NST Part III Experimental and Theoretical Physics — Michaelmas 2025

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Particle Physics Major Option

EXAMPLES SHEET QUESTIONS (ALL)

NATURAL UNITS AND HEAVISIDE-LORENTZ UNITS

1. (a) In the units he normally uses, your particle-physics lecturer was $10^{16}/\,\mathrm{GeV}$ tall and had a mass of $4.40 \times 10^{28}\ \mathrm{GeV}$ when aged $2.11 \times 10^{33}/\,\mathrm{GeV}$. Calculate his Body Mass Index (BMI) and determine whether he was obese at this point in his life.

   (b) Show that charge can indeed be measured in units of $(\varepsilon_0 \hbar c)^{\frac{1}{2}}$. [You may wish to consider dimensional analysis of the Coulomb force law $F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$.]

SOLUTION

(a) The laborious way of working out the height L and mass of M of the lecturer would be to insert all the right powers of $\hbar$ and c and use $\hbar \approx 1.055 \times 10^{-34}\ Js$ and $c = 3.00 \times\ 10^8\ m/s$. This requires many numbers and lots of use of the calculator. Using this bad way to calculate L we might write something like:

$$
\begin{aligned}
L &= 10^{16}\ \hbar c/\,\mathrm{GeV} && (1)\\
&= \frac{(10^{16}) \times (1.055 \times 10^{-34} Js) \times (3.00 \times 10^8\ m/s)}{10^9 \times (1.60 \times 10^{-19} J)} && (2)\\
&= \frac{(10^{16}) \times (1.055 \times 10^{-34}) \times (3.00 \times 10^8)}{10^9 \times (1.60 \times 10^{-19})}\ m && (3)\\
&= \frac{1.055 \times 3.00}{1.60} \times 10^{16-34+8+10}\ m && (4)\\
&= 1.97 \times 10^0\ m && (5)\\
&= 1.97\ m. && (6)
\end{aligned}
$$

Much better would be to use $1 = \hbar c = 197\ \mathrm{MeV} \cdot \mathrm{fm}$. This nicer approach would give us:

$$
\begin{aligned}
L &= 10^{16}/\,\mathrm{GeV} && (7)\\
&= 10^{16}/\,\mathrm{GeV} \times 1 && (8)\\
&= 10^{16}/\,\mathrm{GeV} \times (197\ \mathrm{MeV} \cdot fm) && (9)\\
&= 197 \times 10^{16-9+6-15}\ m && (10)\\
&= 197 \times 10^{-2}\ m && (11)\\
&= 1.97\ m && (12)\\
& && (13)
\end{aligned}
$$

The mass of the lecturer in S.I. units is easier to calculate as $E \sim mc^2$ reminds us that masses are only a factor of c^2 away from energies, and everyone knows c. Therefore

$$\begin{aligned} M &= 4.40 \times 10^{28}\ (\text{GeV}/c^2) && (14) \\ &= (4.40 \times 10^{28}) \times (10^9 \times (1.60 \times 10^{-19}\ J))/(3.00 \times 10^8\ m/s)^2 && (15) \\ &= (4.40 * 1.60/9.00) * 10^{28-10-16}\ \text{kg} && (16) \\ &= 78\ \text{kg}. && (17) \end{aligned}$$

Hence the BMI (which is mass in kg divided by square of height in metres) is

$$\text{BMI} = 78/(1.97)^2 = 20.1. \qquad (18)$$

According to Wikipedia (`https://en.wikipedia.org/wiki/Body_mass_index`) the WHO defines obsedity as a BMI over 25 if the person is more than 20 years old, so he is not obese given the age supplied (44 years).

(b)

$$\begin{aligned} [q_1 q_2] &= [4pi\varepsilon_0 F r^2] && (19) \\ &= [\varepsilon_0 F L^2] && (20) \\ &= [\varepsilon_0 (FL) L] && (21) \\ &= [\varepsilon_0 E L] && (22) \\ &= [\varepsilon_0 (ET)(L/T)] && (23) \\ &= [\varepsilon_0 \hbar c]. && (24) \end{aligned}$$

SPECIAL RELATIVITY

2. a) Draw the two leading-order Feynman diagrams for $e^+e^- \to e^+e^-$ involving single photon exchange, and write q, the 4-momentum of the exchanged virtual photon, in terms of the 4-momenta of the initial and/or final state particles. By evaluating q^2 in the centre of mass frame, or otherwise, determine whether q is *timelike* ($q^2 > 0$) or *spacelike* ($q^2 < 0$) in each case.

b) The *Mandelstam variables* s, t, u in the scattering process $a + b \to 1 + 2$ are defined in terms of the particle 4-vectors as

$$s = (p_a + p_b)^2, \qquad t = (p_a - p_1)^2, \qquad u = (p_a - p_2)^2 \ .$$

Show that $s + t + u = {m_a}^2 + {m_b}^2 + {m_1}^2 + {m_2}^2$.

c) Show that $\sqrt{s}$ is the total energy of the collision in the centre of mass frame.

d) At the HERA accelerator in Hamburg, 27.5 GeV electrons are brought into head-on collision with 820 GeV protons. Calculate the centre of mass energy, $\sqrt{s}$, of $e^-\mathrm{p}$ collisions at HERA, and determine the beam energy that would be needed to produce $e^-\mathrm{p}$ collisions with this value of $\sqrt{s}$ using electrons incident on a stationary proton target.

e) Show that, in the laboratory frame with particle X at rest, the reaction $\nu + X \to \ell + Y$ can only proceed if the incoming neutrino has an energy above a threshold given by

$$E_\nu > \frac{(m_l + m_Y)^2 - m_X^2}{2m_X} \ .$$

[Aside: when revising at the end of the course you may wish to consider reviewing Question 1 of the January 2017 past Tripos paper for this course as looks more deelply into the connections between Mandelstam variables and the characteristics of different scattering processes.]

SOLUTION

a) The two leading order Feynman diagrams for $e^+e^- \to e^+e^-$ scattering are:

e^+ p_1 p_3 e^+

e^+ e^+

p_2 p_4

q

q

p_1 p_3

e^- e^-

e^- p_2 p_4 e^-

For diagram 1, the 4-momentum of the virtual photon is $q = p_1 + p_2$. In the centre of mass frame, we have $q = p_1 + p_2 = (2E, 0, 0, 0)$, and hence

$$q^2 = 4E^2 > 0 \qquad \Rightarrow \quad q^2 \text{ is timelike.}$$

For diagram 2, $q = p_1 - p_3$. In the centre of mass frame, we have $E_1 = E_3$ (elastic scattering) and hence $q = (0, \boldsymbol{p}_1 - \boldsymbol{p}_3)$. Therefore

$$q^2 = -(\boldsymbol{p}_1 - \boldsymbol{p}_3)^2 < 0 \qquad \Rightarrow \quad q^2 \text{ is spacelike}$$

b) Since $p_a^2 = m_a^2$ *etc.*:

$$\begin{aligned} s + t + u &= (p_a + p_b)^2 + (p_a - p_1)^2 + (p_a - p_2)^2 \\ &= 3p_a^2 + p_b^2 + p_1^2 + p_2^2 + 2p_a.p_b - 2p_a.p_1 - 2p_a.p_2 \\ &= 3m_a^2 + m_b^2 + m_1^2 + m_2^2 + 2p_a.(p_b - p_1 - p_2) \\ &= 3m_a^2 + m_b^2 + m_1^2 + m_2^2 + 2p_a. - p_a \\ &= m_a^2 + m_b^2 + m_1^2 + m_2^2 \end{aligned}$$

where energy-momentum conservation, $p_a + p_b = p_1 + p_2$, has been used in the last line but one.

