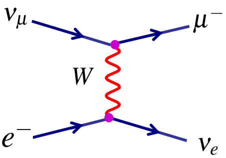
$$E_{\nu_e}>0 \qquad E_{\nu_\mu}>110\,\mathrm{MeV} \qquad E_{\nu_\tau}>3.5\,\mathrm{GeV}$$

★ Electron neutrinos from the sun and nuclear reactors $E_\nu\sim1\,\mathrm{MeV}$ which oscillate into muon or tau neutrinos cannot interact via charged current interactions – “they effectively disappear”

★ Atmospheric muon neutrinos $E_\nu\sim1\,\mathrm{GeV}$ which oscillate into tau neutrinos cannot interact via charged current interactions – “disappear”

•To date, most experimental signatures for neutrino oscillation are a deficit of neutrino interactions (with the exception of SNO) because below threshold for produce lepton of different flavour from original neutrino

- In Handout 10 derived expressions for CC neutrino-quark cross sections in ultra-relativistic limit (neglecting masses of neutrinos/quarks)
- For high energy **muon** neutrinos can directly use the results from p page 340 5



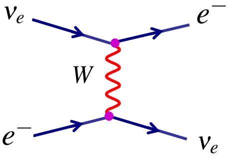
$$\sigma_{\nu_\mu e^-} = \frac{G_F^2 s}{\pi}$$

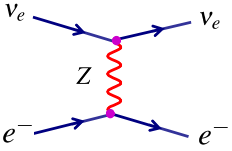
with $s = (E_\nu + m_e)^2 - E_\nu^2 \approx 2m_e E_\nu$

$$\boxed{\sigma_{\nu_\mu e^-} = \frac{2m_e G_F^2 E_\nu}{\pi}}$$

Cross section increases linearly with lab. frame neutrino energy

- For electron neutrinos there is another lowest order diagram with the same final state





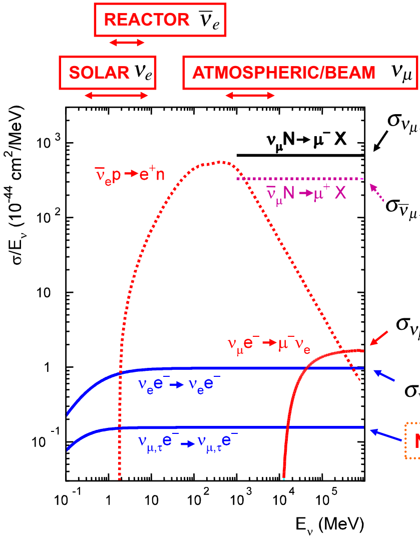
It turns out that the cross section is lower than the pure CC cross section due to negative interference when summing matrix elements $|M_{CC} + M_{NC}|^2 < |M_{CC}|^2$

$$\boxed{\sigma_{\nu_e e} \approx 0.6\sigma_{\nu_e e}^{CC}}$$

- In the high energy limit the CC neutrino-nucleon cross sections are larger due to the higher centre-of-mass energy: $s = (E_\nu + m_n)^2 - E_\nu^2 \approx 2m_n E_\nu$

Neutrino Detection

★ **The detector technology/interaction process depends on type of neutrino and energy**



$$\sigma_{\nu_\mu N} = 0.22 \frac{2 m_p G_F^2 E_\nu}{\pi}$$

$$\sigma_{\overline{\nu}_\mu N} = 0.33 \frac{2 m_p G_F^2 E_\nu}{3\pi}$$

Deep Inelastic Scattering ((p. 347))

$$\sigma_{\nu_\mu e^-} = \frac{2 m_e G_F^2 E_\nu}{\pi}$$

CC only Threshold 11 GeV

$$\sigma_{\nu_e e^-} \approx 0.6 \frac{2 m_e G_F^2 E_\nu}{\pi}$$

NC + CC

NC only (see handout 13)

Atmospheric/Beam Neutrinos

$$\nu_e, \nu_\mu, \overline{\nu}_e, \overline{\nu}_\mu : E_\nu > 1\,\mathrm{GeV}$$

➊ **Water Čerenkov:** e.g. Super Kamiokande

➋ **Iron Calorimeters:** e.g. MINOS, CDHS (see handout 10)

- Produce high energy charged lepton – relatively easy to detect

Solar Neutrinos

$$\nu_e \;:\; E_\nu < 20\,\mathrm{MeV}$$

➊ **Water Čerenkov:** e.g. Super Kamiokande

- Detect Čerenkov light from electron produced in $\nu_e + e^- \rightarrow \nu_e + e^-$
- Because of background from natural radioactivity limited to $E_\nu > 5\,\mathrm{MeV}$
- Because Oxygen is a doubly magic nucleus don't get $\nu_e + n \rightarrow e^- + p$

➋ **Radio-Chemical:** e.g. Homestake, SAGE, GALLEX

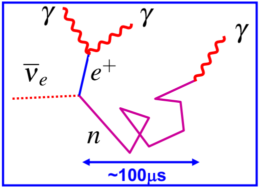
- Use inverse beta decay process, e.g. $\nu_e + {}^{71}\mathrm{Ga} \rightarrow e^- + {}^{71}\mathrm{Ge}$
- Chemically extract produced isotope and count decays (only gives a rate)

Reactor Neutrinos

$$\overline{\nu}_e \;:\; E_{\overline{\nu}} < 5\,\mathrm{MeV}$$

➊ **Liquid Scintillator:** e.g. KamLAND

- Low energies ➞ large radioactive background
- Dominant interaction: $\overline{\nu}_e + p \rightarrow e^+ + n$
- **Prompt** positron annihilation signal **+ delayed** signal from ***n*** (space/time correlation reduces background)
- electrons produced by photons excite scintillator which produces light

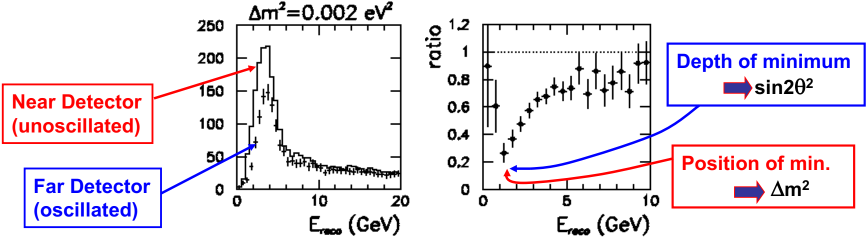


1) Long Baseline Neutrino Experiments

- Initial studies of neutrino oscillations from atmospheric and solar neutrinos
  - atmospheric neutrinos discussed in **examinable** appendix
- Emphasis of neutrino research now on neutrino beam experiments
- Allows the physicist to take control – design experiment with specific goals
- In the last few years, long baseline neutrino oscillation experiments have started taking data: K2K, MINOS, CNGS, T2K

Basic Idea:

- ★ Intense neutrino beam
- ★ Two detectors: one close to beam the other hundreds of km away
- ★ Measure ratio of the neutrino energy spectrum in far detector (oscillated) to that in the near detector (unoscillated)
- ★ Partial cancellation of systematic biases



Near Detector (unoscillated)

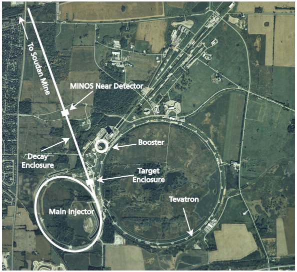
Far Detector (oscillated)

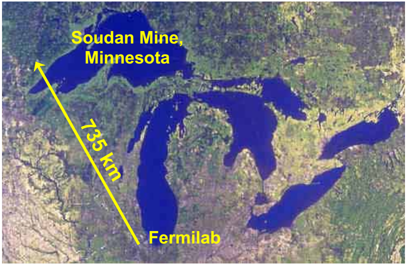
Depth of minimum ➡ $\sin 2\theta^2$

Position of min. ➡ Δm^2

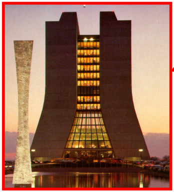
MINOS

- 120 GeV protons extracted from the MAIN INJECTOR at Fermilab (see p. 271)
- $2.5x10^{13}$ protons per pulse hit target → very intense beam - 0.3 MW on target





Two detectors:

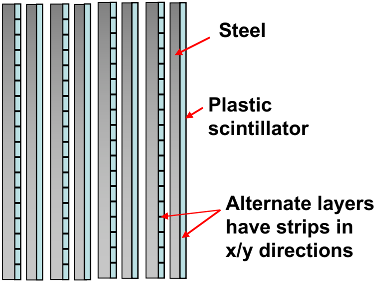


★ 1000 ton, NEAR Detector at Fermilab : **1 km** from beam

★ 5400 ton FAR Detector, 720m underground in Soudan mine, N. Minnesota: **735 km** from beam

The MINOS Detectors:

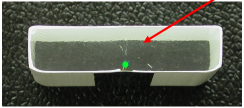
- Dealing with high energy neutrinos $E_\nu > 1\,\mathrm{GeV}$
- The muons produced by ν_μ interactions travel several metres
- Steel-Scintillator sampling calorimeter
- Each plane: 2.54 cm steel +1 cm scintillator
- A charged particle crossing the scintillator produces light – detect with PMTs

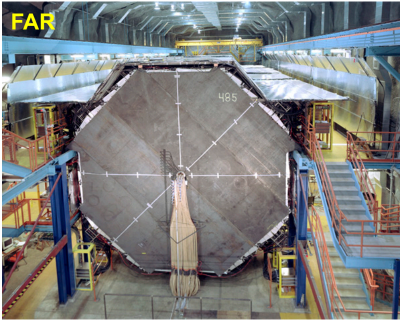


Steel

Plastic scintillator

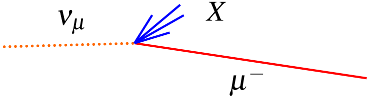
Alternate layers have strips in x/y directions



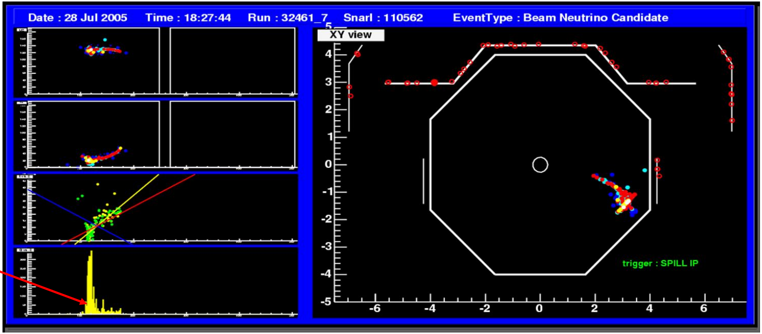


- **Neutrino detection via CC interactions on nucleon**

$$\nu_\mu + N \rightarrow \mu^- + X$$



Example event:



Signal from hadronic shower

- **The main feature of the MINOS detector is the very good neutrino energy resolution**

$$\boxed{E_\nu = E_\mu + E_X}$$

- **Muon energy from range/curvature in B-field**
- **Hadronic energy from amount of light observed**

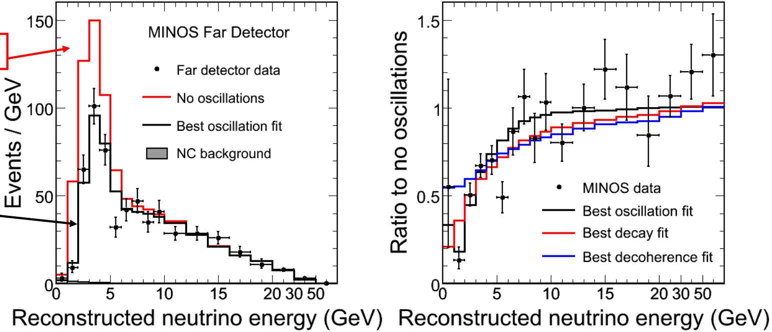
MINOS Results

- **For the MINOS experiment L is fixed and observe oscillations as function of** E_ν
- **For** $|\Delta m^2_{32}| \sim 2.5 \times 10^{-3}\,\mathrm{eV}^2$ **first oscillation minimum at** $E_\nu = 1.5\,\mathrm{GeV}$
- **To a very good approximation can use two flavour formula as oscillations corresponding to** $|\Delta m^2_{21}| \sim 8 \times 10^{-5}\,\mathrm{eV}^2$ **occur at** $E_\nu = 50\,\mathrm{MeV}$, **beam contains very few neutrinos at this energy + well below detection threshold**

MINOS Collaboration, Phys. Rev. Lett. 101, 131802, 2008

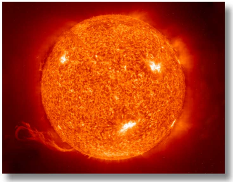
no oscillations

best fit osc



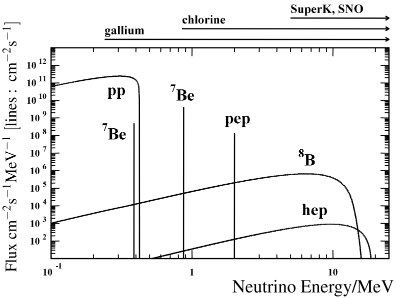
$$\boxed{|\Delta m^2_{32}| = (2.43 \pm 0.12) \times 10^{-3}\,\mathrm{eV}^2}$$

2) Solar Neutrinos



- The Sun is powered by the weak interaction – producing a very large flux of electron neutrinos

$$\boxed{2 \times 10^{38}\, \nu_e\, \mathrm{s}^{-1}}$$



- Several different nuclear reactions in the sun ⇨ complex neutrino energy spectrum

$$\boxed{p + p \to d + e^+ + \nu_e} \quad E_\nu < 0.5\,\mathrm{MeV}$$

$$\boxed{{}^8B \to {}^8Be^* + e^+ + \nu_e} \quad E_\nu \sim 5\mathrm{MeV}$$

$$p + e^- + p \to d + \nu_e$$

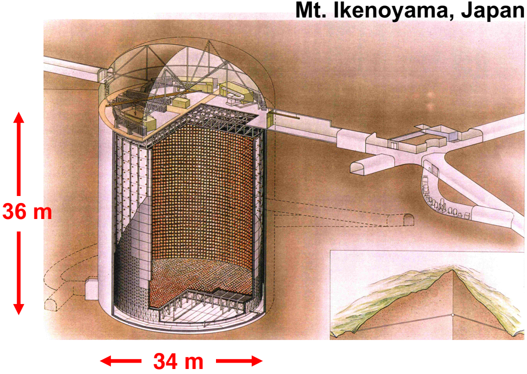
$$^7Be + e^- \to {}^7Li + \nu_e$$

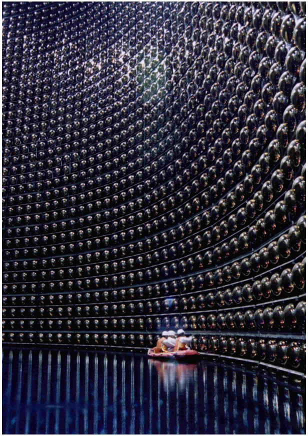
$$^3He + p \to {}^4He + e^+ + \nu_e$$

- All experiments saw a deficit of electron neutrinos compared to experimental prediction – the SOLAR NEUTRINO PROBLEM
- e.g. Super Kamiokande

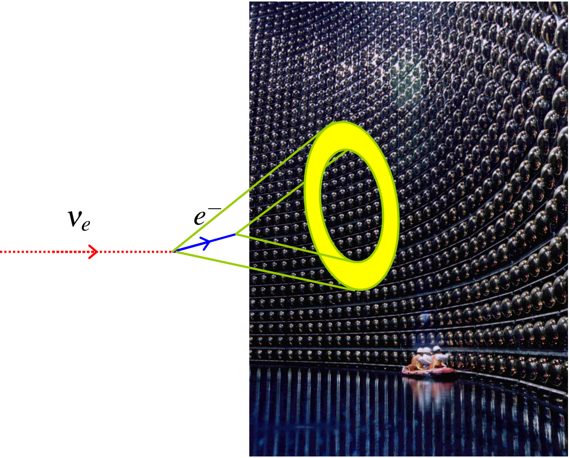
Solar Neutrinos I: Super Kamiokande

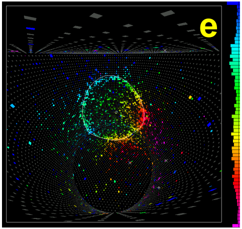
- 50000 ton water Čerenkov detector
- Water viewed by 11146 Photo-multiplier tubes
- Deep underground to filter out cosmic rays otherwise difficult to detect rare neutrino interactions

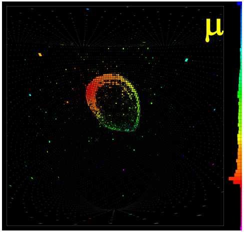




- Detect neutrinos by observing Čerenkov radiation from charged particles which travel faster than speed of light in water *c/n*

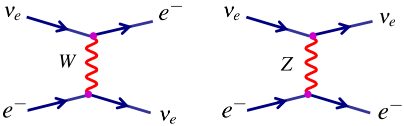






- Can distinguish electrons from muons from pattern of light – muons produce clean rings whereas electrons produce more diffuse “fuzzy” rings

- Sensitive to solar neutrinos with $E_\nu > 5\,\mathrm{MeV}$
- For lower energies too much background from natural radioactivity (β-decays)
- Hence detect mostly neutrinos from $^8B \rightarrow {}^8Be^* + e^+ + \nu_e$

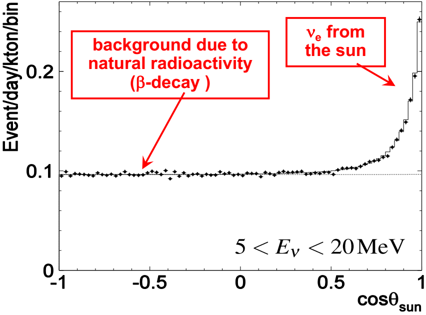


- Detect electron Čerenkov rings from

$$\nu_e + e^- \rightarrow \nu_e + e^-$$

- In LAB frame the electron is produced preferentially along the ν_e direction

S.Fukada et al., Phys. Rev. Lett. 86 5651-5655, 2001



background due to natural radioactivity (β-decay)

ν_e from the sun

Results:

- Clear signal of neutrinos from the sun
- However too few neutrinos

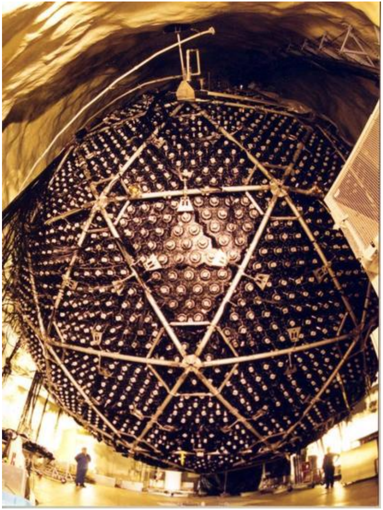
DATA/SSM = 0.45±0.02

SSM = "Standard Solar Model" Prediction

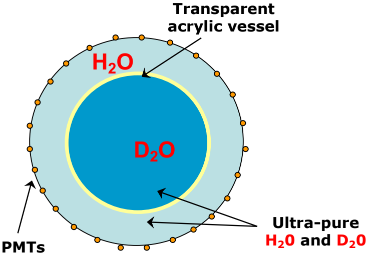
The Solar Neutrino "Problem"

Solar Neutrinos II: SNO

•**S**udbury **N**eutrino **O**bservatory located in a deep mine in Ontario, Canada



- 1000 ton heavy water (D_2O) Čerenkov detector
- D_2O inside a 12m diameter acrylic vessel
- Surrounded by 3000 tons of normal water
- Main experimental challenge is the need for very low background from radioactivity
- Ultra-pure H_2O and D_2O
- Surrounded by 9546 PMTs

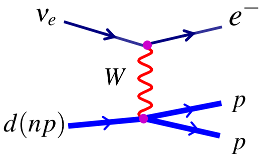


★ **Detect Čerenkov light from three different reactions:**

CHARGE CURRENT

- Detect Čerenkov light from electron
- Only sensitive to ν_e (thresholds)
- Gives a measure of ν_e flux

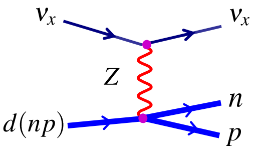
$$\boxed{\text{CC Rate} \propto \phi(\nu_e)}$$



NEUTRAL CURRENT

- Neutron capture on a deuteron gives **6.25 MeV**
- Detect Čerenkov light from electrons scattered by γ
- Measures total neutrino flux

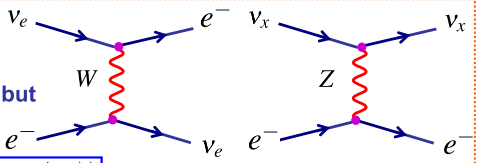
$$\boxed{\text{NC Rate} \propto \phi(\nu_e) + \phi(\nu_\mu) + \phi(\nu_\tau)}$$



ELASTIC SCATTERING

- Detect Čerenkov light from electron
- Sensitive to all neutrinos (NC part) – but larger cross section for ν_e

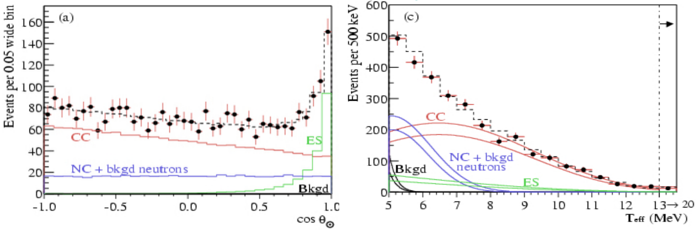
$$\boxed{\text{ES Rate} \propto \phi(\nu_e) + 0.154(\phi(\nu_\mu) + \phi(\nu_\tau))}$$



★ **Experimentally can determine rates for different interactions from:**

- **angle with respect to sun: electrons from ES point back to sun**
- **energy: NC events have lower energy – 6.25 MeV photon from neutron capture**
- **radius from centre of detector: gives a measure of background from neutrons**

SNO Collaboration, Q.R. Ahmad et al., Phys. Rev. Lett. 89:011301, 2002



★**Using different distributions obtain a measure of numbers of events of each type:**

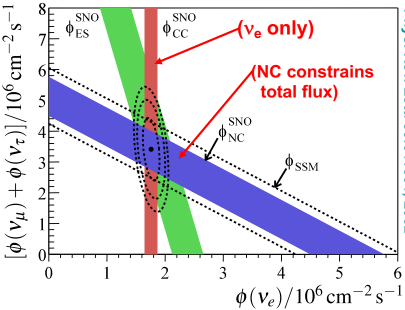
$$\boxed{\text{CC : } 1968 \pm 61} \propto \phi(\nu_e)$$

$$\boxed{\text{ES : } 264 \pm 26} \propto \phi(\nu_e) + 0.154[\phi(\nu_\mu) + \phi(\nu_\tau)]$$

$$\boxed{\text{NC : } 576 \pm 50} \propto \phi(\nu_e) + \phi(\nu_\mu) + \phi(\nu_\tau)$$

→ **Measure of electron neutrino flux + total flux !**

- ★ **Using known cross sections can convert observed numbers of events into fluxes**
- ★ **The different processes impose different constraints**
- ★ **Where constraints meet gives separate measurements of ν_e and $\nu_\mu + \nu_\tau$ fluxes**



(ν_e only)

(NC constrains total flux)

SNO Collaboration, Q.R. Ahmad et al., Phys. Rev. Lett. 89:011301, 2002

SNO Results:

$$\phi(\nu_e) = (1.8 \pm 0.1) \times 10^{-6}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$$

$$\phi(\nu_\mu) + \phi(\nu_\tau) = (3.4 \pm 0.6) \times 10^{-6}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$$

SSM Prediction:

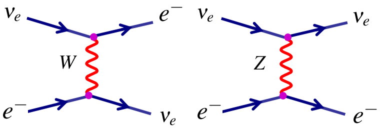
$$\phi(\nu_e) = 5.1 \times 10^{-6}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$$

- **Clear evidence for a flux of ν_μ and/or ν_τ from the sun**
- **Total neutrino flux is consistent with expectation from SSM**
- **Clear evidence of $\nu_e \rightarrow \nu_\mu$ and/or $\nu_e \rightarrow \nu_\tau$ neutrino transitions**

Interpretation of Solar Neutrino Data

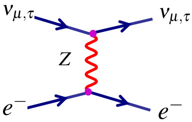
★ The interpretation of the solar neutrino data is complicated by **MATTER EFFECTS**

- The quantitative treatment is non-trivial and is not given here
- Basic idea is that as a neutrino leaves the sun it crosses a region of high electron density
- The coherent forward scattering process ($\nu_e \to \nu_e$) for an electron neutrino



CC + **NC**

is different to that for a muon or tau neutrino



NC

- Can enhance oscillations – “MSW effect”

★ A combined analysis of all solar neutrino data gives:

$$\Delta m^2_{\text{solar}} \approx 8 \times 10^{-5}\,\text{eV}^2,\ \sin^2 2\theta_{\text{solar}} \approx 0.85$$

3) Reactor Experiments

- To explain reactor neutrino experiments we need the full three neutrino expression for the electron neutrino survival probability (11) which depends on U_{e1}, U_{e2}, U_{e3}
- Substituting these PMNS matrix elements in Equation (11):

$$\begin{aligned} P(\nu_e \to \nu_e) &\approx 1 - 4U_{e1}^2 U_{e2}^2 \sin^2\Delta_{21} - 4(1-U_{e3}^2)U_{e3}^2 \sin^2\Delta_{32} \\ &= 1 - 4(c_{12}c_{13})^2(s_{12}c_{13})^2 \sin^2\Delta_{21} - 4(1-s_{13}^2)s_{13}^2 \sin^2\Delta_{32} \\ &= 1 - c_{13}^4(2s_{12}c_{12})^2 \sin^2\Delta_{21} - (2c_{13}s_{13})^2 \sin^2\Delta_{32} \\ &= 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \sin^2 2\theta_{13}\sin^2\Delta_{32} \end{aligned}$$

- Contributions with short wavelength (atmospheric) and long wavelength (solar)
- For a 1 MeV neutrino

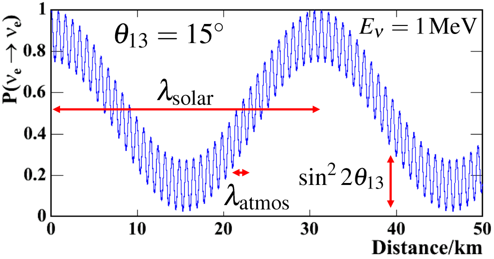
$$\lambda_{\text{osc}}(\text{km}) = 2.47\frac{E(\text{GeV})}{\Delta m^2(\text{eV}^2)}$$

$$\Rightarrow \quad \lambda_{21} = 30.0\,\text{km}$$

$$\lambda_{32} = 0.8\,\text{km}$$

- Amplitude of short wavelength oscillations given by

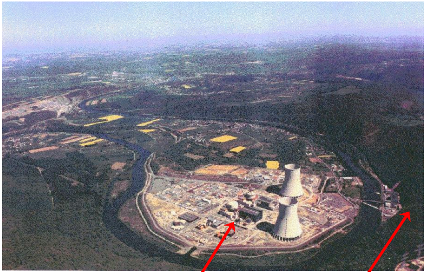
$$\sin^2 2\theta_{13}$$



Reactor Experiments I : CHOOZ

- Two nuclear reactors, each producing 4.2 GW
- Place detector 1km from reactor cores
- Reactors produce intense flux of $\overline{\nu}_e$

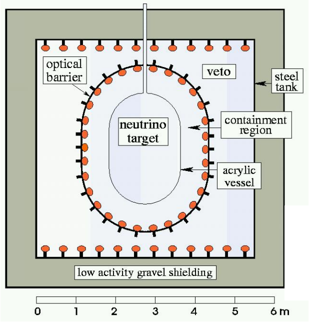
France



reactors

Detector
150m underground

Detector



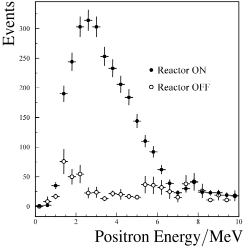
- Anti-neutrinos interact via inverse beta decay $\overline{\nu}_e + p \rightarrow e^+ + n$
- Detector is liquid scintillator loaded with Gadolinium (large n capture cross section)
- Detect photons from positron annihilation and a delayed signal from photons from neutron capture on Gadolinium

$$e^+ + e^- \rightarrow \gamma + \gamma$$
$$n + \mathrm{Gd} \rightarrow \mathrm{Gd}^* \rightarrow \mathrm{Gd} + \gamma + \gamma + \ldots$$

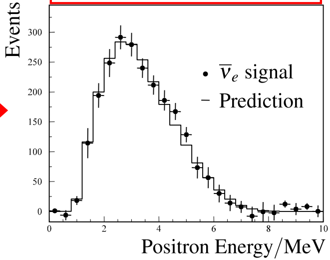
- At 1km and energies > 1 MeV, only the **short** wavelength component matters

$$P(\nu_e \to \nu_e) \quad = \quad 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \sin^2 2\theta_{13}\sin^2\Delta_{32} \approx 1 - \sin^2 2\theta_{13}\sin^2\Delta_{32}$$

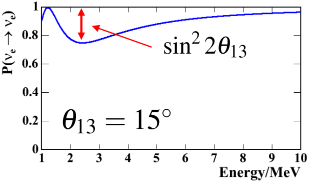
CHOOZ Raw Data



Background subtracted



Compare to effect of oscillations



★ Data agree with unoscillated prediction both in terms of rate and energy spectrum

$$\boxed{N_{\text{data}}/N_{\text{expect}} = 1.01 \pm 0.04}$$

CHOOZ Collaboration,
M.Apollonio et al.,
Phys. Lett. B420, 397-404, 1998

★ Hence $\sin^2 2\theta_{13}$ must be small !

