UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics
MASTER OF ADVANCED STUDY IN PHYSICS

Monday 15th January 2024 10:00 to 12:00

MAJOR TOPICS
Paper 1/PP (Particle Physics)

*Answer **two** questions only. The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate. The paper has content on 5 sides, including this one, and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

*You should use a **separate Answer Book** for each question.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2x20-page answer books	Mathematical Formulae Handbook
Rough workpad	Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

The information in this box may be used in any question.

The Pauli-matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Dirac representation of the gamma matrices is:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

The Dirac representation of the gamma matrices has the following properties:

$$(\gamma^0)^* = \gamma^0, \ \ (\gamma^1)^* = \gamma^1, \ \ (\gamma^2)^* = -\gamma^2, \ \ (\gamma^3)^* = \gamma^3 \ \ \textit{and} \ \ \gamma^2(\gamma^\mu)^* = -\gamma^\mu\gamma^2.$$

Using the above representation, the Part III Particles lecture course defined the following particle and anti-particle spinors:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix}, \qquad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix},$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix}, \qquad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

for objects whose three-momentum $\vec{p}$ is given by $|\vec{p}|(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ where $c = \cos\frac{\theta}{2}$ and $s = \sin\frac{\theta}{2}$. The normalising constant is $N = \sqrt{E+m}$.

$\hbar \approx 1.05 \times 10^{-34}$ kg m^2/s, $\qquad c \approx 3.00 \times 10^8$ m/s, $\qquad e \approx 1.60 \times 10^{-19}$ C.

$m_e = 5.11 \times 10^{-4}$ GeV. $\qquad m_p = m_n = 1.67 \times 10^{-27}$ kg.

1 In this question the Dirac Hamiltonian H for a particle of mass m may be taken to be

$$H = \vec{p} \cdot \vec{\alpha} + m\beta$$

in which $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, and α_1, α_2, α_3 and β are four appropriately chosen matrices while $\vec{p}$ is the three-momentum operator. The gamma matrices γ^μ for $\mu \in \{0, 1, 2, 3\}$ may be assumed to be defined by $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$ for $i \in \{1, 2, 3\}$.

The Dirac representation of the gamma matrices given on page 2 satisfies:

$$2g^{\mu\nu} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu, \qquad \text{and} \tag{1}$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{2}$$

in which $g^{\mu\nu}$ is a Minkowski metric of unspecified signature.

(a) Explain whether the relation (1) must be satisfied by **all** representations of the gamma matrices, or only by **some** representations. If only satisfied by **some**, you should state the condition(s) under which the relation would hold. State also the physical principle(s) or objective(s) that relation (1) encodes, if any. [5]

(b) Repeat (a) but this time for relation (2). [5]

In the rest of the question, a metric with a $(+, -, -, -)$ signature must always be used!

(c) Explain briefly why, when using the Dirac representation of the gamma matrices,

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

is a good operator to use to measure the intrinsic spin of a Dirac fermion. [5]

Using the summation convention with indices $i, j, k \in \{1, 2, 3\}$ the operator $\vec{L} = \vec{x} \times \vec{p}$ may be written $L^i = \epsilon^{ijk} x^j p^k$.

(d) Using the canonical commutation relation for the $\vec{x}$ and $\vec{p}$ operators show that the following holds for ***every*** representation of the gamma matrices:

$$[H, L^i] = i\hbar\epsilon^{ijk} p^j \gamma^0 \gamma^k.$$ [5]

Using the summation convention with indices $i, j, k \in \{1, 2, 3\}$, define the operator Q^i by

$$Q^i = i\hbar\epsilon^{ijk}\gamma^j\gamma^k.$$

(e) For a general representation of the gamma matrices: (i) prove that Q^i is always Hermitian, (ii) find the commutator $[H, Q^i]$, and (iii) hence or otherwise give a physical interpretation to the Q^i operator. [10]

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2 The neutrino weak and mass eigenstates are related by the unitary PMNS matrix:

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}=\begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix},\qquad \begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}=\begin{pmatrix}U^*_{e1} & U^*_{\mu1} & U^*_{\tau1}\\ U^*_{e2} & U^*_{\mu2} & U^*_{\tau2}\\ U^*_{e3} & U^*_{\mu3} & U^*_{\tau3}\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$$

(a) What are the weak and mass eigenstates? How do they differ from each other? [3]