c) In the centre of mass frame, the 4-momenta of particles a and b can be taken to be $p_a = (E_a, 0, 0, p)$, $p_b = (E_b, 0, 0, -p)$. Hence $p_a + p_b = (E_a + E_b, 0, 0, 0)$ and $s = (p_a + p_b)^2 = (E_a + E_b)^2$. Hence $\sqrt{s} = E_a + E_b$, the total collision energy in the centre of mass frame.

d) HERA: electron and proton masses can be neglected, so 4-momenta are:

$$p_a = (E_a, 0, 0, E_a) \qquad p_b = (E_b, 0, 0, -E_b) \qquad \Rightarrow \qquad p_a + p_b = (E_a + E_b, 0, 0, E_a - E_b)$$

Hence

$$s = (p_a + p_b)^2 = (E_a + E_b)^2 - (E_a - E_b)^2 = 4E_aE_b \, ,$$

which gives

$$\sqrt{s} = 2\sqrt{E_a E_b} = 2\sqrt{27.5\,\mathrm{GeV} * 820\,\mathrm{GeV}} = 300\,\mathrm{GeV}\ .$$

For electrons incident on a stationary proton target:

$$p_a = (E_a, 0, 0, E_a) \qquad p_b = (m_p, 0, 0, 0) \qquad \Rightarrow \qquad p_a + p_b = (E_a + m_p, 0, 0, E_a)\ .$$

Hence

$$s = (p_a + p_b)^2 = (E_a + m_p)^2 - E_a^2 = 2E_a m_p + m_p^2\ ,$$

which gives

$$E_a = \frac{s - m_p^2}{2m_p} = \frac{(300\,\mathrm{GeV})^2 - (0.938\,\mathrm{GeV})^2}{2 \times (0.938\,\mathrm{GeV})} = 47974\,\mathrm{GeV}\ .$$

e) For the scattering process $\nu + \mathrm{X} \to \ell + \mathrm{Y}$ to be kinematically allowed, we must have

$$\sqrt{s} > m_l + m_Y\ . \tag{25}$$

This is easily seen by considering the centre of mass frame: at threshold, the particles ℓ and Y are both produced at rest. Equation (25) involves only Lorentz-invariant quantities, and so can be applied to *any* reference frame. In particular, in the lab frame, with X at rest, we have

$$s = m_X^2 + 2p_\nu \cdot p_X = m_X^2 + 2E_\nu m_X\ .$$

Hence we need

$$m_X^2 + 2E_\nu m_X > (m_l + m_Y)^2$$

which gives a threshold neutrino energy in the lab frame of

$$\boxed{E_\nu > \frac{(m_l + m_Y)^2 - m_X^2}{2m_X}}\ .$$

3. a) For a particle of four-momentum $p^\mu = (E, p_x, p_y, p_z)$, show that the scalar product

$$p^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

is Lorentz invariant by explicitly transforming the four components of p^μ.

b) Use the Lorentz transformations to show that the volume element d^3p/E in momentum space is Lorentz invariant, *i.e.* that

$$\frac{\mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z}{E} = \frac{\mathrm{d}p'_x \mathrm{d}p'_y \mathrm{d}p'_z}{E'}\ .$$

SOLUTION

a) Lorentz transformation (with $c = 1$):

$$E' = \gamma(E - \beta p_x) \qquad p'_y = p_y$$
$$p'_x = \gamma(p_x - \beta E) \qquad p'_z = p_z$$

where $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v/c = v$. Hence

$$\begin{aligned}(p')^2 &= (E')^2 - (p'_x)^2 - (p'_y)^2 - (p'_z)^2 \\ &= \gamma^2(E - \beta p_x)^2 - \gamma^2(p_x - \beta E)^2 - p_y^2 - p_z^2 \\ &= \gamma^2(1-\beta^2)E^2 - \gamma^2(1-\beta^2)p_x^2 - p_y^2 - p_z^2 \\ &= E^2 - p_x^2 - p_y^2 - p_z^2 \\ &= p^2\end{aligned}$$

b) Since $\mathrm{d}p'_y = \mathrm{d}p_y$ and $\mathrm{d}p'_z = \mathrm{d}p_z$ we have

$$\mathrm{d}^3p' = \mathrm{d}p'_x \mathrm{d}p'_y \mathrm{d}p'_z = \frac{\mathrm{d}p'_x}{\mathrm{d}p_x} \cdot \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z = \frac{\mathrm{d}p'_x}{\mathrm{d}p_x} \mathrm{d}^3p$$

where $p'_x = \gamma(p_x - \beta E)$ and E is to be understood as $E = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}$. The derivative is

$$\frac{\mathrm{d}p'_x}{\mathrm{d}p_x} = \frac{\mathrm{d}}{\mathrm{d}p_x}\left[\gamma(p_x - \beta E)\right] = \gamma\left(1 - \beta\frac{\mathrm{d}E}{\mathrm{d}p_x}\right) .$$

The components p_y and p_z remain unchanged in the transformation, and so can be treated as constants. Hence

$$\frac{\mathrm{d}E}{\mathrm{d}p_x} = \frac{\mathrm{d}}{\mathrm{d}p_x}\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2} = \frac{p_x}{\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}} = \frac{p_x}{E} .$$

This gives

$$\frac{\mathrm{d}p'_x}{\mathrm{d}p_x} = \gamma\left(1 - \beta\frac{p_x}{E}\right) = \gamma\frac{E - \beta p_x}{E} = \frac{E'}{E} ,$$

and therefore

$$\frac{\mathrm{d}^3p'}{E'} = \frac{1}{E'} \cdot \frac{E'}{E}\mathrm{d}^3p = \frac{\mathrm{d}^3p}{E}$$

4. In a 2-body decay, $a \to 1 + 2$, show that the three-momentum of the final state particles in the centre of mass frame has magnitude

$$p^* = \frac{1}{2m_a}\sqrt{\left[m_a^2 - (m_1 + m_2)^2\right]\left[m_a^2 - (m_1 - m_2)^2\right]} .$$

SOLUTION

Decay $a \to 1 + 2$: energy conservation gives

$$m_a = E_1 + E_2 = \sqrt{m_1^2 + p^{*2}} + \sqrt{m_2^2 + p^{*2}}$$

Squaring:

$$m_a^2 = E_1^2 + E_2^2 + 2E_1E_2 = m_1^2 + m_2^2 + 2p^{*2} + 2\sqrt{(m_1^2 + p^{*2})(m_2^2 + p^{*2})}$$

$$\Rightarrow \qquad 2\sqrt{(m_1^2 + p^{*2})(m_2^2 + p^{*2})} = m_a^2 - m_1^2 - m_2^2 - 2p^{*2} .$$

