NATURAL SCIENCES TRIPOS Part III

Monday 12 January 2009 9:00am to 10:30am

EXPERIMENTAL AND THEORETICAL PHYSICS (1)
Advanced Quantum Condensed Matter Physics

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
20 page answer book
Rough workpad
Metric graph paper

SPECIAL REQUIREMENTS
Mathematical formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

(10 December 2008)

A

1 Write brief notes on **two** of the following:

(a) Density functional theory (including discussions of the basic principles of the theory, its strengths and weaknesses, and its application to molecular dynamics simulations). [15]

(b) Magnetic exchange interactions in solids (including discussions of direct, double, super- and RKKY exchange). [15]

(c) Optical properties of Frenkel and Wannier excitons in bulk semiconductors (including schematic diagrams of how to construct the optical absorption spectrum in the two cases). [15]

2 The expression for the Bloch state energies calculated in $\boldsymbol{k}\cdot\boldsymbol{p}$ theory is

$$E_j(\boldsymbol{k}) = E_j^0 + \frac{\hbar^2 k^2}{2m_e} + \frac{\hbar^2}{m_e^2}\sum_{n\neq j}\frac{|\boldsymbol{k}\cdot\langle U_n^0|\hat{\boldsymbol{p}}|U_j^0\rangle|^2}{E_j^0 - E_n^0}.$$

Explain the meaning of all the symbols in this equation. [5]

Explain with the help of this expression how the negative curvature of the valence band in common semiconductors arises. [5]

The band dispersion in the vicinity of the valence band maximum of a particular semiconductor is given by

$$E(k_x, k_y, k_z) = E_{VBM} - \frac{\hbar^2 k_x^2}{2m_{11}} - \frac{\hbar^2 k_y^2}{2m_{22}} - \frac{\hbar^2 k_z^2}{2m_{33}}.$$

Discuss how such an anisotropic valence band dispersion could arise. [4]

What is the significance of the joint density of states in the context of optical absorption and how do van-Hove singularities arise? [4]

For the system above calculate the joint density of states between the valence and conduction band. The dispersion relationship of the conduction band is assumed to be isotropic with effective mass m_e^*. [12]

[*Hint: The volume of an ellipsoid with principal axes a,b,c is* $V = \frac{4\pi}{3}abc$.]

(10 December 2008)

3 What are Cooper pairs and how does their formation lead to the phenomenon of superconductivity ? [8]

The BCS ground state wavefunction is given by

$$|\Psi_{BCS}\rangle = \prod_k (u_k^* + v_k^* c_{k\uparrow}^+ c_{-k\downarrow}^+)|0\rangle.$$

Show that

$$\langle\Psi_{BCS}|c_{k\uparrow}^+ c_{-k\downarrow}^+|\Psi_{BCS}\rangle = u_k^* v_k.$$

What would the expectation value of this operator be if evaluated for the filled Fermi sea of the normal state? [10]

By using

$$u_k^* v_k = \frac{1}{2}\frac{\Delta_0}{\sqrt{(\epsilon_k - \mu)^2 + \Delta_0^2}}$$

estimate the typical energy and wavevector range around the Fermi level over which electrons are paired into Cooper pairs. [4]

Estimate the density of Cooper pairs and the average distance between them for a metal with electron density n. Compare this to the "correlation length" of the superconductor, which you can estimate from the above wavevector range for Cooper pair formation by using the Heisenberg uncertainty relationship. For the comparison use numerical values for the different parameters that are typical for metals and conventional superconductors. [8]

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Monday 12 January 2009 11:30am to 1:00pm

EXPERIMENTAL AND THEORETICAL PHYSICS (2)
Soft Matter

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

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Rough workpad	Approved calculator allowed

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(15 December 2008)

B

1 Write brief notes on **two** of the following:

(a) Crystallisation of polymers, their structure and crystallisation kinetics. [15]

(b) Three experimental methods to quantitatively determine, with a nanometre resolution, the interfacial structure and composition of polymers. [15]

(c) Reversible adhesion found in geckos, insects, and spiders. [15]

2 Consider a semi-dilute solution of polystyrene (PS) with a degree of polymerisation N and a statistical segment length b in cyclohexane under Theta conditions.

