UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics
MASTER OF ADVANCED STUDY IN PHYSICS

Monday 15th January 2024 10:00 to 12:00

MAJOR TOPICS
Paper 1/PP (Particle Physics)

*Answer **two** questions only. The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate. The paper has content on 17 sides, including this one, and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

*You should use a **separate Answer Book** for each question.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2x20-page answer books	Mathematical Formulae Handbook
Rough workpad	Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

The information in this box may be used in any question.

The Pauli-matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Dirac representation of the gamma matrices is:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

The Dirac representation of the gamma matrices has the following properties:

$$(\gamma^0)^* = \gamma^0, \ \ (\gamma^1)^* = \gamma^1, \ \ (\gamma^2)^* = -\gamma^2, \ \ (\gamma^3)^* = \gamma^3 \ \ \textit{and} \ \ \gamma^2(\gamma^\mu)^* = -\gamma^\mu\gamma^2.$$

Using the above representation, the Part III Particles lecture course defined the following particle and anti-particle spinors:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix}, \qquad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix},$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix}, \qquad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

for objects whose three-momentum $\vec{p}$ is given by $|\vec{p}|(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ where $c = \cos\frac{\theta}{2}$ and $s = \sin\frac{\theta}{2}$. The normalising constant is $N = \sqrt{E+m}$.

$\hbar \approx 1.05 \times 10^{-34}$ kg m^2/s, $\qquad c \approx 3.00 \times 10^8$ m/s, $\qquad e \approx 1.60 \times 10^{-19}$ C.

$m_e = 5.11 \times 10^{-4}$ GeV. $\qquad m_p = m_n = 1.67 \times 10^{-27}$ kg.

1 In this question the Dirac Hamiltonian H for a particle of mass m may be taken to be

$$H = \vec{p} \cdot \vec{\alpha} + m\beta$$

in which $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, and α_1, α_2, α_3 and β are four appropriately chosen matrices while $\vec{p}$ is the three-momentum operator. The gamma matrices γ^μ for $\mu \in \{0, 1, 2, 3\}$ may be assumed to be defined by $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$ for $i \in \{1, 2, 3\}$.

The Dirac representation of the gamma matrices given on page 2 satisfies:

$$2g^{\mu\nu} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu, \qquad \text{and} \tag{1}$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{2}$$

in which $g^{\mu\nu}$ is a Minkowski metric of unspecified signature.

(a) Explain whether the relation (1) must be satisfied by **all** representations of the gamma matrices, or only by **some** representations. If only satisfied by **some**, you should state the condition(s) under which the relation would hold. State also the physical principle(s) or objective(s) that relation (1) encodes, if any. [5]

[Mostly bookwork – the exception being critical consideration of the effect of metric signature which is unseen.]

The following argument shows that (1) is true if the metric is $(+, -, -, -)$, but that we'd need an extra minus sign in (1) if our metric was $(-, +, +, +)$. The reason we get (1) (with or without an extra minus sign) is because we are interested in Lorentz Covariance and can get it by ensuring that the Klein Gordon is trapped within the Dirac equation as the following arugment shows:

Want Dirac Equation to have KLEIN GORDON "within" it so that solns are Lorentz Covariant.

$H\psi = E\psi \quad \Rightarrow \quad H^2\psi = E^2\psi = (|\underline{p}|^2 + m^2)\psi \quad$ so need $H^2 = |\underline{p}|^2 + m^2$

$$H = \underline{p}\cdot\underline{\alpha} + m\beta \quad \Rightarrow \quad H^2 = (p_i\alpha_i + m\beta)^2$$

$$\begin{aligned} \Rightarrow \quad H^2 &= \left(\sum_{i=1}^{3} p_i^2\alpha_i^2\right) + m^2\beta^2 \\ &\quad + \sum_{i<j}(p_ip_j\alpha_i\alpha_j) + \sum_{i>j} p_ip_j\alpha_i\alpha_j \\ &\quad + \sum_i p_i m\,\alpha_i\beta + \sum_i p_i m\beta\alpha_i \overset{\text{want}}{\equiv} p^2 + m^2 \quad \forall\, \underline{p}, m \end{aligned}$$

$\therefore$ need $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = \mathbb{1}$ ⓪

& $\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \quad \forall\, i \neq j$ ①

$\alpha_i\beta + \beta\alpha_i = 0 \quad \forall\, i$ ②

$\gamma^0 = \beta$ & $\gamma^i = \beta\alpha^i \Rightarrow (\gamma^0)^2 = \mathbb{1}$

$$\begin{aligned} \Rightarrow (\gamma^i)^2 &= (\beta\alpha^i)(\beta\alpha^i) && \text{(no sum)} \\ &= -\beta\beta\alpha^i\alpha^i && \text{(no sum) using ②} \\ &= -\mathbb{1}\,\mathbb{1} && \text{(using ⓪)} \\ &= -\mathbb{1} \end{aligned}$$

$\therefore \quad \gamma^0\gamma^0 + \gamma^0\gamma^0 = 2 \qquad \gamma^0\gamma^1 + \gamma^1\gamma^0 = \beta\beta\alpha^1 + \beta\alpha^1\beta$

$\gamma^1\gamma^1 + \gamma^1\gamma^1 = -2 \qquad\qquad = \alpha^1 - \alpha^1\beta\beta$

$\vdots \qquad\qquad\qquad\qquad\qquad = 0$ etc

$\therefore \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ is correct if $g^{\mu\nu} = \text{diag}(+,-,-,-)$

But if $g^{\mu\nu} = \text{diag}(-,+,+,+)$ then we would need instead $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}$.