⇨ $\boxed{\sin^2 2\theta_{13} < 0.12 - 0.2}$

Exact limit depends on $|\Delta m^2_{32}|$

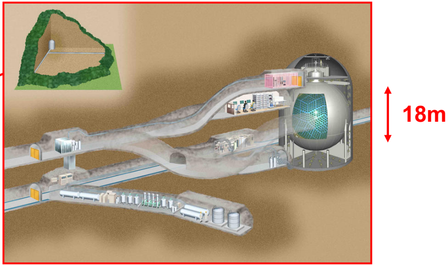
★ From atmospheric neutrinos (see appendix) can exclude $\theta_{13} \sim \frac{\pi}{2}$

- Hence the CHOOZ limit: $\sin^2 2\theta_{13} < 0.2$ can be interpreted as $\sin^2\theta_{13} < 0.05$

Reactor Experiments II : KamLAND

International Nuclear Safety Center at ANL, Oct 2002

- Detector located in same mine as Super Kamiokande



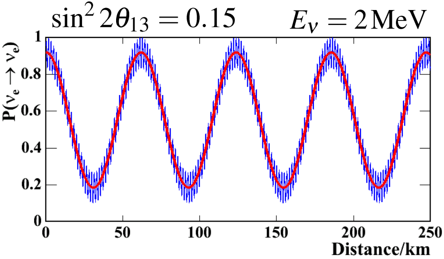
- 70 GW from nuclear power (7% of World total) from reactors within **130-240 km**
- Liquid scintillator detector, 1789 PMTs
- Detection via inverse beta decay: $\overline{\nu}_e + p \to e^+ + n$

  Followed by $e^+ + e^- \to \gamma + \gamma$ prompt

  $n + p \to d + \gamma(2.2\,\mathrm{MeV})$ delayed

- **For MeV neutrinos at a distance of 130-240 km oscillations due to Δm^2_{32} are very rapid**
- **Experimentally, only see average effect**

$$\langle \sin^2 \Delta_{32} \rangle = 0.5$$



★ **Here:**

$$\begin{aligned} P(\nu_e \to \nu_e) &= 1 - \cos^4\theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21} - \sin^2 2\theta_{13} \sin^2 \Delta_{32} \\ &\approx 1 - \cos^4\theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21} - \frac{1}{2} \sin^2 2\theta_{13} \\ &= \cos^4\theta_{13} + \sin^4\theta_{13} - \cos^4\theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21} \\ &\approx \cos^4\theta_{13} (1 - \sin^2 2\theta_{12} \sin^2 \Delta_{21}) \end{aligned}$$

Averaging over rapid oscillations

neglect $\sin^4\theta_{13}$

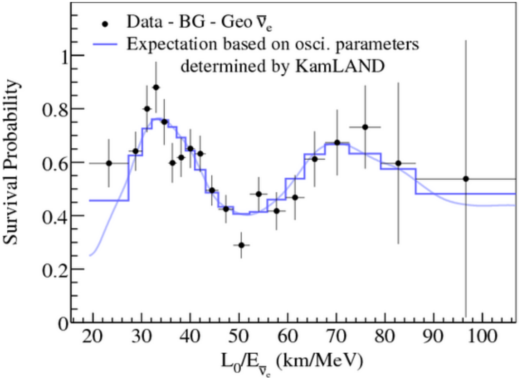
- **Obtain two-flavour oscillation formula multiplied by** $\cos^4\theta_{13}$
- **From CHOOZ** $\cos^4\theta_{13} > 0.9$

(Try Question 21)

KamLAND RESULTS:

Observe: 1609 events
Expect: 2179±89 events (if no oscillations)

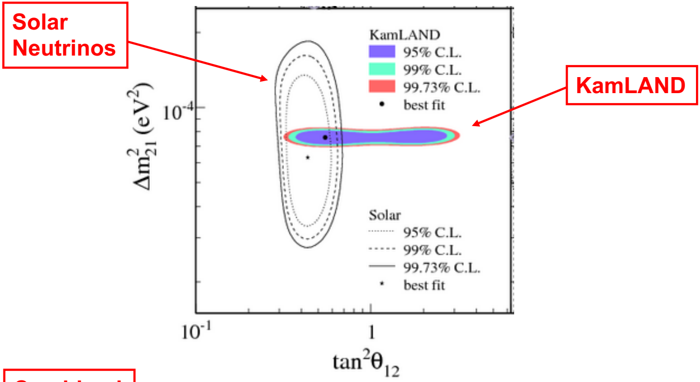
KamLAND Collaboration, Phys. Rev. Lett., 221803, 2008



- ★ Clear evidence of electron anti-neutrino oscillations consistent with the results from solar neutrinos
- ★ Oscillatory structure clearly visible
- ★ Compare data with expectations for different osc. parameters and perform χ^2 fit to extract measurment

Combined Solar Neutrino and KamLAND Results

- ★ **KamLAND data provides strong constraints on** $|\Delta m^2_{21}|$
- ★ **Solar neutrino data (especially SNO) provides a strong constraint on** θ_{12}



Combined

$$|\Delta m^2_{21}| = (7.59 \pm 0.21) \times 10^{-5}\,\mathrm{eV}^2$$

$$\tan^2\theta_{12} = 0.47^{+0.06}_{-0.05}$$

Recent work …

★ **Increasing evidence for non-zero value of non-zero θ_{13}**

- **T2K:** $\nu_\mu \longrightarrow \nu_e$ **appearance (2.5 σ)**
- **MINOS:** $\nu_\mu \longrightarrow \nu_e$ **appearance (2 σ)**
- **Double-CHOOZ:** $\overline{\nu}_e$ **disappearance (2 σ)**

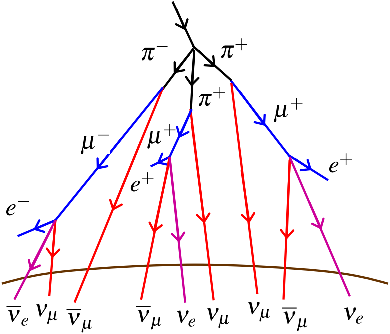
$$\boxed{\sin^2 2\theta_{13} \approx 0.04 - 0.08\,?}$$

- **in 2013/2014 Daya Bay experiment (see question 21) measured:**

$$\boxed{\sin^2 2\theta_{13} = 0.092 \pm 0.016}$$

Atmospheric Neutrinos

- **High energy cosmic rays** (up to 10^{20} eV) interact in the upper part of the Earth's atmosphere
- The cosmic rays (~86% protons, 11% He Nuclei, ~1% heavier nuclei, 2% electrons) mostly interact hadronically giving showers of mesons (mainly pions)



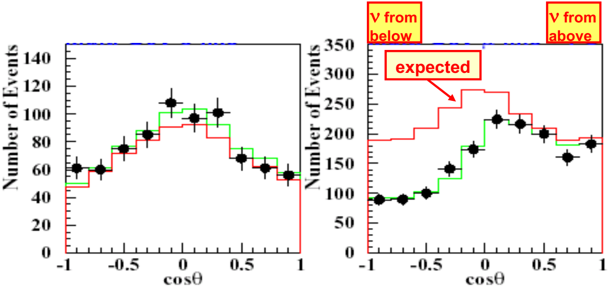
- Neutrinos produced by:

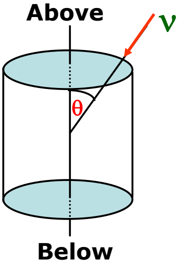
$$\pi^+ \to \mu^+ \nu_\mu \qquad \pi^- \to \mu^- \overline{\nu}_\mu$$
$$\hookrightarrow e^+ \nu_e \overline{\nu}_\mu \qquad \hookrightarrow e^- \overline{\nu}_e \nu_\mu$$

- Flux $\sim 1\,\mathrm{cm}^{-2}\mathrm{sr}^{-1}\mathrm{s}^{-1}$
- Typical energy : $\mathrm{E}_\nu \sim 1\,\mathrm{GeV}$
- Expect $$\frac{N(\nu_\mu + \overline{\nu}_\mu)}{N(\nu_e + \overline{\nu}_e)} \approx 2$$
- Observe a lower ratio with deficit of $\nu_\mu / \overline{\nu}_\mu$ coming from below the horizon, i.e. large distance from production point on other side of the Earth

Super Kamiokande Atmospheric Results

- **Typical energy: $E_\nu \sim 1\,\mathrm{GeV}$ (much greater than solar neutrinos – no confusion)**
- **Identify ν_e and ν_μ interactions from nature of Čerenkov rings**
- **Measure rate as a function of angle with respect to local vertical**
- **Neutrinos coming from above travel ~20 km**
- **Neutrinos coming from below (i.e. other side of the Earth) travel ~12800 km**

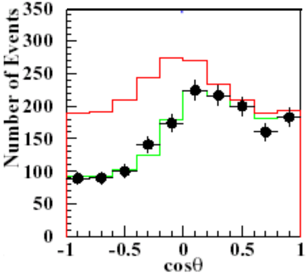




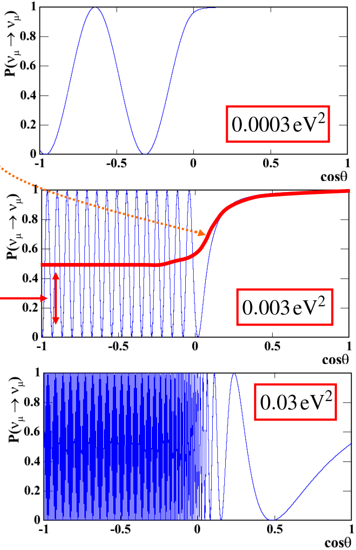
- ★ **Prediction for ν_e rate agrees with data**
- ★ **Strong evidence for disappearance of ν_μ for large distances**
- ★ **Consistent with $\nu_\mu \rightarrow \nu_\tau$ oscillations**
- ★ **Don't detect the oscillated ν_τ as typically below interaction threshold of 3.5 GeV**

Interpretation of Atmospheric Neutrino Data

- Measure muon direction and energy not neutrino direction/energy
- Don't have E/θ resolution to see oscillations
- Oscillations "smeared" out in data
- Compare data to predictions for $|\Delta m^2|$



$$1 - \frac{1}{2}\sin^2 2\theta$$



★ **Data consistent with:**

$$|\Delta m^2_{\mathrm{atmos}}| \approx 0.0025\,\mathrm{eV}^2$$
$$\sin^2 2\theta_{\mathrm{atmos}} \approx 1$$

Summary of Current Knowledge

SOLAR Neutrinos/KamLAND

KamLAND + Solar: $|\Delta m^2_{21}| \approx (7.6 \pm 0.2) \times 10^{-5}\,\mathrm{eV}^2$

SNO + KamLAND + Solar: $\tan^2\theta_{12} \approx 0.47 \pm 0.05$

$$\rightarrow \boxed{\sin\theta_{12} \approx 0.56; \qquad \cos\theta_{12} \approx 0.82}$$

Atmospheric Neutrinos/Long Baseline experiments

MINOS: $|\Delta m^2_{32}| \approx (2.4 \pm 0.1) \times 10^{-3}\,\mathrm{eV}^2$

Super Kamiokande: $\sin^2 2\theta_{23} > 0.92$

$$\boxed{\cos\theta_{23} \approx \sin\theta_{23} \approx \frac{1}{\sqrt{2}}}$$

CHOOZ + (atmospheric)

$$\boxed{\sin^2 2\theta_{13} < 0.15}$$

2014 Daya Bay

$$\boxed{\sin^2 2\theta_{13} = 0.092 \pm 0.016}$$

★**Currently no knowledge about CP violating phase** δ

- **Regardless of uncertainty in** θ_{13}

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \approx \begin{pmatrix} c_{12} & s_{12} & ? \\ -s_{12}c_{23} & c_{12}c_{23} & s_{23} \\ s_{12}s_{23} & -c_{12}s_{23} & c_{23} \end{pmatrix}$$

- **For the approximate values of the mixing angles on the previous page obtain:**

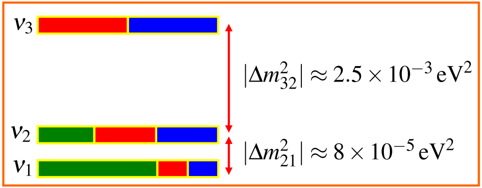
$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \approx \begin{pmatrix} 0.85 & 0.53 & 0.1e^{i\delta}? \\ -0.37 & 0.60 & 0.71 \\ 0.37 & -0.60 & 0.71 \end{pmatrix}$$

★ **Have approximate expressions for mass eigenstates in terms of weak eigenstates:**

$$|\nu_3\rangle \approx \frac{1}{\sqrt{2}}(|\nu_\mu\rangle + |\nu_\tau\rangle)$$

$$|\nu_2\rangle \approx 0.53|\nu_e\rangle + 0.60(|\nu_\mu\rangle - |\nu_\tau\rangle)$$

$$|\nu_1\rangle \approx 0.85|\nu_e\rangle - 0.37(|\nu_\mu\rangle - |\nu_\tau\rangle)$$



Final Words: Neutrino Masses

- Neutrino oscillations require non-zero neutrino masses
- But only determine mass-squared differences – not the masses themselves
- No direct measure of neutrino mass – only mass limits:

$$m_\nu(e) < 2\,\text{eV}; \quad m_\nu(\mu) < 0.17\,\text{MeV}; \quad m_\nu(\tau) < 18.2\,\text{MeV}$$

Note the e, μ, τ refer to charged lepton flavour in the experiment, e.g. $m_\nu(e) < 2\,\text{eV}$ refers to the limit from tritium beta-decay

- Also from cosmological evolution infer that the sum

$$\boxed{\sum_i m_{\nu_i} < \text{few eV}}$$

★ 10 years ago – assumed massless neutrinos + hints that neutrinos might oscillate !

★ Now, know a great deal about massive neutrinos

★ But many unknowns: θ_{13}, δ, mass hierarchy, absolute values of neutrino masses

★ Measurements of these SM parameters is the focus of the next generation of expts.

Appendix: 3-Flavour Treatment of Atmospheric Neutrinos

non-examinable

- ★ The energies of the detected atmospheric neutrinos are of order 1 GeV
- ★ The wavelength of oscillations associated with $|\Delta m^2_{21}| = 8 \times 10^{-5}\,\mathrm{eV}^2$ is

$$\lambda_{21} = 31000\,\mathrm{km}$$

$$\boxed{\lambda_{\mathrm{osc}}(\mathrm{km}) = 2.47\frac{E(\mathrm{GeV})}{\Delta m^2(\mathrm{eV}^2)}}$$

- If we neglect the corresponding term in the expression for $P(\nu_\mu \to \nu_\tau)$ - equation (16)

$$\begin{aligned} P(\nu_\mu \to \nu_\tau) &\approx -4U_{\mu 1}U_{\tau 1}U_{\mu 2}U_{\tau 2}\sin^2\Delta_{21} + 4U^2_{\mu 3}U^2_{\tau 3}\sin^2\Delta_{32} \\ &\approx 4U^2_{\mu 3}U^2_{\tau 3}\sin^2\Delta_{32} \\ &= 4\sin^2\theta_{23}\cos^2\theta_{23}\cos^4\theta_{13}\sin^2\Delta_{32} \\ &= \cos^4\theta_{13}\sin^2 2\theta_{23}\sin^2\Delta_{32} \end{aligned}$$

- The Super-Kamiokande data are consistent with $\nu_\mu \to \nu_\tau$ which excludes the possibility of $\cos^4\theta_{13}$ being small
- Hence the CHOOZ limit: $\sin^2 2\theta_{13} < 0.2$ can be interpreted as $\sin^2\theta_{13} < 0.05$

<u>NOTE:</u> the three flavour treatment of atmospheric neutrinos is discussed below. The oscillation parameters in nature conspire in such a manner that the two flavour treatment provides a good approximation of the observable effects of atmospheric neutrino oscillations

3-Flavour Treatment of Atmospheric Neutrinos

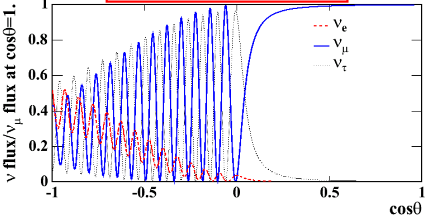
non-examinable

- Previously stated that the long-wavelength oscillations due to Δm^2_{21} have little effect on atmospheric neutrino oscillations because for a the wavelength for a 1 GeV neutrino is approx 30000 km.
- However, maximum oscillation probability occurs at $\lambda/2$
- This is not small compared to diameter of Earth and cannot be neglected
- As an example, take the oscillation parameters to be

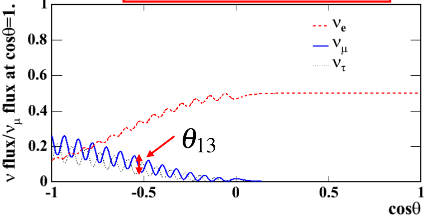
$$\theta_{12} = 32^\circ;\ \theta_{23} = 45^\circ;\ \theta_{13} = 7.5^\circ$$

- Predict neutrino flux as function of $\cos\theta$
- Consider what happens to muon and electron neutrinos separately

muon neutrinos only



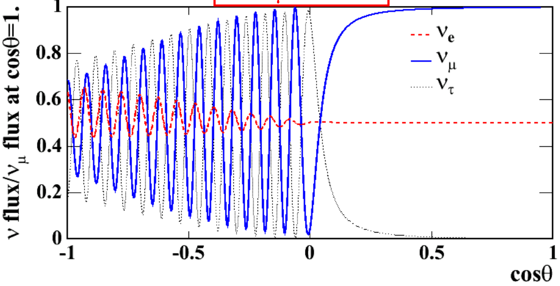
electron neutrinos only



- Δm^2_{21} has a big effect at $\cos\theta \sim -1$

non-examinable

- From previous page it is clear that the two neutrino treatment of oscillations of atmospheric muon neutrinos is a very poor approximation
- However, in atmosphere produce two muon neutrinos for every electron neutrino
- Need to consider the combined effect of oscillations on a mixed “beam” with both ν_μ and ν_e

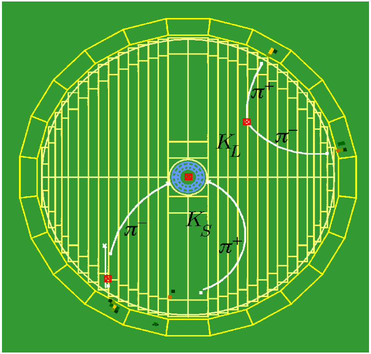


2/3 ν_μ + 1/3 ν_e

- At large distances the average muon neutrino flux is still approximately half the initial flux, but only because of the oscillations of the original electron neutrinos and the fact that $\sin^2 2\theta_{23} \sim 1$
- Because the atmospheric neutrino experiments do not resolve fine structure, the **observable** effects of oscillations approximated by two flavour formula

Particle Physics

Dr Lester



Handout 12 : The CKM Matrix and CP Violation

CP Violation in the Early Universe

- Very early in the universe might expect equal numbers of baryons and anti-baryons
- However, today the universe is matter dominated (no evidence for anti-galaxies, etc.)
- From "Big Bang Nucleosynthesis" obtain the matter/anti-matter asymmetry

$$\xi = \frac{n_B - n_{\overline{B}}}{n_\gamma} \approx \frac{n_B}{n_\gamma} \approx 10^{-9}$$

i.e. for every baryon in the universe today there are 10^9 photons

- How did this happen?

★ Early in the universe need to create a very small asymmetry between baryons and anti-baryons

e.g. for every 10^9 anti-baryons there were 10^9+1 baryons

baryons/anti-baryons annihilate ⇒

1 baryon + ~10^9 photons + no anti-baryons

★ To generate this initial asymmetry three conditions must be met (Sakharov, 1967):

❶ "Baryon number violation", i.e. $n_B - n_{\overline{B}}$ is not constant

❷ "C and CP violation", if CP is conserved for a reaction which generates a net number of baryons over anti-baryons there would be a CP conjugate reaction generating a net number of anti-baryons

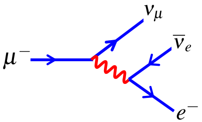
❸ "Departure from thermal equilibrium", in thermal equilibrium any baryon number violating process will be balanced by the inverse reaction

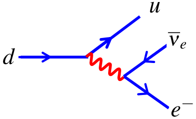
- CP Violation is an essential aspect of our understanding of the universe
- A natural question is whether the SM of particle physics can provide the necessary CP violation?
- There are two places in the SM where CP violation enters: the **PMNS matrix** and the **CKM matrix**
- **To date CP violation has been observed only in the quark sector**
- Because we are dealing with quarks, which are only observed as **bound states**, this is a fairly complicated subject. Here we will approach it in two steps:
  - i) Consider **particle – anti-particle oscillations** without CP violation
  - ii) Then discuss the effects of **CP violation**

★ Many features in common with neutrino oscillations – except that we will be considering the oscillations of decaying particles (i.e. mesons) !

The Weak Interaction of Quarks

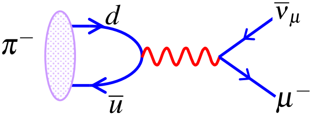
★ **Slightly different values of G_F measured in μ decay and nuclear β decay:**

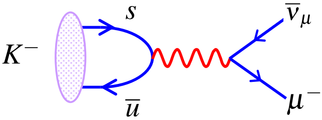




$$G_F^\mu = (1.16632 \pm 0.00002) \times 10^{-5}\,\mathrm{GeV}^{-2} \qquad G_F^\beta = (1.136 \pm 0.003) \times 10^{-5}\,\mathrm{GeV}^{-2}$$

★ **In addition, certain hadronic decay modes are observed to be suppressed, e.g. compare $K^- \to \mu^- \overline{\nu}_\mu$ and $\pi^- \to \mu^- \overline{\nu}_\mu$. Kaon decay rate suppressed factor 20 compared to the expectation assuming a universal weak interaction for quarks.**



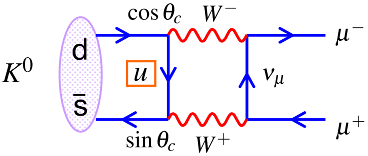


- **Both observations explained by Cabibbo hypothesis (1963): weak eigenstates are different from mass eigenstates, i.e. weak interactions of quarks have same strength as for leptons but a u-quark couples to a linear combination of s and d**

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

GIM Mechanism

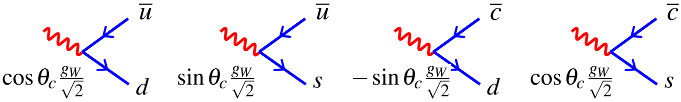
★ In the weak interaction have couplings between both ud and us which implies that neutral mesons can decay via box diagrams, e.g.



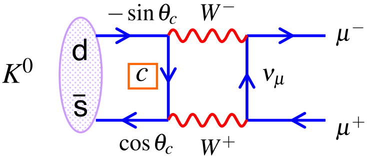
$$\boxed{M_1 \propto g_W^4 \cos\theta_c \sin\theta_c}$$

- Historically, the observed branching was **much smaller** than predicted

★ Led **Glashow**, **Illiopoulos** and **Maiani** to postulate existence of an extra quark - **before** discovery of charm quark in **1974**. Weak interaction couplings become



★ Gives another box diagram for $K^0 \rightarrow \mu^+\mu^-$



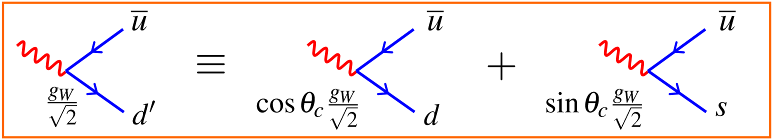
$$\boxed{M_2 \propto -g_W^4 \cos\theta_c \sin\theta_c}$$

- Same final state so sum amplitudes

$$|M|^2 = |M_1 + M_2|^2 \approx 0$$

- Cancellation not exact because $m_u \neq m_c$

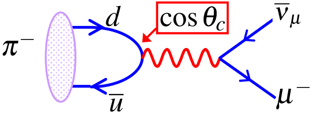
i.e. weak interaction couples different generations of quarks

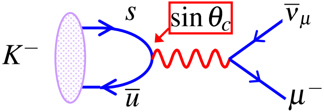


(The same is true for leptons e.g. $e^-\nu_1$, $e^-\nu_2$, $e^-\nu_3$ couplings – connect different generations)

★ Can explain the observations on the previous pages with $\theta_c = 13.1^\circ$

- Kaon decay suppressed by a factor of $\tan^2\theta_c \approx 0.05$ relative to pion decay

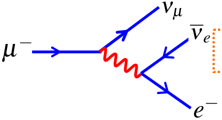




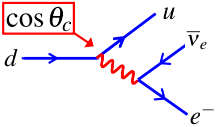
$$\Gamma(\pi^- \rightarrow \mu^-\overline{\nu}_\mu) \propto |M|^2 \propto \cos^2\theta_c$$

$$\Gamma(K^- \rightarrow \mu^-\overline{\nu}_\mu) \propto |M|^2 \propto \sin^2\theta_c$$

- Hence expect $G_F^\beta = G_F^\mu \cos\theta_c$



$\propto G_F.G_F$



$\propto G_F.(G_F\cos^2\theta_c)$

CKM Matrix

★ **Extend ideas to three quark flavours (analogue of three flavour neutrino treatment)**

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

By convention CKM matrix defined as acting on quarks with charge $-\frac{1}{3}e$

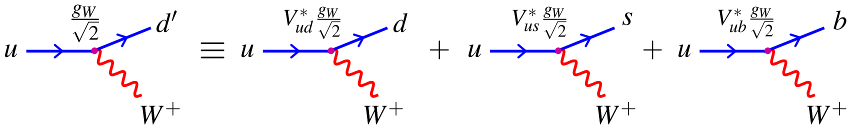
Weak eigenstates

CKM Matrix

Mass Eigenstates

(Cabibbo, Kobayashi, Maskawa)

★ **e.g. Weak eigenstate** d' **is produced in weak decay of an up quark:**

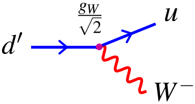


- **The CKM matrix elements** V_{ij} **are complex constants**
- **The CKM matrix is unitary**
- **The** V_{ij} **are not predicted by the SM – have to determined from experiment**

Feynman Rules

- Depending on the order of the interaction, $u \rightarrow d$ or $d \rightarrow u$, the CKM matrix enters as either V_{ud} or V^*_{ud}
- Writing the interaction in terms of the WEAK eigenstates

NOTE: $\overline{u}$ is the adjoint spinor not the anti-up quark

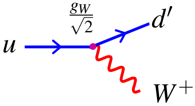


$$j_{d'u} = \overline{u}\left[-i\frac{g_W}{\sqrt{2}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\right]d'$$

- Giving the $\boxed{d \rightarrow u}$ weak current:

$$\boxed{j_{du} = \overline{u}\left[-i\frac{g_W}{\sqrt{2}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\right]V_{ud}d}$$

- For $u \rightarrow d'$ the weak current is:



$$j_{ud'} = \overline{d}'\left[-i\frac{g_W}{\sqrt{2}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\right]u$$

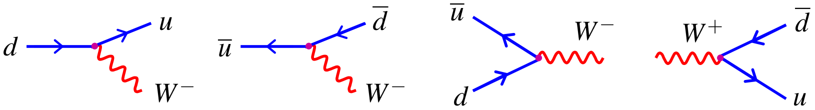
- In terms of the mass eigenstates $\overline{d}' = d'^\dagger\gamma^0 \rightarrow (V_{ud}d)^\dagger\gamma^0 = V^*_{ud}d^\dagger\gamma^0 = V^*_{ud}\overline{d}$

- Giving the $\boxed{u \rightarrow d}$ weak current:

$$\boxed{j_{ud} = \overline{d}V^*_{ud}\left[-i\frac{g_W}{\sqrt{2}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\right]u}$$

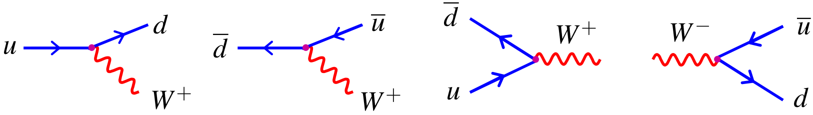
- Hence, when the charge $-\frac{1}{3}$ quark enters as the adjoint spinor, the complex conjugate of the CKM matrix is used

★ **The vertex factor the following diagrams:**



is $$\boxed{-i\frac{g_W}{\sqrt{2}}V_{ud}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)}$$

★ **Whereas, the vertex factor for:**



is $$\boxed{-i\frac{g_W}{\sqrt{2}}V^*_{ud}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)}$$

★ Experimentally (see Appendix I) determine

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} \approx \begin{pmatrix} 0.974 & 0.226 & 0.004 \\ 0.23 & 0.96 & 0.04 \\ ? & ? & ? \end{pmatrix}$$

★ Currently little direct experimental information on V_{td}, V_{ts}, V_{tb}

★ Assuming unitarity of CKM matrix, e.g. $|V_{ub}|^2 + |V_{cb}|^2 + |V_{tb}|^2 = 1$
gives:

Cabibbo matrix

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} \approx \begin{pmatrix} 0.974 & 0.226 & 0.004 \\ 0.23 & 0.96 & 0.04 \\ 0.01 & 0.04 & 0.999 \end{pmatrix}$$

Near diagonal – very different from PMNS

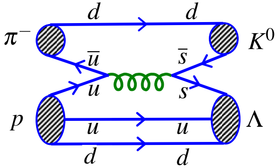
★ NOTE: within the SM, the charged current, $W^{\pm}$, weak interaction:
① Provides the only way to change flavour !
② only way to change from one generation of quarks or leptons to another !

★ However, the off-diagonal elements of the CKM matrix are relatively small.