(a) Using a scaling model, derive the chain size (end-to-end distance) R_0 and derive the expression for the correlation length (mesh size) as a function of polymer volume fraction ϕ. Give scaling expressions for the size of sections of the chain for all length scales from b to R_0. [5]

(b) Give an estimate for the reptation tube diameter and the number of monomers between entanglements, N_e, in an entangled semi-dilute solution. [3]

(c) Using the result from (a), derive an expression for the average number of entanglements each chain makes with its neighbours as a function of ϕ. [3]

(d) PS with the same value of N is end-grafted to a surface to form a brush with an average inter-anchor spacing D and a grafting density $\Sigma = 1/D^2$. This brush is brought into contact with the semi-dilute solution. Depending on ϕ and Σ, the grafted polymer chains assume conformations that can be divided into four regimes. (*i*) Qualitatively describe the conformation of the grafted chains in the four regimes, (*ii*) derive expressions for the average thickness of the grafted polymer layer as a function of ϕ and Σ, and (*iii*) derive expressions for the cross-overs between the four regimes in terms of ϕ and Σ. [16]

(e) Explain which of the four brush regimes are likely to enhance lubrication when two brush-covered surfaces are sliding past each other? [3]

(15 December 2008)

3 The surface of a lotus leaf is covered by protruding wax pillars that can be modelled as standing cylinders with a diameter of $10\,\mu$m, the tops of which are terminated by hemispherical caps. The cylinders are arranged on a hexagonal lattice (every cylinder is surrounded by six other cylinders) and the gap between neighbouring cylinders is $10\,\mu$m. The pillars and the planar surface from which they protrude are made from a wax which has a surface energy of 20×10^{-3} J m^{-2}. Water has a surface energy of 72×10^{-3} J m^{-2} and the interfacial energy of the wax-water interface is 45×10^{-3} J m^{-2}.

(a) Calculate the Young's contact angle of water on a planar surface of wax, and the Wenzel and Cassie-Baxter contact angles of a drop of water on the surface described above for a cylinder height (including the spherical cap) of $10\,\mu$m. [5]

(b) Sketch and label an individual pillar in contact with water. For comparison also sketch a pillar with a flat top and describe the difference between the two in terms of their local contact angle with water. [5]

(c) For a flat-topped pillar array with the dimensions given above, calculate the effective interfacial energy per unit area of fully wetted surface (Wenzel) and a surface in the Cassie-Baxter state (air partially trapped between the water and the surface). What is the consequence of this difference in effective interfacial energies? [5]

(d) Extend the calculation in (c) to calculate the pillar height for the cross-over from Wenzel to Cassie-Baxter wetting. [5]

(e) Describe the difference in contact angle hysteresis for the three different wetting scenarios (Young, Wenzel, Cassie-Baxter). Give a qualitative reason for this difference and explain the role of the microstructure in determining the difference in contact line dynamics. [5]

(f) Qualitatively, what happens if a water-ethanol mixture is used instead of water? [5]

END OF PAPER

(15 December 2008)

NATURAL SCIENCES TRIPOS   Part III

Monday 12 January 2009   2:30pm to 4:00pm

EXPERIMENTAL AND THEORETICAL PHYSICS (3)
Astrophysics and Cosmology

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

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(10 December 2008)

C

1 Show that a high matter-to-energy conversion efficiency can be expected from accretion onto a compact object such as a neutron star or black hole. [3]

By differentiating $r\Omega$, derive the tangential velocity gradient in a Keplerian accretion disc and thus show that the stress

$$s_\phi = -\frac{3}{2}\eta\Omega,$$

where η is the viscosity and Ω is the angular velocity at radius r. [3]

Show that the rate of change of angular momentum at radius r is equal to

$$\dot{M}\left(\sqrt{GMr} - \sqrt{GMr_0}\right),$$

where r_0 is the inner radius of the disc. [3]

Discuss the last term in the expression above, and then show, by considering the energy production rate per unit volume in a disc of thickness h, where $h \ll r$, that the surface flux of the disc

$$F(r) = \frac{3\dot{M}}{8\pi r^2}\frac{GM}{r}\left(1 - \left(\frac{r_0}{r}\right)^{1/2}\right).$$

[8]

Integrate this to give the total luminosity, explaining the factor of 1/2. Estimate the mass-to-energy conversion efficiency down to 3 Schwarzschild radii (assuming Newtonian gravity). At what radius does the surface flux peak? [5]

Assuming the surface emission is blackbody radiation, find the ratio of the maximum disc temperatures of discs around non-spinning black holes of 10 and 10^9 solar masses if their accretion rates are such that their luminosities correspond to 10% of the Eddington limit. [5]

