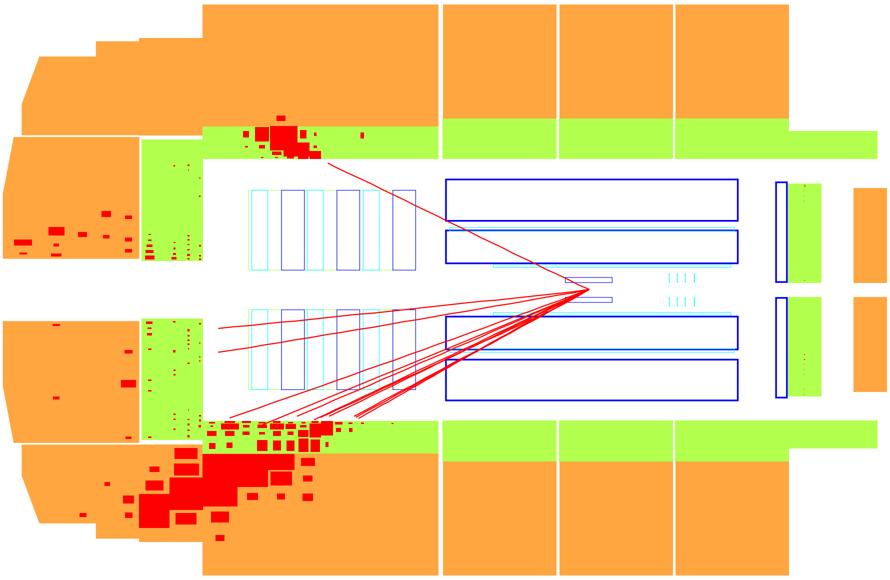
Particle Physics Major Option Exam, January 2001

SOLUTIONS

2. Deep-inelastic scattering at HERA



The isolated particle in the upper part of the diagram is the scattered positron, with a large signal in the electromagnetic calorimeter followed by a negligible signal in the hadronic calorimeter.

Taking the x-axis to point vertically upwards and the z-axis to point horizontally to the right in the diagram, the e^+ beam must enter from the left along $+z$ and the proton beam from the right along $-z$; otherwise longitudinal momentum is not conserved. (Also, the detector is asymmetric, being deeper on the $-z$ side to contain the more energetic proton fragments). The scattering angle of the e^+ (of energy 240 GeV) can be estimated from the diagram to be $\theta \approx 154°$. Hence the 4-momenta p_1, p_2, p_3 of the incoming e^+, the incoming proton and the scattered e^+, in units of GeV, are:

$$p_1 = (27.5, 0, 0, 27.5) \qquad p_2 = (820, 0, 0, -820)$$
$$p_3 = (240, 240 \times \sin 154°, 0, 240 \times \cos 154°) = (240, 105.2, 0, -215.7)$$

This gives a four-momentum transfer

$$q = p_1 - p_3 = (-212.5, -105.2, 0, 243.2) \ ,$$

and hence

$$q^2 = (-212.5)^2 - (-105.2)^2 - (243.2)^2 = \boxed{-25057\,\mathrm{GeV}^2} \ .$$

The scalar product $p_2.q$ is

$$p_2.q = 820 \times (-212.5) - (-820 \times 243.2) = 25174\,\mathrm{GeV}^2 \ ,$$

giving a Bjorken x value of

$$x = -\frac{q^2}{2M\nu} = -\frac{q^2}{2p_2.q} = \frac{25057\,\mathrm{GeV}^2}{2 \times (25174\,\mathrm{GeV}^2)} = \boxed{0.498} \ .$$

(Note that the numerical value of the proton mass, $M = 0.938\,\mathrm{GeV}$, is not in fact needed).

For ep scattering, given

$$2xF_1^{\mathrm{ep}} = F_2^{\mathrm{ep}} = \sum_i z_i^2 x q_i(x, q^2)$$

we have

$$F_2^{\mathrm{ep}} = \frac{4}{9}xu(x) + \frac{1}{9}xd(x) + \frac{4}{9}x\overline{u}(x) + \frac{1}{9}x\overline{d}(x) \ .$$

For en scattering, we interchange $u(x)$ and $d(x)$:

$$F_2^{\mathrm{en}} = \frac{4}{9}xd(x) + \frac{1}{9}xu(x) + \frac{4}{9}x\overline{d}(x) + \frac{1}{9}x\overline{u}(x) \ .$$