(b) Repeat (a) but this time for relation (2). [5]

[Mostly bookwork – but not completely since lecture notes were focused on Dirac representation rather than general representations.]

The following argument shows that our desire that the Hamiltonian be Hermitian (so that Energies are real and observable) forces us to require (2) in all cases.

$\gamma^0 = \beta$, $\gamma^i = \beta\alpha^i = \gamma^0\alpha^i$

From (a) know that $(\gamma^0)^2 = 1$ always $\therefore \alpha^i = \gamma^0\gamma^i$

$\Rightarrow H = m\gamma^0 + p^i\gamma^0\gamma^i = \gamma^0(m + p^i\gamma^i)$

Want $H = H^\dagger \; \forall p, m$ (so that energies are real).

Considering $p = 0$, $m \neq 0$ shows
$H = H^\dagger \Rightarrow \gamma^0 = (\gamma^0)^\dagger$ always. $\circledast$

$\Rightarrow (\gamma^0)^\dagger = \gamma^0\gamma^0\gamma^0$

Considering $p = (1, 0, 0)$ $m = 0$ shows
$$H = H^\dagger \Rightarrow \gamma^0\gamma^i = (\gamma^0\gamma^i)^\dagger$$
$$= (\gamma^i)^\dagger(\gamma^0)^\dagger = (\gamma^i)^\dagger\gamma^0 \quad \text{by } \circledast$$

$\Rightarrow (\gamma^i)^\dagger = \gamma^0\gamma^i\gamma^0$ ALWAYS.

$\Rightarrow (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ ALWAYS.

In the rest of the question, a metric with a $(+, -, -, -)$ signature must always be used!

The purpose of the above requirement is so that we realise that throughout the rest of the question we satisfy the preconditions for both (1) and (2) and so can freely use either for ANY representation of the gamma matrices we might later consider.

(c) Explain briefly why, when using the Dirac representation of the gamma matrices,
$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$
is a good operator to use to measure the intrinsic spin of a Dirac fermion. [5]

This part of this question serves two purposes. The first purpose is to see if candidates can recall from memory the main parts of argument used in lectures. The main points of that argument were:

- Ehrenfest theorem requires $[H, X] = 0$ if X is to be an operator whose value is to be a constant of the motion.
- Orbital angular momentum is not conserved because $[H, \vec{L}]$ is shown to take the (non-zero) value $-i\vec{\alpha} \wedge \vec{p}$.

(TURN OVER

- But then we went on to show that $[H, \vec{S}] = +i\vec{\alpha} \wedge \vec{p}$ which is exactly the opposite of $[H, \vec{L}]$ and so
- the mystery quantity $\vec{L} + \vec{S}$ has been shown to be a quantity of the motion.
- Since most of it ($\vec{L}$) is orbital angular momentum and only a very small bit of it ($\vec{S}$) needs to be added on to conserve it, we call that new part an extra bit of ('intrinsic') angular momentum.

An argument set out at a level of detail similar to the above would get full marks, as the question does not demand a proof but just asks for a brief explanation.

Using the summation convention with indices $i, j, k \in \{1, 2, 3\}$ the operator $\vec{L} = \vec{x} \times \vec{p}$ may be written $L^i = \epsilon^{ijk} x^j p^k$.

(d) Using the canonical commutation relation for the $\vec{x}$ and $\vec{p}$ operators show that the following holds for ***every*** representation of the gamma matrices:

$$[H, L^i] = i\hbar\epsilon^{ijk} p^j \gamma^0 \gamma^k. \qquad [5]$$

$$[H, L_i] = [H, \varepsilon^{ijk} x^j p^k]$$

$$= [\gamma^0(m + p^b\gamma^b), \varepsilon^{ijk} x^j p^k]$$

$$= \varepsilon^{ijk}\gamma^0 [m + p^b\gamma^b, x^j p^k]$$

$$= \varepsilon^{ijk}\gamma^0 [p^b\gamma^b, x^j p^k]$$

$$= \varepsilon^{ijk}\gamma^0 [p^b, x^j p^k]\gamma^b$$

$$= \varepsilon^{ijk}\gamma^0 \Big(\underbrace{[p^b, x^j]}_{= -i\hbar\delta^{bj}} p^k + x^j \cancelto{0}{[p^b, p^k]}\Big)\gamma^b$$

$$= -i\hbar\, \varepsilon^{ijk}\gamma^0 \delta^{bj} p^k \gamma^b$$

$$= -i\hbar\, \varepsilon^{ibk}\gamma^0 p^k \gamma^b$$

rename $b \to k$, $k \to j$

$$= -i\hbar\, \varepsilon^{ikj}\gamma^0 p^j \gamma^k$$

$$= i\hbar\, \varepsilon^{ijk} p^j \gamma^0 \gamma^k$$

Using the summation convention with indices $i, j, k \in \{1, 2, 3\}$, define the operator Q^i by

$$Q^i = i\hbar\epsilon^{ijk}\gamma^j\gamma^k.$$

(e) For a general representation of the gamma matrices: (i) prove that Q^i is always Hermitian, (ii) find the commutator

$[H, Q^i]$, and (iii) hence or otherwise give a physical interpretation to the Q^i operator. [10]

[Not bookwork, but (hopefully) scaffolded/prepared by earlier parts of this question, particularly parts (a) and (b).]