- Weak interaction largest between quarks of the same generation.
- Coupling between first and third generation quarks is very small !

★ Just as for the PMNS matrix – the CKM matrix allows CP violation in the SM

The Neutral Kaon System

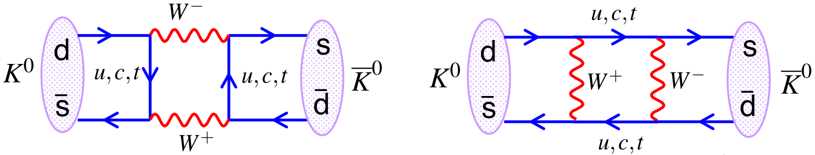


- **Neutral Kaons** are **produced** copiously in **strong interactions**, e.g.

$$\pi^-(d\bar{u}) + p(uud) \rightarrow \Lambda(uds) + K^0(d\bar{s})$$

$$\pi^+(u\bar{d}) + p(uud) \rightarrow K^+(u\bar{s}) + \overline{K}^0(s\bar{d}) + p(uud)$$

- **Neutral Kaons decay** via the **weak interaction**
- The Weak Interaction also allows **mixing** of neutral kaons via **"box diagrams"**



- This allows **transitions** between the strong eigenstates states $K^0, \overline{K}^0$
- Consequently, the neutral kaons propagate as eigenstates of the overall strong + weak interaction **(Appendix II)**; i.e. as linear combinations of $K^0, \overline{K}^0$
- These neutral kaon states are called the **"K-short"** K_S and the **"K-long"** K_L
- These states have approximately the same mass $m(K_S) \approx m(K_L) \approx 498\,\text{MeV}$
- But very different lifetimes: $\boxed{\tau(K_S) = 0.9 \times 10^{-10}\,\text{s}}$ $\boxed{\tau(K_L) = 0.5 \times 10^{-7}\,\text{s}}$

CP Eigenstates

★The K_S and K_L are closely related to eigenstates of the combined charge conjugation and parity operators: CP

- The strong eigenstates $K^0(d\bar{s})$ and $\overline{K}^0(s\bar{d})$ have $J^P = 0^-$

with
$$\hat{P}|K^0\rangle = -|K^0\rangle, \quad \hat{P}|\overline{K}^0\rangle = -|\overline{K}^0\rangle$$

- The charge conjugation operator changes particle into anti-particle and *vice versa*

$$\hat{C}|K^0\rangle = \hat{C}|d\bar{s}\rangle = +|s\bar{d}\rangle = |\overline{K}^0\rangle$$

similarly
$$\hat{C}|\overline{K}^0\rangle = |K^0\rangle$$

The + sign is purely conventional, could have used a - with no physical consequences

- Consequently

$$\hat{C}\hat{P}|K^0\rangle = -|\overline{K}^0\rangle \qquad \hat{C}\hat{P}|\overline{K}^0\rangle = -|K^0\rangle$$

i.e. neither K^0 or $\overline{K}^0$ are eigenstates of CP

- Form CP eigenstates from linear combinations:

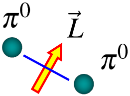
$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\overline{K}^0\rangle)$$
$$|K_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\overline{K}^0\rangle)$$

$$\hat{C}\hat{P}|K_1\rangle = +|K_1\rangle$$
$$\hat{C}\hat{P}|K_2\rangle = -|K_2\rangle$$

Decays of CP Eigenstates

- Neutral kaons often decay to pions (the lightest hadrons)
- The kaon masses are approximately **498 MeV** and the pion masses are approximately **140 MeV**. Hence neutral kaons can decay to either 2 or 3 pions

Decays to Two Pions:



☆ $K^0 \rightarrow \pi^0\pi^0 \qquad J^P: \quad 0^- \rightarrow 0^- + 0^-$

- Conservation of angular momentum ➡ $\vec{L} = 0$

$$\Rightarrow \quad \hat{P}(\pi^0\pi^0) = -1.-1.(-1)^L = +1$$

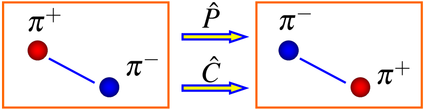
- The $\pi^0 = \frac{1}{\sqrt{2}}(u\overline{u} - d\overline{d})$ is an eigenstate of $\hat{C}$

$$C(\pi^0\pi^0) = C\pi^0.C\pi^0 = +1.+1 = +1$$

$$\Rightarrow \quad \boxed{CP(\pi^0\pi^0) = +1}$$

☆ $K^0 \rightarrow \pi^+\pi^-$ as before $\hat{P}(\pi^+\pi^-) = +1$

★ Here the **C** and **P** operations have the identical effect



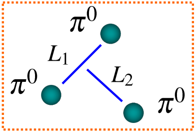
Hence the combined effect of $\hat{C}\hat{P}$ is to leave the system unchanged

$$\boxed{\hat{C}\hat{P}(\pi^+\pi^-) = +1}$$

Neutral kaon decays to two pions occur in CP even (i.e. +1) eigenstates

Decays to Three Pions:

★ $K^0 \to \pi^0\pi^0\pi^0$ $\qquad J^P: \quad 0^- \to 0^- + 0^- + 0^-$



- **Conservation of angular momentum:**

$$L_1 \oplus L_2 = 0 \quad \Rightarrow \quad L_1 = L_2$$

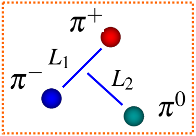
Remember L is magnitude of angular momentum vector

$$P(\pi^0\pi^0\pi^0) = -1.-1.-1.(-1)^{L_1}.(-1)^{L_2} = -1$$

$$C(\pi^0\pi^0\pi^0) = +1.+1.+1$$

$$\Rightarrow \quad \boxed{CP(\pi^0\pi^0\pi^0) = -1}$$

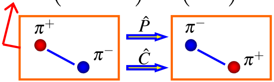
★ $K^0 \to \pi^+\pi^-\pi^0$



- **Again** $\quad L_1 = L_2$

$$P(\pi^+\pi^-\pi^0) = -1.-1.-1.(-1)^{L_1}.(-1)^{L_2} = -1$$

$$C(\pi^+\pi^-\pi^0) = +1.C(\pi^+\pi^-) = P(\pi^+\pi^-) = (-1)^{L_1}$$



Hence: $\quad \boxed{CP(\pi^+\pi^-\pi^0) = -1.(-1)^{L_1}}$

- **The small amount of energy available in the decay,** $m(K) - 3m(\pi) \approx 70\,\text{MeV}$ **means that the L>0 decays are strongly suppressed by the angular momentum barrier effects (recall QM tunnelling in alpha decay)**

Neutral kaon decays to three pions occur in CP odd (i.e. -1) eigenstates

- ★ **If** CP were conserved in the Weak decays of neutral kaons, would expect decays to pions to occur from states of definite CP (i.e. the CP eigenstates K_1 , K_2)

$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\overline{K}^0\rangle) \qquad \hat{C}\hat{P}|K_1\rangle = +|K_1\rangle \qquad K_1 \rightarrow \pi\pi$$

CP EVEN

$$|K_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\overline{K}^0\rangle) \qquad \hat{C}\hat{P}|K_2\rangle = -|K_2\rangle \qquad K_2 \rightarrow \pi\pi\pi$$

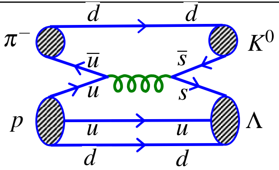
CP ODD

- ★ Expect lifetimes of CP eigenstates to be very different
  - For two pion decay energy available: $m_K - 2m_\pi \approx 220\,\mathrm{MeV}$
  - For three pion decay energy available: $m_K - 3m_\pi \approx 80\,\mathrm{MeV}$
- ★ Expect decays to two pions to be more rapid than decays to three pions due to increased phase space
- ★ This is exactly what is observed: a short-lived state **"K-short"** which decays to (mainly) to two pions and a long-lived state **"K-long"** which decays to three pions
- ★ **In the absence of CP violation we can identify**

$$|K_S\rangle = |K_1\rangle \equiv \frac{1}{\sqrt{2}}(|K^0\rangle - |\overline{K}^0\rangle) \qquad \text{with decays:} \qquad K_S \rightarrow \pi\pi$$

$$|K_L\rangle = |K_2\rangle \equiv \frac{1}{\sqrt{2}}(|K^0\rangle + |\overline{K}^0\rangle) \qquad \text{with decays:} \qquad K_L \rightarrow \pi\pi\pi$$

Neutral Kaon Decays to pions



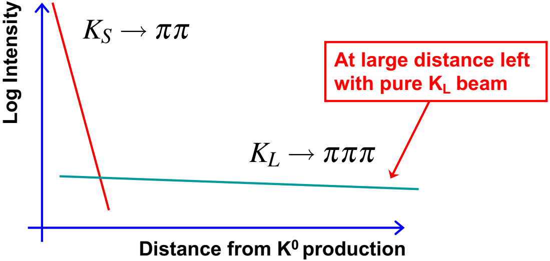
- Consider the decays of a beam of K^0
- The decays to pions occur in states of definite CP
- If CP is conserved in the decay, need to express K^0 in terms of K_S and K_L

$$|K_0\rangle = \frac{1}{\sqrt{2}}(|K_S\rangle + |K_L\rangle)$$

- Hence from the point of view of decays to pions, a K^0 beam is a linear combination of CP eigenstates:

a rapidly decaying CP-even component and a long-lived CP-odd component

- Therefore, expect to see predominantly two-pion decays near start of beam and predominantly three pion decays further downstream



At large distance left with pure K_L beam

★To see how this works algebraically:

- Suppose at time t=0 make a beam of pure K^0

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|K_S\rangle + |K_L\rangle)$$

- Put in the time dependence of wave-function

$$|K_S(t)\rangle = |K_S\rangle e^{-im_S t - \Gamma_S t/2}$$

K

NOTE the term $e^{-\Gamma_S t/2}$ ensures the K$_S$ probability density decays exponentially

i.e. $|\psi_S|^2 = \langle K_S(t)|K_S(t)\rangle = e^{-\Gamma_S t} = e^{-t/\tau_S}$

- Hence wave-function evolves as

$$|\psi(t)\rangle \quad = \quad \frac{1}{\sqrt{2}}\left[|K_S\rangle e^{-(im_S+\frac{\Gamma_S}{2})t} + |K_L\rangle e^{-(im_L+\frac{\Gamma_L}{2})t}\right]$$

- Writing $\theta_S(t) = e^{-(im_S+\frac{\Gamma_S}{2})t}$ and $\theta_L(t) = e^{-(im_L+\frac{\Gamma_L}{2})t}$

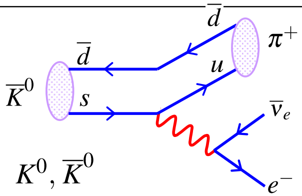
$$|\psi(t)\rangle \quad = \quad \frac{1}{\sqrt{2}}(\theta_S(t)|K_S\rangle + \theta_L(t)|K_L\rangle)$$

- The **decay rate to two pions** for a state which was produced as K^0:

$$\Gamma(K^0_{t=0} \to \pi\pi) \propto |\langle K_S|\psi(t)\rangle|^2 \propto |\theta_S(t)|^2 = e^{-\Gamma_S t} = e^{-t/\tau_S}$$

which is as anticipated, i.e. decays of the short lifetime component **K$_S$**

Neutral Kaon Decays to Leptons



- Neutral kaons can also decay to leptons

$$\overline{K}^0 \to \pi^+ e^- \overline{\nu}_e \qquad \overline{K}^0 \to \pi^+ \mu^- \overline{\nu}_\mu$$
$$K^0 \to \pi^- e^+ \nu_e \qquad K^0 \to \pi^- \mu^+ \nu_\mu$$

- **Note:** the final states are not CP eigenstates which is why we express these decays in terms of $K^0, \overline{K}^0$

- Neutral kaons propagate as combined eigenstates of weak + strong interaction i.e. the K_S, K_L . The main decay modes/branching fractions are:

K_S	$\to$	$\pi^+\pi^-$	$BR = 69.2\%$
	$\to$	$\pi^0\pi^0$	$BR = 30.7\%$
	$\to$	$\pi^- e^+ \nu_e$	$BR = 0.03\%$
	$\to$	$\pi^+ e^- \overline{\nu}_e$	$BR = 0.03\%$
	$\to$	$\pi^- \mu^+ \nu_\mu$	$BR = 0.02\%$
	$\to$	$\pi^+ \mu^- \overline{\nu}_\mu$	$BR = 0.02\%$

K_L	$\to$	$\pi^+\pi^-\pi^0$	$BR = 12.6\%$
	$\to$	$\pi^0\pi^0\pi^0$	$BR = 19.6\%$
	$\to$	$\pi^- e^+ \nu_e$	$BR = 20.2\%$
	$\to$	$\pi^+ e^- \overline{\nu}_e$	$BR = 20.2\%$
	$\to$	$\pi^- \mu^+ \nu_\mu$	$BR = 13.5\%$
	$\to$	$\pi^+ \mu^- \overline{\nu}_\mu$	$BR = 13.5\%$

- Leptonic decays are more likely for the K-long because the three pion decay modes have a lower decay rate than the two pion modes of the K-short

Strangeness Oscillations (neglecting CP violation)

- The "semi-leptonic" decay rate to $\pi^- e^+ \nu_e$ occurs from the K^0 state. Hence to calculate the expected decay rate, need to know the K^0 component of the wave-function. For example, for a beam which was initially K^0 we have **(1)**

$$|\psi(t)\rangle \quad = \quad \frac{1}{\sqrt{2}}(\theta_S(t)|K_S\rangle + \theta_L(t)|K_L\rangle)$$

- Writing K_S, K_L in terms of $K^0, \overline{K}^0$

$$\begin{aligned} |\psi(t)\rangle &= \tfrac{1}{2}\left[\theta_S(t)(|K^0\rangle - |\overline{K}^0\rangle) + \theta_L(t)(|K^0\rangle + |\overline{K}^0\rangle)\right] \\ &= \tfrac{1}{2}(\theta_S + \theta_L)|K^0\rangle + \tfrac{1}{2}(\theta_L - \theta_S)|\overline{K}^0\rangle \end{aligned}$$

- Because $\theta_S(t) \neq \theta_L(t)$ a state that was initially a K^0 evolves with time into a mixture of K^0 and $\overline{K}^0$ - "strangeness oscillations"

- The K^0 intensity (i.e. K^0 fraction):

$$\Gamma(K^0_{t=0} \to K^0) = |\langle K^0|\psi(t)\rangle|^2 = \tfrac{1}{4}|\theta_S + \theta_L|^2 \qquad \textbf{(2)}$$

- Similarly $$\Gamma(K^0_{t=0} \to \overline{K}^0) = |\langle \overline{K}^0|\psi(t)\rangle|^2 = \tfrac{1}{4}|\theta_S - \theta_L|^2 \qquad \textbf{(3)}$$

- Using the identity $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\Re(z_1 z_2^*)$

$$\begin{aligned}
|\theta_S \pm \theta_L|^2 &= |e^{-(im_S+\frac{1}{2}\Gamma_S)t} \pm e^{-(im_L+\frac{1}{2}\Gamma_L)t}|^2 \\
&= e^{-\Gamma_S t} + e^{-\Gamma_L t} \pm 2\Re\{e^{-im_S t} e^{-\frac{1}{2}\Gamma_S t}.e^{+im_L t} e^{-\frac{1}{2}\Gamma_L t}\} \\
&= e^{-\Gamma_S t} + e^{-\Gamma_L t} \pm 2e^{-\frac{\Gamma_S+\Gamma_L}{2}t}\Re\{e^{-i(m_S-m_L)t}\} \\
&= e^{-\Gamma_S t} + e^{-\Gamma_L t} \pm 2e^{-\frac{\Gamma_S+\Gamma_L}{2}t}\cos(m_S-m_L)t \\
&= e^{-\Gamma_S t} + e^{-\Gamma_L t} \pm 2e^{-\frac{\Gamma_S+\Gamma_L}{2}t}\cos\Delta m t
\end{aligned}$$

- Oscillations between neutral kaon states with frequency given by the mass splitting $\Delta m = m(K_L) - m(K_S)$
- Reminiscent of neutrino oscillations ! Only this time we have decaying states.
- Using equations (2) and (3):

$$\Gamma(K^0_{t=0} \to K^0) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right] \qquad (4)$$

$$\Gamma(K^0_{t=0} \to \overline{K}^0) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right] \qquad (5)$$

- **Experimentally we find:** $\boxed{\tau(K_S) = 0.9 \times 10^{-10}\,\mathrm{s}}$ $\boxed{\tau(K_L) = 0.5 \times 10^{-7}\,\mathrm{s}}$

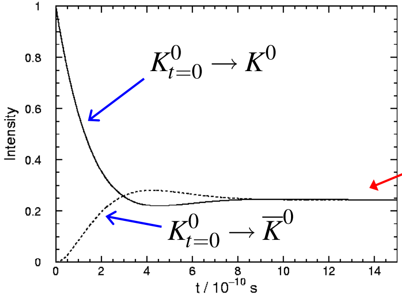
  and $\boxed{\Delta m = (3.506 \pm 0.006) \times 10^{-15}\,\mathrm{GeV}}$

  i.e. the K-long mass is greater than the K-short by 1 part in 10^{16}

- **The mass difference corresponds to an oscillation period of**

$$T_{osc} = \frac{2\pi\hbar}{\Delta m} \approx 1.2 \times 10^{-9}\,\mathrm{s}$$

- **The oscillation period is relatively long compared to the K$_S$ lifetime and consequently, do not observe very pronounced oscillations**

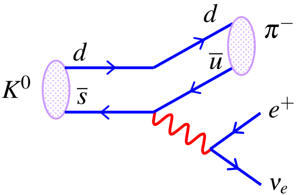


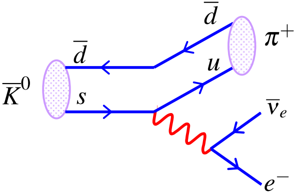
$$\Gamma(K^0_{t=0} \to K^0) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma(K^0_{t=0} \to \overline{K}^0) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

After a few K$_S$ lifetimes, left with a pure K$_L$ beam which is half K^0 and half $\overline{\mathrm{K}}^0$

★ **Strangeness oscillations can be studied by looking at semi-leptonic decays**





★ **The charge of the observed pion (or lepton) tags the decay as from either a $\overline{K}^0$ or K^0 because**

$$K^0 \to \pi^- e^+ \nu_e \qquad \overline{K}^0 \not\to \pi^- e^+ \nu_e$$
$$\overline{K}^0 \to \pi^+ e^- \overline{\nu}_e \qquad K^0 \not\to \pi^+ e^- \overline{\nu}_e$$

but … **NOT ALLOWED** **(see Question 23)**

•**So for an initial K^0 beam, observe the decays to both charge combinations:**

$$K^0_{t=0} \to K^0 \qquad\qquad K^0_{t=0} \to \overline{K}^0$$

↳ $\pi^- e^+ \nu_e$ ↳ $\pi^+ e^- \overline{\nu}_e$

which provides a way of measuring strangeness oscillations

The CPLEAR Experiment

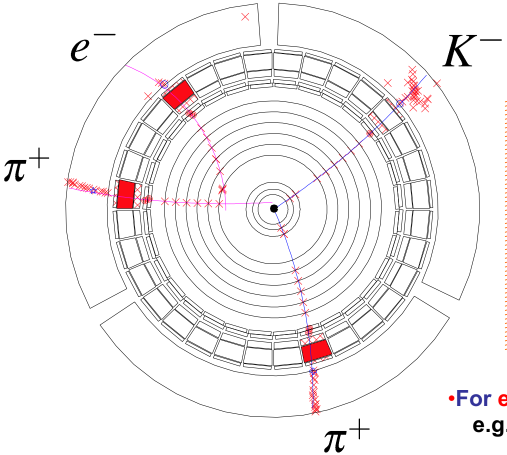
- **CERN : 1990-1996**
- **Used a low energy anti-proton beam**
- **Neutral kaons produced in reactions**

$$\overline{p}p \rightarrow K^-\pi^+K^0$$

$$\overline{p}p \rightarrow K^+\pi^-\overline{K}^0 \qquad \text{(Question 24)}$$

- **Low energy, so particles produced almost at rest**
- **Observe production process and decay in the same detector**
- **Charge of $K^\pm\pi^\mp$ in the production process tags the initial neutral kaon as either K^0 or $\overline{K}^0$**
- **Charge of decay products tags the decay as either as being either K^0 or $\overline{K}^0$**
- **Provides a direct probe of strangeness oscillations**

An example of a CPLEAR event



$K^-(s\bar{u})$
$K^0(d\bar{s})$
$\overline{K}^0(s\bar{d})$

Production:

$$\bar{p}p \rightarrow \boxed{K^-\pi^+} K^0$$

Mixing

Decay:

$$\overline{K}^0 \rightarrow \boxed{\pi^+ e^-} \overline{\nu}_e$$

- **For each event know initial wave-function,**
  e.g. here: $|\psi(t=0)\rangle = |K^0\rangle$

- Can measure decay rates as a function of time for all combinations:

e.g. $R^+ = \Gamma(K^0_{t=0} \to \pi^- e^+ \overline{\nu}_e) \propto \Gamma(K^0_{t=0} \to K^0)$

- From equations (4), (5) and similar relations:

$$R_+ \equiv \Gamma(K^0_{t=0} \to \pi^- e^+ \nu_e) = N_{\pi e \nu} \tfrac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$R_- \equiv \Gamma(K^0_{t=0} \to \pi^+ e^- \overline{\nu}_e) = N_{\pi e \nu} \tfrac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\overline{R}_- \equiv \Gamma(\overline{K}^0_{t=0} \to \pi^+ e^- \overline{\nu}_e) = N_{\pi e \nu} \tfrac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\overline{R}_+ \equiv \Gamma(\overline{K}^0_{t=0} \to \pi^- e^+ \nu_e) = N_{\pi e \nu} \tfrac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

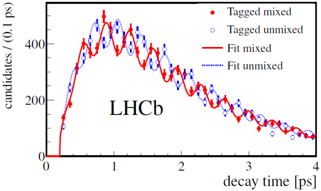
where $N_{\pi e \nu}$ is some overall normalisation factor

- Express measurements as an “asymmetry” to remove dependence on $N_{\pi e \nu}$

$$A_{\Delta m} = \frac{(R_+ + \overline{R}_-) - (R_- + \overline{R}_+)}{(R_+ + \overline{R}_-) + (R_- + \overline{R}_+)}$$

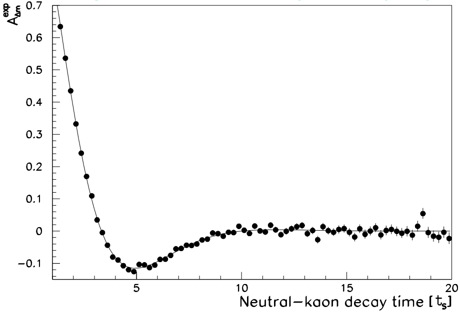
- Using the above expressions for R_+ etc., obtain

$$A_{\Delta m} = \frac{2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t}{e^{-\Gamma_S t}+e^{-\Gamma_L t}}$$



(See also B^0_s-B^0_s bar mixing plots in arXiv:1304.4741)

A. Angelopoulos et al., Eur. Phys. J. C22 (2001) 55



- ★ **Points show the data**
- ★ **The line shows the theoretical prediction for the value of Δm most consistent with the CPLEAR data:**

$$\boxed{\Delta m = 3.485 \times 10^{-15}\,\mathrm{GeV}}$$

- The sign of Δm is not determined here but is known from other experiments
- When the CPLEAR results are combined with experiments at FermiLab obtain:

$$\boxed{\Delta m = m(K_L) - m(K_S) = (3.506 \pm 0.006) \times 10^{-15}\,\mathrm{GeV}}$$

CP Violation in the Kaon System

★ **So far we have ignored CP violation in the neutral kaon system**

★ **Identified the K-short as the CP-even state and the K-long as the CP-odd state**

$$|K_S\rangle = |K_1\rangle \equiv \frac{1}{\sqrt{2}}(|K^0\rangle - |\overline{K}^0\rangle)$$ **with decays:** $K_S \to \pi\pi$ **CP = +1**

$$|K_L\rangle = |K_2\rangle \equiv \frac{1}{\sqrt{2}}(|K^0\rangle + |\overline{K}^0\rangle)$$ **with decays:** $K_L \to \pi\pi\pi$ **CP = -1**

★ **At a long distance from the production point a beam of neutral kaons will be 100% K-long (the K-short component will have decayed away). Hence, if CP is conserved, would expect to see only three-pion decays.**

★ **In 1964 Fitch & Cronin (joint Nobel prize) observed 45 $K_L \to \pi^+\pi^-$ decays in a sample of 22700 kaon decays a long distance from the production point**

⇨ **Weak interactions violate CP**

•**CP is violated in hadronic weak interactions, but only at the level of 2 parts in 1000**

K_L to pion BRs:

K_L	$\to$	$\pi^+\pi^-\pi^0$	$BR = 12.6\%$	$CP = -1$
	$\to$	$\pi^0\pi^0\pi^0$	$BR = 19.6\%$	$CP = -1$
	$\to$	$\pi^+\pi^-$	$BR = 0.20\%$	$CP = +1$
	$\to$	$\pi^0\pi^0$	$BR = 0.08\%$	$CP = +1$

★ **Two possible explanations of CP violation in the kaon system:**

i) The K_S and K_L do not correspond exactly to the CP eigenstates K_1 and K_2

$$\boxed{|K_S\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_1\rangle + \varepsilon|K_2\rangle\right]} \qquad \boxed{|K_L\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_2\rangle + \varepsilon|K_1\rangle\right]}$$

with $|\varepsilon| \sim 2\times 10^{-3}$

- In this case the observation of $K_L \rightarrow \pi\pi$ is accounted for by:

$$|K_L\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_2\rangle + \varepsilon|K_1\rangle\right]$$

$|K_1\rangle \rightarrow \pi\pi$ **CP = +1**

$|K_2\rangle \rightarrow \pi\pi\pi$ **CP = -1**

ii) and/or CP is violated in the decay

$|K_L\rangle = |K_2\rangle$ **CP = -1**

$\rightarrow \pi\pi\pi$ **CP = -1**

$\rightarrow \pi\pi$ **CP = +1**

Parameterised by ε'

★ Experimentally both known to contribute to the mechanism for CP violation in the kaon system but <u>i)</u> dominates: $\varepsilon'/\varepsilon = (1.7\pm0.3)\times 10^{-3}$ { NA48 (CERN) / KTeV (FermiLab)

★ The dominant mechanism is discussed in **non-examinable** Appendix III

CP Violation in Semi-leptonic decays

- If observe a neutral kaon beam a long time after production (i.e. a large distances) it will consist of a pure **K$_L$** component

$$|K_L\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)\underset{\hookrightarrow \pi^- e^+ \nu_e}{|K_0\rangle} + (1-\varepsilon)\underset{\hookrightarrow \pi^+ e^- \overline{\nu}_e}{|\overline{K}^0\rangle}\right]$$

- Decays to $\pi^- e^+ \nu_e$ must come from the K^0 component, and decays to $\pi^+ e^- \overline{\nu}_e$ must come from the $\overline{K}^0$ component

$$\Gamma(K_L \to \pi^+ e^- \overline{\nu}_e) \propto |\langle \overline{K}^0 | K_L \rangle|^2 \propto |1-\varepsilon|^2 \approx 1 - 2\Re\{\varepsilon\}$$

$$\Gamma(K_L \to \pi^- e^+ \nu_e) \propto |\langle K^0 | K_L \rangle|^2 \propto |1+\varepsilon|^2 \approx 1 + 2\Re\{\varepsilon\}$$

- Results in a small difference in decay rates: the decay to $\pi^- e^+ \nu_e$ is **0.7 % more likely** than the decay to $\pi^+ e^- \overline{\nu}_e$
  - This difference has been observed and thus provides the first direct evidence for an absolute difference between matter and anti-matter.

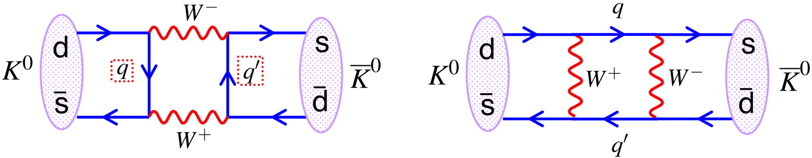
- It also provides an unambiguous definition of matter which could, for example, be transmitted to aliens in a distant galaxy

> "The electrons in our atoms have the same charge as those emitted least often in the decays of the long-lived neutral kaon"

CP Violation and the CKM Matrix

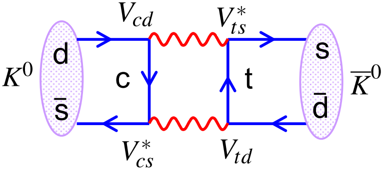
★ How can we explain $\Gamma(\overline{K}^0_{t=0} \to K^0) \neq \Gamma(K^0_{t=0} \to \overline{K}^0)$ in terms of the CKM matrix ?