Hence

$$\int_0^1 \frac{1}{x}\left(F_2^{\mathrm{ep}} - F_2^{\mathrm{en}}\right) \mathrm{d}x = \int_0^1 \frac{1}{3}\left[u(x) - d(x) + \overline{u}(x) - \overline{d}(x)\right] \mathrm{d}x$$

Breaking each distribution function into "valence" and "sea" components and assuming the sea components are all identical, we can write

$$u = u_V(x) + S(x) \quad d = d_V(x) + S(x) \quad \overline{u} = S(x) \quad \overline{d} = S(x) \ .$$

The distribution functions are normalised to the total number of that parton type in the proton:

$$\int_0^1 u_V(x)dx = 2 \qquad \int_0^1 d_V(x)dx = 1 \ .$$

Hence

$$\int_0^1 \frac{1}{x}\left(F_2^{\mathrm{ep}} - F_2^{\mathrm{en}}\right) \mathrm{d}x = \frac{1}{3} \times (2 - 1) = \frac{1}{3} \ .$$

In terms of $u_V(x)$, $d_V(x)$, $S(x)$, we have

$$F_2^{\mathrm{ep}} = \frac{4}{9}x(u_V + S) + \frac{1}{9}x(d_V + S) + \frac{4}{9}xS + \frac{1}{9}xS = x\left[\frac{4}{9}u_V + \frac{1}{9}d_V + \frac{10}{9}S\right]$$

$$F_2^{\mathrm{en}} = \frac{4}{9}x(d_V + S) + \frac{1}{9}x(u_V + S) + \frac{4}{9}xS + \frac{1}{9}xS = x\left[\frac{4}{9}d_V + \frac{1}{9}u_V + \frac{10}{9}S\right]$$

The ratio R of these two structure functions is

$$R = \frac{F_2^{\mathrm{en}}}{F_2^{\mathrm{ep}}} = \frac{4d_V + u_V + 10S}{4u_V + d_V + 10S}$$

As $x \to 0$, the sea component $S(x)$ completely dominates and $R \to 1$. As $x \to 1$, $S(x)$ becomes negligible and the ratio depends on the relative magnitude of u_V and d_V. Experimentally, d_V becomes very small and $R \to \frac{1}{4}$.

3. **Helicity and Handedness:**

Without loss of generality, choose the direction of motion of the particle to be along the $+z$-axis. In the limit $E \gg m$, the free particle spinors become

$$u_1 = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_2 = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},$$

Operating on these with γ^5 gives

$$\gamma^5 u_1 = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = u_1$$
$$\gamma^5 u_2 = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} = -\sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} = -u_2$$

Therefore, the left-handed and right-handed chiral components $\frac{1}{2}(1-\gamma^5)u$ and $\frac{1}{2}(1+\gamma^5)u$ are

$$\tfrac{1}{2}(1-\gamma^5)u_1 = 0, \qquad \tfrac{1}{2}(1+\gamma^5)u_1 = u_1$$
$$\tfrac{1}{2}(1-\gamma^5)u_2 = u_2, \qquad \tfrac{1}{2}(1+\gamma^5)u_1 = 0\ .$$

Any free particle spinor u can be expressed as a linear combination $u = \alpha_1 u_1 + \alpha_2 u_2$. This has left- and right-handed chiral components

$$u_{\rm L} \equiv \tfrac{1}{2}(1-\gamma^5)u = \alpha_1\tfrac{1}{2}(1-\gamma^5)u_1 + \alpha_2\tfrac{1}{2}(1-\gamma^5)u_2 = \alpha_2 u_2,$$
$$u_{\rm R} \equiv \tfrac{1}{2}(1+\gamma^5)u = \alpha_1\tfrac{1}{2}(1+\gamma^5)u_1 + \alpha_2\tfrac{1}{2}(1+\gamma^5)u_2 = \alpha_1 u_1\ .$$

But, for motion along the $+z$-axis, u_1 and u_2 are the positive and negative helicity eigenstates, respectively:

$$\widehat{S}_z u_1 = +\tfrac{1}{2}u_1, \qquad \widehat{S}_z u_2 = -\tfrac{1}{2}u_2$$