(i):

$$Q_i \stackrel{\text{def}}{=} i\hbar\, \varepsilon^{iab} \gamma^a \gamma^b \quad \leftarrow a,b,i \in \{1,2,3\}$$

$$\therefore \quad Q_i^\dagger = -i\hbar\, \varepsilon^{iab} \gamma^{b\dagger} \gamma^{a\dagger}$$

$$= -i\hbar\, \varepsilon^{iab} \gamma^0 \gamma^b \cancel{\gamma^0} \cancel{\gamma^0} \gamma^a \gamma^0$$

$\leftarrow (\gamma^0)^2 = \mathbb{1}$ follows from $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ and $(+,-,-,-)$ metric.

$$= i\hbar\, \varepsilon^{iab} \gamma^0 \gamma^a \gamma^b \gamma^0$$

$$= i\hbar\, \varepsilon^{iab} \gamma^a \gamma^b \gamma^0 \gamma^0 \quad (2 \text{ flips})$$

$$= i\hbar\, \varepsilon^{iab} \gamma^a \gamma^b = Q_i \qquad \therefore\ Q_i \text{ is Hermitian.}$$

(ii):

(TURN OVER

$$[H, Q_i] = i\hbar \cdot \epsilon^{iab}\left[\gamma^0(m+p^j\gamma^j),\ \gamma^a\gamma^b\right]$$

$$= i\hbar \cdot \epsilon^{iab}\left(\gamma^0(m+p^j\gamma^j)\gamma^a\gamma^b - \gamma^a\gamma^b\gamma^0(m+p^j\gamma^j)\right)$$

2 flips no change

$$= i\hbar \cdot \epsilon^{iab}\gamma^0\left(m\cancel{\gamma^a\gamma^b} + p^j\gamma^j\gamma^a\gamma^b - \cancel{\gamma^a\gamma^b m} - \gamma^a\gamma^b p^j\gamma^j\right)$$

$$= i\hbar \cdot \epsilon^{iab}\gamma^0 p^j\left(\gamma^j\gamma^a\gamma^b - \gamma^a\gamma^b\gamma^j\right)$$

but $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$

$\Rightarrow \{\gamma^a,\gamma^b\} = -2\delta^{ab}$

ie $\gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta^{ab}$

$\Rightarrow \gamma^a\gamma^b = -\gamma^b\gamma^a - 2\delta^{ab}$

$$= i\hbar \cdot \epsilon^{iab}\gamma^0 p^j\left((-\gamma^a\gamma^j - 2\delta^{aj})\gamma^b + \gamma^a(\gamma^j\gamma^b + 2\delta^{jb})\right)$$

$$= i\hbar \cdot 2\epsilon^{iab}\gamma^0 p^j\left(\gamma^a\delta^{jb} - \gamma^b\delta^{aj}\right)$$

$$= i\hbar \cdot 2\epsilon^{iab}\gamma^0\left(p^b\gamma^a - p^a\gamma^b\right)$$

rename a, b in second term

$$= i\hbar \cdot 2\gamma^0\left[\epsilon^{iab}p^b\gamma^a - \epsilon^{iba}p^b\gamma^a\right]$$

ϵ^{iab} antisym

$$= i\hbar \cdot 2\gamma^0\left[\epsilon^{iab}p^b\gamma^a + \epsilon^{iab}p^b\gamma^a\right]$$

$$= i\hbar \cdot 4\gamma^0\epsilon^{iab}p^b\gamma^a$$

rename $b \to j$, $a \to k$

$$= 4i\hbar\gamma^0\epsilon^{ikj}p^j\gamma^k$$

$$= -4i\hbar\epsilon^{ijk}p^j\gamma^0\gamma^k.$$

(iii):

Hence $[H, L_i + \frac{1}{4}Q_i] = 0$ or more simply

$$[H, \vec{L} + \frac{1}{4}\vec{Q}] = 0$$

and so, since $\vec{L}$ is the orbital angular momentum operator for a general representation of the gamma matrices, then $\frac{1}{4}\vec{Q}$ must be the general representation of the spin operator. In other words, it does the same thing as $\vec{S}$ but works in all representations (at least those with signature $(+,-,-,-)$) not just the Dirac representation.

---

2 The neutrino weak and mass eigenstates are related by the unitary PMNS matrix:

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3}\\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}, \qquad \begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = \begin{pmatrix}U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1}\\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2}\\ U^*_{e3} & U^*_{\mu 3} & U^*_{\tau 3}\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$$

(a) What are the weak and mass eigenstates? How do they differ from each other? [3]

The weak eigenstates are ν_e, ν_μ, ν_τ and are defined to be the states which are created in association with a W boson and a lepton of the named flavour. The mass eigenstates are ν_1, ν_2, ν_3 and are defined to be eigenstates of the Hamiltonian that have well defined masses.

(b) Assuming a neutrino propagation speed of $v = c$, show that the wavefunction at a distance z, for a neutrino beam which initially consisted entirely of ν_μ is given by:

$$|\psi_\nu(z)\rangle = U_{\mu 1}\,|\nu_1\rangle\, e^{-i\phi_1} + U_{\mu 2}\,|\nu_2\rangle\, e^{-i\phi_2} + U_{\mu 3}\,|\nu_3\rangle\, e^{-i\phi_3}$$

with $\phi_i \approx m_i^2 z/2E_i$, where m_i and E_i are the mass and energy of ν_i. [2]

---

natural units

$c = \frac{z}{t} \Rightarrow t = \frac{z}{c} = z$

Bookwork

$\nu_1(t) = \nu_1(0)\, e^{-ip_\mu x^\mu} = \nu_1(0) e^{-iE_1 t + i\underline{p}_1 \cdot \underline{x}}$

If $\underline{p} \propto \underline{k}$, and Det @ $z\underline{k}$, then $\nu_1(t) = \nu_1(0) e^{-iE_1 z + ip_1 z}$

$\therefore \nu_1(t) = \nu_1(0)\, e^{-i\phi_1}$ where $\phi_i = E_i - p_i$.