Squaring again:

$$\Rightarrow \qquad 4(m_1^2 + p^{*\,2})(m_2^2 + p^{*\,2}) = (m_a^2 - m_1^2 - m_2^2 - 2p^{*\,2})^2\ .$$

Multiplying out and rearranging gives

$$\begin{aligned} 4m_a^2 p^{*\,2} &= (m_a^2 - m_1^2 - m_2^2)^2 - (2m_1 m_2)^2 \\ &= (m_a^2 - m_1^2 - m_2^2 - 2m_1 m_2)(m_a^2 - m_1^2 - m_2^2 + 2m_1 m_2) \\ &= \left[m_a^2 - (m_1 + m_2)^2\right]\left[m_a^2 - (m_1 - m_2)^2\right]\ . \end{aligned}$$

Hence

$$\boxed{p^* = \frac{1}{2m_a}\sqrt{\left[m_a^2 - (m_1+m_2)^2\right]\left[m_a^2 - (m_1-m_2)^2\right]}}\ .$$

TWO BODY DECAY

5. According to the hypothesis of SU(3) symmetry (*i.e.* uds flavour independence) of invariant matrix elements, the two-body decay processes $\rho \to \pi\pi$ and $\mathrm{K}^* \to \mathrm{K}\pi$ have invariant matrix elements of the form

$$M_{\mathrm{fi}} = C p_\pi$$

where $C_\rho / C_{K^*} = 2/\sqrt{3}$ and p_π is the final state centre of mass momentum. Show that the predicted ratio of decay rates agrees with experiment to within about 15%.

[Use the result of Question 4 to obtain p_π. Take the π, ρ, K and K* meson masses to be 139, 770, 494 and 892 MeV respectively. The measured widths are $\Gamma(\rho \to \pi\pi) = 153 \pm 2\,\mathrm{MeV}$ and $\Gamma(\mathrm{K}^* \to \mathrm{K}\pi) = 51.3 \pm 0.8\,\mathrm{MeV}$.]

SOLUTION

a) The matrix element $M_{\mathrm{fi}} = Cp_\pi$ depends only on the centre of mass momentum $p_\pi = p^*$ of the final state particles, not on their directions, *i.e.* the decays are isotropic. For any isotropic two-body decay $a \to 1 + 2$, the decay rate is

$$\Gamma = \frac{p^*}{8\pi m_a^2}|M_{\mathrm{fi}}|^2 = \frac{p^*}{8\pi m_a^2}\cdot (Cp^*)^2 = \frac{C^2 p^{*\,3}}{8\pi m_a^2}\ .$$

From question 3, the centre of mass momentum is given by

$$p^* = \frac{1}{2m_a}\left[(m_a + m_1 + m_2)(m_a - m_1 + m_2)(m_a + m_1 - m_2)(m_a - m_1 - m_2)\right]^{1/2}\ .$$

For $\rho \to \pi\pi$, we have $m_a = m_\rho = 770\,\mathrm{MeV}$, $m_1 = m_2 = m_\pi \approx 140\,\mathrm{MeV}$:

$$p^* = \frac{1}{2m_\rho}\sqrt{(m_\rho + 2m_\pi).m_\rho.m_\rho.(m_\rho - 2m_\pi)} = \tfrac{1}{2}\sqrt{m_\rho^2 - 4m_\pi^2} = 359\,\mathrm{MeV}$$

For $\mathrm{K}^* \to \mathrm{K}\pi$, we have $m_a = m_{K^*} = 892\,\mathrm{MeV}$, $m_1 = m_K \approx 494\,\mathrm{MeV}$, $m_2 = m_\pi \approx 140\,\mathrm{MeV}$ giving $p^* = 288\,\mathrm{MeV}$.

$$\Rightarrow \frac{\Gamma(\rho \to \pi\pi)}{\Gamma(\mathrm{K}^* \to \mathrm{K}\pi)} = \frac{C_\rho^2}{C_{\mathrm{K}^*}^2}\cdot\frac{m_{\mathrm{K}^*}^2}{m_\rho^2}\cdot\left(\frac{p_\rho^*}{p_{\mathrm{K}^*}^*}\right)^3 = \left(\frac{2}{\sqrt{3}}\right)^2\cdot\left(\frac{892}{770}\right)^2\cdot\left(\frac{359}{288}\right)^3 = 3.46$$

Data:

$$\Gamma(\rho \to \pi\pi) = 153 \pm 2\,\text{MeV}, \qquad \Gamma(\text{K}^* \to \text{K}\pi) = 51.3 \pm 0.8\,\text{MeV}$$

giving a measured ratio of 2.98 .

6. The π^+ meson decays almost entirely via the two body decay process $\pi^+ \to \mu^+\nu_\mu$, with an invariant matrix element given by

$$|M_{\text{fi}}|^2 = 2G_\text{F}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)$$

where $G_\text{F} = 1.166 \times 10^{-5}\,\text{GeV}^{-2}$ is the Fermi constant, and f_π is related to the size of the pion wavefunction (the pion being a composite object).

a) Obtain a formula for the $\pi^+ \to \mu^+\nu_\mu$ decay rate. Assuming $f_\pi \sim m_\pi$, calculate the pion lifetime in natural units and in seconds, and compare to measurement.

[$m_\pi = 139.6\,\text{MeV}$, $m_\mu = 105.7\,\text{MeV}$.]

b) By replacing m_μ by m_e, show that the rate of $\pi^+ \to e^+\nu_e$ decay is 1.28×10^{-4} times smaller than the corresponding decay rate to muons. Show also that, on the basis of phase space alone (*i.e.* neglecting the factor $|M_{\text{fi}}|^2$), the decay rate to electrons would be expected to be *greater* than the rate to muons.

SOLUTION

a) From question 3, the momentum of the μ^+ or ν_μ from a $\pi^+ \to \mu^+\nu_\mu$ decay, in the π^+ rest frame, is

$$p^* = \frac{(m_\pi + m_\mu)(m_\pi - m_\mu)}{2m_\pi} = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

and hence the decay rate is

$$\begin{aligned}
\Gamma = \frac{p^*}{8\pi m_\pi^2}|M_{\text{fi}}|^2 &= \frac{m_\pi^2 - m_\mu^2}{16\pi m_\pi^3} \cdot 2G_\text{F}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2) \\
&= \frac{G_\text{F}^2}{8\pi}\frac{m_\mu^2}{m_\pi}(m_\pi^2 - m_\mu^2)^2 \\
&= \frac{(1.166 \times 10^{-5})^2}{8\pi} \cdot \frac{0.105^2}{0.140}(0.140^2 - 0.105^2)^2 \\
&= 3.34 \times 10^{-17}\,\text{GeV}
\end{aligned}$$