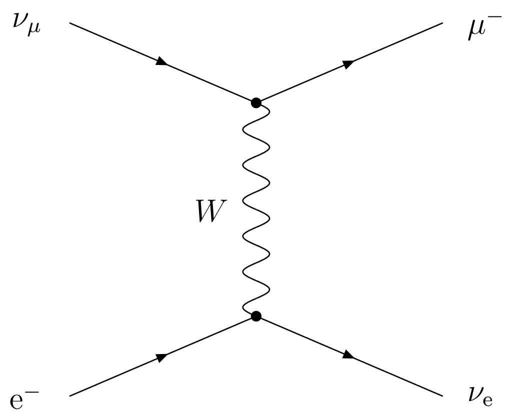
Hence:

$$\widehat{S}_z u_{\rm L} = \alpha_2\widehat{S}_z u_2 = -\tfrac{1}{2}\alpha_2 u_2 = -\tfrac{1}{2}u_{\rm L}$$
$$\widehat{S}_z u_{\rm R} = \alpha_1\widehat{S}_z u_1 = +\tfrac{1}{2}\alpha_1 u_1 = +\tfrac{1}{2}u_{\rm R}$$

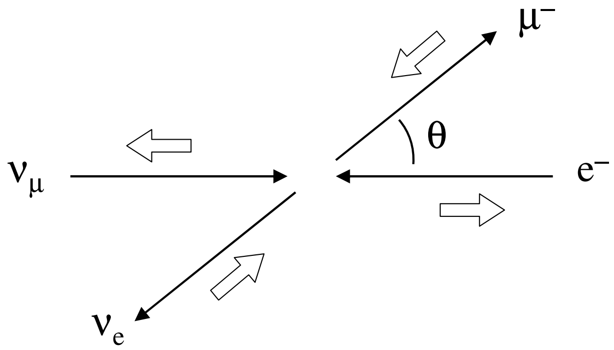
demonstrating that, in the relativistic limit, $\frac{1}{2}(1-\gamma^5)u$ is a spin-down eigenstate (helicity -1) and $\frac{1}{2}(1+\gamma^5)u$ is a spin-up eigenstate (helicity $+1$).

Neutrino scattering:

Feynman diagram for $\nu_\mu e^- \to \mu^- \nu_e$:

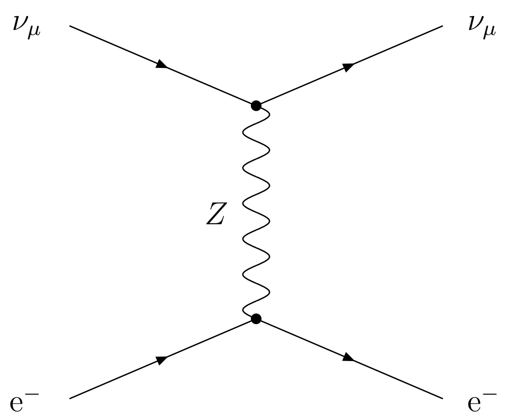


Spin diagram: the ν_μ and ν_e both have negative helicity. Since the interaction is mediated by a $W^\pm$ boson, only the left-handed chiral components of the e^- and μ^- can contribute. In the relativistic limit, the left-handed chiral component of a particle is a negative helicity eigenstate. Since the e^- and μ^- masses can be neglected ($E \gg m$), the e^- and μ^- therefore both have negative helicity.

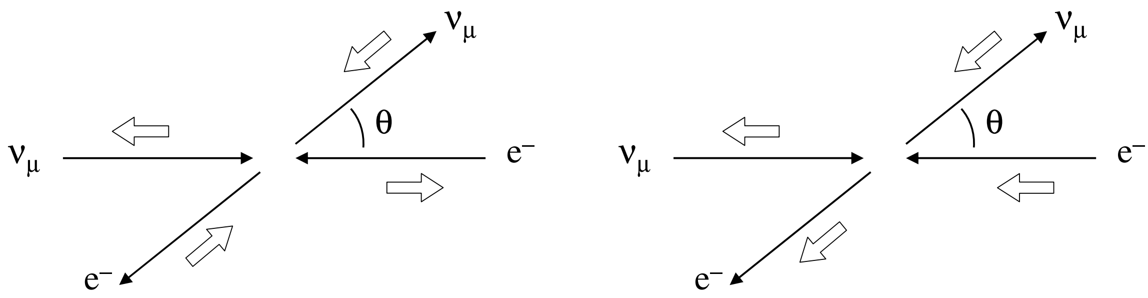


The total spin in both the initial and final states is zero. In the centre of mass frame, the total 3-momentum is also zero. Hence there is no preferred spatial direction and the scattering is isotropic.