How does energy production alter in a disc around a spinning black hole? What observations of discs can be used to measure the spin? [3]

2 The cosmological field equations can be written as

$$\frac{\ddot{R}}{R} + \frac{4\pi G\rho}{3}(1+\epsilon) - \frac{\Lambda}{3} = 0 \quad (A),$$

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda}{3} = -\frac{kc^2}{R^2} \quad (B).$$

Explain briefly the meaning of the quantities appearing in these equations. [3]

A spherical overdensity of mass M in the early universe is collapsing under gravity. Let $R_s(t)$ be the radius of the collapsing object as a function of time and let $\bar{R}(t)$ be the radius of an unperturbed sphere encompassing the same mass M.

Explain how equations (A) and (B) can be used to model the behaviour of $R_s(t)$ and $\bar{R}(t)$. [3]

Define the difference of radii of the perturbed and unperturbed spheres as

$$\delta R(t) \equiv \bar{R}(t) - R_s(t).$$

Assuming zero pressure, a zero cosmological constant, and that the background model is matter-dominated Einstein-de-Sitter, use equation (A) to demonstrate that for $\delta R/\bar{R} \ll 1$,

$$\frac{d^2}{dt^2}(\delta R) \approx \bar{H}^2 \delta R,$$

where $\bar{H}(t)$ is the Hubble parameter of the background universe. [6]

By substituting in this equation a trial solution for δR of the form At^n, with n and A constants, and using the expression for the Hubble parameter in a matter-dominated Einstein-de-Sitter universe, show that the density perturbation

$$\delta\rho(t) \equiv \rho_s(t) - \bar{\rho}(t),$$

where $\rho_s(t)$ and $\bar{\rho}(t)$ are the densities of the perturbed and unperturbed spheres respectively, obeys

$$\frac{\delta\rho}{\bar{\rho}} \propto t^m$$

for values of m which you should determine. [9]

Comment on the significance of these results for the development of structure in the universe. [4]

The spherical overdensity goes on to form a cluster of galaxies. Use approximate arguments, based on energy, to show that the radius of the cluster after virialization is one half of the maximum attained by the object in the process of formation. [5]

3 Write brief notes on **three** of the following:

(a) jets and radio sources; [10]

(b) heating and cooling of astrophysical gas; [10]

(c) the structure and stability of isolated self-gravitating gas clouds; [10]

(d) distance measures; [10]

(e) pulsars. [10]

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Tuesday 13 January 2009 9:00am to 10:30am

EXPERIMENTAL AND THEORETICAL PHYSICS (4)
Particle Physics

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(10 December 2008)

D

1 The vertex factor for the interaction between a Higgs boson and a fermion is

$$-i\frac{g_W m_f}{2m_W},$$

where g_W is the weak decay constant, and m_W and m_f are the masses of the W-boson and the fermion, f. Write down the matrix element for the decay $H \to f\bar{f}$. [3]

Consider the decay of the Higgs boson in its rest frame where the fermion is produced with polar angle θ and azimuthal angle ϕ. Assuming $m_H \gg m_f$, evaluate the $H \to f\bar{f}$ matrix elements for the four possible combinations of particle and anti-particle helicities and comment on your results. [8]

Given the expression for the decay rate:

$$\frac{d\Gamma}{d\Omega} = \frac{p^*}{32\pi^2 m_H^2}\langle|M_{fi}|^2\rangle,$$

where p^* is the centre-of-mass momentum of either final state particle, show that the partial decay width for the Higgs boson to $\tau^+\tau^-$ is

$$\Gamma_\tau = \frac{G_F}{\sqrt{2}}\frac{m_\tau^2 m_H}{4\pi}.$$ [4]

Assuming $m_H = 120\,\text{GeV}$, and neglecting decays to $\nu\bar{\nu}$, e^+e^-, $\mu^+\mu^-$, $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$, obtain values for the total decay width of the Higgs boson, Γ_H, and the branching fraction for $H \to b\bar{b}$. [5]

At a future muon collider operating at the Higgs boson resonance (assumed to be $\sqrt{s} = 120\,\text{GeV}$), compare the cross section for the process $\mu^+\mu^- \to H \to b\bar{b}$ to the cross section for the QED process $\mu^+\mu^- \to \gamma \to b\bar{b}$, $\sigma_{QED} = 4\pi\alpha^2/9s$. Comment on the possible advantages and disadvantages of a muon collider compared to an electron-positron collider. [6]