The pion lifetime is therefore

$$\tau_\pi = \frac{1}{\Gamma} = \frac{1}{3.34 \times 10^{-17}} = 3.0 \times 10^{16}\,\text{GeV}^{-1}$$

which can be converted to SI units using $\hbar = 6.58 \times 10^{-25}$ GeV.s :

$$\tau_\pi = (3.0 \times 10^{16}).(6.58 \times 10^{-25}) = 1.97 \times 10^{-8}\,\text{s}$$

b) Ratio of decay rates:

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{m_e^2}{m_\mu^2} \cdot \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = \left(\frac{0.511}{105.6}\right)^2 \cdot \left(\frac{139.6^2 - 0.511^2}{139.6^2 - 105.6^2}\right)^2 = 1.28 \times 10^{-4}$$

On the basis of phase space alone, *i.e.* neglecting the contribution to the decay rate from $|M_{\mathrm{fi}}|^2$, we have

$$\Gamma = \frac{p^*}{8\pi m_\pi^2} \propto p^* \ .$$

Hence the ratio of decay rates is just the ratio of the centre of mass momenta appropriate to each decay:

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{p^*(\pi^+ \to e^+\nu_e)}{p^*(\pi^+ \to \mu^+\nu_\mu)} = \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} = 2.34$$

THE DIRAC EQUATION

7. Write down a simplified form of the Dirac equation for a spinor $\psi(t)$ describing a particle of mass m at rest. For the standard Pauli-Dirac representation of the γ matrices, obtain a differential equation for each component ψ_i of the spinor ψ, and hence write down a general solution for the evolution of ψ. Comment on your result and on its relation to the standard plane wave solutions involving $u_1(p)$, $u_2(p)$, $v_1(p)$, $v_2(p)$.

SOLUTION

For a particle of mass m at rest ($\boldsymbol{p} = \boldsymbol{0}$), since $\boldsymbol{p} = -i\boldsymbol{\nabla}$, we have $\partial\psi/\partial x = \partial\psi/\partial y = \partial\psi/\partial z = 0$. Hence $\psi = \psi(t)$ only, and the Dirac equation simplifies to

$$\boxed{i\gamma^0 \frac{\partial \psi}{\partial t} = m\psi} \, .$$

In the Pauli-Dirac representation, this is

$$i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \\ \dot{\psi}_4 \end{pmatrix} = m \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} ,$$

which gives

$$i\dot{\psi}_1 = m\psi_1, \qquad i\dot{\psi}_2 = m\psi_2, \qquad -i\dot{\psi}_3 = m\psi_3, \qquad -i\dot{\psi}_4 = m\psi_4 \, .$$

These equations have the solutions

$$\psi_1 = A_1 e^{-imt}, \qquad \psi_2 = A_2 e^{-imt}, \qquad \psi_3 = A_3 e^{+imt}, \qquad \psi_4 = A_4 e^{+imt} ,$$

where the A_i are complex constants. The general solution for ψ is therefore

$$\boxed{\psi = \begin{pmatrix} A_1 e^{-imt} \\ A_2 e^{-imt} \\ A_3 e^{+imt} \\ A_4 e^{+imt} \end{pmatrix}} \, .$$

This can be expressed as a linear combination of the four independent solutions

$$\psi = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \quad N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \quad N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt}, \quad N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt} , \tag{26}$$

where $N = \sqrt{2m}$ to normalise to $2E = 2m$ particles per unit volume.

Thus both positive energy, e^{-imt}, and negative energy, e^{+imt}, solutions unambiguously emerge.

The spinors in Equation (26) can be obtained by setting $E = m$, $p_x = p_y = p_z = 0$ in the standard plane wave solutions $u_1 e^{i(\boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{r}-Et)}$, $u_2 e^{i(\boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{r}-Et)}$, $v_2 e^{-i(\boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{r}-Et)}$, $v_1 e^{-i(\boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{r}-Et)}$, as expected.

8. a) For the standard Pauli-Dirac representation of the γ matrices, and for an arbitrary pair of spinors ψ and ϕ with components ψ_i and ϕ_i, show that the current $\overline{\psi}\gamma^\mu\phi$ is given by

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\
\overline{\psi}\gamma^1\phi &= \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\overline{\psi}\gamma^2\phi &= -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\overline{\psi}\gamma^3\phi &= \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2
\end{aligned}$$

b) For a particle or antiparticle with four-momentum $p^\mu = (E, p_x, p_y, p_z)$, show that

$$\overline{u}_1\gamma^\mu u_1 = \overline{u}_2\gamma^\mu u_2 = \overline{v}_1\gamma^\mu v_1 = \overline{v}_2\gamma^\mu v_2 = 2p^\mu$$

and that

$$\overline{u}_1\gamma^\mu u_2 = \overline{u}_2\gamma^\mu u_1 = \overline{v}_1\gamma^\mu v_2 = \overline{v}_2\gamma^\mu v_1 = 0\ .$$

c) Hence show that the current $j^\mu = \overline{\psi}(p)\gamma^\mu\psi(p)$ corresponding to a general free particle spinor $\psi(p) = u(p)e^{i(\boldsymbol{p}\cdot\boldsymbol{r}-Et)}$ or antiparticle spinor $\psi(p) = v(p)e^{-i(\boldsymbol{p}\cdot\boldsymbol{r}-Et)}$ is given by $j^\mu = 2p^\mu$. Write down the particle density and flux represented by j^μ.

SOLUTION

a) For an arbitrary pair of spinors ψ and ϕ say, with spinor components ψ_i and ϕ_i, standard matrix multiplication gives, for $\mu = 0$,

$$\overline{\psi}\gamma^0\phi = \begin{pmatrix}\psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^*\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix} = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4\ .$$

Similarly, for $\mu = 1, 2, 3$, we obtain

$$\begin{aligned}
\overline{\psi}\gamma^1\phi &= \begin{pmatrix}\psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^*\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix} = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\overline{\psi}\gamma^2\phi &= \begin{pmatrix}\psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^*\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix} = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\overline{\psi}\gamma^3\phi &= \begin{pmatrix}\psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^*\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix} = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2\ .
\end{aligned}$$

In summary:

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\
\overline{\psi}\gamma^1\phi &= \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\overline{\psi}\gamma^2\phi &= -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\overline{\psi}\gamma^3\phi &= \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2
\end{aligned}$$

b) For the free particle spinor u_1, the first element of the current 4-vector is

$$\begin{aligned}\overline{u}_1\gamma^0 u_1 &= (E+m)\left[1 + \frac{p_z^2}{(E+m)^2} + \frac{(p_x^2+p_y^2)}{(E+m)^2}\right] \\ &= (E+m)\left[1 + \frac{p^2}{(E+m)^2}\right] \\ &= \frac{(E+m)^2+p^2}{E+m} = \frac{2E^2+2Em}{E+m} = 2E\ ,\end{aligned}$$

where, in the last line, we have made use of the relation $E^2 = p^2 + m^2$.