★Consider the box diagrams responsible for mixing, i.e.



where $q = \{u, c, t\}, \quad q' = \{u, c, t\}$

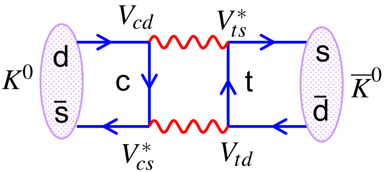
★ Have to sum over all possible quark exchanges in the box. For simplicity consider just one diagram



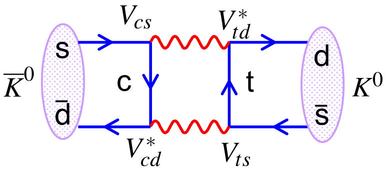
$$M_{fi} \propto A_{ct} V_{cd} V^*_{cs} V_{td} V^*_{ts}$$

A constant related to integrating over virtual momenta

- ★ Compare the equivalent box diagrams for $K^0 \to \overline{K}^0$ and $\overline{K}^0 \to K^0$



$$M_{fi} \propto A_{ct} V_{cd} V^*_{cs} V_{td} V^*_{ts}$$



$$M'_{fi} \propto A_{ct} V^*_{cd} V_{cs} V^*_{td} V_{ts} = M^*_{fi}$$

- ★ Can be shown that CP violation is driven by terms like $M_{fi} - M^*_{fi} = 2\Im\{M_{fi}\}$
- ★ Hence the rates can only be different if the CKM matrix has imaginary component

$$|\varepsilon| \propto \Im\{M_{fi}\}$$

- ★ A more formal derivation is given in **Thomson's "Modern Particle Physics", chap 14.**
- ★ In the kaon system we can show (question 25)

$$|\varepsilon| \propto A_{ut}.\Im\{V_{ud}V^*_{us}V_{td}V^*_{ts}\} + A_{ct}.\Im\{V_{cd}V^*_{cs}V_{td}V^*_{ts}\} + A_{tt}.\Im\{V_{td}V^*_{ts}V_{td}V^*_{ts}\}$$

Shows that CP violation is related to the imaginary parts of the CKM matrix

Summary

- ★ The weak interactions of quarks are described by the **CKM** matrix
- ★ Similar structure to the lepton sector, although unlike the **PMNS** matrix, the **CKM** matrix is nearly diagonal
- ★ CP violation enters through via a complex phase in the **CKM** matrix
- ★ A great deal of experimental evidence for CP violation in the weak interactions of quarks
- ★ CP violation is needed to explain matter – anti-matter asymmetry in the Universe
- ★ **HOWEVER**, CP violation in the SM is **not sufficient** to explain the matter – anti-matter asymmetry. There is probably another mechanism.

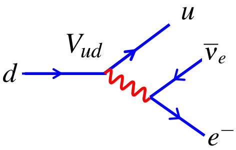
Appendix I: Determination of the CKM Matrix

Non-examinable

- The experimental determination of the **CKM matrix** elements comes mainly from measurements of leptonic decays (the leptonic part is well understood).
- It is easy to produce/observe meson decays, however theoretical uncertainties associated with the decays of bound states often limits the precision
- Contrast this with the measurements of the **PMNS matrix**, where there are few theoretical uncertainties and the experimental difficulties in dealing with neutrinos limits the precision.

➊ **$|V_{ud}|$** **from nuclear beta decay**

$$\begin{pmatrix} \times & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$



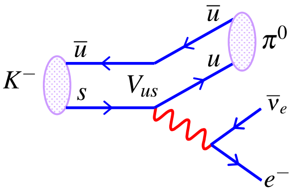
Super-allowed $0^+ \rightarrow 0^+$ beta decays are relatively free from theoretical uncertainties

$$\Gamma \propto |V_{ud}|^2$$

$$\boxed{|V_{ud}| = 0.97377 \pm 0.00027} \qquad (\approx \cos\theta_c)$$

❷ **$|V_{us}|$** **from semi-leptonic kaon decays**

$$\begin{pmatrix} \cdot & \times & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$



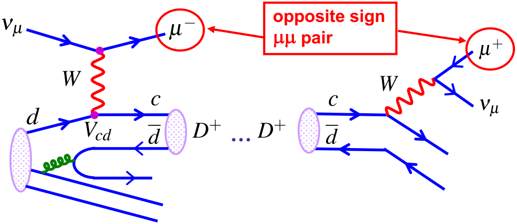
$$\Gamma \propto |V_{us}|^2$$

$$\boxed{|V_{us}| = 0.2257 \pm 0.0021} \qquad (\approx \sin\theta_c)$$

❸ **$|V_{cd}|$** **from neutrino scattering** $\nu_\mu + N \to \mu^+\mu^- X$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \times & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

Look for opposite charge di-muon events in ν_μ scattering from production and decay of a $D^+(c\bar{d})$ meson


opposite sign μμ pair

$$\text{Rate} \propto \underbrace{|V_{cd}|^2 \text{Br}(D^+ \to X\mu^+\nu_\mu)}$$

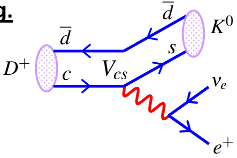
Measured in various collider experiments

$$\Rightarrow \quad \boxed{|V_{cd}| = 0.230 \pm 0.011}$$

❹ **$|V_{cs}|$** **from semi-leptonic charmed meson decays**

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \times & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

e.g.



$$\Gamma \propto |V_{cs}|^2$$

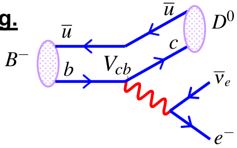
- **Precision limited by theoretical uncertainties**

$$\boxed{|V_{cs}| = 0.957 \pm \underbrace{0.017}_{\text{experimental error}} \pm \underbrace{0.093}_{\text{theory uncertainty}}}$$

❺ **$|V_{cb}|$** **from semi-leptonic B hadron decays**

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \times \\ \cdot & \cdot & \cdot \end{pmatrix}$$

e.g.



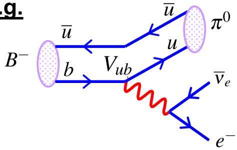
$$\Gamma \propto |V_{cb}|^2$$

$$\boxed{|V_{cb}| = 0.0416 \pm 0.0006}$$

❻ **$|V_{ub}|$** **from semi-leptonic B hadron decays**

$$\begin{pmatrix} \cdot & \cdot & \times \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

e.g.



$$\Gamma \propto |V_{ub}|^2$$

$$\boxed{|V_{ub}| = 0.0043 \pm 0.0003}$$

Appendix II: Particle – Anti-Particle Mixing

Non-examinable

- The wave-function for a single particle with lifetime $\tau = 1/\Gamma$ evolves with time as:

$$\psi(t) = Ne^{-\Gamma t/2} e^{-iMt}$$

which gives the appropriate exponential decay of

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\psi(0)\rangle e^{-t/\tau}$$

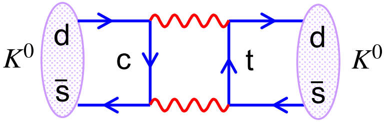
- The wave-function satisfies the time-dependent wave equation:

$$\hat{H}|\psi(t)\rangle = \boxed{(M - \tfrac{1}{2}i\Gamma)}|\psi(t)\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle \qquad \textbf{(A1)}$$

- For a bound state such as a K^0 the mass term includes the "mass" from the weak interaction "potential" $\hat{H}_{\text{weak}}$

$$M = m_{K^0} + \langle K^0|\hat{H}_{\text{weak}}|K^0\rangle + \sum_j \frac{|\langle K^0|\hat{H}_{\text{weak}}|j\rangle|^2}{m_{K^0} - E_j}$$

Sum over intermediate states j



The third term is the 2nd order term in the perturbation expansion corresponding to box diagrams resulting in $K^0 \rightarrow \bar{K}^0$

- The total decay rate is the sum over all possible decays $K^0 \longrightarrow f$

$$\Gamma = 2\pi \sum_f |\langle f|\hat{H}_{weak}|K^0\rangle|^2 \rho_F$$

Density of final states

- ★ Because there are also diagrams which allow $K^0 \leftrightarrow \overline{K}^0$ mixing need to consider the time evolution of a mixed stated

$$\psi(t) = a(t)K^0 + b(t)\overline{K}^0 \qquad \text{(A2)}$$

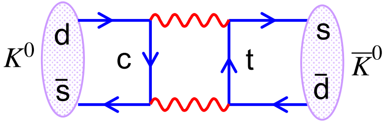
- ★ The time dependent wave-equation of (A1) becomes

$$\begin{pmatrix} M_{11} - \frac{1}{2}i\Gamma_{11} & M_{12} - \frac{1}{2}i\Gamma_{12} \\ M_{21} - \frac{1}{2}i\Gamma_{21} & M_{22} - \frac{1}{2}i\Gamma_{22} \end{pmatrix} \begin{pmatrix} |K^0(t)\rangle \\ |\overline{K}^0(t)\rangle \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} |K^0(t)\rangle \\ |\overline{K}^0(t)\rangle \end{pmatrix} \qquad \text{(A3)}$$

the diagonal terms are as before, and the off-diagonal terms are due to mixing.

$$M_{11} = m_{K^0} + \langle K^0|\hat{H}_{\rm weak}|K^0\rangle + \sum_n \frac{|\langle K^0|\hat{H}_{\rm weak}|K^0\rangle|^2}{m_{K^0} - E_n}$$

$$M_{12} = \sum_j \frac{\langle K^0|\hat{H}_{\rm weak}|j\rangle^* \langle j|\hat{H}_{\rm weak}|\overline{K}^0\rangle}{m_{K^0} - E_j}$$



- The off-diagonal decay terms include the effects of interference between decays to a common final state

$$\Gamma_{12} = 2\pi \sum_f \langle f|\hat{H}_{weak}|K^0\rangle^* \langle f|\hat{H}_{weak}|\overline{K}^0\rangle \rho_F$$

- In terms of the time dependent coefficients for the kaon states, **(A3)** becomes

$$\left[\mathbf{M} - i\tfrac{1}{2}\Gamma\right] \begin{pmatrix} a \\ b \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix}$$

where the Hamiltonian can be written:

$$\mathbf{H} = \mathbf{M} - i\tfrac{1}{2}\Gamma = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} - \tfrac{1}{2} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}$$

- Both the mass and decay matrices represent observable quantities and are Hermitian

$$M_{11} = M_{11}^*, \quad M_{22} = M_{22}^*, \quad M_{12} = M_{21}^*$$

$$\Gamma_{11} = \Gamma_{11}^*, \quad \Gamma_{22} = \Gamma_{22}^*, \quad \Gamma_{12} = \Gamma_{21}^*$$

- Furthermore, if CPT is conserved then the masses and decay rates of the $\overline{K}^0$ and K^0 are identical:

$$M_{11} = M_{22} = M; \qquad \Gamma_{11} = \Gamma_{22} = \Gamma$$

- Hence the time evolution of the system can be written:

$$\boxed{\begin{pmatrix} M-\frac{1}{2}i\Gamma & M_{12}-\frac{1}{2}i\Gamma_{12} \\ M_{12}^*-\frac{1}{2}i\Gamma_{12}^* & M-\frac{1}{2}i\Gamma \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = i\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix}} \qquad \textbf{(A4)}$$

- To solve the coupled differential equations for a(t) and b(t), first find the eigenstates of the Hamiltonian (the K_L and K_S) and then transform into this basis. The eigenvalue equation is:

$$\begin{pmatrix} M-\frac{1}{2}i\Gamma & M_{12}-\frac{1}{2}i\Gamma_{12} \\ M_{12}^*-\frac{1}{2}i\Gamma_{12}^* & M-\frac{1}{2}i\Gamma \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \textbf{(A5)}$$

- Which has non-trivial solutions for

$$|\mathbf{H}-\lambda I| = 0$$

$$\Rightarrow \quad (M-\tfrac{1}{2}i\Gamma-\lambda)^2-(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*)(M_{12}-\tfrac{1}{2}i\Gamma_{12}) = 0$$

with eigenvalues

$$\lambda = M-\tfrac{1}{2}i\Gamma \pm \sqrt{(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*)(M_{12}-\tfrac{1}{2}i\Gamma_{12})}$$

- The eigenstates can be obtained by substituting back into **(A5)**

$$(M-\tfrac{1}{2}i\Gamma)x_1+(M_{12}-\tfrac{1}{2}i\Gamma_{12})=(M-\tfrac{1}{2}i\Gamma\pm\sqrt{(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*)(M_{12}-\tfrac{1}{2}i\Gamma_{12})})x_1$$

⇨
$$\frac{x_2}{x_1}=\pm\sqrt{\frac{M_{12}^*-\frac{1}{2}i\Gamma_{12}^*}{M_{12}-\frac{1}{2}i\Gamma_{12}}}$$

★ **Define**

$$\eta=\sqrt{\frac{M_{12}^*-\frac{1}{2}i\Gamma_{12}^*}{M_{12}-\frac{1}{2}i\Gamma_{12}}}$$

★ **Hence the normalised eigenstates are**

$$|K_\pm\rangle=\frac{1}{\sqrt{1+|\eta|^2}}\begin{pmatrix}1\\ \pm\eta\end{pmatrix}=\frac{1}{\sqrt{1+|\eta|^2}}(|K^0\rangle\pm\eta|\overline{K}^0\rangle)$$

★ **Note, in the limit where M_{12},Γ_{12} are real, the eigenstates correspond to the CP eigenstates K$_1$ and K$_2$. Hence we can identify the general eigenstates as as the long and short lived neutral kaons:**

$$\boxed{|K_L\rangle=\frac{1}{\sqrt{1+|\eta|^2}}(|K^0\rangle+\eta|\overline{K}^0\rangle)}\quad\boxed{|K_S\rangle=\frac{1}{\sqrt{1+|\eta|^2}}(|K^0\rangle-\eta|\overline{K}^0\rangle)}$$

★ **Substituting these states back into (A2):**

$$\begin{aligned}|\psi(t)\rangle &= a(t)|K^0\rangle + b(t)|\overline{K}^0\rangle \\ &= \sqrt{1+|\eta|^2}\left[\frac{a(t)}{2}(K_L+K_S)+\frac{b(t)}{2\eta}(K_L-K_S)\right] \\ &= \sqrt{1+|\eta|^2}\left[\left(\frac{a(t)}{2}+\frac{b(t)}{2\eta}\right)K_L+\left(\frac{a(t)}{2}-\frac{b(t)}{2\eta}\right)K_S\right] \\ &= \frac{\sqrt{1+|\eta|^2}}{2}\left[a_L(t)K_L+a_S(t)K_S\right]\end{aligned}$$

with

$$\boxed{a_L(t) \equiv a(t) + \frac{b(t)}{\eta}} \qquad \boxed{a_S(t) \equiv a(t) - \frac{b(t)}{\eta}}$$

★ **Now consider the time evolution of** $a_L(t)$

$$i\frac{\partial a_L}{\partial t} = i\frac{\partial a}{\partial t} + \frac{i}{\eta}\frac{\partial b}{\partial t}$$

★ **Which can be evaluated using (A4) for the time evolution of a(t) and b(t):**

$$
\begin{aligned}
i\frac{\partial a_L}{\partial t} &= \left[(M-\tfrac{1}{2}i\Gamma_{12})a+(M_{12}-\tfrac{1}{2}i\Gamma_{12})b\right]+\frac{1}{\eta}\left[(M_{12}^{*}-\tfrac{1}{2}i\Gamma^{*}_{12})a+(M-\tfrac{1}{2}i\Gamma)b\right]\\
&= (M-\tfrac{1}{2}i\Gamma)\left(a+\frac{b}{\eta}\right)+(M_{12}-\tfrac{1}{2}i\Gamma_{12})b+\frac{1}{\eta}(M_{12}^{*}-\tfrac{1}{2}i\Gamma^{*}_{12})a\\
&= (M-\tfrac{1}{2}i\Gamma)a_L+(M_{12}-\tfrac{1}{2}i\Gamma_{12})b+\left(\sqrt{(M_{12}^{*}-\tfrac{1}{2}i\Gamma^{*}_{12})(M_{12}-\tfrac{1}{2}i\Gamma_{12})}\right)a\\
&= (M-\tfrac{1}{2}i\Gamma)a_L+\left(\sqrt{(M_{12}^{*}-\tfrac{1}{2}i\Gamma^{*}_{12})(M_{12}-\tfrac{1}{2}i\Gamma_{12})}\right)\left(a+\frac{b}{\eta}\right)\\
&= (M-\tfrac{1}{2}i\Gamma)a_L+\left(\sqrt{(M_{12}^{*}-\tfrac{1}{2}i\Gamma^{*}_{12})(M_{12}-\tfrac{1}{2}i\Gamma_{12})}\right)a_L\\
&= (m_L-\tfrac{1}{2}i\Gamma_L)a_L
\end{aligned}
$$

★ **Hence:**

$$\boxed{i\frac{\partial a_L}{\partial t}=(m_L-\tfrac{1}{2}i\Gamma_L)a_L}$$

with $m_L = M+\Re\left\{\sqrt{(M_{12}^{*}-\tfrac{1}{2}i\Gamma^{*}_{12})(M_{12}-\tfrac{1}{2}i\Gamma_{12})}\right\}$

and $\Gamma_L = \Gamma-2\Im\left\{\sqrt{(M_{12}^{*}-\tfrac{1}{2}i\Gamma^{*}_{12})(M_{12}-\tfrac{1}{2}i\Gamma_{12})}\right\}$

- ★ **Following the same procedure obtain:**

$$\boxed{i\frac{\partial a_S}{\partial t} = (m_S - \tfrac{1}{2}i\Gamma_S)a_S}$$

with $$m_S = M - \Re\left\{\sqrt{(M_{12}^* - \tfrac{1}{2}i\Gamma_{12}^*)(M_{12} - \tfrac{1}{2}i\Gamma_{12})}\right\}$$

and $$\Gamma_S = \Gamma + 2\Im\left\{\sqrt{(M_{12}^* - \tfrac{1}{2}i\Gamma_{12}^*)(M_{12} - \tfrac{1}{2}i\Gamma_{12})}\right\}$$

- ★ **In matrix notation we have**

$$\begin{pmatrix} M_L - \frac{1}{2}i\Gamma_L & 0 \\ 0 & M_S - \frac{1}{2}i\Gamma_S \end{pmatrix}\begin{pmatrix} a_L \\ a_S \end{pmatrix} = i\frac{\partial}{\partial t}\begin{pmatrix} a_L \\ a_S \end{pmatrix}$$

- ★ **Solving we obtain**

$$a_L(t) \propto e^{-im_L t - \Gamma_L t/2} \qquad a_S(t) \propto e^{-im_S t - \Gamma_S t/2}$$

- ★ **Hence in terms of the K$_L$ and K$_S$ basis the states propagate as independent particles with definite masses and lifetimes (the mass eigenstates). The time evolution of the neutral kaon system can be written**

$$|\psi(t)\rangle = A_L e^{-im_L t - \Gamma_L t/2}|K_L\rangle + A_S e^{-im_S t - \Gamma_S t/2}|K_S\rangle$$

where A$_L$ and A$_S$ are constants

Appendix III: CP Violation : $\pi\pi$ decays

Non-examinable

- ★ Consider the development of the $K^0 - \overline{K}^0$ system **now** including CP violation
- ★ Repeat previous derivation using

$$|K_S\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_1\rangle + \varepsilon|K_2\rangle\right] \qquad |K_L\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_2\rangle + \varepsilon|K_1\rangle\right]$$

- Writing the CP eigenstates in terms of $K^0, \overline{K}^0$

$$|K_L\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)|K_0\rangle + (1-\varepsilon)|\overline{K}^0\rangle\right]$$

$$|K_S\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)|K_0\rangle - (1-\varepsilon)|\overline{K}^0\rangle\right]$$

- Inverting these expressions obtain

$$|K^0\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1+\varepsilon}\left(|K_L\rangle + |K_S\rangle\right) \qquad |\overline{K}^0\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1-\varepsilon}\left(|K_L\rangle - |K_S\rangle\right)$$

- Hence a state that was produced as a K^0 evolves with time as:

$$|\psi(t)\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1+\varepsilon}\left(\theta_L(t)|K_L\rangle + \theta_S(t)|K_S\rangle\right)$$

where as before $\theta_S(t) = e^{-(im_S+\frac{\Gamma_S}{2})t}$ and $\theta_L(t) = e^{-(im_L+\frac{\Gamma_L}{2})t}$

- If we are considering the decay rate to ππ need to express the wave-function in terms of the CP eigenstates (remember we are neglecting CP violation in the decay)

$$
\begin{aligned}
|\psi(t)\rangle &= \frac{1}{\sqrt{2}}\frac{1}{1+\varepsilon}\left[(|K_2\rangle+\varepsilon|K_1\rangle)\theta_L(t)+(|K_1\rangle+\varepsilon|K_2\rangle)\theta_S(t)\right] \\
&= \frac{1}{\sqrt{2}}\frac{1}{1+\varepsilon}\left[(\theta_S+\varepsilon\theta_L)|K_1\rangle+(\theta_L+\varepsilon\theta_S)|K_2\rangle\right]
\end{aligned}
$$

CP Eigenstates

- Two pion decays occur with CP = +1 and therefore arise from decay of the CP = +1 kaon eigenstate, i.e. K_1

$$
\Gamma(K^0_{t=0}\rightarrow\pi\pi)\propto|\langle K_1|\psi(t)\rangle|^2=\frac{1}{2}\left|\frac{1}{1+\varepsilon}\right|^2|\theta_S+\varepsilon\theta_L|^2
$$

- Since $|\varepsilon|\ll 1$

$$
\left|\frac{1}{1+\varepsilon}\right|^2=\frac{1}{(1+\varepsilon^*)(1+\varepsilon)}\approx\frac{1}{1+2\Re\{\varepsilon\}}\approx 1-2\Re\{\varepsilon\}
$$

- Now evaluate the $|\theta_S+\varepsilon\theta_L|^2$ term again using

$$
|z_1\pm z_2|^2=|z_1|^2+|z_2|^2\pm 2\Re(z_1z_2^*)
$$

$$\begin{aligned}|\theta_S+\varepsilon\theta_L|^2 &= |e^{-im_S t-\frac{\Gamma_S}{2}t}+\varepsilon e^{-im_L t-\frac{\Gamma_L}{2}t}|^2\\ &= e^{-\Gamma_S t}+|\varepsilon|^2 e^{-\Gamma_L t}+2\Re\{e^{-im_S t-\frac{\Gamma_S}{2}t}.\varepsilon^* e^{+im_L t-\frac{\Gamma_L}{2}t}\}\end{aligned}$$

- **Writing** $\varepsilon = |\varepsilon|e^{i\phi}$

$$\begin{aligned}|\theta_S+\varepsilon\theta_L|^2 &= e^{-\Gamma_S t}+|\varepsilon|^2 e^{-\Gamma_L t}+2|\varepsilon|e^{-(\Gamma_S+\Gamma_L)t/2}\Re\{e^{i(m_L-m_S)t-\phi}\}\\ &= e^{-\Gamma_S t}+|\varepsilon|^2 e^{-\Gamma_L t}+2|\varepsilon|e^{-(\Gamma_S+\Gamma_L)t/2}\cos(\Delta m.t-\phi)\end{aligned}$$

- **Putting this together we obtain:**

$$\boxed{\Gamma(K^0_{t=0}\to\pi\pi) \quad = \quad \frac{1}{2}(1-2\Re\{\varepsilon\})N_{\pi\pi}\left[e^{-\Gamma_S t}+|\varepsilon|^2 e^{-\Gamma_L t}+2|\varepsilon|e^{-(\Gamma_S+\Gamma_L)t/2}\cos(\Delta m.t-\phi)\right]}$$

Short lifetime component $K_S\to\pi\pi$

CP violating long lifetime component $K_L\to\pi\pi$

Interference term

- **In exactly the same manner obtain for a beam which was produced as** $\overline{K}^0$

$$\boxed{\Gamma(\overline{K}^0_{t=0}\to\pi\pi) \quad = \quad \frac{1}{2}(1+2\Re\{\varepsilon\})N_{\pi\pi}\left[e^{-\Gamma_S t}+|\varepsilon|^2 e^{-\Gamma_L t}-2|\varepsilon|e^{-(\Gamma_S+\Gamma_L)t/2}\cos(\Delta m.t-\phi)\right]}$$

Interference term changes sign

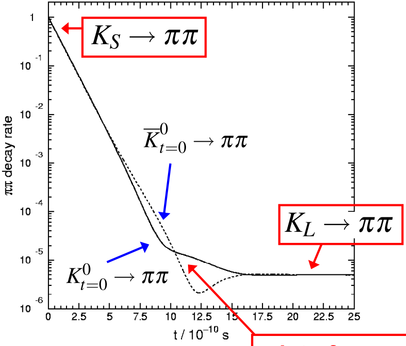
★ **At large proper times only the long lifetime component remains :**

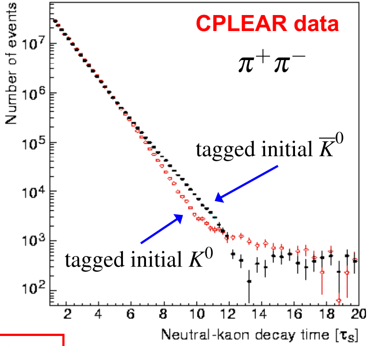
$$\Gamma(K^0_{t=0}\to\pi\pi) \quad \to \quad \frac{1}{2}(1-2\Re\{\varepsilon\})N_{\pi\pi}.|\varepsilon|^2 e^{-\Gamma_L t}$$

i.e. CP violating $K_L \to \pi\pi$ decays

★ **Since CPLEAR can identify whether a K^0 or $\overline{K}^0$ was produced, able to measure $\Gamma(K^0_{t=0}\to\pi\pi)$ and $\Gamma(\overline{K}^0_{t=0}\to\pi\pi)$**

Prediction with CP violation





★**The CPLEAR data shown previously can be used to measure** $\varepsilon = |\varepsilon| e^{i\phi}$

- **Define the asymmetry:**

$$A_{+-} = \frac{\Gamma(\overline{K}^0_{t=0} \to \pi\pi) - \Gamma(K^0_{t=0} \to \pi\pi)}{\Gamma(\overline{K}^0_{t=0} \to \pi\pi) + \Gamma(K^0_{t=0} \to \pi\pi)}$$

- **Using expressions on page 443**

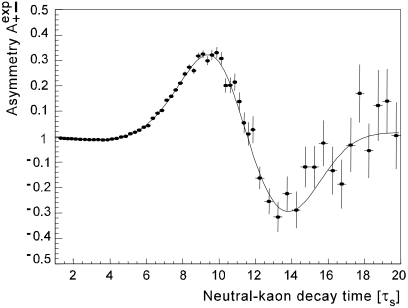
$$A_{+-} = \frac{4\Re\{\varepsilon\}\left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t}\right] - 4|\varepsilon| e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t - \phi)}{2\left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t}\right] - \underbrace{8\Re\{\varepsilon\}|\varepsilon| e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t - \phi)}_{\boxed{\propto |\varepsilon|\Re\{\varepsilon\} \text{ i.e. two small quantities and can safely be neglected}}}}$$

$$A_{+-} \approx \frac{2\Re\{\varepsilon\}\left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t}\right] - 2|\varepsilon| e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t - \phi)}{e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t}}$$

$$= 2\Re\{\varepsilon\} - \frac{2|\varepsilon| e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t - \phi)}{e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t}}$$

$$= 2\Re\{\varepsilon\} - \frac{2|\varepsilon| e^{(\Gamma_S-\Gamma_L)t/2}\cos(\Delta m.t - \phi)}{1 + |\varepsilon|^2 e^{(\Gamma_S-\Gamma_L)t}}$$

A.Apostolakis et al., Eur. Phys. J. C18 (2000) 41



Best fit to the data:

$$\boxed{\begin{aligned} |\varepsilon| &= (2.264 \pm 0.035) \times 10^{-3} \\ \phi &= (43.19 \pm 0.73)^\circ \end{aligned}}$$

Appendix IV: CP Violation via Mixing

Non-examinable

★ A full description of the SM origin of CP violation in the kaon system is beyond the level of this course, nevertheless, the relation to the box diagrams is illustrated below

★ The K-long and K-short wave-functions depend on η

$$\boxed{|K_L\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(|K^0\rangle + \eta|\overline{K}^0\rangle)} \quad \boxed{|K_S\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(|K^0\rangle - \eta|\overline{K}^0\rangle)}$$

with $$\eta = \sqrt{\frac{M_{12}^* - \frac{1}{2}i\Gamma_{12}^*}{M_{12} - \frac{1}{2}i\Gamma_{12}}}$$

★ If $M_{12}^* = M_{12}; \quad \Gamma_{12}^* = \Gamma_{12}$ then the K-long and K-short correspond to the CP eigenstates K_1 and K_2

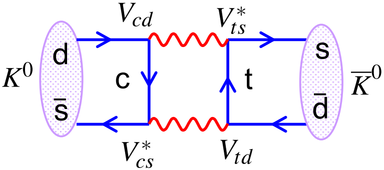
•CP violation is therefore associated with imaginary off-diagonal mass and decay elements for the neutral kaon system

•Experimentally, CP violation is small and $\eta \approx 1$

•Define: $\varepsilon = \frac{1-\eta}{1+\eta}$ ⇨ $\eta = \frac{1-\varepsilon}{1+\varepsilon}$

- Consider the mixing term M_{12} which arises from the sum over all possible intermediate states in the mixing box diagrams

e.g.



$$M_{12} = A_{ct} V_{cd} V_{cs}^* V_{ts}^* V_{td} + \ldots$$

- In the Standard Model, CP violation is associated with the imaginary components of the CKM matrix, and it can be shown that mixing leads to CP violation with

$$|\varepsilon| \propto \Im\{M_{12}\}$$

- The differences in masses of the mass eigenstates can be shown to be:

$$\Delta m_K = m_{K_L} - m_{K_S} \approx \sum_{q,q'} \frac{G_F^2}{3\pi^2} f_K^2 m_K |V_{qd} V_{qs}^* V_{q'd} V_{q's}^*| m_q m_{q'}$$

where q and q' are the quarks in the loops and f_K is a constant

- **In terms of the small parameter** ε

$$|K_L\rangle = \frac{1}{2\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)|K^0\rangle + (1-\varepsilon)|\overline{K}^0\rangle\right]$$
$$|K_S\rangle = \frac{1}{2\sqrt{1+|\varepsilon|^2}}\left[(1-\varepsilon)|K^0\rangle + (1+\varepsilon)|\overline{K}^0\rangle\right]$$

★ **If epsilon is non-zero we have CP violation in the neutral kaon system**

Writing $\eta = \sqrt{\frac{M_{12}^* - \frac{1}{2}i\Gamma_{12}^*}{M_{12} - \frac{1}{2}i\Gamma_{12}}} = \sqrt{\frac{z^*}{z}}$ **and** $z = ae^{i\phi}$

gives $\eta = e^{-i\phi}$

★ **From which we can find an expression for** ε

$$\varepsilon.\varepsilon^* = \frac{1-e^{-i\phi}}{1+e^{-i\phi}}.\frac{1-e^{+i\phi}}{1+e^{i\phi}} = \frac{2-\cos\phi}{2+\cos\phi} = \tan^2\frac{\phi}{2}$$
$$|\varepsilon| = |\tan\tfrac{\phi}{2}|$$

★ **Experimentally we know** ε **is small, hence** ϕ **is small**

$$|\varepsilon| \approx \frac{1}{2}\phi = \frac{1}{2}\arg z \approx \frac{1}{2}\frac{\Im\{M_{12} - \frac{1}{2}i\Gamma_{12}\}}{|M_{12} - \frac{1}{2}i\Gamma_{12}|}$$

Appendix V: Time Reversal Violation

- **Previously, equations (4) and (5), obtained expressions for strangeness oscillations in the absence of CP violation, e.g.**

$$\Gamma(K^0_{t=0} \rightarrow K^0) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

- **This analysis can be extended to include the effects of CP violation to give the following rates (see question 24):**

$$\Gamma(K^0_{t=0} \rightarrow K^0) \propto \tfrac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma(\overline{K}^0_{t=0} \rightarrow \overline{K}^0) \propto \tfrac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma(\overline{K}^0_{t=0} \rightarrow K^0) \propto \boxed{\tfrac{1}{4}(1+4\Re\{\varepsilon\})}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma(K^0_{t=0} \rightarrow \overline{K}^0) \propto \boxed{\tfrac{1}{4}(1-4\Re\{\varepsilon\})}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

★ **Including the effects of CP violation find that**

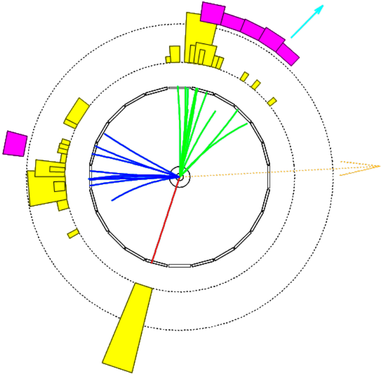
$$\boxed{\Gamma(\overline{K}^0_{t=0} \rightarrow K^0) \neq \Gamma(K^0_{t=0} \rightarrow \overline{K}^0)}$$

Violation of time reversal symmetry !