Assuming that a muon collider could operate with fully polarized beams, where the helicities of the μ^+ and μ^- can be chosen, explain how one could distinguish between (i) a Higgs boson with the Standard Model scalar couplings to fermions, and (ii) an exotic Higgs boson with scalar minus pseudo-scalar couplings, $(1-\gamma^5)$. Briefly discuss whether it would be possible to distinguish a Standard Model Higgs boson from one with pure pseudo-scalar couplings (γ^5). [4]

(10 December 2008)

[You may require the following information:

In the limit of $E \gg m$, the Dirac spinors for the helicity eigenstates are

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix},\ u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ e^{i\phi}c \\ s \\ -e^{i\phi}c \end{pmatrix},\ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix},\ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix},$$

where $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$.

The Dirac matrices are given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$G_{\mathrm{F}} = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$, $G_{\mathrm{F}}/\sqrt{2} = g_{\mathrm{W}}^2/8m_{\mathrm{W}}^2$.

The second and third generation fermion masses are $m_\mu = 0.106\,\mathrm{GeV}$, $m_\tau = 1.8\,\mathrm{GeV}$, $m_s = 0.1\,\mathrm{GeV}$, $m_c = 1.5\,\mathrm{GeV}$, $m_b = 4.5\,\mathrm{GeV}$, and $m_t = 175\,\mathrm{GeV}$.

The relativistic Breit-Wigner formula for a resonance of mass m and spin J is

$$\sigma = \frac{4\pi(2J+1)}{m^2}\frac{s\Gamma_i\Gamma_f}{(s-m^2)^2 + m^2\Gamma^2},$$

where Γ_i, Γ_f and Γ are the appropriate partial decay widths and the total decay width.]

2 The ν weak and mass eigenstates are related by the unitary PMNS matrix

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}, \quad \begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = \begin{pmatrix}U^*_{e1} & U^*_{\mu1} & U^*_{\tau1}\\ U^*_{e2} & U^*_{\mu2} & U^*_{\tau2}\\ U^*_{e3} & U^*_{\mu3} & U^*_{\tau3}\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}.$$

Briefly discuss the significance of *mass* and *weak* neutrino eigenstates. [4]

Assuming a neutrino propagation speed of $v = c$, show that the wavefunction at a distance z, for a neutrino beam which initially consisted of entirely ν_μ is given by:

$$|\psi_\nu(z)\rangle = U_{\mu1}|\nu_1\rangle e^{-i\phi_1} + U_{\mu2}|\nu_2\rangle e^{-i\phi_2} + U_{\mu3}|\nu_3\rangle e^{-i\phi_3},$$

with $\phi_i \approx m_i^2 z/2E_i$, where m_i and E_i are the mass and energy of ν_i. [5]

Explain why in the corresponding expression for $\overline{\nu}_\mu$, the matrix elements U are replaced by U^*. [2]

In the T2K experiment, neutrinos are detected in the Super-Kamiokande detector at a distance of $L = 295$ km from the origin of the beam. The beam is almost mono-energetic, with the energy chosen such that $\phi_3 - \phi_2 = \pi$. Assuming $|m_3^2 - m_2^2| \gg |m_2^2 - m_1^2|$, show that the $\nu_\mu \to \nu_e$ oscillation probability is given by

$$P(\nu_\mu \to \nu_e) \approx -2\pi\frac{\Delta m^2_{21}}{\Delta m^2_{32}}\mathrm{Im}(U_{\mu1}U^*_{e1}U^*_{\mu2}U_{e2}) - 4\mathrm{Re}(U_{\mu1}U^*_{e1}U^*_{\mu3}U_{e3} + U_{\mu2}U^*_{e2}U^*_{\mu3}U_{e3}),$$

where $\Delta m^2_{ji} = m_j^2 - m_i^2$. State clearly any assumptions used. [8]

[*You may wish to use the unitarity relation* $|U_{\mu1}U^*_{e1} + U_{\mu2}U^*_{e2} + U_{\mu3}U^*_{e3}|^2 = 0$ *and the identity* $|z_1 + z_2 + z_3|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\mathrm{Re}(z_1 z_2^* + z_1 z_3^* + z_2 z_3^*)$.]