Repeating this exercise for the remaining terms in the 4-vector current gives, altogether,

$$\overline{u}_1\gamma^0 u_1 = 2E; \qquad \overline{u}_1\gamma^1 u_1 = 2p_x; \qquad \overline{u}_1\gamma^2 u_1 = 2p_y; \qquad \overline{u}_1\gamma^3 u_1 = 2p_z$$

which can be expressed more compactly as

$$\overline{u}_1\gamma^\mu u_1 = (2E, 2p_x, 2p_y, 2p_z) = 2p^\mu\ .$$

Repeating the above exercise for u_2, v_1 and v_2 in place of u_1 gives

$$\overline{u}_1\gamma^\mu u_1 = \overline{u}_2\gamma^\mu u_2 = \overline{v}_1\gamma^\mu v_1 = \overline{v}_2\gamma^\mu v_2 = 2p^\mu\ ,$$

while the cross-terms are easily seen to vanish:

$$\overline{u}_1\gamma^\mu u_2 = \overline{u}_2\gamma^\mu u_1 = \overline{v}_1\gamma^\mu v_2 = \overline{v}_2\gamma^\mu v_1 = 0\ .$$

c) For a particle, with $\psi = u(p)e^{ip.x}$, we have

$$\overline{\psi} = \psi^\dagger\gamma^0 = u(p)^\dagger\gamma^0 e^{-ip.x} = \overline{u}(p)e^{-ip.x}\ ,$$

and hence

$$j^\mu = \overline{\psi}\gamma^\mu\psi = \overline{u}\gamma^\mu u\ .$$

For an antiparticle, we have similarly $j^\mu = \overline{v}\gamma^\mu v$.

A particle spinor $u(p)$ can always be expressed as a linear combination of the basis spinors u_1, u_2:

$$u = \alpha_1 u_1 + \alpha_2 u_2, \qquad |\alpha_1|^2 + |\alpha_2|^2 = 1\ .$$

Hence

$$\overline{u}\gamma^\mu u = |\alpha_1|^2\overline{u}_1\gamma^\mu u_1 + |\alpha_2|^2\overline{u}_2\gamma^\mu u_2 = 2p^\mu\ .$$

Thus

$$\boxed{j^\mu = 2p^\mu}\ .$$

The current 4-vector is $j^\mu = (\rho, \boldsymbol{j})$ so

$$\rho = 2E, \qquad \boldsymbol{j} = 2\boldsymbol{p}\ ,$$

ρ being the particle density and $\boldsymbol{j}$ being the flux.

9. a) For a particle with 4-momentum $p^\mu = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$, show that the spinors $(1+\gamma^5)u_1$ and $(1+\gamma^5)u_2$ are not in general proportional to $u_\uparrow$ but become so in the relativistic limit $E \gg m$.

   b) Define the terms *helicity* and *chirality*. How are chirality and helicity related to the spinors and result described in part (a) ?

   c) What would be the equivalent result to that described in (a) for the corresponding antiparticle spinors $(1+\gamma^5)v_1$ and $(1+\gamma^5)v_2$?

SOLUTION

a) For $p^\mu = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$, we have

$$u_\uparrow(p) = \sqrt{E+m}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \\ \frac{p}{E+m}\cos\theta/2 \\ \frac{p}{E+m}e^{i\phi}\sin\theta/2 \end{pmatrix}, \qquad u_\downarrow(p) = \sqrt{E+m}\begin{pmatrix} -\sin\theta/2 \\ e^{i\phi}\cos\theta/2 \\ \frac{p}{E+m}\sin\theta/2 \\ -\frac{p}{E+m}e^{i\phi}\cos\theta/2 \end{pmatrix}$$

But

$$\begin{aligned}(1+\gamma^5)u_1 &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} \\ &= \sqrt{E+m}\begin{pmatrix} 1+p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1+p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix}\end{aligned}$$

which, in general, is clearly not proprtional to $u_\uparrow$.

In the limit $E \gg m$, the spinors u_1 and u_2 become

$$\begin{aligned} u_1 &= \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} \rightarrow \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} \\ u_2 &= \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix} \rightarrow \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ e^{-i\phi}\sin\theta \\ -\cos\theta \end{pmatrix}. \end{aligned}$$

Hence

$$(1+\gamma^5)u_1 \to \begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}\sqrt{E}\begin{pmatrix}1\\0\\\cos\theta\\e^{i\phi}\sin\theta\end{pmatrix} = \sqrt{E}\begin{pmatrix}1+\cos\theta\\e^{i\phi}\sin\theta\\1+\cos\theta\\e^{i\phi}\sin\theta\end{pmatrix}$$

$$= 2\sqrt{E}\cos\theta/2\begin{pmatrix}\cos\theta/2\\e^{i\phi}\sin\theta/2\\\cos\theta/2\\e^{i\phi}\sin\theta/2\end{pmatrix} = 2\cos\theta/2\cdot u_\uparrow \quad (27)$$

and similarly:

$$(1+\gamma^5)u_2 \to \begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}\sqrt{E}\begin{pmatrix}0\\1\\e^{-i\phi}\sin\theta\\-\cos\theta\end{pmatrix} = \sqrt{E}\begin{pmatrix}e^{-i\phi}\sin\theta\\1-\cos\theta\\e^{-i\phi}\sin\theta\\1-\cos\theta\end{pmatrix}$$

$$= 2\sqrt{E}\sin\theta/2\begin{pmatrix}e^{-i\phi}\cos\theta/2\\\sin\theta/2\\e^{-i\phi}\cos\theta/2\\\sin\theta/2\end{pmatrix} = 2e^{-i\phi}\sin\theta/2\cdot u_\uparrow \quad (28)$$

b) The *helicity* operator $h = \boldsymbol{\Sigma}.\hat{\boldsymbol{p}}$ gives the projection of the particle spin along the direction of motion. A particle or antiparticle with the spin vector aligned along (opposite to) the direction of motion has $h = +1$ ($h = -1$) and is said to be *right-handed* (*left-handed*).

Any (particle or antiparticle) spinor ψ can be expressed as the sum of its *left-handed* and *right-handed chiral components*

$$\psi = \psi_{\rm L} + \psi_{\rm R}; \qquad \psi_{\rm L} \equiv \tfrac{1}{2}(1-\gamma^5)\psi \qquad \psi_{\rm R} \equiv \tfrac{1}{2}(1+\gamma^5)\psi\ .$$

In the extreme relativistic limit ($E \gg m$), the left-handed and right-handed chiral components are also eigenstates of the helicity operator:

For a *particle*:	$\psi_{\rm L}$ has helicity -1	$\psi_{\rm R}$ has helicity $+1$
For an *antiparticle*:	$\psi_{\rm L}$ has helicity $+1$	$\psi_{\rm R}$ has helicity -1

The results in part a) show that, in the relativistic limit, and *only* in the relativistic limit, the right-handed chiral components $(1+\gamma^5)u_1$ and $(1+\gamma^5)u_2$ are both proportional to $u_\uparrow$, *i.e.* are both positive helicity eigenstates. Since any particle spinor u can be expressed as a linear combination of u_1 and u_2, this result holds quite generally *i.e.* in the relativistic limit, the right-handed chiral component $(1+\gamma^5)u$ becomes a right-handed helicity eigenstate for any particle spinor u.

c) For antiparticles, the right-handed chiral component $\frac{1}{2}(1+\gamma^5)\psi$ becomes a left-handed helicity eigenstate in the relativistic limit. Hence $(1+\gamma^5)v_1$ and $(1+\gamma^5)v_2$ will both become proportional to $v_\downarrow$ in the relativsitic limit.