Note $\phi = \sqrt{E^2} - p = \sqrt{m^2 + p^2} - p = p\left(\sqrt{1 + \frac{m^2}{p^2}} - 1\right)$

$= p\left(\frac{1}{2}\frac{m^2}{p^2} + O\left(\frac{m^4}{p^4}\right)\right)$ (Taylor series)

$\approx \frac{m^2}{2p} \approx \frac{m^2}{2E}$ (since $p \approx E$)

$\therefore \phi_i \approx \frac{m_i^2}{2E_i}$ QED.

(c) Explain why in the corresponding expression for the anti-neutrino, $\bar{\nu}_\mu$, the matrix elements U are replaced by U^*. [1]

This is less a question (since it is expected that most candidates will get get the mark) and more a scaffold to ensure that candidates who go further don't forget to complex

(TURN OVER

conjugate their Us where needed. Nonetheless, to the extent that it is answers, we will accept any mention of the fact that (as mentioned many times in lectured verbally, with this question in mind) universally complex conjugation is a feature that is preserved in all description of anti-particles. This was shown explicitly when we constructed the charge-conjugation operator in the Dirac part of the course. Nonetheless, a student can also get full marks if they relate the answer to the complex conjugation in adjoint spinors, which can make the difference between a particle entering or leaving a Feynman diagram, which we showed introduced a complex conjugate onto elements of the CKM matrix when comparing (say) the 'cost' of of $W^+ \to u\bar{d}$ to the 'cost' of $W^- \to d\bar{u}$.

In the T2K experiment, neutrinos are detected in the Super-Kamiokande detector at a distance of $L = 295$ km from the origin of the beam. The beam is almost mono-energetic, with the energy chosen such that $\phi_3 - \phi_2 = \pi$.

(d) Assuming $\left|m_3^2 - m_2^2\right| \gg \left|m_2^2 - m_1^2\right|$, show that the $\nu_\mu \to \nu_e$ oscillation probability is given by

$$P\left(\nu_\mu \to \nu_e\right) \approx -2\pi \frac{\Delta m_{21}^2}{\Delta m_{32}^2} \operatorname{Im}\left(U_{\mu 1} U_{e1}^* U_{\mu 2}^* U_{e2}\right) - 4 \operatorname{Re}\left(U_{\mu 1} U_{e1}^* U_{\mu 3}^* U_{e3} + U_{\mu 2} U_{e2}^* U_{\mu 3}^* U_{e3}\right)$$

where $\Delta m_{ji}^2 = m_j^2 - m_i^2$. State clearly any assumptions used. [12]

$\left[\right.$*You may wish to use the unitarity relation* $\left|U_{\mu 1} \dot{U}_{e1}^* + U_{\mu 2} U_{e2}^* + U_{\mu 3} U_{e3}^*\right|^2 = 0$ *and the identity* $|z_1 + z_2 + z_3|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2 \operatorname{Re}\left(z_1 z_2^* + z_1 z_3^* + z_2 z_3^*\right)$.$\left.\right]$

UNSEEN:

For neutrino produced as ν_μ, the later-time wavefunc is:

$$\nu(t) = U_{\mu 1}|\nu_1\rangle e^{-i\phi_1} + U_{\mu 2}|\nu_2\rangle e^{-i\phi_2} + U_{\mu 3}|\nu_3\rangle e^{-i\phi_3}$$

$$\nu(t) = U_{\mu 1}\left(U_{e1}^*|\nu_e\rangle + U_{\mu 1}^*|\nu_\mu\rangle + U_{\tau 1}^*|\nu_\tau\rangle\right)e^{-i\phi_1} + U_{\mu 2}\left(U_{e2}^*|\nu_e\rangle + U_{\mu 2}^*|\nu_\mu\rangle + U_{\tau 2}^*|\nu_\tau\rangle\right)e^{-i\phi_1} + U_{\mu 3}\left(U_{e3}^*|\nu_e\rangle + U_{\mu 3}^*|\nu_\mu\rangle + U_{\tau 3}^*|\nu_\tau\rangle\right)e^{-i\phi_1}$$

$$= \underbrace{\left(U_{\mu 1}U_{e1}^* e^{-i\phi_1} + U_{\mu 2}U_{e2}^* e^{-i\phi_2} + U_{\mu 3}U_{e3}^* e^{-i\phi_3}\right)}_{A_{\mu\to e}}|\nu_e\rangle + \underbrace{\left(U_{\mu 1}U_{\mu 1}^* e^{-i\phi_1} + U_{\mu 2}U_{\mu 2}^* e^{-i\phi_2} + U_{\mu 3}U_{\mu 3}^* e^{-i\phi_3}\right)}_{A_{\mu\to\mu}}|\nu_\mu\rangle + \underbrace{\left(U_{\mu 1}U_{\tau 1}^* e^{-i\phi_1} + U_{\mu 2}U_{\tau 2}^* e^{-i\phi_2} + U_{\mu 3}U_{\tau 3}^* e^{-i\phi_3}\right)}_{A_{\mu\to\tau}}|\nu_\tau\rangle$$

$$= A_{\mu\to e}|\nu_e\rangle + A_{\mu\to\mu}|\nu_\mu\rangle + A_{\mu\to\tau}|\nu_\tau\rangle.$$

$$\therefore P(\nu_\mu\to\nu_e) = |A_{\mu\to e}|^2$$

$$= \left|U_{\mu 1}U_{e1}^* e^{-i\phi_1} + U_{\mu 2}U_{e2}^* e^{-i\phi_2} + U_{\mu 3}U_{e3}^* e^{-i\phi_3}\right|^2$$