★ **No surprise, as CPT is conserved, CP violation implies T violation**

Particle Physics

Dr Lester



Handout 13 : Electroweak Unification and the W and Z Bosons

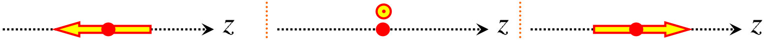
Boson Polarization States

- ★ In this handout we are going to consider the decays of W and Z bosons, for this we will need to consider the polarization. Here simply quote results although the justification was given in Appendices A1-A4 of Handout 8 (pages 290-298)
- ★ A real (i.e. not virtual) <u>massless</u> spin-1 boson can exist in two transverse polarization states, a massive spin-1 boson also can be longitudinally polarized
- ★ Boson wave-functions are written in terms of the polarization four-vector ε^μ

$$\boxed{B^\mu = \varepsilon^\mu e^{-ip.x} = \varepsilon^\mu e^{i(\vec{p}.\vec{x}-Et)}}$$

- ★ For a spin-1 boson travelling along the z-axis, the polarization four vectors are:

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_L = \frac{1}{m}(p_z,0,0,E) \quad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$



$$\boxed{S_z = -1} \quad \boxed{S_z = 0} \quad \boxed{S_z = +1}$$

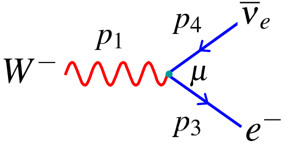
transverse | **longitudinal** | **transverse**

Longitudinal polarization isn′t present for on-shell massless particles, the photon can exist in two helicity states $h = \pm 1$ (LH and RH circularly polarized light)

W-Boson Decay

★To calculate the W-Boson decay rate first consider $W^- \to e^- \overline{\nu}_e$

★ Want matrix element for :



Incoming W-boson : $\epsilon_\mu(p_1)$

Out-going electron : $\overline{u}(p_3)$

Out-going $\overline{\nu}_e$: $v(p_4)$

Vertex factor : $-i\frac{g_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5)$

$$-iM_{fi} = \epsilon_\mu(p_1).\overline{u}(p_3).-i\frac{g_W}{\sqrt{2}}\gamma^\mu\tfrac{1}{2}(1-\gamma^5).v(p_4)$$

Note, no propagator

$$\Rightarrow \quad \boxed{M_{fi} = \frac{g_W}{\sqrt{2}}\epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(1-\gamma^5)v(p_4)}$$

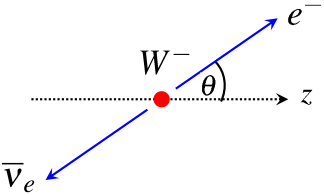
★ This can be written in terms of the four-vector scalar product of the W-boson polarization $\epsilon_\mu(p_1)$ and the weak charged current j^μ

$$\boxed{M_{fi} = \frac{g_W}{\sqrt{2}}\epsilon_\mu(p_1).j^\mu} \quad \text{with} \quad \boxed{j^\mu = \overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(1-\gamma^5)v(p_4)}$$

W-Decay : The Lepton Current

★ First consider the lepton current $j^{\mu} = \overline{u}(p_3)\gamma^{\mu}\frac{1}{2}(1-\gamma^5)v(p_4)$

★ Work in Centre-of-Mass frame



$$p_1 = (m_W, 0, 0, 0);$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

with $E = \dfrac{m_W}{2}$

★ In the ultra-relativistic limit only LH particles and RH anti-particles participate in the weak interaction so

$$j^{\mu} = \overline{u}(p_3)\gamma^{\mu}\frac{1}{2}(1-\gamma^5)v(p_4) = \overline{u}_{\downarrow}(p_3)\gamma^{\mu}v_{\uparrow}(p_4)$$

Note: $\frac{1}{2}(1-\gamma^5)v(p_4) = v_{\uparrow}(p_4)$ $\qquad$ $\overline{u}(p_3)\gamma^{\mu}v_{\uparrow}(p_4) = \overline{u}_{\downarrow}(p_3)\gamma^{\mu}v_{\uparrow}(p_4)$

Chiral projection operator, e.g. see p.150 or p.318.

"Helicity conservation", e.g. see p.151 or p.319

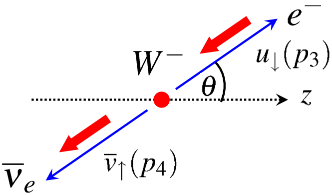
- We have already calculated the current

$$j^\mu = \overline{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4)$$

when considering $e^+e^- \rightarrow \mu^+\mu^-$

- From page 139 we have for $\mu_L^-\mu_R^+$

$$\boxed{j^\mu_{\uparrow\downarrow} = 2E(0, -\cos\theta, -i, \sin\theta)}$$

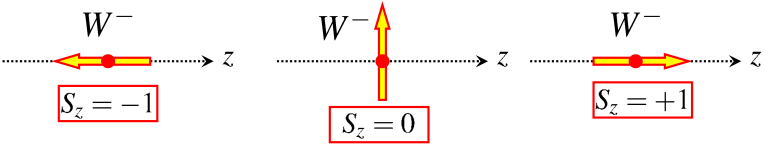


- For the charged current weak Interaction we only have to consider this single combination of helicities

$$j^\mu = \overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_4) = \overline{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4) = 2E(0, -\cos\theta, -i, \sin\theta)$$

and the three possible W-Boson polarization states:

$$\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0, 1, -i, 0); \quad \varepsilon_L = \frac{1}{m}(p_z, 0, 0, E) \quad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0, 1, i, 0)$$



- ★ **For a W-boson at rest these become:**

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_L = (0,0,0,1) \quad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

- ★ **Can now calculate the matrix element for the different polarization states**

$$M_{fi} = \frac{g_W}{\sqrt{2}}\varepsilon_\mu(p_1)j^\mu \quad \textbf{with} \quad j^\mu = 2\frac{m_W}{2}(0,-\cos\theta,-i,\sin\theta)$$

Decay at rest : $E_e = E_\nu = m_W/2$

- ★ **giving**

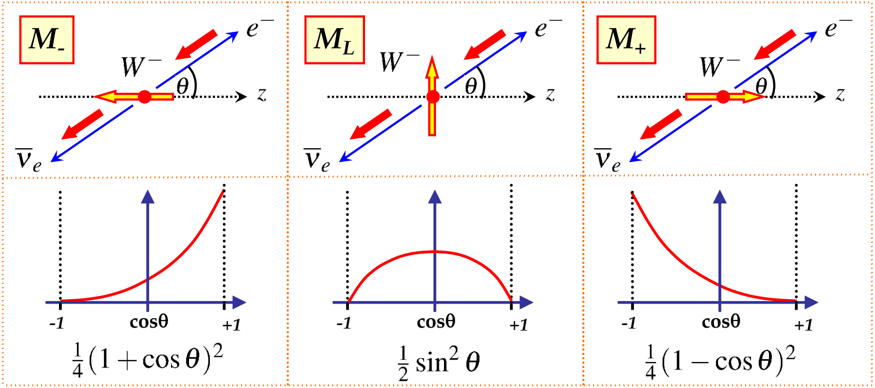
$$\boxed{\varepsilon_-} \quad M_- = \frac{g_W}{\sqrt{2}}\frac{1}{\sqrt{2}}(0,1,-i,0).m_W(0,-\cos\theta,-i,\sin\theta) = \tfrac{1}{2}g_W m_W(1+\cos\theta)$$

$$\boxed{\varepsilon_L} \quad M_L = \frac{g_W}{\sqrt{2}}(0,0,0,1).m_W(0,-\cos\theta,-i,\sin\theta) = -\frac{1}{\sqrt{2}}g_W m_W \sin\theta$$

$$\boxed{\varepsilon_+} \quad M_+ = -\frac{g_W}{\sqrt{2}}\frac{1}{\sqrt{2}}(0,1,i,0).m_W(0,-\cos\theta,-i,\sin\theta) = \tfrac{1}{2}g_W m_W(1-\cos\theta)$$

$$\Rightarrow \quad \boxed{\begin{aligned} |M_-|^2 &= g_W^2 m_W^2 \tfrac{1}{4}(1+\cos\theta)^2 \\ |M_L|^2 &= g_W^2 m_W^2 \tfrac{1}{2}\sin^2\theta \\ |M_+|^2 &= g_W^2 m_W^2 \tfrac{1}{4}(1-\cos\theta)^2 \end{aligned}}$$

★ **The angular distributions can be understood in terms of the spin of the particles**



$$\frac{1}{4}(1+\cos\theta)^2 \qquad \frac{1}{2}\sin^2\theta \qquad \frac{1}{4}(1-\cos\theta)^2$$

★ **The differential decay rate (s(see page 31)7) can be found using:**

$$\frac{d\Gamma}{d\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}|M|^2$$

where p* is the C.o.M momentum of the final state particles, here $p^* = \frac{m_W}{2}$

- **Hence for the three different polarisations we obtain:**

$$\frac{\mathrm{d}\Gamma_-}{\mathrm{d}\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{4}(1+\cos\theta)^2 \qquad \frac{\mathrm{d}\Gamma_L}{\mathrm{d}\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{2}\sin^2\theta \qquad \frac{\mathrm{d}\Gamma_+}{\mathrm{d}\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{4}(1-\cos\theta)^2$$

- **Integrating over all angles using**

$$\int \tfrac{1}{4}(1\pm\cos\theta)^2 \mathrm{d}\phi\,\mathrm{d}\cos\theta = \int \tfrac{1}{2}\sin^2\theta\,\mathrm{d}\phi\,\mathrm{d}\cos\theta = \frac{4\pi}{3}$$

- **Gives**

$$\boxed{\Gamma_- = \Gamma_L = \Gamma_+ = \frac{g_W^2 m_W}{48\pi}}$$

- **The total W-decay rate is independent of polarization; this has to be the case as the decay rate cannot depend on the arbitrary definition of the z-axis**

- **For a sample of unpolarized W boson each polarization state is equally likely, for the average matrix element sum over all possible matrix elements and average over the three initial polarization states**

$$\begin{aligned} \left\langle |M_{fi}|^2 \right\rangle &= \tfrac{1}{3}\left(|M_-|^2 + |M_L|^2 + |M_+|^2\right) \\ &= \tfrac{1}{3} g_W^2 m_W^2 \left[\tfrac{1}{4}(1+\cos\theta)^2 + \tfrac{1}{2}\sin^2\theta + \tfrac{1}{4}(1-\cos\theta)^2\right] \\ &= \tfrac{1}{3} g_W^2 m_W^2 \end{aligned}$$

- **For a sample of unpolarized W-bosons, the decay is isotropic (as expected)**

★ **For this isotropic decay**

$$\frac{d\Gamma}{d\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}\langle|M|^2\rangle \Rightarrow \Gamma = \frac{4\pi|p^*|}{32\pi^2 m_W^2}\langle|M|^2\rangle$$

$$\Rightarrow \boxed{\Gamma(W^- \to e^-\overline{\nu}) = \frac{g_W^2 m_W}{48\pi}}$$

★ **The calculation for the other decay modes (neglecting final state particle masses) is same. For quarks need to account for colour and CKM matrix. No decays to top – the top mass (175 GeV) is greater than the W-boson mass (80 GeV)**

$$\begin{array}{lll|l|ll|l|} W^- \to e^-\overline{\nu}_e & & W^- \to d\overline{u} & \times 3|V_{ud}|^2 & & W^- \to d\overline{c} & \times 3|V_{cd}|^2 \\ W^- \to \mu^-\overline{\nu}_\mu & & W^- \to s\overline{u} & \times 3|V_{us}|^2 & & W^- \to s\overline{c} & \times 3|V_{cs}|^2 \\ W^- \to \tau^-\overline{\nu}_\tau & & W^- \to b\overline{u} & \times 3|V_{ub}|^2 & & W^- \to b\overline{c} & \times 3|V_{cb}|^2 \end{array}$$

★ **Unitarity of CKM matrix gives, e.g.** $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$

★ **Hence** $BR(\mathrm{W} \to qq') = 6BR(\mathrm{W} \to \mathrm{e}\nu)$

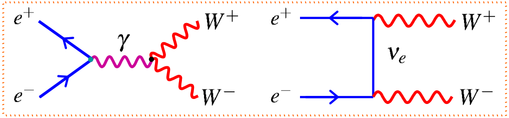
and thus the total decay rate :

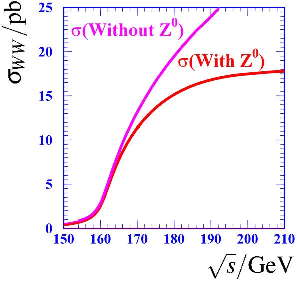
$$\boxed{\Gamma_W = 9\Gamma_{W\to e\nu} = \frac{3g_W^2 m_W}{16\pi} = 2.07\,\mathrm{GeV}}$$

Experiment: 2.14±0.04 GeV
(our calculation neglected a 3% QCD correction to decays to quarks)

From W to Z

- The $W^{\pm}$ bosons carry the EM charge - suggestive Weak are EM forces are related.
- W bosons can be produced in e^+e^- annihilation

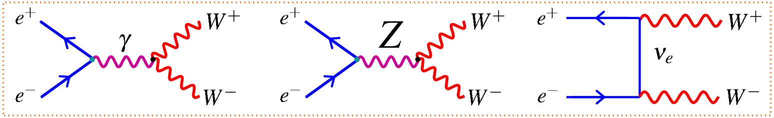




- With just these two diagrams there is a problem: the cross section increases with C.o.M energy and at some point violates **QM unitarity**

> **UNITARITY VIOLATION: when QM calculation gives larger flux of W bosons than incoming flux of electrons/positrons**

- Problem can be "fixed" by introducing a new boson, the Z. The new diagram interferes negatively with the above two diagrams fixing the unitarity problem



$$|M_{\gamma WW}+M_{ZWW}+M_{\nu WW}|^2 < |M_{\gamma WW}+M_{\nu WW}|^2$$

- Only works if **Z, γ, W** couplings are related: need **ELECTROWEAK UNIFICATION**

SU(2)$_L$: The Weak Interaction

★ The Weak Interaction arises from SU(2) local phase transformations

$$\psi \to \psi' = \psi e^{i\vec{\alpha}(x).\frac{\vec{\sigma}}{2}}$$

where the $\vec{\sigma}$ are the generators of the SU(2) symmetry, i.e the three Pauli spin matrices

⇒ **3 Gauge Bosons** $W_1^\mu,\ W_2^\mu,\ W_3^\mu$

★ The wave-functions have two components which, in analogy with isospin, are represented by "weak isospin"

★ The fermions are placed in isospin doublets and the local phase transformation corresponds to

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix} \to \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}' = e^{i\vec{\alpha}(x).\frac{\vec{\sigma}}{2}}\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}$$

★ Weak Interaction only couples to LH particles/RH anti-particles, hence only place LH particles/RH anti-particles in weak isospin doublets: $I_W = \frac{1}{2}$
RH particles/LH anti-particles placed in weak isospin singlets: $I_W = 0$

Weak Isospin

$$\boxed{I_W = \frac{1}{2}} \quad \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L, \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L, \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L, \begin{pmatrix}u\\ d'\end{pmatrix}_L, \begin{pmatrix}c\\ s'\end{pmatrix}_L, \begin{pmatrix}t\\ b'\end{pmatrix}_L \begin{matrix}\leftarrow \boxed{I_W^3 = +\frac{1}{2}}\\ \leftarrow \boxed{I_W^3 = -\frac{1}{2}}\end{matrix}$$

$$\boxed{I_W = 0} \quad (\nu_e)_R,\ (e^-)_R, \ldots(u)_R,\ (d)_R, \ldots$$

Note: RH/LH refer to chiral states

★ For simplicity only consider $\chi_L = \begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L$

- The gauge symmetry specifies the form of the interaction: one term for each of the 3 generators of SU(2) – [note: here include interaction strength in current]

$$j^1_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_1 \chi_L \qquad j^2_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_2 \chi_L \qquad j^3_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_3 \chi_L$$

★ The charged current W^+/W^- interaction enters as a linear combinations of W_1, W_2

$$W^{\pm\mu} = \tfrac{1}{\sqrt{2}}(W_1^\mu \mp i W_2^\mu)$$

★ The $W^\pm$ interaction terms

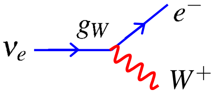
$$j^\mu_\pm = \tfrac{g_W}{\sqrt{2}}(j_1^\mu \mp i j_2^\mu) = \tfrac{g_W}{\sqrt{2}}\overline{\chi}_L \gamma^\mu \tfrac{1}{2}(\sigma_1 \mp i\sigma_2)\chi_L$$

★ Express in terms of the weak isospin ladder operators $\sigma_\pm = \tfrac{1}{2}(\sigma_1 \pm i\sigma_2)$

$$\boxed{j^\mu_\pm = \tfrac{g_W}{\sqrt{2}}\overline{\chi}_L \gamma^\mu \sigma_\mp \chi_L} \Big\}$$

Origin of $\frac{1}{\sqrt{2}}$ in Weak CC

W⁺



corresponds to

$$\boxed{j^\mu_+ = \tfrac{g_W}{\sqrt{2}}\overline{\chi}_L \gamma^\mu \sigma_- \chi_L}$$

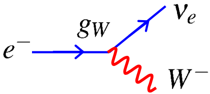
Bars indicates adjoint spinors

which can be understood in terms of the weak isospin doublet

$$j^\mu_+ = \frac{g_W}{\sqrt{2}}\overline{\chi}_L \gamma^\mu \sigma_- \chi_L = \frac{g_W}{\sqrt{2}}(\overline{\nu}_L, \overline{e}_L)\gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \nu \\ e \end{pmatrix}_L = \frac{g_W}{\sqrt{2}}\overline{e}_L \gamma^\mu \nu_L = \boxed{\frac{g_W}{\sqrt{2}}\overline{e}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\nu}$$

★ **Similarly**

W^-



corresponds to

$$\boxed{j^\mu_- = \frac{g_W}{\sqrt{2}}\overline{\chi}_L\gamma^\mu\sigma_+\chi_L}$$

$$j^\mu_- = \frac{g_W}{\sqrt{2}}\overline{\chi}_L\gamma^\mu\sigma_+\chi_L = \frac{g_W}{\sqrt{2}}(\overline{\nu}_L,\overline{e}_L)\gamma^\mu\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}\nu\\e\end{pmatrix}_L = \frac{g_W}{\sqrt{2}}\overline{\nu}_L\gamma^\mu e_L = \frac{g_W}{\sqrt{2}}\overline{\nu}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)e$$

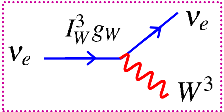
★ **However have an additional interaction due to W^3**

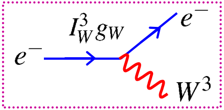
$$\boxed{j^\mu_3 = g_W\overline{\chi}_L\gamma^\mu\tfrac{1}{2}\sigma_3\chi_L}$$

expanding this:

$$\boxed{I^3_W = \pm\tfrac{1}{2}}$$

$$j^\mu_3 = g_W\frac{1}{2}(\overline{\nu}_L,\overline{e}_L)\gamma^\mu\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\nu\\e\end{pmatrix}_L = g_W\tfrac{1}{2}\overline{\nu}_L\gamma^\mu\nu_L - g_W\tfrac{1}{2}\overline{e}_L\gamma^\mu e_L$$





NEUTRAL CURRENT INTERACTIONS !

Electroweak Unification

- ★Tempting to identify the W^3 as the Z
- ★However this is not the case, have two physical neutral spin-1 gauge bosons, γ, Z and the W^3 is a mixture of the two,
- ★ Equivalently write the photon and Z in terms of the W^3 and a new neutral spin-1 boson the B
- ★The **physical** bosons (the Z and photon field, A) are:

$$A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W$$

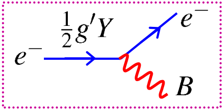
$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W$$

θ_W **is the weak mixing angle**

- ★The new boson is associated with a new gauge symmetry similar to that of electromagnetism : $U(1)_Y$
- ★The charge of this symmetry is called **WEAK HYPERCHARGE** Y

$$\boxed{Y = 2Q - 2I^3_W}$$

Q is the EM charge of a particle
I^3_W is the third comp. of weak isospin



- •By convention the coupling to the $\mathbf{B}_\mu$ is $\frac{1}{2}g'Y$

$$e_L : \; Y = 2(-1) - 2(-\tfrac{1}{2}) = -1 \qquad \nu_L : \; Y = -1$$

$$e_R : \; Y = 2(-1) - 2(0) = -2 \qquad \nu_R : \; Y = 0$$

(this identification of hypercharge in terms of Q and I_3 makes all of the following work out)

★ For this to work the coupling constants of the **W³**, **B**, and **photon** must be related

e.g. consider contributions involving the neutral interactions of electrons:

γ $$j^{em}_{\mu} = e\overline{\psi}Q_e\gamma_\mu\psi = e\bar{\mathrm{e}}_L Q_{\mathrm{e}}\gamma_\mu \mathrm{e}_L + e\bar{\mathrm{e}}_R Q_e \gamma_\mu \mathrm{e}_R$$

W³ $$j^{W^3}_{\mu} = -\frac{g_W}{2}\bar{\mathrm{e}}_L\gamma_\mu \mathrm{e}_L$$

B $$j^{Y}_{\mu} = \frac{g'}{2}\overline{\psi}Y_e\gamma_\mu\psi = \frac{g'}{2}\bar{\mathrm{e}}_L Y_{\mathrm{e}_L}\gamma_\mu \mathrm{e}_L + \frac{g'}{2}\bar{\mathrm{e}}_R Y_{\mathrm{e}_R}\gamma_\mu \mathrm{e}_R$$

★ The relation $A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W$ is equivalent to requiring

$$\boxed{j^{em}_{\mu} = j^{Y}_{\mu}\cos\theta_W + j^{W^3}_{\mu}\sin\theta_W}$$

•Writing this in full:

$$e\bar{\mathrm{e}}_L Q_{\mathrm{e}}\gamma_\mu \mathrm{e}_L + e\bar{\mathrm{e}}_R Q_e\gamma_\mu \mathrm{e}_R = \tfrac{1}{2}g'\cos\theta_W[\bar{\mathrm{e}}_L Y_{\mathrm{e}_L}\gamma_\mu \mathrm{e}_L + \bar{\mathrm{e}}_R Y_{\mathrm{e}_R}\gamma_\mu \mathrm{e}_R] - \tfrac{1}{2}g_W\sin\theta_W[\bar{\mathrm{e}}_L\gamma_\mu e_L]$$

$$-e\bar{\mathrm{e}}_L\gamma_\mu \mathrm{e}_L - e\bar{\mathrm{e}}_R\gamma_\mu \mathrm{e}_R = \tfrac{1}{2}g'\cos\theta_W[-\bar{\mathrm{e}}_L\gamma_\mu \mathrm{e}_L - 2\bar{\mathrm{e}}_R\gamma_\mu \mathrm{e}_R] - \tfrac{1}{2}g_W\sin\theta_W[\bar{\mathrm{e}}_L\gamma_\mu e_L]$$

which works if: $$\boxed{e = g_W\sin\theta_W = g'\cos\theta_W}$$ (i.e. equate coefficients of L and R terms)

★ Couplings of electromagnetism, the weak interaction and the interaction of the U(1)$_Y$ symmetry are therefore related.