Feynman diagram for $\nu_\mu e^- \to \nu_\mu e^-$:



Spin diagrams: the initial and final state ν_μ both have negative helicity. Since the interaction is now mediated by a Z^0 boson, both the left-handed *and* right-handed chiral components, *i.e.* both helicity eigenstates, of the e^- can contribute. Because of helicity conservation, the initial and final helicity state of the e^- must be the same. Thus there are two possible spin configurations, one with both e^- spins down (negative helicity, left-handed) and one with both e^- spins up (positive helicity, right-handed).



The left-handed case has relative interaction strength c_L^e and gives isotropic scattering. The right-handed case has relative interaction strength c_R^e and gives an extra factor of $\frac{1}{4}(1+\cos\theta)^2$ because the initial and final states both have total spin $+1$ along the particle axis.

For $\nu_\mu e^- \to \mu^- \nu_e$, considering just the contributions from the vertex factors, we have

$$M_{fi} \sim \frac{g_W}{\sqrt{2}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\cdot\frac{g_W}{\sqrt{2}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5) = \frac{g_W^2}{2}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\cdot\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\ .$$

This is given to result in a differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{g_W^2}{8\pi m_W^2}\right)^2 s\ .$$

For $\nu_\mu e^- \to \nu_\mu e^-$, the vertex factors contribute

$$\begin{aligned} M_{fi} &\sim \frac{g_Z}{2}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\cdot\frac{g_Z}{2}\gamma^\mu \tfrac{1}{2}(c_V^e - c_A^e\gamma^5) \\ &= \frac{g_Z^2}{2}\left[c_L^e\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\gamma^\mu\tfrac{1}{2}(1-\gamma^5) + c_R^e\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\gamma^\mu\tfrac{1}{2}(1+\gamma^5)\right]\ . \end{aligned}$$

The first term on the right-hand side is identical to the $\nu_\mu e^- \to \mu^-\nu_e$ case, except that g_W is replaced by g_Z and there is an extra factor of c_L^e. The second term has an extra factor of c_R^e, and $1+\gamma^5$ in place of $1-\gamma^5$. The latter gives rise to the different allowed spin configuration discussed above, with angular distribution $\frac{1}{4}(1+\cos\theta)^2$. The $\nu_\mu e^- \to \nu_\mu e^-$ differential cross section can therefore be written down directly from the $\nu_\mu e^- \to \mu^-\nu_e$ cross section above as

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{g_Z^2}{8\pi m_Z^2}\right)^2 s\left[(c_L^e)^2 + \tfrac{1}{4}(1+\cos\theta)^2(c_R^e)^2\right]$$

where we have also replaced m_W by m_Z. Integrating over all angles using

$$\int \tfrac{1}{4}(1+\cos\theta)^2 d\Omega = 2\pi\int_{-1}^{+1}\tfrac{1}{4}(1+x)^2 dx = \frac{1}{3}\times 4\pi$$

and using $g_{\rm W} = g_{\rm Z}\cos\theta_{\rm W}$, $m_{\rm W} = m_{\rm Z}\cos\theta_{\rm W}$ gives

$$\frac{\sigma(\nu_\mu {\rm e}^- \to \nu_\mu {\rm e}^-)}{\sigma(\nu_\mu {\rm e}^- \to \mu^- \nu_{\rm e})} = (c_{\rm L}^{\rm e})^2 + \frac{1}{3}\,(c_{\rm R}^{\rm e})^2 \;\; .$$

Since the electron has $I_W^{(3)} = -\frac{1}{2}$ and $Q = -1$, we have

$$c_{\rm L}^{\rm e} = -\tfrac{1}{2} + \sin^2\theta_{\rm W} \qquad c_{\rm R}^{\rm e} = \sin^2\theta_{\rm W} \;\; .$$

Substituting then gives a quadratic equation for $\sin^2\theta_{\rm W}$:

$$R = \left(-\tfrac{1}{2} + \sin^2\theta_{\rm W}\right)^2 + \tfrac{1}{3}\sin^4\theta_{\rm W} = \tfrac{1}{4} - \sin^2\theta_{\rm W} + \frac{4}{3}\sin^4\theta_{\rm W} = 0.09 \;\; ,$$

which can be solved to give $\sin^2\theta_{\rm W} \approx 0.52$ or $\sin^2\theta_{\rm W} \approx 0.23$, the latter being the correct result (consistent with all other data).