(By remark in our answer to (b))

$$= |U_{\mu 1}U_{e1}^*|^2 + |U_{\mu 2}U_{e2}^*|^2 + |U_{\mu 3}U_{e3}^*|^2 + 2\,\mathrm{Re}\Big[\underbrace{U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2}\, e^{i(\phi_2-\phi_1)}}_{\text{"123"}} + \underbrace{U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3}\, e^{\overbrace{i(\phi_3-\phi_2)}^{\pi}}}_{\text{"231"}} + \underbrace{U_{\mu 3}U_{e3}^*U_{\mu 1}^*U_{e1}\, e^{\overbrace{i(\phi_1-\phi_3)}^{-\pi}}}_{\text{"312"}}\Big]$$

$$= \Big|\overbrace{U_{\mu 1}U_{e1}^* + U_{\mu 2}U_{e2}^* + U_{\mu 3}U_{e3}^*}^{=0}\Big|^2 - 2\mathrm{Re}\left[U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2} + U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3} + U_{\mu 3}U_{e3}^*U_{\mu 1}^*U_{e1}\right] + 2\mathrm{Re}\left[U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2}\, e^{i\delta\phi_{21}} + U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3}\, e^{i\pi} + U_{\mu 3}U_{e3}^*U_{\mu 1}^*U_{e1}\, e^{-i\pi}\right]$$

$$= 2\mathrm{Re}\left[U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2}\left(e^{i\delta\phi_{21}} - 1\right) - 2U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3} - 2U_{\mu 3}U_{e3}^*U_{\mu 1}^*U_{e1}\right]$$

We are aiming for:

$P(\nu_\mu \to \nu_e) \approx -2\pi \frac{\Delta m_{21}^2}{\Delta m_{32}^2} \mathrm{Im}\left(\underbrace{U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2}}_{A}\right) - 4\,\mathrm{Re}\left(\underbrace{U_{\mu 1}U_{e1}^*U_{\mu 3}^*U_{e3}}_{B} + \underbrace{U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3}}_{C}\right)$

where $\Delta m_{ji}^2 = m_j^2 - m_i^2$. State clearly any assumptions used.

You may wish to use the unitarity relation $|U_{\mu 1}U_{e1}^* + U_{\mu 2}U_{e2}^* + U_{\mu 3}U_{e3}^*|^2 = 0$ *and the identity* $|z_1+z_2+z_3|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\,\mathrm{Re}\left(z_1 z_2^* + z_1 z_3^* + z_2 z_3^*\right)$.

COMPLEX CONJUGATE

Given where we are asked to head, let us use $\mathrm{Re}(z) = \mathrm{Re}(\bar{z})$ on third term:

CHANGES

$$= 2\mathrm{Re}\Big[\underbrace{U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2}}_{A}\left(e^{i\delta\phi_{21}} - 1\right) - 2\underbrace{U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3}}_{C} - 2\underbrace{U_{\mu 3}^*U_{e3}U_{\mu 1}U_{e1}^*}_{B}\Big]$$

SMALL ANGLE APPROX ... we neglect $O((\delta\phi_{21})^2)$ term in Taylor series for exp. This is OK because $\delta\phi_{21} \ll \pi$ since:

ASIDE:

$$\phi_2 - \phi_1 \approx \frac{m_2^2 z}{2E_2} - \frac{m_1^2 z}{2E_1} \approx \frac{z\,\Delta m_{21}^2}{2E}$$

$E \gg m_1, m_2, m_3$ $\therefore E_1 = E_2 = E_3 \equiv E$

$$\pi = \phi_3 - \phi_2 \approx \frac{m_3^2 z}{2E_3} - \frac{m_2^2 z}{2E_2} \approx \frac{z\,\Delta m_{32}^2}{2E}$$

$$\therefore \frac{\phi_2-\phi_1}{\pi} = \frac{\Delta m_{21}^2}{\Delta m_{32}^2} \Rightarrow \phi_2 - \phi_1 = \pi\frac{\Delta m_{21}^2}{\Delta m_{32}^2} = \delta\phi_{21} \ll \pi \text{ since}$$

$$|\Delta m_{32}^2| \gg |\Delta m_{21}^2| \Rightarrow |\phi_2 - \phi_1| \ll |\phi_3 - \phi_2| = \pi.$$

$$\approx 2\mathrm{Re}\left[A(i\delta\phi_{21} + \ldots) - 2(C+B)\right]$$

$$= -2\,\mathrm{Im}[A]\,\delta\phi_{21} - 4\mathrm{Re}[B+C]$$

$$= -2\,\mathrm{Im}[A]\,\pi\frac{\Delta m_{21}^2}{\Delta m_{32}^2} - 4\mathrm{Re}[B+C] \quad (\text{using } \circledast)$$

$$= -2\pi\frac{\Delta m_{21}^2}{\Delta m_{32}^2}\,\mathrm{Im}[A] - 4\mathrm{Re}[B+C]$$

QED.