10. a) Without resorting to an explicit representation of the Dirac gamma matrices, show that the matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ has the following properties:

$$(\gamma^5)^2 = 1, \qquad \gamma^{5\dagger} = \gamma^5, \qquad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5 .$$

b) Show that the adjoint spinors $\overline{\psi_{\mathrm{L}}}$ and $\overline{\psi_{\mathrm{R}}}$ corresponding to the left-handed and right-handed components $\psi_{\mathrm{L}} \equiv \frac{1}{2}(1-\gamma^5)\psi$ and $\psi_{\mathrm{R}} \equiv \frac{1}{2}(1+\gamma^5)\psi$ are:

$$\begin{aligned}\overline{\psi_{\mathrm{L}}} &= \overline{\psi}\tfrac{1}{2}(1+\gamma^5)\\ \overline{\psi_{\mathrm{R}}} &= \overline{\psi}\tfrac{1}{2}(1-\gamma^5) .\end{aligned}$$

c) Show that $\overline{\phi_{\mathrm{L}}}\gamma^\mu\psi_{\mathrm{R}} = \overline{\phi_{\mathrm{R}}}\gamma^\mu\psi_{\mathrm{L}} = 0$, and that the current $\overline{\phi}\gamma^\mu\psi$ can be decomposed as

$$\overline{\phi}\gamma^\mu\psi = \overline{\phi_{\mathrm{L}}}\gamma^\mu\psi_{\mathrm{L}} + \overline{\phi_{\mathrm{R}}}\gamma^\mu\psi_{\mathrm{R}} .$$

SOLUTION

a) Repeatedly use the fact that the γ matrices anticommute and satisfy $(\gamma^0)^2 = 1,\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$:

$$\begin{aligned}(\gamma^5)^2 = (i\gamma^0\gamma^1\gamma^2\gamma^3)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 & &\\ &= \ \ \gamma^0\gamma^1\gamma^2\gamma^0\gamma^3\gamma^1\gamma^2\gamma^3 & &\text{since}\quad \gamma^3\gamma^0 = -\gamma^0\gamma^3\\ &= -\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 & &\text{since}\quad \gamma^2\gamma^0 = -\gamma^0\gamma^2\\ &= \ \ \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 & &\text{since}\quad \gamma^1\gamma^0 = -\gamma^0\gamma^1\\ &= \ \ \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 & &\text{since}\quad (\gamma^0)^2 = 1\\ &= -\gamma^1\gamma^2\gamma^1\gamma^3\gamma^2\gamma^3 & &\text{since}\quad \gamma^3\gamma^1 = -\gamma^1\gamma^3\\ &= \ \ \gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3 & &\text{since}\quad \gamma^2\gamma^1 = -\gamma^1\gamma^2\\ &= -\gamma^2\gamma^3\gamma^2\gamma^3 & &\text{since}\quad (\gamma^1)^2 = -1\\ &= \ \ \gamma^3\gamma^2\gamma^2\gamma^3 & &\text{since}\quad \gamma^2\gamma^3 = -\gamma^3\gamma^2\\ &= -\gamma^3\gamma^3 & &\text{since}\quad (\gamma^2)^2 = -1\\ &= \ \ 1 & &\end{aligned}$$

Using $\gamma^{0\dagger} = \gamma^0$, $\gamma^{1\dagger} = -\gamma^1$, $\gamma^{2\dagger} = -\gamma^2$, $\gamma^{3\dagger} = -\gamma^3$:

$$\begin{aligned}\gamma^{5\dagger} = (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger = -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} = i\gamma^3\gamma^2\gamma^1\gamma^0&\\ = -i\gamma^2\gamma^1\gamma^0\gamma^3 = -i\gamma^1\gamma^0\gamma^2\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5&\end{aligned}$$

Consider $\gamma^5\gamma^2$ for example:

$$\gamma^5\gamma^2 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = i\gamma^0\gamma^2\gamma^1\gamma^2\gamma^3 = -i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^2\gamma^5$$

and similarly: $\gamma^5\gamma^0 = -\gamma^0\gamma^5,\ \gamma^5\gamma^1 = -\gamma^1\gamma^5,\ \gamma^5\gamma^3 = -\gamma^3\gamma^5$ giving altogether $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$.

b) An adjoint spinor is defined as $\overline{\psi} \equiv \psi^\dagger\gamma^0$, so that

$$\begin{aligned}\overline{\psi_{\mathrm{L}}} = \psi_L^\dagger\gamma^0 &= \left[\tfrac{1}{2}(1-\gamma^5)\psi\right]^\dagger\gamma^0 & &\\ &= \psi^\dagger\tfrac{1}{2}(1-\gamma^5)\gamma^0 & &\text{since}\quad \gamma^{5\dagger} = \gamma^5\\ &= \psi^\dagger\gamma^0\tfrac{1}{2}(1+\gamma^5) & &\text{since}\quad \gamma^0\gamma^5 = -\gamma^5\gamma^0\\ &= \overline{\psi}\tfrac{1}{2}(1+\gamma^5) & &\end{aligned}$$

and similarly:

$$\overline{\psi_R} = \overline{\psi}\tfrac{1}{2}(1-\gamma^5)\ .$$

c) Separate the spinor ψ into its left- and right-handed components via

$$\psi = \tfrac{1}{2}(1-\gamma^5)\psi + \tfrac{1}{2}(1+\gamma^5)\psi = \psi_L + \psi_R$$

For the adjoint spinor:

$$\overline{\psi} = \psi^\dagger\gamma^0 = (\psi_L+\psi_R)^\dagger\gamma^0 = \psi_L^\dagger\gamma^0 + \psi_R^\dagger\gamma^0 = \overline{\psi_L} + \overline{\psi_R}$$

Hence

$$\begin{aligned}\overline{\phi}\gamma^\mu\psi &= \left[\,\overline{\phi_L} + \overline{\phi_R}\,\right]\gamma^\mu\left[\psi_L+\psi_R\right] \\ &= \overline{\phi_L}\gamma^\mu\psi_L + \overline{\phi_L}\gamma^\mu\psi_R + \overline{\phi_R}\gamma^\mu\psi_L + \overline{\phi_R}\gamma^\mu\psi_R\end{aligned}$$