Taking $\theta_{23} = \pi/4$, ignoring the effects of CP violation by taking U to be real ($\delta = 0$), and assuming θ_{13} is small, show that $P(\nu_\mu \to \nu_e) \approx 2\theta_{13}^2$. [5]

[*In the limit where θ_{13} is small, the PMNS matrix can be written*

$$\begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{pmatrix} = \begin{pmatrix} c_{12} & s_{12} & \theta_{13}e^{-i\delta}\\ -s_{12}c_{23} - c_{12}s_{23}\theta_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}\theta_{13}e^{i\delta} & s_{23}\\ s_{12}s_{23} - c_{12}s_{23}\theta_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}\theta_{13}e^{i\delta} & c_{23}\end{pmatrix}$$

where $s_{ij} = \sin\theta_{ij}$ *and* $c_{ij} = \cos\theta_{ij}$.]

At T2K the effects of CP violation are expected to be small. Show that for maximal CP violation, $\delta = \pi/2$, the CP asymmetry,

$$A_{CP} = \frac{P(\nu_\mu \to \nu_e) - P(\overline{\nu}_\mu \to \overline{\nu}_e)}{P(\nu_\mu \to \nu_e) + P(\overline{\nu}_\mu \to \overline{\nu}_e)} \propto \frac{\Delta m^2_{21}}{\Delta m^2_{32}}\frac{1}{\theta_{13}}.$$ [6]

3 Write brief notes on **two** of the following:

(a) Electroweak Unification. [15]

(b) Parton distribution functions and their determination in deep inelastic scattering experiments. [15]

(c) SU(3) flavour symmetry and the observed *uds* baryons. [15]

(d) The Dirac equation and its role in particle physics. [15]

END OF PAPER

(10 December 2008)

NATURAL SCIENCES TRIPOS Part III

Tuesday 13 January 2009 11:30am to 1:00pm

EXPERIMENTAL AND THEORETICAL PHYSICS (5)
Physics of the Earth as a Planet

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

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(10 December 2008)

1 Define the *Rayleigh number*, *aspect ratio* and *planform* of a convecting layer of fluid. [6]

Describe how the aspect ratio and planform vary with the Rayleigh number when the fluid layer is heated from below. [10]

What constraints do seismic tomography and other observations provide on the aspect ratio and planform of mantle convection? [14]

2 Consider an isotropic point source at depth h km in a homogeneous half-space with compressional wave velocity α. Formulate expressions for the ray parameter p, distance x, travel time t, delay time τ, and $\mathrm{d}x/\mathrm{d}p$. x, t, τ, and $\mathrm{d}x/\mathrm{d}p$ can be written in terms of h, p, and η (the vertical slowness). Make approximate plots of $x(p)$, $\tau(p)$, $t(p)$, and $\mathrm{d}x(p)/\mathrm{d}p$. [10]

The complete set of reflection coefficients for P-SV waves incident on a free surface are:

$$R_F^{PP} = \frac{4\rho^2\eta_\alpha\eta_\beta - (\eta_\beta^2 - p^2)^2}{4\rho^2\eta_\alpha\eta_\beta + (\eta_\beta^2 - p^2)^2} \qquad R_F^{PS} = \frac{4\rho\eta_\alpha\,(p^2 - \eta_\beta^2)}{4\rho^2\eta_\alpha\eta_\beta + (\eta_\beta^2 - p^2)^2}$$

$$R_F^{SP} = \frac{4\rho\eta_\beta\,(p^2 - \eta_\beta^2)}{4\rho^2\eta_\alpha\eta_\beta + (\eta_\beta^2 - p^2)^2} \qquad R_F^{SS} = \frac{4\rho^2\eta_\alpha\eta_\beta - (\eta_\beta^2 - p^2)^2}{4\rho^2\eta_\alpha\eta_\beta + (\eta_\beta^2 - p^2)^2}$$

where $\eta_v = (1/v^2 - p^2)^{1/2}$. Make an accurate, labelled sketch of the relative amplitudes *vs.* ray parameter of the waves generated by a shear wave incident on the free surface of a Poisson solid with compressional wave speed 6 km s^{-1}. [10]

Describe and sketch the wavefront pattern set up for the case given above. [10]

3 Answer both (a) and (b):

(a) Write brief notes on the Earth's free oscillations. [15]