(e) Taking $\theta_{23} = \pi/4$, ignoring the effects of CP violation by taking U to be real ($\delta = 0$), and assuming θ_{13} is small, show that $P\left(\nu_\mu \to \nu_e\right) \approx 2\theta_{13}^2$. [4]

In the limit where θ_{13} is small, the PMNS matrix can be written

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12} & s_{12} & \theta_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}\theta_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}\theta_{13}e^{i\delta} & s_{23} \\ s_{12}s_{23} - c_{12}s_{23}\theta_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}\theta_{13}e^{i\delta} & c_{23} \end{pmatrix}$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$.

Technically this * matters

$=0$ in (e)

can ignore all these *'s since starred quantities are real!

(e)

$$P\left(\nu_\mu \to \nu_e\right) \approx -2\pi \frac{\Delta m_{21}^2}{\Delta m_{32}^2} \operatorname{Im}\left(U_{\mu 1} U_{e1}^* U_{\mu 2}^* U_{e2}\right) - 4 \operatorname{Re}\left(U_{\mu 1} U_{e1}^* \boxed{U_{\mu 3}^* U_{e3}} + U_{\mu 2} U_{e2}^* \boxed{U_{\mu 3}^* U_{e3}}\right)$$

where $\Delta m_{ji}^2 = m_j^2 - m_i^2$. State clearly any assumptions used.

[*You may wish to use the unitarity relation* $\left|U_{\mu 1} U_{e1}^* + U_{\mu 2} U_{e2}^* + U_{\mu 3} U_{e3}^*\right|^2 = 0$ *and the identity* $|z_1 + z_2 + z_3|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\operatorname{Re}\left(z_1 z_2^* + z_1 z_3^* + z_2 z_3^*\right)$.]

(e) Taking $\theta_{23} = \pi/4$, ignoring the effects of CP violation by taking U to be real ($\delta = 0$), and assuming θ_{13} is small, show that $P\left(\nu_\mu \to \nu_e\right) \approx 2\theta_{13}^2$.

[*In the limit where θ_{13} is small, the PMNS matrix can be written*

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12} & s_{12} & \theta_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{23} \theta_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} \theta_{13} e^{i\delta} & s_{23} \\ s_{12} s_{23} - c_{12} s_{23} \theta_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} \theta_{13} e^{i\delta} & c_{23} \end{pmatrix}$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$.]

$s_{23} = \frac{1}{\sqrt{2}} = c_{23}$

Under the stated conditions:

$$P(\nu_\mu \to \nu_e) \underset{\text{NO CP VIOL}}{=} -4\left(U_{\mu 1} U_{e1} + U_{\mu 2} U_{e2}\right) \boxed{U_{\mu 3} U_{e3}}$$

$$= -4\left(-c_{12}\left(s_{12} \cancel{c_{23}} + c_{12} s_{23} \theta_{13}\right) + s_{12}\left(c_{12} \cancel{c_{23}} - s_{12} s_{23} \theta_{13}\right)\right) \theta_{13} s_{23}$$

$$= 4\theta_{13}^2 s_{23}\left(s_{12}^2 s_{23} + c_{12}^2 s_{23}\right)$$

$$= 4\theta_{13}^2 s_{23}^2 \qquad \left(\text{since } s_{12}^2 + c_{12}^2 = 1\right)$$

$$= 4\theta_{13}^2 \left(\tfrac{1}{\sqrt{2}}\right)^2 \qquad \left(\text{since } \theta_{23} = \tfrac{\pi}{4}\right)$$

$$= 2\theta_{13}^2 \qquad \text{QED.}$$

At T2K the effects of CP violation are expected to be small.

(f) Show that for maximal CP violation, $\delta = \pi/2$, the CP asymmetry,

$$A_{CP} = \frac{P\left(\nu_\mu \to \nu_e\right) - P\left(\bar{\nu}_\mu \to \bar{\nu}_e\right)}{P\left(\nu_\mu \to \nu_e\right) + P\left(\bar{\nu}_\mu \to \bar{\nu}_e\right)} \propto \frac{\Delta m_{21}^2}{\Delta m_{32}^2} \frac{1}{\theta_{13}}.$$

[8]

Under maximal CP viol $e^{i\delta}=i,\; e^{-i\delta}=-i$.

$$-4\,\mathrm{Re}\left(\underline{U_{\mu1}U_{e1}}+\underline{U_{\mu2}U_{e2}}\right)U_{\mu3}U_{e3} \underset{\text{MAX CP VIOL}}{=} -4\,\mathrm{Re}\left(-c_{12}\left(s_{12}\cancel{s_{23}}+c_{12}s_{23}\theta_{13}\,i\right)+s_{12}\left(c_{12}\cancel{c_{23}}-s_{12}s_{23}\theta_{13}\,i\right)\right)(-i)\,\theta_{13}\,s_{23}$$

$$= 4\,\mathrm{Re}\left(\theta_{13}^2 s_{23}\left(s_{12}^2 s_{23}+c_{12}^2 s_{23}\right)\right) \quad \text{= SAME AS FOR NO CP VIOL}$$