The Z Boson

★**In this model we can now derive the couplings of the Z Boson**

$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W$$

$\boxed{I^3_W}$

for the electron $I^3_W = -\frac{1}{2}$

$$j^Z_\mu = -\tfrac{1}{2} g' \sin\theta_W [\bar{e}_L Y_{e_L} \gamma_\mu e_L + \bar{e}_R Y_{e_R} \gamma_\mu e_R] - \tfrac{1}{2} g_W \cos\theta_W [\bar{e}_L \gamma_\mu e_L]$$

- **Writing this in terms of weak isospin and charge:**

$$j^Z_\mu = -\tfrac{1}{2} g' \sin\theta_W [\bar{e}_L \boxed{(2Q - 2I^3_W)} \gamma_\mu e_L + \bar{e}_R \underbrace{\boxed{(2Q)}} \gamma_\mu e_R] + \boxed{I^3_W} g_W \cos\theta_W [\bar{e}_L \gamma_\mu e_L]$$

For RH chiral states $I_3=0$

- **Gathering up the terms for LH and RH chiral states:**

$$j^Z_\mu = \left[g' I^3_W \sin\theta_W - g' Q \sin\theta_W + g_W I^3_W \cos\theta_W\right] \bar{e}_L \gamma_\mu e_L - \left[g' Q \sin\theta_W\right] \bar{e}_R \gamma_\mu e_R$$

- **Using:** $e = g_W \sin\theta_W = g' \cos\theta_W$ **gives**

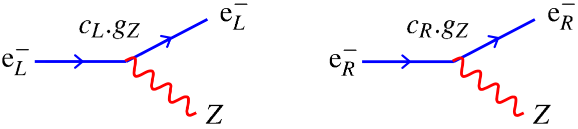
$$j^Z_\mu = \left[g' \frac{(I^3_W - Q\sin^2\theta_W)}{\sin\theta_W}\right] \bar{e}_L \gamma_\mu e_L - \left[g' \frac{Q\sin^2\theta_W}{\sin\theta_W}\right] \bar{e}_R \gamma_\mu e_R$$

$$\boxed{j^Z_\mu = g_Z (I^3_W - Q\sin^2\theta_W)[\bar{e}_L \gamma_\mu e_L] - g_Z Q \sin^2\theta_W [\bar{e}_R \gamma_\mu e_R]}$$

with $\boxed{e = g_Z \cos\theta_W \sin\theta_W}$ **i.e.** $\boxed{g_Z = \frac{g_W}{\cos\theta_W}}$

- ★ Unlike for the Charged Current Weak interaction (W) the Z Boson couples to both LH and RH chiral components, **but not equally…**

$$\begin{aligned} j^Z_\mu &= g_Z(I^3_W - Q\sin^2\theta_W)[\bar{e}_L\gamma_\mu e_L] - g_Z Q\sin^2\theta_W[\bar{e}_R\gamma_\mu e_R] \\ &= g_Z c_L[\bar{e}_L\gamma_\mu e_L] + g_Z c_R[\bar{e}_R\gamma_\mu e_R] \end{aligned}$$



$$\boxed{c_L = I^3_W - Q\sin^2\theta_W} \qquad \boxed{c_R = -Q\sin^2\theta_W}$$

W³ part of Z couples only to LH components (like W±)

B$_\mu$ part of Z couples equally to LH and RH components

- ★ Use projection operators to obtain vector and axial vector couplings

$$\bar{u}_L\gamma_\mu u_L = \bar{u}\gamma_\mu\tfrac{1}{2}(1-\gamma_5)u \qquad \bar{u}_R\gamma_\mu u_R = \bar{u}\gamma_\mu\tfrac{1}{2}(1+\gamma_5)u$$

$$j^Z_\mu = g_Z\bar{u}\gamma_\mu\left[c_L\tfrac{1}{2}(1-\gamma_5) + c_R\tfrac{1}{2}(1+\gamma_5))\right]u$$

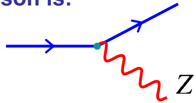
$$j^Z_\mu = \frac{g_Z}{2}\overline{u}\gamma_\mu\left[(c_L + c_R) + (c_R - c_L)\gamma_5\right]u$$

- **Which in terms of V and A components gives:** $j^Z_\mu = \frac{g_Z}{2}\overline{u}\gamma_\mu\left[c_V - c_A\gamma_5\right]u$

with $\boxed{c_V = c_L + c_R = I^3_W - 2Q\sin^2\theta_W}$ $\boxed{c_A = c_L - c_R = I^3_W}$

- **Hence the vertex factor for the Z boson is:**

$$\boxed{-ig_Z\tfrac{1}{2}\gamma_\mu\left[c_V - c_A\gamma_5\right]}$$

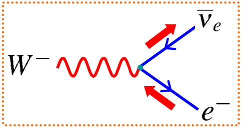


- **Using the experimentally determined value of the weak mixing angle:** $\sin^2\theta_W \approx 0.23$

Fermion	Q	I^3_W L	I^3_W R	c_L	c_R	c_V	c_A
ν_e, ν_μ, ν_τ	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$
e^-, μ^-, τ^-	-1	$-\frac{1}{2}$	0	-0.27	0.23	-0.04	$-\frac{1}{2}$
u, c, t	$+\frac{2}{3}$	$+\frac{1}{2}$	0	0.35	-0.15	$+0.19$	$+\frac{1}{2}$
d, s, b	$-\frac{1}{3}$	$-\frac{1}{2}$	0	-0.42	0.08	-0.35	$-\frac{1}{2}$

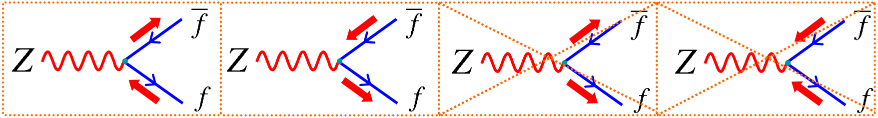
Z Boson Decay : Γ_Z

- ★ In W-boson decay only had to consider one helicity combination of (assuming we can neglect final state masses: helicity states = chiral states)



W-boson couples:
to LH particles
and RH anti-particles

- ★ But Z-boson couples to LH and RH particles (with different strengths)
- ★ Need to consider only two helicity (or more correctly chiral) combinations:

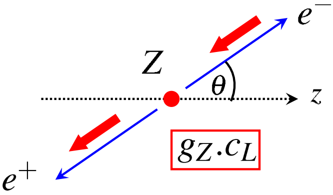


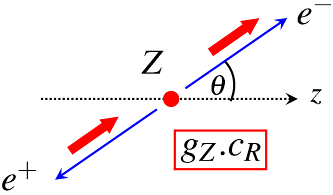
This can be seen by considering either of the combinations which give zero

e.g.

$$\begin{aligned}\bar{u}_R\gamma^\mu(c_V+c_A\gamma_5)v_R &= u^\dagger\tfrac{1}{2}(1+\gamma^5)\gamma^0\gamma^\mu(c_V+c_A\gamma^5)\tfrac{1}{2}(1-\gamma^5)v\\ &= \tfrac{1}{4}u^\dagger\gamma^0(1-\gamma^5)\gamma^\mu(1-\gamma^5)(c_V+c_A\gamma^5)v\\ &= \tfrac{1}{4}\bar{u}\gamma^\mu(1+\gamma^5)(1-\gamma^5)(c_V+c_A\gamma_5)v=0\end{aligned}$$

- **In terms of left and right-handed combinations need to calculate:**





- **For unpolarized Z bosons:** **(Question 26)**

$$\langle|M_{fi}|^2\rangle = \frac{1}{3}[2c_L^2 g_Z^2 m_Z^2 + 2c_R^2 g_Z^2 m_Z^2] = \frac{2}{3} g_Z^2 m_Z^2 (c_L^2 + c_R^2)$$

average over polarization

- **Using** $c_V^2 + c_A^2 = 2(c_L^2 + c_R^2)$ **and** $\dfrac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \dfrac{|p^*|}{32\pi^2 m_W^2}|M|^2$

$$\boxed{\Gamma(Z \to e^+e^-) = \frac{g_Z^2 m_Z}{48\pi}(c_V^2 + c_A^2)}$$

Z Branching Ratios

(Question 27)

★ (Neglecting fermion masses) obtain the same expression for the other decays

$$\Gamma(Z \to f\overline{f}) = \frac{g_Z^2 m_Z}{48\pi}(c_V^2 + c_A^2)$$

- Using values for c_V and c_A on page 471 obtain:

$$Br(Z \to e^+e^-) = Br(Z \to \mu^+\mu^-) = Br(Z \to \tau^+\tau^-) \boxed{\approx 3.5\%}$$

$$Br(Z \to \nu_1\overline{\nu}_1) = Br(Z \to \nu_2\overline{\nu}_2) = Br(Z \to \nu_3\overline{\nu}_3) \boxed{\approx 6.9\%}$$

$$Br(Z \to d\overline{d}) = Br(Z \to s\overline{s}) = Br(Z \to b\overline{b}) \boxed{\approx 15\%}$$

$$Br(Z \to u\overline{u}) = Br(Z \to c\overline{c}) \boxed{\approx 12\%}$$

- The Z Boson therefore predominantly decays to hadrons

$$Br(Z \to \text{hadrons}) \approx 69\%$$

Mainly due to factor 3 from colour

- Also predict total decay rate (total width)

$$\Gamma_Z = \sum_i \Gamma_i = 2.5\,\text{GeV}$$

Experiment:

$$\boxed{\Gamma_Z = 2.4952 \pm 0.0023\,\text{GeV}}$$

Summary

- The Standard Model interactions are mediated by spin-1 **gauge bosons**
- The form of the interactions are completely specified by the assuming an underlying local phase transformation ➡ **GAUGE INVARIANCE**

$U(1)_{em}$ ⇨ **QED**

$SU(2)_L$ ⇨ **Charged Current Weak Interaction + W^3**

$SU(3)_{col}$ ⇨ **QCD**

- In order to "unify" the electromagnetic and weak interactions, introduced a new symmetry gauge symmetry : U(1) hypercharge

$U(1)_Y$ ⇨ **B_μ**

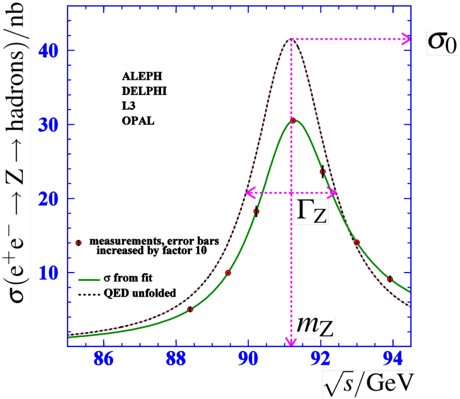
- The physical Z boson and the photon are mixtures of the neutral W boson and B determined by the **Weak Mixing angle**

$$\boxed{\sin^2\theta_W \approx 0.23}$$

- **Have we really unified the EM and Weak interactions ? Well not really…**
  - Started with two independent theories with coupling constants g_W, e
  - Ended up with coupling constants which are related but at the cost of introducing a new parameter in the Standard Model θ_W
  - Interactions not unified from any higher theoretical principle… **but it works!**

Particle Physics

Dr Lester

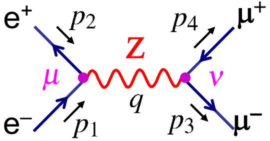


Handout 14 : Precision Tests of the Standard Model

The Z Resonance

★ **Want to calculate the cross-section for** $e^+e^- \to Z \to \mu^+\mu^-$

- **Feynman rules for the diagram below give:**



e⁺e⁻ vertex: $\overline{v}(p_2)\cdot -ig_Z\gamma^\mu\,\frac{1}{2}(c_V^e - c_A^e\gamma^5)\cdot u(p_1)$

Z propagator: $\dfrac{-ig_{\mu\nu}}{q^2-m_Z^2}$

μ⁺μ⁻ vertex: $\overline{u}(p_3)\cdot -ig_Z\gamma^\nu\,\frac{1}{2}(c_V^\mu - c_A^\mu\gamma^5)\cdot v(p_4)$

➡ $$-iM_{fi} = [\overline{v}(p_2)\cdot -ig_Z\gamma^\mu\,\tfrac{1}{2}(c_V^e - c_A^e\gamma^5)\cdot u(p_1)].\frac{-ig_{\mu\nu}}{q^2-m_Z^2}.[\overline{u}(p_3)\cdot -ig_Z\gamma^\nu\,\tfrac{1}{2}(c_V^\mu - c_A^\mu\gamma^5)\cdot v(p_4)]$$

➡ $$\boxed{M_{fi} = -\frac{g_Z^2}{q^2-m_Z^2}g_{\mu\nu}[\overline{v}(p_2)\gamma^\mu\,\tfrac{1}{2}(c_V^e - c_A^e\gamma^5)\cdot u(p_1)].[\overline{u}(p_3)\gamma^\nu\,\tfrac{1}{2}(c_V^\mu - c_A^\mu\gamma^5)\cdot v(p_4)]}$$

★ **Convenient to work in terms of helicity states by explicitly using the Z coupling to LH and RH chiral states (ultra-relativistic limit so helicity = chirality)**

$$\tfrac{1}{2}(c_V - c_A\gamma^5) = c_L\,\underbrace{\tfrac{1}{2}(1-\gamma^5)} + c_R\,\underbrace{\tfrac{1}{2}(1+\gamma^5)}$$

LH and RH projections operators

hence $c_V = (c_L + c_R), \;\; c_A = (c_L - c_R)$

and

$$\begin{aligned} \tfrac{1}{2}(c_V - c_A\gamma^5) &= \tfrac{1}{2}(c_L + c_R - (c_L - c_R)\gamma^5) \\ &= c_L\tfrac{1}{2}(1-\gamma^5) + c_R\tfrac{1}{2}(1+\gamma^5) \end{aligned}$$

with $c_L = \frac{1}{2}(c_V + c_A), \;\; c_R = \frac{1}{2}(c_V - c_A)$

★ **Rewriting the matrix element in terms of LH and RH couplings:**

$$\begin{aligned} M_{fi} = -\frac{g_Z^2}{q^2 - m_Z^2} g_{\mu\nu} & [c_L^e \overline{v}(p_2)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_1) + c_R^e \overline{v}(p_2)\gamma^\mu \tfrac{1}{2}(1+\gamma^5)u(p_1)] \\ & \times [c_L^\mu \overline{u}(p_3)\gamma^\nu \tfrac{1}{2}(1-\gamma^5)v(p_4) + c_R^\mu \overline{u}(p_3)\gamma^\nu \tfrac{1}{2}(1+\gamma^5)v(p_4)] \end{aligned}$$

★ **Apply projection operators remembering that in the ultra-relativistic limit**

$$\tfrac{1}{2}(1-\gamma^5)u = u_\downarrow; \quad \tfrac{1}{2}(1+\gamma^5)u = u_\uparrow, \quad \tfrac{1}{2}(1-\gamma^5)v = v_\uparrow, \quad \tfrac{1}{2}(1+\gamma^5)v = v_\downarrow$$

⇨

$$\begin{aligned} M_{fi} = -\frac{g_Z}{q^2 - m_Z^2} g_{\mu\nu} & [c_L^e \overline{v}(p_2)\gamma^\mu u_\downarrow(p_1) + c_R^e \overline{v}(p_2)\gamma^\mu u_\uparrow(p_1)] \\ & \times [c_L^\mu \overline{u}(p_3)\gamma^\nu v_\uparrow(p_4) + c_R^\mu \overline{u}(p_3)\gamma^\nu v_\downarrow(p_4)] \end{aligned}$$

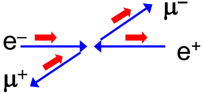
★ **For a combination of V and A currents,** $\overline{u}_\uparrow \gamma^\mu v_\uparrow = 0$ **etc, gives four orthogonal contributions**

⇨

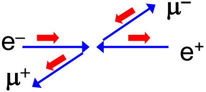
$$\boxed{\begin{aligned} -\frac{g_Z^2}{q^2 - m_Z^2} g_{\mu\nu} & [c_L^e \overline{v}_\uparrow(p_2)\gamma^\mu u_\downarrow(p_1) + c_R^e \overline{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1)] \\ & \times [c_L^\mu \overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) + c_R^\mu \overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)] \end{aligned}}$$

★ **Sum of 4 terms**

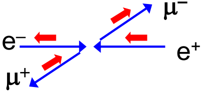
$$M_{RR} = -\frac{g_Z^2}{q^2 - m_Z^2} c_R^e c_R^\mu g_{\mu\nu} [\overline{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1)][\overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)]$$



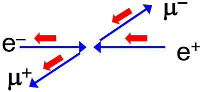
$$M_{RL} = -\frac{g_Z^2}{q^2 - m_Z^2} c_R^e c_L^\mu g_{\mu\nu} [\overline{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1)][\overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4)]$$



$$M_{LR} = -\frac{g_Z^2}{q^2 - m_Z^2} c_L^e c_R^\mu g_{\mu\nu} [\overline{v}_\uparrow(p_2)\gamma^\mu u_\downarrow(p_1)][\overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)]$$



$$M_{LL} = -\frac{g_Z^2}{q^2 - m_Z^2} c_L^e c_L^\mu g_{\mu\nu} [\overline{v}_\uparrow(p_2)\gamma^\mu u_\downarrow(p_1)][\overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4)]$$



Remember: the L/R refer to the helicities of the initial/final state <u>particles</u>

★ **Fortunately we have calculated these terms before when considering**

$e^+e^- \rightarrow \gamma \rightarrow \mu^+\mu^-$ **giving:** (p(pages 142-143)3)

$$[\overline{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1)][\overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)] = s(1+\cos\theta)$$ **etc.**

- ★ **Applying the QED results to the Z exchange with**

$$\frac{e^2}{q^2} \rightarrow \frac{g_Z^2}{q^2 - m_Z^2} c^e c^\mu$$

where $q^2 = s = 4E_e^2$

gives:

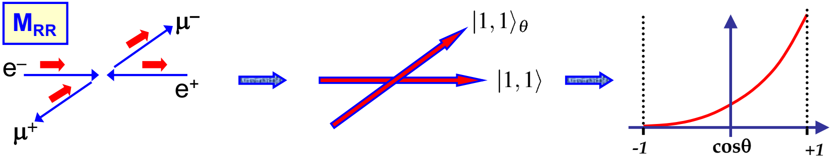
$$|M_{RR}|^2 = s^2 \left| \frac{g_Z^2}{s - m_Z^2} \right|^2 (c_R^e)^2 (c_R^\mu)^2 (1 + \cos\theta)^2$$

$$|M_{RL}|^2 = s^2 \left| \frac{g_Z^2}{s - m_Z^2} \right|^2 (c_R^e)^2 (c_L^\mu)^2 (1 - \cos\theta)^2$$

$$|M_{LR}|^2 = s^2 \left| \frac{g_Z^2}{s - m_Z^2} \right|^2 (c_L^e)^2 (c_R^\mu)^2 (1 - \cos\theta)^2$$

$$|M_{LL}|^2 = s^2 \left| \frac{g_Z^2}{s - m_Z^2} \right|^2 (c_L^e)^2 (c_L^\mu)^2 (1 + \cos\theta)^2$$

- ★ **As before, the angular dependence of the matrix elements can be understood in terms of the spins of the incoming and outgoing particles e.g.**



The Breit-Wigner Resonance

★ Need to consider carefully the propagator term $1/(s-m_Z^2)$ which diverges when the C.o.M. energy is equal to the rest mass of the Z boson

★ To do this need to account for the fact that the Z boson is an unstable particle

- For a stable particle at rest the time development of the wave-function is:

$$\psi \sim e^{-imt}$$

- For an unstable particle this must be modified to

$$\psi \sim e^{-imt} e^{-\Gamma t/2}$$

so that the particle probability decays away exponentially

$$\psi^* \psi \sim e^{-\Gamma t} = e^{-t/\tau} \quad \text{with} \quad \tau = \frac{1}{\Gamma_Z}$$

- Equivalent to making the replacement

$$m \rightarrow m - i\Gamma/2$$

★ In the Z boson propagator make the substitution:

$$m_Z \rightarrow m_Z - i\Gamma_Z/2$$

★ Which gives:

$$(s-m_Z^2) \longrightarrow [s-(m_Z - i\Gamma_Z/2)^2] = s - m_Z^2 + im_Z\Gamma_Z + \tfrac{1}{4}\Gamma_Z^2 \approx s - m_Z^2 + im_Z\Gamma_Z$$

where it has been assumed that $\Gamma_Z \ll m_Z$

★ Which gives

$$\left|\frac{1}{s-m_Z^2}\right|^2 \rightarrow \left|\frac{1}{s-m_Z^2+im_Z\Gamma_Z}\right|^2 = \frac{1}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}$$

★ **And the Matrix elements become**

$$|M_{RR}|^2 = \frac{g_Z^4 s^2}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}(c_R^e)^2(c_R^\mu)^2(1+\cos\theta)^2$$

etc.

★ **In the limit where initial and final state particle mass can be neglected:**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|M_{fi}|^2$$

(page 37)

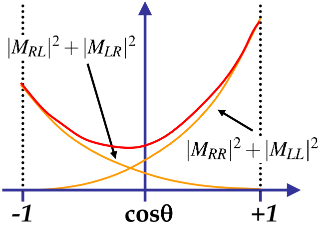
★ **Giving:**

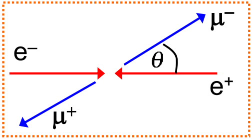
$$\frac{d\sigma_{RR}}{d\Omega} = \frac{1}{64\pi^2}\frac{g_Z^4 s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}(c_R^e)^2(c_R^\mu)^2(1+\cos\theta)^2$$

$$\frac{d\sigma_{LL}}{d\Omega} = \frac{1}{64\pi^2}\frac{g_Z^4 s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}(c_L^e)^2(c_L^\mu)^2(1+\cos\theta)^2$$

$$\frac{d\sigma_{LR}}{d\Omega} = \frac{1}{64\pi^2}\frac{g_Z^4 s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}(c_L^e)^2(c_R^\mu)^2(1-\cos\theta)^2$$

$$\frac{d\sigma_{RL}}{d\Omega} = \frac{1}{64\pi^2}\frac{g_Z^4 s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}(c_R^e)^2(c_L^\mu)^2(1-\cos\theta)^2$$





★ **Because $|M_{LL}|^2+|M_{RR}|^2 \neq |M_{LR}|^2+|M_{RL}|^2$, the differential cross section is asymmetric, i.e. parity violation (although not maximal as was the case for the W boson).**

Cross section with unpolarized beams

★ To calculate the total cross section need to sum over all matrix elements and **average** over the **initial spin states**. Here, assuming unpolarized beams (i.e. both **e⁺** and both **e⁻** spin states equally likely) there a four combinations of initial electron/positron spins, so

$$
\begin{aligned}
\langle |M_{fi}|^2 \rangle &= \frac{1}{2} \cdot \frac{1}{2} . (|M_{RR}|^2 + |M_{LL}|^2 + |M_{LR}|^2 + |M_{RL}|^2) \\
&= \tfrac{1}{2} \cdot \tfrac{1}{2} \frac{g_Z^4 s^2}{(s - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \times \{ [(c_R^e)^2 (c_R^\mu)^2 + (c_L^e)^2 (c_L^2)^2](1 + \cos\theta)^2 \\
&\qquad + [(c_L^e)^2 (c_R^\mu)^2 + (c_R^e)^2 (c_L^2)^2](1 - \cos\theta)^2 \}
\end{aligned}
$$

★ The part of the expression {…} can be rearranged:

$$
\begin{aligned}
\{\ldots\} = {} & [(c_R^e)^2 + (c_L^e)^2][(c_R^\mu)^2 + (c_L^\mu)^2](1 + \cos^2\theta) \\
& + 2[(c_R^e)^2 - (c_L^e)^2][(c_R^\mu)^2 - (c_L^\mu)^2] \cos\theta
\end{aligned}
$$

and using $c_V^2 + c_A^2 = 2(c_L^2 + c_R^2)$ and $c_V c_A = c_L^2 - c_R^2$

$$
\{\ldots\} = \frac{1}{4} [(c_V^e)^2 + (c_A^e)^2][(c_V^\mu)^2 + (c_A^\mu)^2](1 + \cos^2\theta) + 2 c_V^e c_A^e c_V^\mu c_A^\mu \cos\theta
$$

★ **Hence the complete expression for the unpolarized differential cross section is:**

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}\langle|M_{fi}|^2\rangle \\ &= \frac{1}{64\pi^2}\cdot\frac{1}{4}\cdot\frac{g_Z^4 s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\times \\ &\quad \left\{\tfrac{1}{4}[(c_V^e)^2+(c_A^e)^2][(c_V^\mu)^2+(c_A^\mu)^2](1+\cos^2\theta)+2c_V^e c_A^e c_V^\mu c_A^\mu\cos\theta\right\}\end{aligned}$$

★ **Integrating over solid angle** $\mathrm{d}\Omega = \mathrm{d}\phi\,\mathrm{d}(\cos\theta) = 2\pi\mathrm{d}(\cos\theta)$

$\int_{-1}^{+1}(1+\cos^2\theta)\mathrm{d}(\cos\theta) = \int_{-1}^{+1}(1+x^2)dx = \frac{8}{3}$ **and** $\int_{-1}^{+1}\cos\theta\,\mathrm{d}(\cos\theta) = 0$

$$\boxed{\sigma_{e^+e^-\to Z\to\mu^+\mu^-} = \frac{1}{192\pi}\frac{g_Z^4 s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\left[(c_V^e)^2+(c_A^e)^2][(c_V^\mu)^2+(c_A^\mu)^2\right]}$$

★ **Note: the total cross section is proportional to the sums of the squares of the vector- and axial-vector couplings of the initial and final state fermions**

$$(c_V^f)^2+(c_A^f)^2$$

Connection to the Breit-Wigner Formula

★ **Can write the total cross section**

$$\boxed{\sigma_{e^+e^-\to Z\to\mu^+\mu^-} = \frac{1}{192\pi}\frac{g_Z^4 s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\left[(c_V^e)^2+(c_A^e)^2\right]\left[(c_V^\mu)^2+(c_A^\mu)^2\right]}$$

in terms of the Z boson decay rates (partial widths) from p page 496 8 (question 26)

$$\Gamma(Z\to e^+e^-) = \frac{g_Z^2 m_Z}{48\pi}\left[(c_V^e)^2+(c_A^e)^2\right] \quad \textbf{and} \quad \Gamma(Z\to\mu^+\mu^-) = \frac{g_Z^2 m_Z}{48\pi}\left[(c_V^\mu)^2+(c_A^\mu)^2\right]$$

$$\Rightarrow \quad \sigma = \frac{12\pi}{m_Z^2}\frac{s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\Gamma(Z\to e^+e^-)\Gamma(Z\to\mu^+\mu^-)$$

★ **Writing the partial widths as $\Gamma_{ee} = \Gamma(Z\to e^+e^-)$ etc., the total cross section can be written**

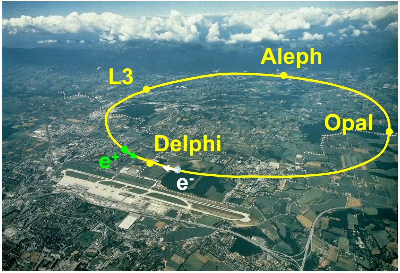
$$\boxed{\sigma(e^+e^- \to Z \to f\overline{f}) = \frac{12\pi}{m_Z^2}\frac{s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\Gamma_{ee}\Gamma_{ff}}$$

where f is the final state fermion flavour:

(The relation to the non-relativistic form of the part II course is given in the appendix)

Electroweak Measurements at LEP

★ The **L**arge **E**lectron **P**ositron (LEP) Collider at CERN (1989-2000) was designed to make precise measurements of the properties of the Z and W bosons.

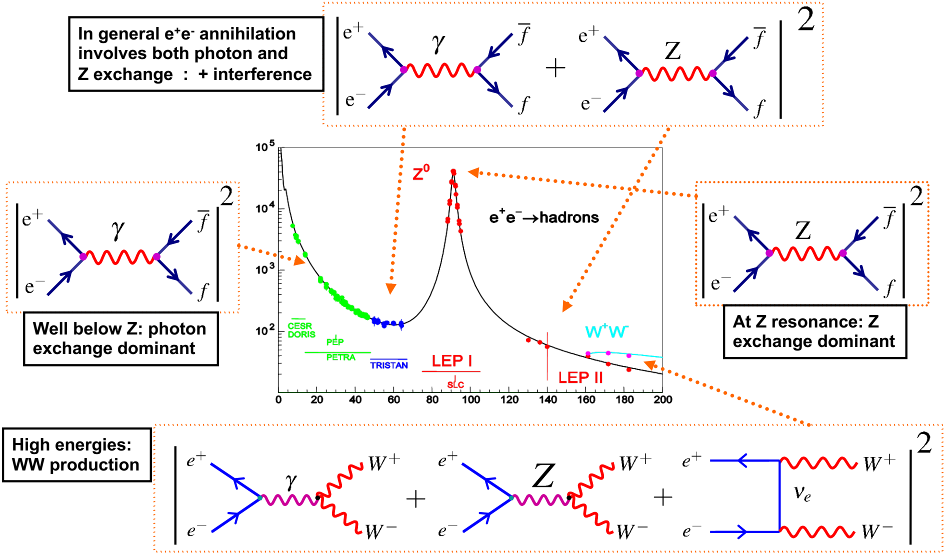


- 26 km circumference accelerator straddling French/Swiss boarder
- Electrons and positrons collided at 4 interaction points
- 4 large detector collaborations (each with 300-400 physicists):
  - ALEPH,
  - DELPHI,
  - L3,
  - OPAL

Basically a large Z and W factory:

- ★ 1989-1995: Electron-Positron collisions at $\sqrt{s}$ = 91.2 GeV
  - 17 Million Z bosons detected
- ★ 1996-2000: Electron-Positron collisions at $\sqrt{s}$ = 161-208 GeV
  - 30000 W^+W^- events detected

e^+e^- Annihilation in Feynman Diagrams



In general e^+e^- annihilation involves both photon and Z exchange : + interference

Well below Z: photon exchange dominant

At Z resonance: Z exchange dominant

High energies: WW production

Cross Section Measurements

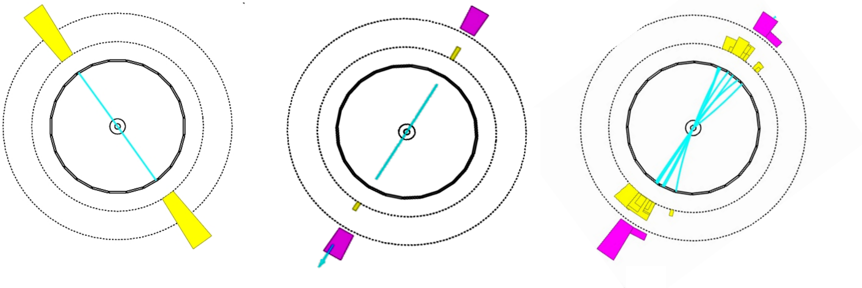
★ **At Z resonance mainly observe four types of event:**

$$e^+e^- \rightarrow Z \rightarrow e^+e^- \qquad e^+e^- \rightarrow Z \rightarrow \mu^+\mu^- \qquad e^+e^- \rightarrow Z \rightarrow \tau^+\tau^-$$

$$e^+e^- \rightarrow Z \rightarrow q\overline{q} \rightarrow \text{hadrons}$$

★ **Each has a distinct topology in the detectors, e.g.**

$$e^+e^- \rightarrow Z \rightarrow e^+e^- \qquad e^+e^- \rightarrow Z \rightarrow \mu^+\mu^- \qquad e^+e^- \rightarrow Z \rightarrow \text{hadrons}$$



★ **To work out cross sections, first count events of each type**

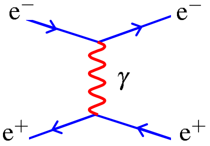
★ **Then need to know "integrated luminosity" of colliding beams, i.e. the relation between cross-section and expected number of interactions**

$$N_{\text{events}} = \mathscr{L}\sigma$$

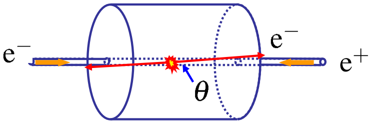
- ★ **To calculate the integrated luminosity need to know numbers of electrons and positrons in the colliding beams and the exact beam profile**

  - very difficult to achieve with precision of better than 10%

- ★ **Instead "normalise" using another type of event:**



- ♦ **Use the QED Bhabha scattering process**
- ♦ **QED, so cross section can be calculated very precisely**
- ♦ **Very large cross section – small statistical errors**
- ♦ **Reaction is very forward peaked – i.e. the electron tends not to get deflected much**



$$\frac{d\sigma}{d\Omega} \propto \frac{1}{q^4} \propto \frac{1}{\sin^4\theta/2} \Rightarrow \boxed{\frac{d\sigma}{d\theta} \propto \frac{1}{\theta^3}}$$

Photon propagator

e.g. see handout 5

- ♦ **Count events where the electron is scattered in the very forward direction**

$$N_{\text{Bhabha}} = \mathscr{L}\sigma_{\text{Bhabha}} \Rightarrow \mathscr{L}$$

σ_{Bhabha} **known from QED calc.**

- ★ **Hence all other cross sections can be expressed as**

$$\boxed{\sigma_i = \frac{N_i}{N_{\text{Bhabha}}}\sigma_{\text{Bhabha}}} \Rightarrow$$

Cross section measurements Involve just event counting !