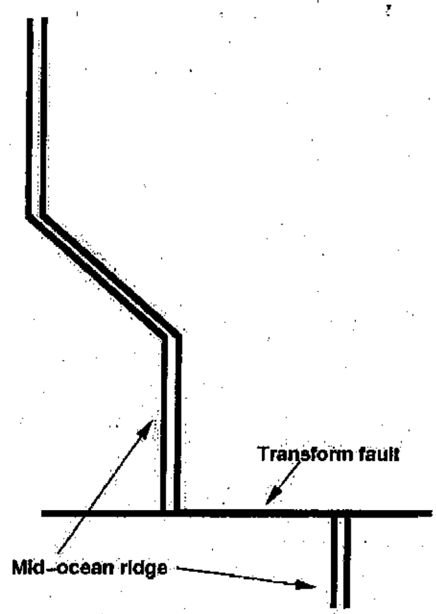
OR

Write brief notes on the forces maintaining plate motions. [15]

(b) Write brief notes on surface waves and the Earth's structure, including in your answer a description of the relationship between the dispersion and the form of the seismic waveform. [15]

OR

Sketch the fault plane solution for earthquakes occuring along the segment of the mid-ocean ridge system shown in the figure below. Label the slip vectors and P- and T-axes of each earthquake plane solution. [15]



END OF PAPER

(10 December 2008)

NATURAL SCIENCES TRIPOS Part III

Tuesday 13 January 2009 2:30pm to 4:00pm

EXPERIMENTAL AND THEORETICAL PHYSICS (6)
Quantum Condensed Matter Field Theory

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

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(10 December 2008)

F

1 Write brief notes on **two** of the following:

(a) Tunnelling and instantons, including examples. [15]

(b) Spin waves in the Heisenberg model of ferro- and antiferro-magnetism. [15]

(c) The Hubbard model and the Mott transition. [15]

2 Sketch, without detailed mathematical proof, the derivation of the Feynman path integral for the propagator from one-dimensional space coordinate q_I at time zero to q_F at time t in its Hamiltonian form: [12]

$$\begin{aligned} G(q_F, q_I; t) &= \langle q_F | \exp(-i\widehat{H}t/\hbar) | q_I \rangle \, \Theta(t) \\ &= \int_{\substack{q(0)=q_I \\ q(t)=q_F}} Dq \int Dp \ \exp\left[\frac{i}{\hbar}\int_0^t dt' \left(p\dot{q} - H(q,p)\right)\right] \Theta(t). \end{aligned}$$

In the case where the Hamiltonian describes free dynamics of a particle in a potential, i.e. $H(q,p) = \frac{p^2}{2m} + V(q)$, derive the Lagrangian form [6]

$$G(q_F, q_I; t) = \int_{\substack{q(0)=q_I \\ q(t)=q_F}} Dq \ \exp\left[\frac{i}{\hbar}\int_0^t dt' L(q,\dot{q})\right] \Theta(t).$$

Explain how in the limit $\hbar \to 0$, classical dynamics is recovered. [4]
Show that for a free particle [8]

$$G(q_F, q_I; t) = \left(\frac{m}{2\pi i \hbar t}\right)^{1/2} \exp\left[\frac{im(q_F - q_I)^2}{2\hbar t}\right] \Theta(t).$$

3 A weakly interacting Bose gas may be described by a Hamiltonian with a repulsive point contact interaction:

$$\widehat{H} = \int d\boldsymbol{x} \left(a^\dagger(\boldsymbol{x}) \left[\frac{\widehat{\boldsymbol{p}}^2}{2m} + V(\boldsymbol{x})\right] a(\boldsymbol{x}) + \frac{g}{2} a^\dagger(\boldsymbol{x}) a^\dagger(\boldsymbol{x}) a(\boldsymbol{x}) a(\boldsymbol{x}) \right),$$

where $[a(\boldsymbol{x}), a^\dagger(\boldsymbol{x}')] = \delta(\boldsymbol{x} - \boldsymbol{x}')$, $\widehat{\boldsymbol{p}} = -i\hbar\nabla$ is the momentum operator, and $V(\boldsymbol{x})$ an external potential. Write down a coherent state path integral representation of the partition function in terms of an action $S[\bar{\psi}, \psi]$ as a functional of a complex field $\psi(\boldsymbol{x}, \tau)$ and its conjugate $\bar{\psi}(\boldsymbol{x}, \tau)$ in the variables of space $\boldsymbol{x}$ and (imaginary) time τ. [5]

Show that this action is minimized by the solutions of the stationary Gross-Pitaevskii equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x}) + g|\psi|^2\right] \psi = \mu\psi ,$$

where μ is the chemical potential, and explain how the solution changes qualitatively depending on the sign of $[\mu - \min(V(\boldsymbol{x}))]$ [5]