(b) Assuming a neutrino propagation speed of $v = c$, show that the wavefunction at a distance z, for a neutrino beam which initially consisted entirely of ν_μ is given by:

$$|\psi_\nu(z)\rangle = U_{\mu1}\,|\nu_1\rangle\, e^{-i\phi_1} + U_{\mu2}\,|\nu_2\rangle\, e^{-i\phi_2} + U_{\mu3}\,|\nu_3\rangle\, e^{-i\phi_3}$$

with $\phi_i \approx m_i^2 z/2E_i$, where m_i and E_i are the mass and energy of ν_i. [2]

(c) Explain why in the corresponding expression for the anti-neutrino, $\bar{\nu}_\mu$, the matrix elements U are replaced by U^*. [1]

In the T2K experiment, neutrinos are detected in the Super-Kamiokande detector at a distance of $L = 295$ km from the origin of the beam. The beam is almost mono-energetic, with the energy chosen such that $\phi_3 - \phi_2 = \pi$.

(d) Assuming $\left|m_3^2 - m_2^2\right| \gg \left|m_2^2 - m_1^2\right|$, show that the $\nu_\mu \to \nu_e$ oscillation probability is given by

$$P\left(\nu_\mu \to \nu_e\right) \approx -2\pi\frac{\Delta m^2_{21}}{\Delta m^2_{32}}\,\mathrm{Im}\left(U_{\mu1}U^*_{e1}U^*_{\mu2}U_{e2}\right) - 4\,\mathrm{Re}\left(U_{\mu1}U^*_{e1}U^*_{\mu3}U_{e3} + U_{\mu2}U^*_{e2}U^*_{\mu3}U_{e3}\right)$$

where $\Delta m^2_{ji} = m_j^2 - m_i^2$. State clearly any assumptions used. [12]

[*You may wish to use the unitarity relation* $\left|U_{\mu1}\dot{U}^*_{e1} + U_{\mu2}U^*_{e2} + U_{\mu3}U^*_{e3}\right|^2 = 0$ *and the identity* $|z_1 + z_2 + z_3|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\,\mathrm{Re}\left(z_1 z_2^* + z_1 z_3^* + z_2 z_3^*\right)$.]

(e) Taking $\theta_{23} = \pi/4$, ignoring the effects of CP violation by taking U to be real ($\delta = 0$), and assuming θ_{13} is small, show that $P\left(\nu_\mu \to \nu_e\right) \approx 2\theta_{13}^2$. [4]

[*In the limit where* θ_{13} *is small, the PMNS matrix can be written*

$$\begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{pmatrix}=\begin{pmatrix}c_{12} & s_{12} & \theta_{13}e^{-i\delta}\\ -s_{12}c_{23} - c_{12}s_{23}\theta_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}\theta_{13}e^{i\delta} & s_{23}\\ s_{12}s_{23} - c_{12}s_{23}\theta_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}\theta_{13}e^{i\delta} & c_{23}\end{pmatrix}$$

where $s_{ij} = \sin\theta_{ij}$ *and* $c_{ij} = \cos\theta_{ij}$.]

At T2K the effects of CP violation are expected to be small.

(f) Show that for maximal CP violation, $\delta = \pi/2$, the CP asymmetry,

$$A_{CP} = \frac{P\left(\nu_\mu \to \nu_e\right) - P\left(\bar{\nu}_\mu \to \bar{\nu}_e\right)}{P\left(\nu_\mu \to \nu_e\right) + P\left(\bar{\nu}_\mu \to \bar{\nu}_e\right)} \propto \frac{\Delta m^2_{21}}{\Delta m^2_{32}}\frac{1}{\theta_{13}}.$$

[8]

3 Although some of the questions below are discursive rather than mathematical, long essays are not required. Indeed, succinct answers which include all the relevant points will be rewarded more highly than lengthy answers which digress into areas of tangential relevance. You are therefore strongly encouraged to keep your answers focused narrowly on what has been asked.

(a) What can neutrino deep inelastic scattering tell us about nucleon structure that electron deep inelastic scattering cannot, and why? [6]

(b) Why is there an $(L = 0, J = \frac{3}{2})$ baryon with sss quark content but no $(L = 0, J = \frac{1}{2})$ baryon with the same quark content? [6]

(c) What are the main differences between parton distribution functions, structure functions and form factors insofar as they are relevant to hadron structure? [6]

(d) Outline six distinct areas (not related to gravity) in which the Standard Model is suspected to be either wrong, or incomplete, or is not yet fully experimentally constrained. [6]

(e) Why is the decay rate of the Z-boson to neutrinos very close to double the decay rate of the Z-boson to charged leptons? [6]

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