$$= 2\theta_{13}^2 \qquad \text{(already calculated)}$$

Separately
$$\mathrm{Im}\left(U_{\mu1}U_{e1}U_{\mu2}^{*}U_{e2}\right) \underset{\text{max cp viol}}{=} \mathrm{Im}\left(-c_{12}\left(s_{12}\underset{=\frac{1}{\sqrt{2}}}{s_{23}}+c_{12}\underset{=\frac{1}{\sqrt{2}}}{s_{23}}\,i\theta_{13}\right)\left(c_{12}\underset{=\frac{1}{\sqrt{2}}}{c_{23}}-s_{12}\underset{=\frac{1}{\sqrt{2}}}{s_{23}}\,i\theta_{13}\right)^{*}s_{12}\right)$$

$$= -\frac{1}{2}c_{12}s_{12}\,\mathrm{Im}\left(\left(s_{12}+c_{12}\,i\theta_{13}\right)\left(c_{12}+s_{12}\,i\theta_{13}\right)\right)$$

$$= -\frac{1}{2}c_{12}s_{12}\left(s_{12}^2\theta_{13}+c_{12}^2\theta_{13}\right)$$

$$= -\frac{1}{2}c_{12}s_{12}\theta_{13}$$

$$\therefore\quad P(\nu_\mu\to\nu_e) \underset{\text{max CP VIOL}}{=} \underbrace{\cancel{2}\pi\frac{\Delta m_{21}^2}{\Delta m_{32}^2}\cancel{\frac{1}{2}}c_{12}s_{12}\theta_{13}}_{A} + \underbrace{2\theta_{13}^2}_{B}$$

$$\therefore\quad P(\bar\nu_\mu\to\bar\nu_e) = -A+B \quad \text{by (c)} \quad \text{since } A \text{ is from "Im" term.}$$

$$\therefore\quad A_{CP} \underset{\text{max CP viol}}{=} \frac{(A+B)-(-A+B)}{(A+B)+(-A+B)} = \frac{A}{B} = \frac{\pi\frac{\Delta m_{21}^2}{\Delta m_{32}^2}c_{12}s_{12}\theta_{13}}{2\theta_{13}^2}$$

$$= \frac{1}{2}\pi c_{12}s_{12}\frac{\Delta m_{21}^2}{\Delta m_{32}^2}\frac{1}{\theta_{13}} \propto \frac{\Delta m_{21}^2}{\Delta m_{32}^2}\frac{1}{\theta_{13}} \qquad \text{QED.}$$

3 Although some of the questions below are discursive rather than mathematical, long essays are not required. Indeed, succinct answers which include all the relevant points will be rewarded more highly than lengthy answers which digress into areas of tangential relevance. You are therefore strongly encouraged to keep your answers focused narrowly on what has been asked.

(a) What can neutrino deep inelastic scattering tell us about nucleon structure that electron deep inelastic scattering cannot, and why? [6]

The main message an answer should convey is that whereas electron DIS uses the photons, neutrino DIS uses the W-boson, and consequently the former constrains the Bjorken-x-distributions of things of given ELECTRIC CHARGE MAGNITUDE whereas the latter constrains the Bjorken-x-distributions of things having given "third component of weak isospin". This means that the electron DIS allows us to separate (say) up-type-quark x-distributions from down-type ones since the former has charge magnitude 2/3 and the latter 1/3 .... yet electron DIS does not let us distinguish up from anti-up, or down from anti-down. Conversely, neutrino DIS cannot distinguish up from down, but does distinguish x-distributions of "up and anti-down" from "down and anti-up". A good answer could/should provide evidence in the form of Feynman diagrams illustrating why an incoming neutrino (conversely anti-neutrino) must cause a quark to make a transition that gain (conversely lose) one unit of positive charge, thus making sensitive to only down and anti-up (or conversely up and anti-down).

(b) Why is there an $(L = 0, J = \frac{3}{2})$ baryon with sss quark content but no $(L = 0, J = \frac{1}{2})$ baryon with the same quark content? [6]

Here the main thing we would like to see in an answer is a description of the interplay between the colour confinement hypothesis, the antisymmetric nature of three-quark colour wavefunctions, the overall antisymmetry due to Fermi-Dirac statistics under fermion interchange, and the rules for combination of spin or $SU(3)$-flavour or $SU(3)$-colour wave functions leads to it being possible to create a sss ground-state baryons only if the spin and flavour combinations are in $J = 3/2$ states. A sketch of the route this could be argued would be to say that:

- need overall antisymmetric wavefunc on exchange of any two quarks
- ground state has $L = 0$ so space wave func symmetric ($(-1)^L$).
- $3 \otimes 3 \otimes 3$ for colour decomposes to $10 + 8 + 8 + 1$ in which the singleton is $+rgb + gbr + brg - bgr - grb - rbg$ and so is antisymmetric.
- Therefore flavour*spin must be symmetric under quark exchange.
- $2 \otimes 2 \otimes 2$ for $SU(2)$-spin leads to $4 + 2 + 2$ with the 4 being totally symmetric $J = 3/2$ quadruplet, and the other two $J = 1/2$ doublets having more complex symmetries.
- $3 \otimes 3 \otimes 3$ for $SU(3)$-flavour leads to $10 + 8 + 8 + 1$ again, with the 10 being symmetric and the 8+8 each having the same more complex symmetries mentioned earlier.
- That therefore the $J = 3/2$ spin quadruplet can be paired with the flavour decuplet, **or** the $J = 1/2$ spin doublets can be combined with the flavour octets, to make valid

hadron wavefunctions.

- That since the sss is in the flavour decuplet, it needs $J = 3/2$.