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Assuming that $V(\boldsymbol{x})$ may be neglected, use the Gross-Pitaevskii equation to determine the amplitude of the ground state order parameter $|\psi_0|$. By expanding to quadratic order in the fluctuations $\delta\psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - \psi_0$, show that the action may be written in a matrix form as [10]

$$S = S[|\psi_0|] + \frac{1}{2}\iint d\tau d\boldsymbol{x} \left(\delta\bar{\psi}(\boldsymbol{x},\tau)\,\delta\psi(\boldsymbol{x},\tau)\right) \begin{pmatrix} \partial_\tau + \frac{\hat{\boldsymbol{p}}^2}{2m} - \mu + 2g|\psi_0|^2 & g\psi_0^2 \\ g\bar{\psi}_0^2 & -\partial_\tau + \frac{\hat{\boldsymbol{p}}^2}{2m} - \mu + 2g|\psi_0|^2 \end{pmatrix} \begin{pmatrix} \delta\psi(\boldsymbol{x},\tau) \\ \delta\bar{\psi}(\boldsymbol{x},\tau) \end{pmatrix}$$

Hence show that the spectrum of fluctuations is Bogoliubov-like, with eigenfrequencies

$$E_{\boldsymbol{p}} = \pm\sqrt{\varepsilon_{\boldsymbol{p}}{}^2 + 2\mu\varepsilon_{\boldsymbol{p}}}\,,$$

where $\varepsilon_{\boldsymbol{p}} = \hbar^2 p^2/2m$. [5]

Give a physical interpretation of this result and its consequences. [5]

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Wednesday 14 January 2009 9:00am to 10:30am

EXPERIMENTAL AND THEORETICAL PHYSICS (7)
Atomic and Optical Physics

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
20 page answer book
Rough workpad
Metric graph paper

SPECIAL REQUIREMENTS
Mathematical formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

(10 December 2008)

G

1 Summarise the selection rules for the electric dipole transitions between atomic states. Include in your summary the selection rules for transitions between gross, fine, and hyperfine energy levels, and the role of the polarisation of the associated radiation. Wherever possible comment briefly on the physical origin of the selection rules. [10]

For an allowed dipole transition between two states with energies E_1, and $E_2 = E_1 + \hbar\omega_0$, how does the rate of *spontaneous* decay Γ from the excited state depend on the relevant dipole matrix element M and the frequency ω_0? Give a physical interpretation of the ω_0 dependence. [5]

The lifetime of the $2p$ state in hydrogen is 1.6 ns. What is the spectrum (i.e. the central frequency and the linewidth) of the spontaneously emitted radiation? [4]

Briefly discuss the possible decay processes and the lifetimes of the $2s, 3s$, and $3p$ hydrogen states. Which one is the shortest and which one the longest lived? [6]

If a two-level system is *driven* on resonance with a laser at frequency $\omega = \omega_0$, how does the cross section for light absorption by the ground state atoms depend on M and ω? Explain the M dependence. [5]

2 A laser of frequency ω interacts with a two-level atom with internal states $|1\rangle$ and $|2\rangle$. The two bare energy levels of the atom are split by $E_2 - E_1 = \hbar\omega_0$. The strength of the atom-laser coupling is given by the Rabi frequency Ω. In a reference frame rotating at the frequency of the laser and in the rotating wave approximation the Hamiltonian is given by

$$\hat{H} = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{pmatrix}.$$

Here $\Delta = \omega - \omega_0$ with $|\Delta| \gg \Omega$ denotes the detuning of the laser field from the atomic resonance and we have chosen the basis $\{|1\rangle, |2\rangle\}$. Derive the a.c. Stark shift potential (or, equivalently, light shift potential) applied by the laser to the atom. [12]

When is the rotating wave approximation applicable? [3]

Explain how the a.c. Stark shift potential can be used for trapping atoms. [5]

Compare the a.c. Stark shift potential of a red-detuned ($\Delta < 0$) laser field to the heating rate due to off-resonant photon scattering assuming $|\Delta| \gg \Omega$ and $|\Delta| \gg \Gamma$, where Γ is the spontaneous decay rate from the excited state. [5]

In ^{87}Rb the electronic ground state is $5S_{1/2}$. Spin-orbit coupling splits the transition to the $5p$ excited state into D_1 ($5S_{1/2} \rightarrow 5P_{1/2}$) and D_2 ($5S_{1/2} \rightarrow 5P_{3/2}$) lines. The wavelengths of the two lines are 795 nm and