Measurements of the Z Line-shape

- ★ **Measurements of the Z resonance lineshape determine:**
  - m_Z : **peak of the resonance**
  - Γ_Z : **FWHM of resonance**
  - Γ_f : **Partial decay widths**
  - N_ν : **Number of light neutrino generations**
- ★ **Measure cross sections to different final states versus C.o.M. energy** $\sqrt{s}$
- ★ **Starting from**

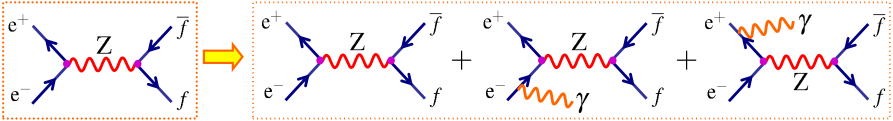
$$\sigma(e^+e^- \to Z \to f\overline{f}) = \frac{12\pi}{m_Z^2}\frac{s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\Gamma_{ee}\Gamma_{ff} \qquad \textbf{(X)}$$

maximum cross section occurs at $\sqrt{s} = m_Z$ **with peak cross section equal to**

$$\boxed{\sigma^0_{f\overline{f}} = \frac{12\pi}{m_Z^2}\frac{\Gamma_{ee}\Gamma_{ff}}{\Gamma_Z^2}}$$

- ★ **Cross section falls to half peak value at** $\sqrt{s} \approx m_z \pm \frac{\Gamma_Z}{2}$ **which can be seen immediately from eqn. (X)**
- ★ **Hence** $\Gamma_Z = \frac{\hbar}{\tau_Z} = \text{FWHM of resonance}$

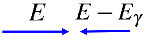
- ★ In practise, it is not that simple, QED corrections distort the measured line-shape
- ★ One particularly important correction: initial state radiation (**ISR**)



- ★ Initial state radiation reduces the centre-of-mass energy of the e^+e^- collision

$$\sqrt{s} = 2E$$

becomes



$$\sqrt{s'} \approx 2E\left(1 - \frac{E_\gamma}{2E}\right)$$

- ★ Measured cross section can be written:

$$\sigma_{\text{meas}}(E) = \int \sigma(E')f(E',E)\mathrm{d}E'$$

Probability of e+e- colliding with C.o.M. energy **E'** when C.o.M energy before radiation is **E**

- ★ Fortunately can calculate $f(E',E)$ very precisely, just QED, and can then obtain Z line-shape from measured cross section

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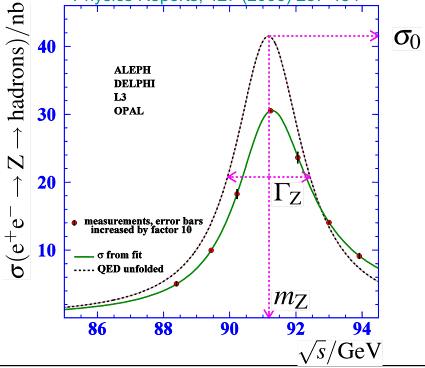


★ **In principle the measurement of m_Z and Γ_Z is rather simple:**
run accelerator at different energies, measure cross sections, account for ISR, then find peak and FWHM

$$\boxed{m_Z = 91.1875 \pm 0.0021\,\mathrm{GeV}} \qquad \boxed{\Gamma_Z = 2.4952 \pm 0.0023\,\mathrm{GeV}}$$

★ **0.002 % measurement of m_Z !**

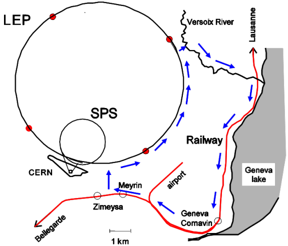
★ **To achieve this level of precision – need to know energy of the colliding beams to better than 0.002 % : sensitive to unusual systematic effects…**

Moon:

- **As the moon orbits the Earth it distorts the rock in the Geneva area very slightly !**
- **The nominal radius of the accelerator of 4.3 km varies by ±0.15 mm**
- **Changes beam energy by ~10 MeV : need to correct for tidal effects !**

Trains:

- **Leakage currents from the TGV railway line return to Earth following the path of least resistance.**
- **Travelling via the Versoix river and using the LEP ring as a conductor.**
- **Each time a TGV train passed by, a small current circulated LEP slightly changing the magnetic field in the accelerator**
- **LEP beam energy changes by ~10 MeV**



Number of generations

- ★ **Total decay width measured from Z line-shape:** $\Gamma_Z = 2.4952 \pm 0.0023\,\mathrm{GeV}$
- ★ **If there were an additional 4th generation would expect $Z \to \nu_4 \overline{\nu}_4$ decays even if the charged leptons and fermions were too heavy (i.e. > $m_Z/2$)**
- ★ **Total decay width is the sum of the partial widths:**

$$\Gamma_Z = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_{\mathrm{hadrons}} + \Gamma_{\nu_1\nu_1} + \Gamma_{\nu_2\nu_2} + \Gamma_{\nu_3\nu_3} + ?$$

- ★ **Although don't observe neutrinos, $Z \to \nu\overline{\nu}$ decays affect the Z resonance shape for all final states**
- ★ **For all other final states can determine partial decay widths from peak cross sections:**

$$\sigma^0_{f\bar{f}} = \frac{12\pi}{m_Z^2}\frac{\Gamma_{ee}\Gamma_{ff}}{\Gamma_Z^2}$$

- ★ **Assuming lepton universality:**

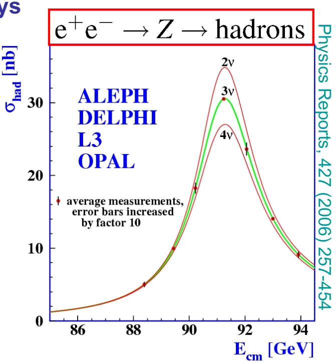
$$\Gamma_Z = 3\Gamma_{\ell\ell} + \Gamma_{\mathrm{hadrons}} + \boxed{N_\nu}\,\Gamma_{\nu\nu}$$

measured from Z lineshape

measured from peak cross sections

calculated, e.g. question 26

⇨ $$\boxed{N_\nu = 2.9840 \pm 0.0082}$$



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- ★ **ONLY 3 GENERATIONS** **(unless a new 4th generation neutrino has very large mass)**

Forward-Backward Asymmetry

★ **On page 495 we obtained the expression for the differential cross section:**

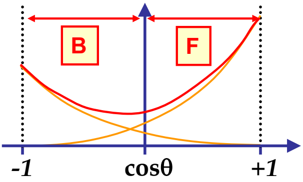
$$\langle|M_{fi}|\rangle^2 \propto [(c_L^e)^2+(c_R^e)^2][(c_L^\mu)^2+(c_R^\mu)^2](1+\cos^2\theta)+\ 2\,[(c_L^e)^2-(c_R^e)^2][(c_L^\mu)^2-(c_R^\mu)^2]\cos\theta$$

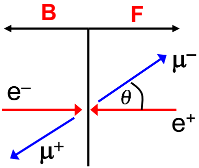
★ **The differential cross sections is therefore of the form:**

(43)
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \kappa\times[A(1+\cos^2\theta)+B\cos\theta] \quad \begin{cases} A = [(c_L^e)^2+(c_R^e)^2][(c_L^\mu)^2+(c_R^\mu)^2] \\ B = 2[(c_L^e)^2-(c_R^e)^2][(c_L^\mu)^2-(c_R^\mu)^2] \end{cases}$$

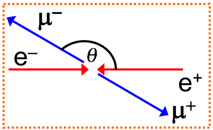
★ **Define the FORWARD and BACKWARD cross sections in terms of angle incoming electron and out-going particle**

$$\boxed{\sigma_F \equiv \int_0^1 \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\mathrm{d}\cos\theta} \qquad \boxed{\sigma_B \equiv \int_{-1}^0 \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\mathrm{d}\cos\theta}$$



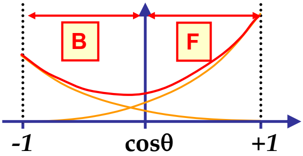


e.g. "backward hemisphere"



★ **The level of asymmetry about cosθ=0 is expressed in terms of the Forward-Backward Asymmetry**

$$A_{\mathrm{FB}} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B}$$



- **Integrating equation (43)**

$$\sigma_F = \kappa \int_0^1 [A(1+\cos^2\theta) + B\cos\theta]\mathrm{d}\cos\theta = \kappa \int_0^1 [A(1+x^2) + Bx]\mathrm{d}x = \kappa\left(\frac{4}{3}A + \frac{1}{2}B\right)$$

$$\sigma_B = \kappa \int_{-1}^0 [A(1+\cos^2\theta) + B\cos\theta]\mathrm{d}\cos\theta = \kappa \int_{-1}^0 [A(1+x^2) + Bx]\mathrm{d}x = \kappa\left(\frac{4}{3}A - \frac{1}{2}B\right)$$

★ **Which gives:**

$$A_{\mathrm{FB}} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{B}{(8/3)A} = \frac{3}{4}\left[\frac{(c_L^e)^2 - (c_R^e)^2}{(c_L^e)^2 + (c_R^e)^2}\right] \cdot \left[\frac{(c_L^\mu)^2 - (c_R^\mu)^2}{(c_L^\mu)^2 + (c_R^\mu)^2}\right]$$

★ **This can be written as**

$$\boxed{A_{\mathrm{FB}} = \frac{3}{4}A_e A_\mu}$$

with

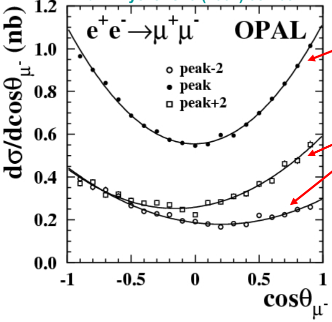
$$\boxed{A_f \equiv \frac{(c_L^f)^2 - (c_R^f)^2}{(c_L^f)^2 + (c_R^f)^2} = \frac{2c_V^f c_A^f}{(c_V^f)^2 + (c_A^f)^2}}$$

★ **Observe a non-zero asymmetry because the couplings of the Z to LH and RH particles are different. Contrast with QED where the couplings to LH and RH particles are the same (parity is conserved) and the interaction is FB symmetric**

Measured Forward-Backward Asymmetries

★ **Forward-backward asymmetries can only be measured for final states where the charge of the fermion can be determined, e.g.** $e^+e^- \rightarrow Z \rightarrow \mu^+\mu^-$

OPAL Collaboration, Eur. Phys. J. C19 (2001) 587-651.



Because $\sin^2\theta_W \approx 0.25$, the value of A_{FB} for leptons is almost zero

For data above and below the peak of the Z resonance interference with $e^+e^- \rightarrow \gamma \rightarrow \mu^+\mu^-$ **leads to a larger asymmetry**

★**LEP data combined:**

$$A_{FB}^{0,e} = 0.0145 \pm 0.0025$$

$$A_{FB}^{0,\mu} = 0.0169 \pm 0.0013$$

$$A_{FB}^{0,\tau} = 0.0188 \pm 0.0017$$

- ★**To relate these measurements to the couplings uses** $A_{\text{FB}} = \frac{3}{4}A_e A_\mu$
- ★ **In all cases asymmetries depend on** A_e
- ★ **To obtain** A_e **could use** $A_{FB}^{0,e} = \frac{3}{4}A_e^2$ **(also see Appendix II for A_{LR})**

Determination of the Weak Mixing Angle

$$\left.\begin{array}{ll} \star\ \textbf{From LEP :} & A_{FB}^{0,f} = \frac{3}{4} A_e A_f \\ \star\ \textbf{From SLC :} & A_{LR} = A_e \end{array}\right\} \quad A_e, A_\mu, A_\tau, \ldots$$

Putting everything together ➡

$$\boxed{\begin{aligned} A_e &= 0.1514 \pm 0.0019 \\ A_\mu &= 0.1456 \pm 0.0091 \\ A_\tau &= 0.1449 \pm 0.0040 \end{aligned}}$$

includes results from other measurements

with $$A_f \equiv \frac{2 c_V^f c_A^f}{(c_V^f)^2 + (c_A^f)^2} = 2 \frac{c_V / c_A}{1 + (c_V / c_A)^2}$$

- ★ **Measured asymmetries give ratio of vector to axial-vector Z coupings.**
- ★ **In SM these are related to the weak mixing angle**

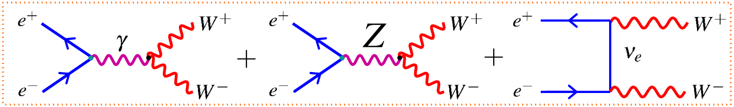
$$\frac{c_V}{c_A} = \frac{I_W^3 - 2Q \sin^2\theta_W}{I_W^3} = 1 - \frac{2Q}{I_3} \sin^2\theta_W = 1 - 4|Q| \sin^2\theta_W$$

- ★ **Asymmetry measurements give precise determination of** $\sin^2\theta_W$

$$\boxed{\sin^2\theta_W = 0.23154 \pm 0.00016}$$

W^+W^- Production

- ★ **From 1995-2000 LEP operated above the threshold for W-pair production**
- ★ **Three diagrams "CC03" are involved**

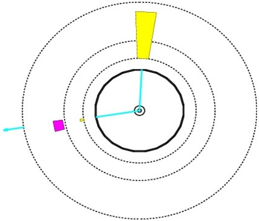


- ★ **W bosons decay (p.459) either to leptons or hadrons with branching fractions:**

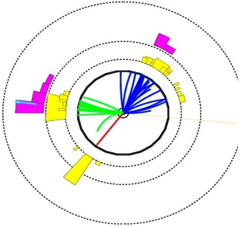
$$Br(W^- \rightarrow \text{hadrons}) \approx 0.67 \qquad Br(W^- \rightarrow \mathrm{e}^-\overline{\nu}_\mathrm{e}) \approx 0.11$$

$$Br(W^- \rightarrow \mu^-\overline{\nu}_\mu) \approx 0.11 \qquad Br(W^- \rightarrow \tau^-\overline{\nu}_\tau) \approx 0.11$$

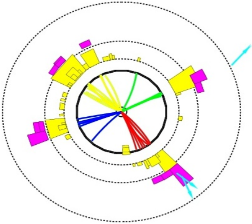
- ★ **Gives rise to three distinct topologies**



$\boxed{W^+W^- \rightarrow \ell^+\nu\ell^-\overline{\nu}}$



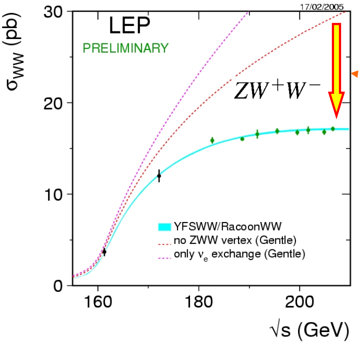
$\boxed{W^+W^- \rightarrow q\overline{q}\ell\nu}$



$\boxed{W^+W^- \rightarrow q\overline{q}q\overline{q}}$

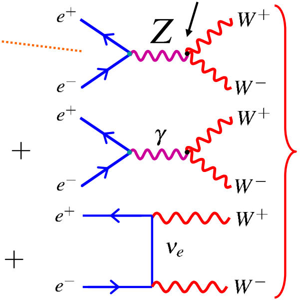
$e^+e^- \rightarrow W^+W^-$ Cross Section

★ Measure cross sections by **counting events** and normalising to low angle Bhabha scattering events



★ Data consistent with SM expectation

★ Provides a direct test of ZW^+W^- vertex



★ Recall that without the Z diagram the cross section violates unitarity

★ Presence of Z fixes this problem

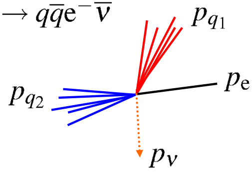
W-mass and W-width

★ **Unlike** $e^+e^- \to Z$ **, the process** $e^+e^- \to W^+W^-$ **is not a resonant process**

⇨ **Different method to measure W-boson Mass**

•**Measure energy and momenta of particles produced in the W boson decays, e.g.**

$$W^+W^- \to q\overline{q}e^-\overline{\nu}$$

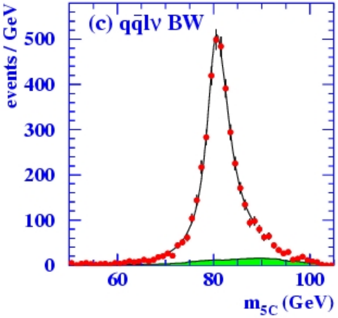


- **Neutrino four-momentum from energy-momentum conservation !**

$$p_{q_1} + p_{q_2} + p_e + p_\nu = (\sqrt{s}, 0)$$

- **Reconstruct masses of two W bosons**

$$M_+^2 = E^2 - \vec{p}^2 = (p_{q_1} + p_{q_2})^2$$
$$M_-^2 = E^2 - \vec{p}^2 = (p_e + p_\nu)^2$$



$$\approx \frac{1}{2}(M_+ + M_-)$$

★ **Peak of reconstructed mass distribution gives**

$$\boxed{m_W = 80.376 \pm 0.033\,\text{GeV}}$$

★ **Width of reconstructed mass distribution gives:**

$$\boxed{\Gamma_W = 2.196 \pm 0.083\,\text{GeV}}$$

Does not include measurements from Tevatron at Fermilab

The Higgs Mechanism

★ Higgs mechanism can be used to give masses to both fermions and gauge bosons – but mechanism is different in the two cases.

★Explaining how the Higgs mechanism gives the W and Z gauge bosons masses, while leaving the photon massless, is (unfortunately) beyond this course. [See, hopefully, Gauge Field Theory minor option)]

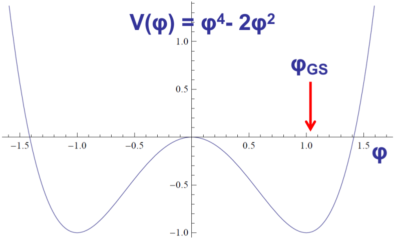
★By way of apology, we instead provide here an attempt to at least describe the way the mechanism gives masses to fermions – that will hopefully whet your appetite.

Higgs Mechanism & Higgs Boson (1)

- Quantum Field Theories (QFTs) are written down in a Lagrangian formalism.
- A scalar field x with a mass m must have a term "½m^2xx" in the Lagrangian.
- A fermionic field ψ with a mass m must have a term "mψψ" in the Lagrangian.
- QFTs that are "Gauge Field Theories" have a Lagrangian which is also invariant under the action of a "Gauge Group".
- The Standard Model "Gauge Group" is chosen to be U(1)xSU(2)$_L$xSU(3) in order to allow it to model EM, weak and strong interactions in accordance with experiment.
- Terms of the type mψψ are (unfortunately!) not invariant under the above gauge group. So one cannot have massive fermions (eg muon) in the Standard Model ☹
- However, interactions between fields enter the Lagrangian as products of three or more fields. For example, a term proportional to "φψψ" leads to the theory having an interaction vertex connecting one φ to two ψ particles. So:
- IF you could contrive to have a term "φψψ" in the Lagrangian AND could guarantee that φ could spend most of its time taking values near some non-zero value "m", THEN (1) the fermion field ψ would act "as if" there were a term "mψψ" in the Lagrangian, and so would look very much like it had mass m, even if it were actually massless, and (2) the field ψ would have an interaction with the field φ, leading to the testable and falsifiable prediction that an excitation of the field φ (i.e. a "φ particle") should couple to, or decay into, the fermions to which it "gives mass".

Higgs Mechanism & Higgs Boson (2)

- **A field φ could spend a lot of time near a non-zero value if it took a non-zero value in its ground state. Most fields take the value of zero in their ground-state, but this need not always be the case:**



- **For example, a field φ having a potential energy $V(\varphi) = a\varphi^4 - b\varphi^2$ has a ground-state located at $\varphi_{GS}=\pm\sqrt{(b/(2a))}$**
- **So by arranging:**
  - (1) for φ to have a non-zero value φ_{GS} in its ground state by ensuring that the potential V(φ) in the Lagrangian is of the right form, and
  - (2) for there to be a (gauge invariant) interaction term "yφψψ" in the Lagrangian ("y" being just a constant of proportionality called the "Yukawa Coupling") ...
- **... then the field ψ will look like it has a mass $m=y\varphi_{GS}$! Call φ the "Higgs Field".**
- **Give different fermions different masses by using different Yukawa Couplings.**
- **Note that in the vicinity of the minimum, the potential V(φ) necessarily takes the form $V(\varphi_{GS}+x) = V_{min}+\lambda x^2+O(x^3)$ for some constants λ and V_{min}. We already said that terms like λx^2 are banned from the Lagrangian if x is a fermionic field as they break gauge invariance. However, these terms are not banned if x is a scalar field. So this excitation x of the Higgs Field must be a scalar. Call it the "Higgs Boson". We recognise λx^2 as a mass-term for a scalar, so the Higgs Boson has a free (and unknown) mass.**

Higgs theory summary for fermions:

Fermions are intrinsically massless, and need to be so to satisfy "Gauge Invariance".

Nevertheless, interactions with the Higgs field make fermions look like they have mass at "low temperature" (i.e. when the Higgs field is near its ground state, below ~10^{15} K)

Apparent fermion masses are controlled by free parameters called Yukawa Couplings (the strength of the coupling to the Higgs field)

A Higgs Boson is an excitation of the Higgs Field.

The Higgs Boson must be a scalar particle to make everything work.

The Higgs Boson has a mass, but the mass is not predicted by the theory – we have to find it experimentally.

The Higgs Boson has couplings to all the particles it gives mass to (and indeed to gauge bosons too!) and so has many ways it could decay, all fully calculable and determined by the theory as a function of its (as yet unknown) mass

(For proper discussion of the Higgs mechanism see the Gauge Field Theory minor option)

Higgs mechanism for gauge bosons:

★ The Higgs mechanism results in absolute **predictions** for **masses of gauge bosons**

★ In the SM, fermion masses are also ascribed to interactions with the Higgs field

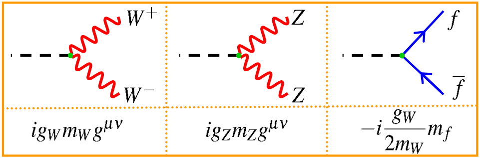
- however, here no prediction of the masses – **just put in by hand**

★ The Higgs is electrically neutral but carries weak hypercharge of **1/2**

★ The photon does not couple to the Higgs field and remains massless

★ The W bosons and the Z couple to weak hypercharge and become massive

Feynman Vertex factors:



$ig_W m_W g^{\mu\nu}$ | $ig_Z m_Z g^{\mu\nu}$ | $-i\frac{g_W}{2m_W}m_f$

★ Within the SM of Electroweak unification with the Higgs mechanism:

Relations between standard model parameters

$$m_W = \left(\frac{\pi\alpha_{em}}{\sqrt{2}G_F}\right)^{\frac{1}{2}}\frac{1}{\sin\theta_W}$$

$$m_Z = \frac{m_W}{\cos\theta_W}$$

★ Hence, if you know **any three** of : $\alpha_{em}, G_F, m_W, m_Z, \sin\theta_W$ predict the other two.

Precision Tests of the Standard Model

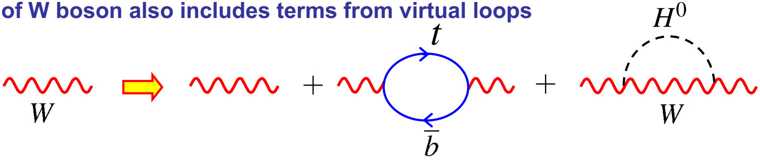
★ **From LEP and elsewhere have precise measurements – can test predictions of the Standard Model !**

- **e.g. predict:** $\boxed{m_W = m_Z \cos\theta_W}$ **measure** $\boxed{m_Z = 91.1875 \pm 0.0021\,\text{GeV}}$ $\boxed{\sin^2\theta_W = 0.23154 \pm 0.00016}$
- **Therefore expect:**

$\boxed{m_W = 79.946 \pm 0.008\,\text{GeV}}$ **but measure** $\boxed{m_W = 80.376 \pm 0.033\,\text{GeV}}$

★ **Close, but not quite right – but have only considered lowest order diagrams**

★ **Mass of W boson also includes terms from virtual loops**



$$\boxed{m_W = m_W^0 + a m_t^2 + b \ln\left(\frac{m_H}{m_W}\right)}$$

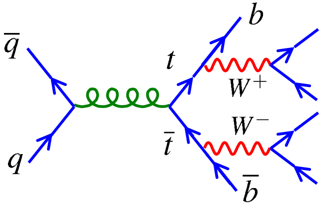
★ **Above "discrepancy" due to these virtual loops, i.e. by making very high precision measurements become sensitive to the masses of particles inside the virtual loops !**

The Top Quark

- ★ **From virtual loop corrections and precise LEP data can predict the top quark mass:**

$$\boxed{m_t^{\text{loop}} = 173 \pm 11\,\text{GeV}}$$

- ★ **In 1994 top quark observed at the Tevatron proton anti-proton collider at Fermilab – with the predicted mass !**



- ★ **The top quark almost exclusively decays to a bottom quark since**

$$|V_{tb}|^2 \gg |V_{td}|^2 + |V_{ts}|^2$$

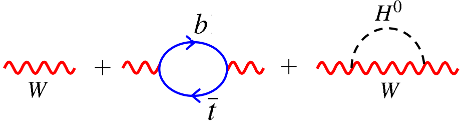
- ★ **Complicated final state topologies:**

$$t\bar{t} \to b\bar{b}q\bar{q}q\bar{q} \to 6\text{ jets}$$
$$t\bar{t} \to b\bar{b}q\bar{q}\ell\nu \to 4\text{ jets} + \ell + \nu$$
$$t\bar{t} \to b\bar{b}\ell\nu\ell\nu \to 2\text{ jets} + 2\ell + 2\nu$$

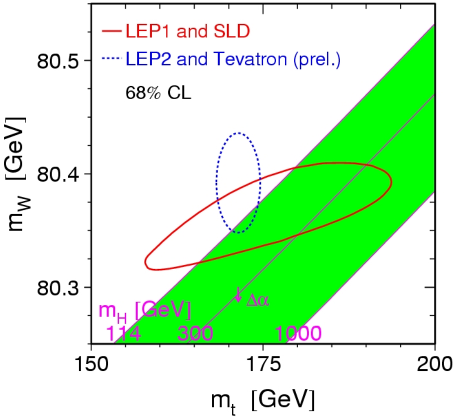
- ★ **Mass determined by direct reconstruction (see W boson mass)**

$$\boxed{m_t^{\text{meas}} = 174.2 \pm 3.3\,\text{GeV}}$$

★ **But the W mass also depends on the Higgs mass (albeit only logarithmically)**



$$m_W = m_W^0 + a m_t^2 + b \ln\left(\frac{m_H}{m_W}\right)$$



★ **Measurements are sufficiently precise to have some sensitivity to the Higgs mass**

★ **Direct and indirect values of the top and W mass can be compared to prediction for different Higgs mass**

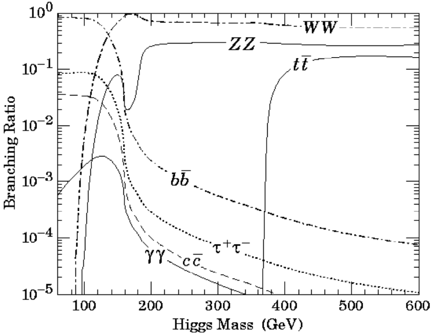
- **Direct:** **W and top masses from direct reconstruction**
- **Indirect:** **from SM interpretation of Z mass, θ_W etc. and**

★ **Data favour a light Higgs:**

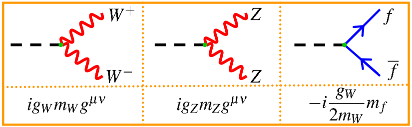
⇨ $\boxed{m_H < 200\,\mathrm{GeV}}$

Hunting the Higgs

- ★ **The Higgs boson is an essential part of the Standard Model – but does it exist ?**
- ★ **Consider the search at LEP. Need to know how the Higgs decays**



- **Higgs boson couplings proportional to mass**



$ig_W m_W g^{\mu\nu}$ $\quad ig_Z m_Z g^{\mu\nu}$ $\quad -i\frac{g_W}{2m_W}m_f$

- **Higgs decays predominantly to heaviest particles which are energetically allowed** **(Question 30)**

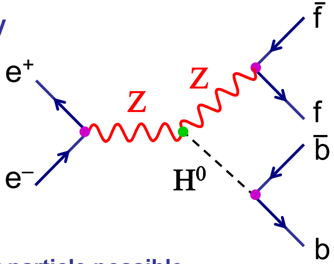
$m_H < 2m_W$ **mainly** $H^0 \rightarrow b\bar{b}$ **+ approx 10%** $H^0 \rightarrow \tau^+\tau^-$

$2m_W < m_H < 2m_t$ **almost entirely** $H^0 \rightarrow W^+W^- \ + \ H^0 \rightarrow ZZ$

$m_H > 2m_t$ **either** $H^0 \rightarrow W^+W^-,\ H^0 \rightarrow ZZ,\ H^0 \rightarrow t\bar{t}$

A Hint from LEP ?