(c) What are the main differences between parton distribution functions, structure functions and form factors insofar as they are relevant to hadron structure? [6]

Parton distribution functions are distributions of Bjorken-x segregated by particle flavour or type. E.g. $u_p(x)dx$ counts the nuber of u-type partons in the p=proton might have Bjorken-x values in $[x, x + dx]$, corresponding (in the infinite momentum frame) to momentum fractions x.

Form factors are fourier transforms of spatial distributions of point like quantities, and hence they provide a measurable handle on the spatial distributions of (say) charges or magnetic moments within a hadron. They are functions of the magnitude of the three momentum transfer, being only really well defined for low energy non-relativistic processes.

Structure functions are functions of the magnitude of the four-momentum transfer q, and are the generalisation of Form Factors to cases that also include relativistic collisions. Though this formally breaks their direct link to spatial distributions, it is nonetheless still the case that $\vec{q}^2 \approx -q^2$ when $q^2/(4M^2) \ll 1$ with M the hadron mass, so the spatial link is maintained in this particular regime of low four-momentum transfer.

Form factors

(d) Outline six distinct areas (not related to gravity) in which the Standard Model is suspected to be either wrong, or incomplete, or is not yet fully experimentally constrained. [6]

For some candidates this might arguably be the hardest of the five section of this question as I suspect that candidates, even if they had access to the lecture notes from the course, would find it hard to find six places in them where there is a clear statement of areas where the SM is wrong or incomplete. Most of the examples of such things that the course introduced were given as explanations verbally during lectures rather than explicitly written in the notes. Nonetheless, this question is not posed as a 'trick' to see who was paying attention in lectures, but rather because the whole emphasis of the course (repeatedly reinforced as we progressed through it) was that we should not accept the SM simply because books tell us that it is 'X' but rather we should critically look at the evidence that forces (or does not force) us to include certain properties within it. As such, I would accept any answers that poke holes in or question the usefulness of any parts of the SM, whether through echoing concerns expressed by the lecturer, or whether echoing concerns that the candidates may have picked up from elsewhere. Possible themes/topics for an answer could include (but need not be limited to) the following:

- Lack of irrefutable evidence that the the colour confinement hypothesis would be derivable from first principles as an emergent feature of raw QCD.
- Lack of constraints on CP violating phases in the PMNS matrix.

(TURN OVER

- Need for more progress on experimental tests for unitarity within the CKM and PMNS matrices.
- Lack of knowledge of the neutrino mass hierarchy (i.e. are the two neutrinos whoch are closer together in mass than to the third neutrino lighter or heavier than that third netutrino?),
- Lack of evidence as to which is the heavier of the two neutrinos that are close together in mass.
- lack of knowledge of whether neutrinos are Dirac particles or Majorana (I do not recall whether this was something I talked about much in the lectures, though I do recall answering questions on this topic posed by students on the course after lectures).
- Lack of understanding for whether the near-diagonal shape of the CKM matrix vs the much more non-diagonal shape of the PMNS matrix is evidence for something else or is just chance.
- Lack of ability to rigorously/accurately determine hadron parton distribution functions or structure functions from first principles (lattice QCD was mentioned as helping to answer this but still not great). Note that the course specifically noted that $2d(x) \neq u(x)$ departures in the proton at high x were not understood, and that also the ratio of $F2en(x)/F2ep(x)$ at $x \to 1$ is not well understood.
- Difficulty simulating any low energy strong processes (such as hadronisation in jet formation) leading to difficulty to predict jet-physics well.
- Questions relating to the strong CP problem (not mentioned in lectures, but would be valid answer if a student gave it from their own knowledge – some did ask me about it at the end of lectures). See:
  `https://en.wikipedia.org/wiki/Strong_CP_problem`
- Questions relating to the heirarchy problem (not mentioned in lectures, but would be valid answer if a student gave it from their own knowledge).
- Are there other Higgs bosons? (This was discussed in the last lecture)
- Does the discovered Higgs have the right spin and couplings to all particles to be the SM Higgs boson in every respect?
- How many generations of matter are there? Evidence was provided that the number is three, but it was mentioned that this is based on the Z-width and that's not sensitive to neutrinos heavier than half the Z-mass.

(e) Why is the decay rate of the Z-boson to neutrinos very close to double the decay rate of the Z-boson to charged leptons? [6]

Many times in the last three lectures it was emphasised that $\sin^2\theta_W$ is very close to (but slightly less than) one quarter. That information when coupled to the parts of the course which gave candidates the ability to quote:

$$\Gamma(Z \to xx) \propto c_V(x)^2 + c_A(x)^2$$

and $c_V = I^3_W(x_L) - 2Q_x \sin^2\theta_W$ and $c_A = I^3_W(x_L)$ should lead candidates to be able to deduce that

$$\Gamma(Z \to \nu\nu) \propto (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2$$

while

$$\Gamma(Z \to ee) \propto (-\tfrac{1}{2} + 2 \times 1 \times 0.23)^2 + (\tfrac{1}{2})^2 = \epsilon^2 + (\tfrac{1}{2})^2$$

(for some small $\epsilon \sim 0.04$) and hence the Z-width to neutrinos is almost (but not quite) double the width to electrons.

[It is not critical that those answering remember the exact values of 0.04 or 0.23, but it is expected that they should have learned that $\sin^2\theta_W$ is close to one quarter, and can give the right signs and magnitudes to I^3_W for leptons and neutrinos. The latter is a key part of the weak interaction part of the course.]

END OF PAPER