780 nm, respectively. For a fixed laser intensity and polarisation sketch the dependence of the a.c. Stark shift potential on the laser wavelength over the range 765 − 810 nm. You may exclude the regions within 1 nm of the D_1 and D_2 lines, and ignore the effects of the hyperfine structure. [5]

3 Write brief notes on **two** of the following:

(a) How laser cooling below the Doppler limit can be achieved. Include requirements regarding the atomic structure and laser fields, and explain limits to the process. Why is the mechanism also referred to as "Sisyphus cooling"? [15]

(b) The concept and the method of evaporative cooling. Your answer should include comments on the basic principle of evaporative cooling, the advantage of this method compared to laser cooling, the requirements for successful evaporation, the experimental implementation of evaporative cooling in atom traps, and both the conceptual and the technical limits to the cooling process. [15]

(c) The basic properties of quantum computation. Include explanations of the terms "qubit", "quantum gate", and "quantum parallelism", and discuss the advantages of quantum computation as compared to classical computation. Describe a single-qubit gate (e.g. the Hadamard gate) and explain its operation principle using the Bloch sphere. Explain the experimental concept and challenges of quantum computation with trapped atomic ions. [15]

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Wednesday 14 January 2008 11:30am to 1:00pm

EXPERIMENTAL AND THEORETICAL PHYSICS (8)
Quantum Field Theory

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
20 page answer book
Rough workpad
Metric graph paper

SPECIAL REQUIREMENTS
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Approved calculator allowed

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(10 December 2008)

H

1 Let $\phi(x)$ be the Heisenberg field operator in Klein-Gordon theory. The field $\phi(x)$ has mode expansion

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a_{\boldsymbol{p}}^{\dagger} e^{ip\cdot x}\right),$$

where $p_0 = E_p = \sqrt{|\boldsymbol{p}|^2 + m^2}$. Write down the commutation relations for the operators $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^{\dagger}$. Explain how particle and multi-particle states are obtained using these operators. [8]

Explain what is meant by the following:

(i) the time-ordered product $T\phi(x)\phi(y)$; [4]

(ii) the normal-ordered product $:\phi(x)\phi(y):$; [4]

(iii) the Feynman propagator $D_F(x-y)$. [4]

Prove Wick's Theorem, explaining how the quantities in (i)–(iii) above are related. [10]

2 The quantised field of Dirac theory can be written in terms of the momentum space operators $a_{\boldsymbol{p}}^s$, $b_{\boldsymbol{p}}^s$ and their Hermitian conjugates. The operators obey the anticommutation relations

$$\left\{a_{\boldsymbol{p}}^s, a_{\boldsymbol{p}'}^{s'\dagger}\right\} = \left\{b_{\boldsymbol{p}}^s, b_{\boldsymbol{p}'}^{s'\dagger}\right\} = (2\pi)^3\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')\, \delta^{ss'},$$

with all other anticommutators vanishing. The number operator is

$$N = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^s + b_{\boldsymbol{p}}^s b_{\boldsymbol{p}}^{s\dagger}\right).$$

Define the vacuum state and determine the normal-ordered form of N. [6]

Show that

$$[N, a_{\boldsymbol{p}}^{s\dagger}] = a_{\boldsymbol{p}}^{s\dagger}.$$

Deduce the related equations involving $a_{\boldsymbol{p}}^s$, $b_{\boldsymbol{p}}^s$, $b_{\boldsymbol{p}}^{s\dagger}$ and discuss the physical interpretation of this set of equations. [12]

Discuss your interpretation above for the case of the electron and compare it with Dirac hole theory. [12]

(10 December 2008)

H

3 The Lagrangian density of quantum electrodynamics (in standard notation) is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(\mathrm{i}\gamma^{\mu}D_{\mu} - m)\psi\ .$$

Write down expressions for $D_{\mu}\psi$ and $F_{\mu\nu}$. [2]

Write down the Feynman rules for quantum electrodynamics. You should state any assumptions you have made. [10]

Write down the Feynman diagrams for the process $e^- + e^+ \to e^- + e^+$, to $\mathcal{O}(e^2)$. Use the Feynman rules to derive the scattering amplitude. [12]

It is also possible to consider the process $e^- + e^+ \to \mu^- + \mu^+$. Write down the Feynman diagram and comment on the difference between the two processes. [6]

END OF PAPER