- ★ **LEP operated with a C.o.M. energy upto 207 GeV**
- ★ **For this energy (assuming the Higgs exists) the main production mechanism would be the "Higgsstrahlung" process**



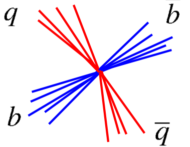
- ★ **Need enough energy to make a Z and H; therefore could produce the Higgs boson if**

$$m_H < 207\,\mathrm{GeV} - m_Z$$

i.e. if $m_H < 116\,\mathrm{GeV}$

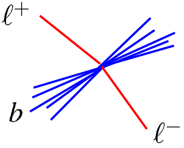
- ★ **The Higgs predominantly decays to the heaviest particle possible**
- ★ **For $m_H < 116\,\mathrm{GeV}$ this is the b-quark (not enough mass to decay to WW/ZZ/tt)**

$\boxed{q\bar{q}b\bar{b}}$



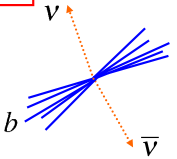
$BR(Z \to q\bar{q}) \approx 70\%$

$\boxed{\ell^+\ell^- b\bar{b}}$ $\ell = \mathrm{e}, \mu, \tau$



$BR(Z \to \ell^+\ell^-) \approx 10\%$

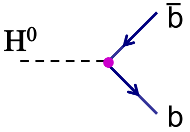
$\boxed{\nu\bar{\nu}b\bar{b}}$

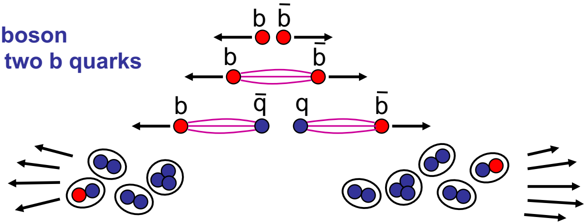


$BR(Z \to \nu\bar{\nu}) \approx 20\%$

Tagging the Higgs Boson Decays

★ One signature for a Higgs boson decay is the production of two b quarks



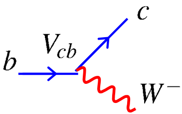


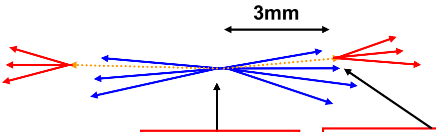
★ Each jet will contain one b-hadron which will decay weakly

★ Because V_{cb} is small $(V_{cb} \approx 0.04)$ hadrons containing b-quarks are relatively long-lived

★ Typical lifetimes of $\tau \sim 1 \times 10^{-12}\,\mathrm{s}$

★ At LEP b-hadrons travel approximately **3mm** before decaying



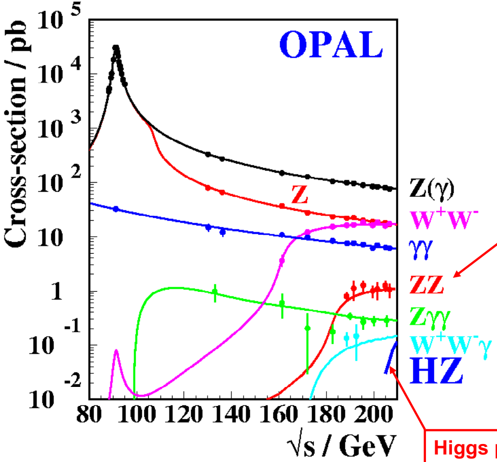


Primary vertex

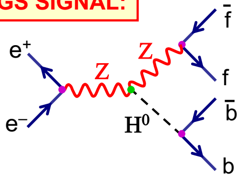
Displaced Secondary Vertex from decay of B hadron

★ Can efficiently identify jets containing b quarks

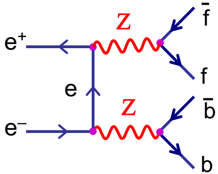
- ★ Clear experimental signature, but small cross section, e.g. for $m_H \approx 115\,\mathrm{GeV}$ would only produce a few tens of $\mathrm{e}^+\mathrm{e}^- \to H^0$ events at LEP
- ★ In addition, there are large “backgrounds”



HIGGS SIGNAL:

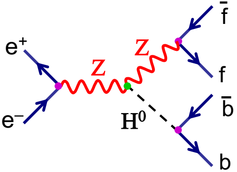


MAIN BACKGROUND:



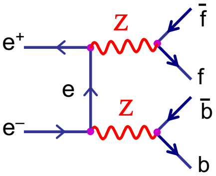
Higgs production cross section (m_H=115 GeV)

★ The only way to distinguish



$\boxed{m_{b\bar{b}} = m_H}$

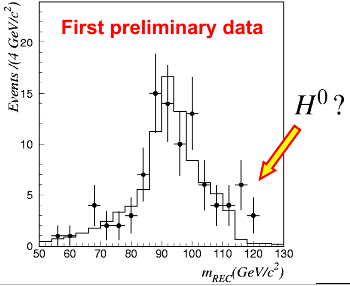
from



$\boxed{m_{b\bar{b}} = m_Z}$

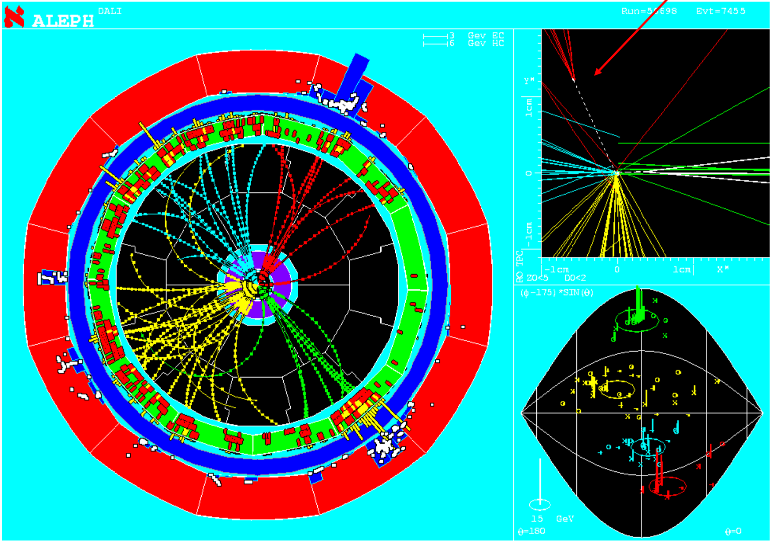
is the from the invariant mass of the jets from the boson decays

★ In 2000 (the last year of LEP running) the **ALEPH** experiment reported an excess of events consistent with being a Higgs boson with mass **115 GeV**



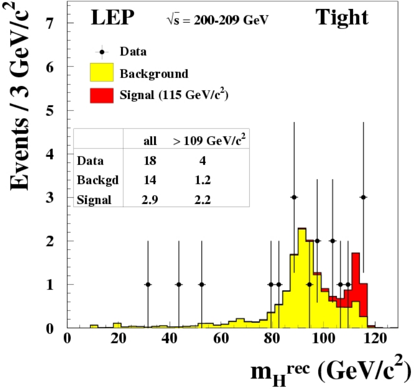
- ALEPH found 3 events which were high relative probability of being signal
- L3 found 1 event with high relative probability of being signal
- OPAL and DELPHI found none

Example event:


Displaced vertex from b-decay

Combined LEP Results

Phys. Lett. B565 (2003) 61-75



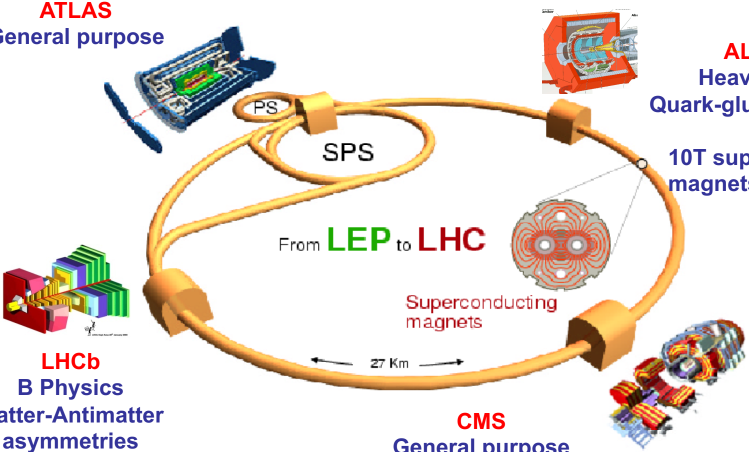
	all	> 109 GeV/c^2
Data	18	4
Backgd	14	1.2
Signal	2.9	2.2

- ★ Final combined LEP results fairly inconclusive
- ★ A hint rather than strong evidence…
- ★ All that can be concluded:

$$\boxed{m_H > 114\,\text{GeV}}$$

The Large Hadron Collider

The LHC is a new proton-proton collider now running in the old LEP tunnel at CERN.



ATLAS
General purpose

ALICE
Heavy ions
Quark-gluon plasma

10T superconducting magnets

LHCb
B Physics
Matter-Antimatter asymmetries

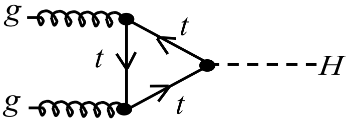
CMS
General purpose

Higgs at Large Hadron Collider

Higgs Production at the LHC

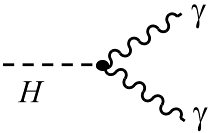
The dominant Higgs production mechanism at the LHC is

"gluon fusion"

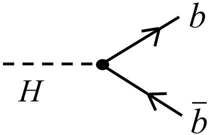


Higgs Decay at the LHC

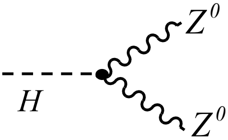
Depending on the mass of the Higgs boson, it will decay in different ways



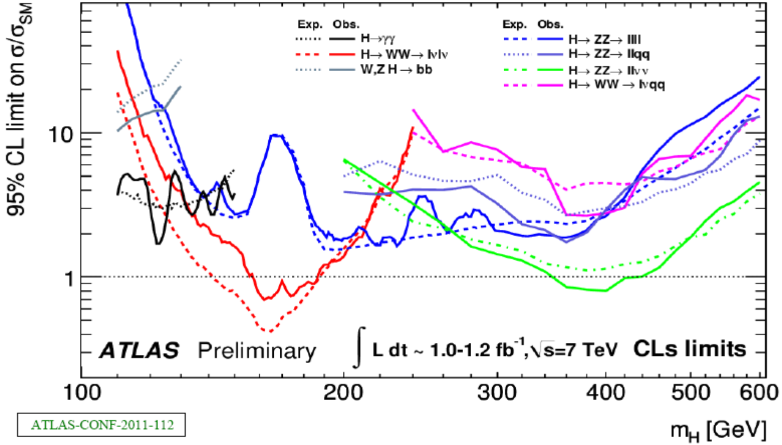
Low Mass



Medium mass



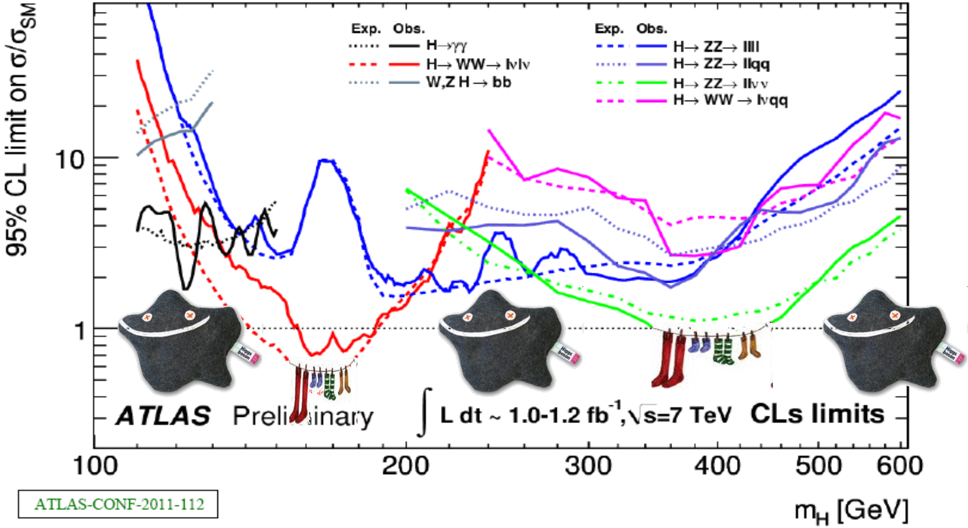
High mass



ATLAS-CONF-2011-112

LHC Higgs data is interpreted in the above plot. For any particular hypothesised Higgs boson mass (shown on the x-axis) the data places (at 95% confidence) an upper bound on the cross section for Higgs-Boson-Like events, in units of "how many would be expected from the Standard Model. In other words, a line level with "10" on the y-axis at mH=125 GeV means "If the Higgs boson has a mass of 125 GeV, then it could have been produced at up to 10 times the rate expected in the Standard Model and could still (just) have gone un-noticed, at 95% confidence".

As data arrives it should lower the curves, unless support from a Higgs boson can prevent curve from passing through dotted line at "1"



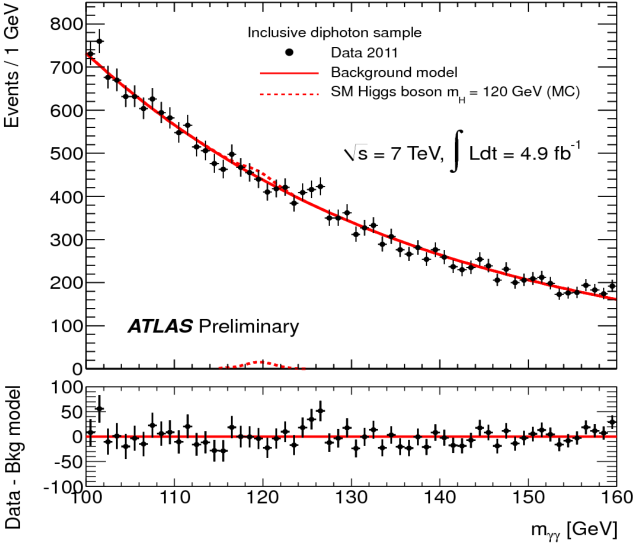
ATLAS-CONF-2011-112

Here is the (unconvincing) data that was shown in Feb 2012

The black blobs are data. The smooth curve is the expected background shape.

The small dotted “bump” indicate how a Higgs signal might change the shape of the distribution if the Higgs boson mass was 120 GeV.

The variable on the x axis is the invariant mass two photons.

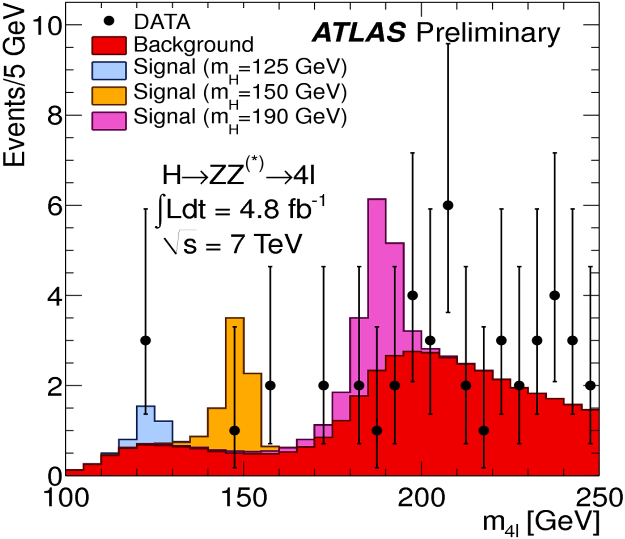


The astonishingly (un?)convincing evidence in the analysis looking for Higgs decays pairs of Z bosons

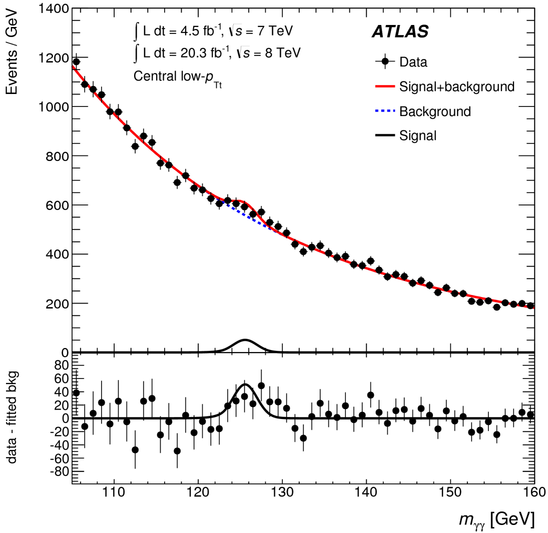
The black blobs are data.

The three triangular lumps indicate what a Higgs signal might look at (for three different Higgs boson masses).

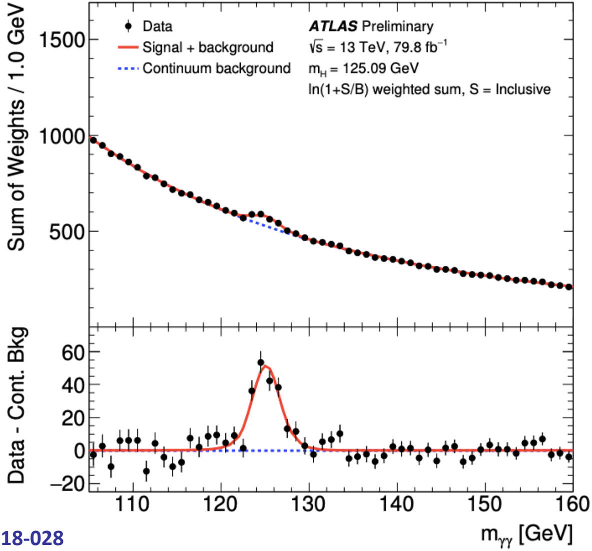
The variable on the x axis is the invariant mass of four leptons which seem to have come from two Z bosons.



The 2015 public ATLAS data for Higgs turning into two photons

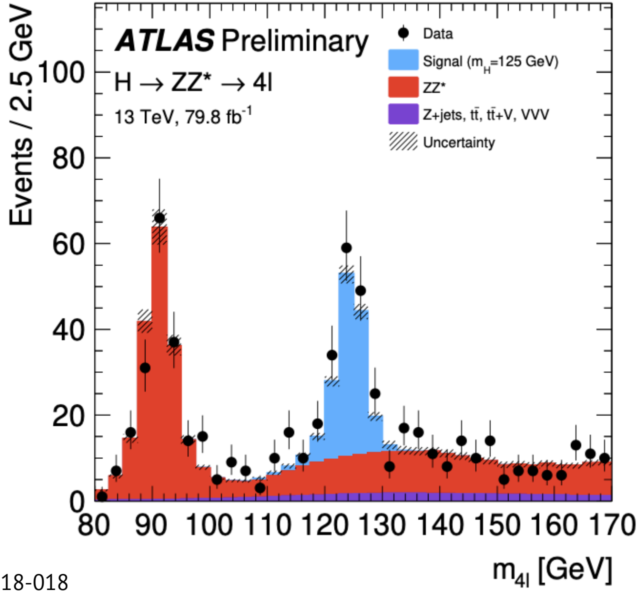


The 2018 public ATLAS data for Higgs turning into two photons



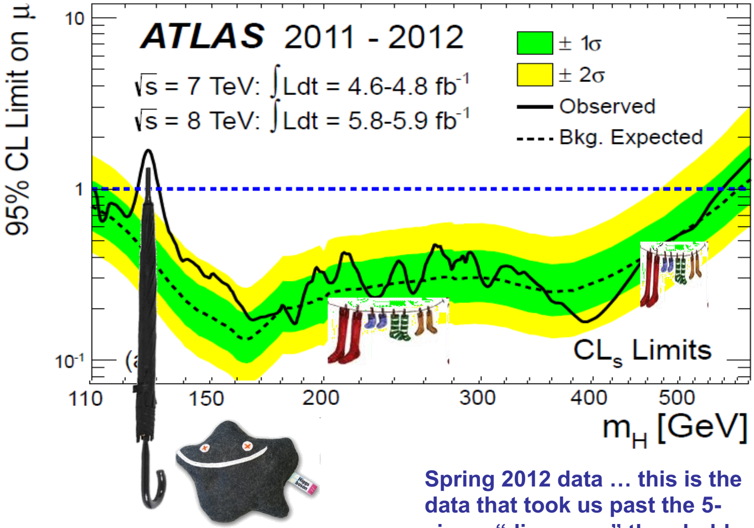
ATLAS-CONF-2018-028

... or turning into four leptons



ATLAS-CONF-2018-018

The discovery plot ... $2*10^{-9}$ = probability of fluctuation



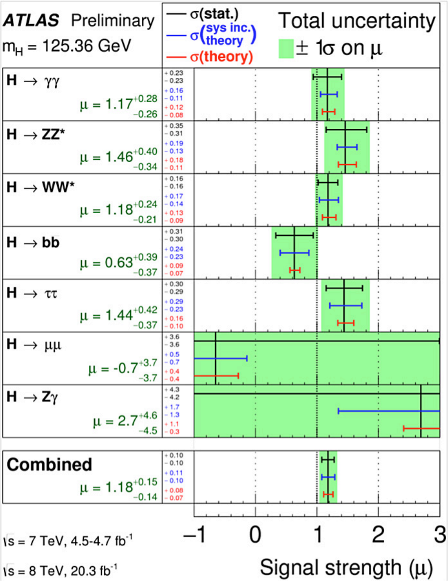
Spring 2012 data … this is the data that took us past the 5-sigma "discovery" threshold

Higgs boson

Now considered to be "discovered". Nobel Prize 2013!

What has been discovered is a bump in the sort of place you'd expect to find a Higgs Boson. In other words, a particle consistent with the Higgs Boson.

To be really sure its "The" Higgs Boson, we are acquiring more information on its spin and couplings (e.g. data shown to the right) . So far everything checks out. The Higgs looks "standard". Nonetheless, other (non-standard) Higgs Bosons could yet be found.

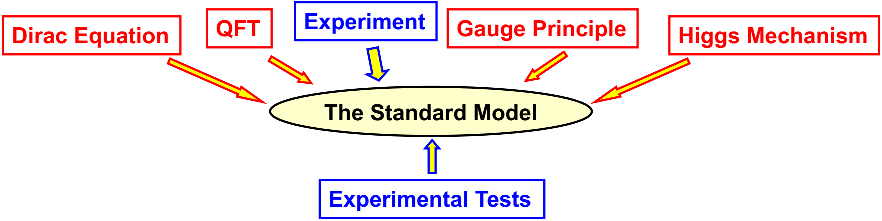


Concluding Remarks

★ In this course (I believe) we have covered almost all aspects of modern particle physics – though in each case we have barely scratched the surface.

★ The Standard Model of Particle Physics is one of the great scientific triumphs of the late 20th century

★ Developed through close interplay of experiment and theory



Dirac Equation

QFT

Experiment

Gauge Principle

Higgs Mechanism

The Standard Model

Experimental Tests

★ Modern experimental particle physics provides many precise measurements. and the **Standard Model** successfully describes all current data !

★ Despite its great success, we should not forget that it is just a model; a collection of beautiful theoretical ideas cobbled together to fit with experimental data.

★ There are many issues / open questions…

The Standard Model : Problems/Open Questions

★ The Standard Model has too many free parameters:

$$m_{\nu_1}, m_{\nu_2}, m_{\nu_3}, m_e, m_\mu, m_\tau, m_d, m_s, m_b, m_u, m_c, m_t$$

$$\theta_{12}, \theta_{13}, \theta_{23}, \delta \quad + \quad \lambda, A, \rho, \eta \qquad e, G_F, \theta_W, \alpha_S \qquad m_H, \theta_{CP}$$

★ Why three generations ?

★ Why $SU(3)_c$ x $SU(2)_L$ x $U(1)$?

★ Unification of the Forces

★ Origin of CP violation in early universe ?

★ What is Dark Matter ?

★ Why is the weak interaction V-A ?

★ Why are neutrinos so light ?

★ Ultimately need to include gravity

★ **Over the last 25 years particle physics has progressed enormously.**

In the next 10 years we will almost certainly have answers to some of the above questions – maybe not the ones we expect…

The End

Appendix I: Non-relativistic Breit-Wigner

- ★ For energies close to the peak of the resonance, can write $\sqrt{s} = m_Z + \Delta$

$$s = m_Z^2 + 2m_Z\Delta + \Delta^2 \approx m_Z^2 + 2m_Z\Delta \qquad \text{for} \quad \Delta \ll m_Z$$

so with this approximation

$$\begin{aligned}(s - m_Z^2)^2 + m_Z^2\Gamma_Z^2 \approx (2m_Z\Delta)^2 + m_Z^2\Gamma_Z^2 &= 4m_Z^2(\Delta + \tfrac{1}{4}\Gamma_Z^2) \\ &= 4m_Z^2[(\sqrt{s} - m_Z)^2 + \tfrac{1}{4}\Gamma_Z^2]\end{aligned}$$

- ★ Giving: $\sigma(e^+e^- \to Z \to f\overline{f}) \approx \dfrac{3\pi}{m_Z^4}\dfrac{s}{(\sqrt{s} - m_Z)^2 + \frac{1}{4}\Gamma_Z^2}\Gamma_e\Gamma_f$

- ★ Which can be written:

$$\sigma(E) = \frac{g\lambda_e^2}{4\pi}\frac{\Gamma_i\Gamma_f}{(E - E_0)^2 + \frac{1}{4}\Gamma^2}$$

$\Gamma_i,\ \Gamma_f$: are the partial decay widths of the initial and final states

$E,\ E_0$: are the centre-of-mass energy and the energy of the resonance

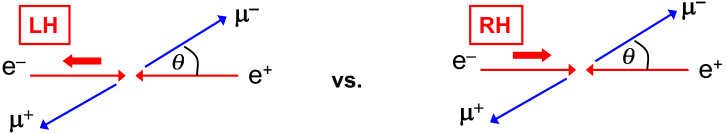
$g = \dfrac{(2J_Z+1)}{(2S_e+1)(2S_e+1)}$ is the spin counting factor $g = \dfrac{3}{2\times 2}$

$\lambda_e = \dfrac{2\pi}{E}$: is the Compton wavelength (natural units) in the C.o.M of either initial particle

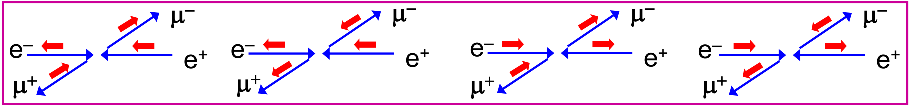
- ★ This is the **non-relativistic** form of the **Breit-Wigner** distribution first encountered in the part II particle and nuclear physics course.

Appendix II: Left-Right Asymmetry, A_{LR}

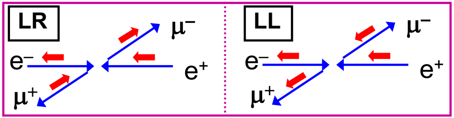
- ★ At an e^+e^- linear collider it is possible to produce polarized electron beams e.g. SLC linear collider at SLAC (California), **1989-2000**
- ★ Measure cross section for any process for LH and RH electrons separately

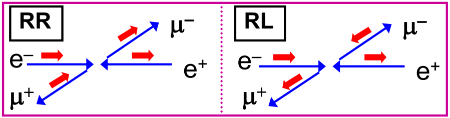


- At LEP measure total cross section: sum of 4 helicity combinations:



- At SLC, by choosing the polarization of the electron beam are able to measure cross sections **separately** for LH / RH electrons





★ **Averaging over the two possible polarization states of the positron for a given electron polarization:**

$$\langle|M_L|\rangle^2 = \tfrac{1}{2}(|M_{LL}|^2 + |M_{LR}|^2) \qquad \langle|M_R|\rangle^2 = \tfrac{1}{2}(|M_{RL}|^2 + |M_{RR}|^2)$$

⇨ $$\boxed{\sigma_L = \tfrac{1}{2}(\sigma_{LR} + \sigma_{LL})} \qquad \boxed{\sigma_R = \tfrac{1}{2}(\sigma_{RR} + \sigma_{RL})}$$

★ **Define cross section asymmetry:**

$$\boxed{A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R}}$$

★ **Integrating the expressions on page 494 gives:**

$$\sigma_{LL} \propto (c_L^e)^2(c_L^\mu)^2 \quad \sigma_{LR} \propto (c_L^e)^2(c_R^\mu)^2 \quad \sigma_{RL} \propto (c_R^e)^2(c_L^\mu)^2 \quad \sigma_{RR} \propto (c_R^e)^2(c_R^\mu)^2$$

⇨ $$\sigma_L \propto (c_L^e)^2[(c_L^\mu)^2 + (c_R^\mu)^2] \qquad \sigma_R \propto (c_R^e)^2[(c_L^\mu)^2 + (c_R^\mu)^2]$$

$$\boxed{A_{LR} = \frac{(c_L^e)^2 - (c_R^e)^2}{(c_L^e)^2 + (c_R^e)^2} = A_e}$$

★ **Hence the Left-Right asymmetry for any cross section depends only on the couplings of the electron**

- ★ **Averaging over the two possible polarization states of the positron for a given electron polarization:**

$$\langle|M_L|\rangle^2 = \tfrac{1}{2}(|M_{LL}|^2 + |M_{LR}|^2) \qquad \langle|M_R|\rangle^2 = \tfrac{1}{2}(|M_{RL}|^2 + |M_{RR}|^2)$$

$$\Rightarrow \quad \boxed{\sigma_L = \tfrac{1}{2}(\sigma_{LR} + \sigma_{LL})} \qquad \boxed{\sigma_R = \tfrac{1}{2}(\sigma_{RR} + \sigma_{RL})}$$

- ★ **Define cross section asymmetry:**

$$\boxed{A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R}}$$

- ★ **Integrating the expressions on page 494 gives:**

$$\sigma_{LL} \propto (c_L^e)^2(c_L^\mu)^2 \quad \sigma_{LR} \propto (c_L^e)^2(c_R^\mu)^2 \quad \sigma_{RL} \propto (c_R^e)^2(c_L^\mu)^2 \quad \sigma_{RR} \propto (c_R^e)^2(c_R^\mu)^2$$

$$\Rightarrow \quad \sigma_L \propto (c_L^e)^2[(c_L^\mu)^2 + (c_R^\mu)^2] \qquad \sigma_R \propto (c_R^e)^2[(c_L^\mu)^2 + (c_R^\mu)^2]$$

$$\boxed{A_{LR} = \frac{(c_L^e)^2 - (c_R^e)^2}{(c_L^e)^2 + (c_R^e)^2} = A_e}$$

- ★ **Hence the Left-Right asymmetry for any cross section depends only on the couplings of the electron**

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