But

$$\begin{aligned}\overline{\phi_L}\gamma^\mu\psi_R &= \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\cdot\gamma^\mu\cdot\tfrac{1}{2}(1+\gamma^5)\psi \\ &= \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\cdot\tfrac{1}{2}(1-\gamma^5)\gamma^\mu\psi \\ &= 0\end{aligned}$$

since $(1+\gamma^5)(1-\gamma^5) = 1-(\gamma^5)^2 = 0$. Similarly: $\overline{\phi_R}\gamma^\mu\psi_L = 0$ giving

$$\overline{\phi}\gamma^\mu\psi = \overline{\phi_L}\gamma^\mu\psi_L + \overline{\phi_R}\gamma^\mu\psi_R$$

as required. Alternatively, show directly that

$$\begin{aligned}\overline{\phi_L}\gamma^\mu\psi_L &= \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\cdot\gamma^\mu\cdot\tfrac{1}{2}(1-\gamma^5)\psi \\ &= \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\cdot\tfrac{1}{2}(1+\gamma^5)\gamma^\mu\psi \\ &= \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\gamma^\mu\psi\end{aligned}$$

and similarly

$$\overline{\phi_R}\gamma^\mu\psi_R = \overline{\phi}\tfrac{1}{2}(1-\gamma^5)\gamma^\mu\psi\ ,$$

again giving

$$\overline{\phi_L}\gamma^\mu\psi_L + \overline{\phi_R}\gamma^\mu\psi_R = \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\gamma^\mu\psi + \overline{\phi}\tfrac{1}{2}(1-\gamma^5)\gamma^\mu\psi = \overline{\phi}\gamma^\mu\psi\ .$$

Thus, for interactions between spin $\frac{1}{2}$ particles (or antiparticles) and photons in QED, the left-handed chiral component of a spinor couples only to another left-handed chiral component ($\overline{\phi_L}\gamma^\mu\psi_L$) and the right-handed chiral component couples only to another right-handed chiral component ($\overline{\phi_R}\gamma^\mu\psi_R$). There is no coupling between the left-handed and right-handed chiral components: ($\overline{\phi_R}\gamma^\mu\psi_L = 0, \overline{\phi_R}\gamma^\mu\psi_L = 0$).

At high energies, the left-handed and right-handed chiral components become helicity eigenstates with definite helicity and we have *helicity conservation* in QED: the *particle* helicity is preserved at a QED vertex.

ELECTRON-MUON ELASTIC SCATTERING

11. a) Show that the general matrix element for $e^-\mu^- \to e^-\mu^-$ scattering via single photon exchange is

$$M_{\mathrm{fi}} = -\frac{e^2}{(p_1-p_3)^2} g_{\mu\nu} \left[\overline{u}(p_3)\gamma^\mu u(p_1)\right] \left[\overline{u}(p_4)\gamma^\nu u(p_2)\right]$$

where p_1 and p_3 are the initial and final e^- four-momenta and p_2 and p_4 are the initial and final μ^- four-momenta.

b) Show that, for scattering in the centre of mass frame with incoming and outgoing e^- four-momenta $p_1^\mu = (E_1, 0, 0, p)$ and $p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta)$, the electron current for the various possible electron spin combinations is

$$\begin{aligned}
\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(E_1 c, ps, -ips, pc)\\
\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(ms, 0, 0, 0)\\
\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= 2(E_1 c, ps, ips, pc)\\
\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) &= -2(ms, 0, 0, 0)
\end{aligned}$$

where m is the electron mass and $s \equiv \sin\theta/2$, $c \equiv \cos\theta/2$.

c) Write down the incoming and outgoing muon 4-momenta p_2 and p_4, and the helicity eigenstate spinors $u_\uparrow(p_2)$, $u_\downarrow(p_2)$, $u_\uparrow(p_4)$ and $u_\downarrow(p_4)$. [Take the muon mass to be M and the muon energy to be E_2]. By comparing the forms of the muon and electron spinors, explain how the muon currents

$$\begin{aligned}
\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(E_2 c, -ps, -ips, -pc)\\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(Ms, 0, 0, 0)\\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) &= 2(E_2 c, -ps, ips, -pc)\\
\overline{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) &= -2(Ms, 0, 0, 0)
\end{aligned}$$

can be written down (up to overall factors of ± 1) without any further calculation.

d) Explain why some of the above currents vanish in the relativistic limit where the electron mass and muon mass can be neglected. Sketch the spin configurations which are allowed in this limit.

e) Show that, in the relativistic limit, the matrix element squared $|M_{\mathrm{LL}}|^2$ for the case where the incoming e^- and incoming μ^- are both left-handed is given by

$$|M_{\mathrm{LL}}|^2 = \frac{4e^4 s^2}{(p_1-p_3)^4}$$

where $s = (p_1+p_2)^2$. Why is the numerator of $|M_{\mathrm{LL}}|^2$ independent of θ ?

f) Find a similar expression for the matrix element $|M_{\mathrm{RL}}|^2$ for a right-handed incoming e^- and a left-handed incoming μ^-, and explain why $|M_{\mathrm{RL}}|^2$ vanishes when $\theta = \pi$. Write down the corresponding results for $|M_{\mathrm{RR}}|^2$ and $|M_{\mathrm{LR}}|^2$.

g) Show that, in the relativistic limit, the differential cross section for unpolarised $e^-\mu^- \to e^-\mu^-$ scattering in the centre of mass frame is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\alpha^2}{s} \cdot \frac{1+\frac{1}{4}(1+\cos\theta)^2}{(1-\cos\theta)^2}\ .$$

h) Show that the spin-averaged matrix element squared (in this ultra-relativistic limit) can be expressed in Lorentz-invariant form as

$$\left\langle |M_{\mathrm{fi}}|^2 \right\rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) \right] ,$$

and that a Lorentz invariant form for the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{2\pi\alpha^2}{q^4} \left[1 + \left(1 + \frac{q^2}{s} \right)^2 \right]$$

where $q^2 = (p_1 - p_3)^2$.

The remainder of this question involves the derivation of a general expression for $\langle |M_{\mathrm{fi}}|^2 \rangle$ *for the case of finite electron and muon masses, and is* optional:

i) Show that the spin-averaged matrix element squared for unpolarised $e^-\mu^- \to e^-\mu^-$ scattering can be written in the form

$$\left\langle |M_{\mathrm{fi}}|^2 \right\rangle = \frac{1}{4} \sum_{\text{spins}} |M_{\mathrm{fi}}|^2 = \frac{1}{4} \frac{e^4}{(p_1 - p_3)^4} L^{\mu\nu} W_{\mu\nu}$$

where the electron and muon tensors $L^{\mu\nu}$ and $W^{\mu\nu}$ are given by

$$L^{\mu\nu} \equiv \sum_{\text{spins}} \left[\overline{u}(p_3) \gamma^\mu u(p_1) \right] \left[\overline{u}(p_3) \gamma^\nu u(p_1) \right]^*$$

$$W_{\mu\nu} \equiv \sum_{\text{spins}} \left[\overline{u}(p_4) \gamma_\mu u(p_2) \right] \left[\overline{u}(p_4) \gamma_\nu u(p_2) \right]^*$$

j) Using the electron currents from part b) above, show that the components of the electron tensor $L^{\mu\nu}$ are

$$\begin{pmatrix} L^{00} & L^{01} & L^{02} & L^{03} \\ L^{10} & L^{11} & L^{12} & L^{13} \\ L^{20} & L^{21} & L^{22} & L^{23} \\ L^{30} & L^{31} & L^{32} & L^{33} \end{pmatrix} = 8 \begin{pmatrix} E_1^2 c^2 + m^2 s^2 & E_1 psc & 0 & E_1 pc^2 \\ E_1 psc & p^2 s^2 & 0 & p^2 sc \\ 0 & 0 & p^2 s^2 & 0 \\ E_1 pc^2 & p^2 sc & 0 & p^2 c^2 \end{pmatrix} ,$$

and hence verify that $L^{\mu\nu}$ has the Lorentz invariant form

$$L^{\mu\nu} = 4 \left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu} \left(m^2 - p_1.p_3 \right) \right] .$$

k) Write down the corresponding expression for $W^{\mu\nu}$ and hence show that

$$\langle |M_{\mathrm{fi}}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_2.p_4)m^2 + 2m^2 M^2 \right]$$

SOLUTION

a) The QED process $e^-\mu^- \to e^-\mu^-$ involves a single Feynman diagram at leading order:

e^- p_1 p_3 e^-

μ

q

ν

μ^- p_2 p_4 μ^-

Applying the Feynman rules gives

$$-iM_{\mathrm{fi}} = [\overline{u}(p_3) \cdot -ie\gamma^\mu \cdot u(p_1)] \cdot \frac{-ig_{\mu\nu}}{(p_1 - p_3)^2} \cdot [\overline{u}(p_4) \cdot -ie\gamma^\nu u(p_2)]$$

and hence

$$\boxed{M_{\mathrm{fi}} = -\frac{e^2}{(p_1 - p_3)^2} g_{\mu\nu} \, [\overline{u}(p_3)\gamma^\mu u(p_1)] \, [\overline{u}(p_4)\gamma^\nu u(p_2)]} \tag{29}$$

b) For a particle of mass m with four-momentum $p^\mu = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$, the helicity eigenstate spinors are

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \\ p/(E+m)\cos\theta/2 \\ p/(E+m)e^{i\phi}\sin\theta/2 \end{pmatrix}; \qquad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -\sin\theta/2 \\ e^{i\phi}\cos\theta/2 \\ p/(E+m)\sin\theta/2 \\ -p/(E+m)e^{i\phi}\cos\theta/2 \end{pmatrix} \tag{30}$$

For the incoming electron, with $p_1 = (E_1, 0, 0, p)$, the two possible spinors are:

$$u_\uparrow(p_1) = \sqrt{E_1+m}\begin{pmatrix} 1 \\ 0 \\ p/(E_1+m) \\ 0 \end{pmatrix}; \quad u_\downarrow(p_1) = \sqrt{E_1+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -p/(E_1+m) \end{pmatrix} \tag{31}$$

For the outgoing electron, with $p_3 = (E_1, p\sin\theta, 0, p\cos\theta)$, the spinors are:

$$u_\uparrow(p_3) = \sqrt{E_1+m}\begin{pmatrix} c \\ s \\ p/(E_1+m)\cdot c \\ p/(E_1+m)\cdot s \end{pmatrix}; \quad u_\downarrow(p_3) = \sqrt{E_1+m}\begin{pmatrix} -s \\ c \\ p/(E_1+m)\cdot s \\ -p/(E_1+m)\cdot c \end{pmatrix} \tag{32}$$

where $c \equiv \cos\theta/2$ and $s \equiv \sin\theta/2$. Noting that the spinors are real, matrix multiplication gives

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi_1\phi_1 + \psi_2\phi_2 + \psi_3\phi_3 + \psi_4\phi_4 \\
\overline{\psi}\gamma^1\phi &= \psi_1\phi_4 + \psi_2\phi_3 + \psi_3\phi_2 + \psi_4\phi_1 \\
\overline{\psi}\gamma^2\phi &= -i(\psi_1\phi_4 - \psi_2\phi_3 + \psi_3\phi_2 - \psi_4\phi_1) \\
\overline{\psi}\gamma^3\phi &= \psi_1\phi_3 - \psi_2\phi_4 + \psi_3\phi_1 - \psi_4\phi_2
\end{aligned}$$

Start with $\overline{u}_\downarrow(p_3)$ and $u_\downarrow(p_1)$:

$$\overline{u}_\downarrow(p_3)\gamma^0 u_\downarrow(p_1) = (E_1+m)\left[c + \frac{p^2}{(E_1+m)^2}c\right] = \frac{(E_1+m)^2+p^2}{(E_1+m)}c = \frac{2E_1^2+2mE_1}{(E_1+m)}c = 2E_1 c$$

where we have used $m^2+p^2=E_1^2$ in the last-but-one step. Similarly, for γ^1, γ^2, γ^3 we have

$$\overline{u}_\downarrow(p_3)\gamma^1 u_\downarrow(p_1) = (E_1+m)\left[\frac{p}{E_1+m}s + \frac{p}{E_1+m}s\right] = 2ps$$

$$\overline{u}_\downarrow(p_3)\gamma^2 u_\downarrow(p_1) = (E_1+m)\left[\frac{-ip}{E_1+m}s - \frac{ip}{E_1+m}s\right] = -2ips$$

$$\overline{u}_\downarrow(p_3)\gamma^3 u_\downarrow(p_1) = (E_1+m)\left[\frac{p}{E_1+m}c + \frac{p}{E_1+m}c\right] = 2pc$$

In summary

$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (2E_1c, 2ps, -2ips, 2pc) \tag{33}$$

Similarly for the other possible spin configurations, giving overall:

$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2(E_1c, ps, -ips, pc) \tag{34}$$
$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2(ms, 0, 0, 0) \tag{35}$$
$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2(E_1c, ps, ips, pc) \tag{36}$$
$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = -2(ms, 0, 0, 0) \tag{37}$$

c) For the incoming μ^-, with four-momentum $p_2 = (E_2, 0, 0, -p)$ and $E_2 = \sqrt{p^2+M^2}$, the helicity eigenstate spinors can be obtained from Equation (30) by setting $\theta = \pi$ and $\phi = 0$:

$$u_\uparrow(p_2) = \sqrt{E_2+M}\begin{pmatrix} 0 \\ 1 \\ 0 \\ p/(E_2+M) \end{pmatrix}; \qquad u_\downarrow(p_2) = \sqrt{E_2+M}\begin{pmatrix} -1 \\ 0 \\ p/(E_2+M) \\ 0 \end{pmatrix} \tag{38}$$

For the outgoing μ^-, with 4-momentum $p_4 = (E_2, -p\sin\theta, 0, -p\cos\theta)$, the helicity eigenstate spinors can be obtained from Equation (30) by setting $\theta \to \pi - \theta$ and $\phi = \pi$:

$$u_\uparrow(p_4) = \sqrt{E_2+M}\begin{pmatrix} s \\ -c \\ p/(E_2+M)\cdot s \\ p/(E_2+M)\cdot -c \end{pmatrix}; \qquad u_\downarrow(p_4) = \sqrt{E_2+M}\begin{pmatrix} -c \\ -s \\ p/(E_2+M)\cdot c \\ -p/(E_2+M)\cdot -s \end{pmatrix} \tag{39}$$

using $\cos(\pi-\theta)/2 = \sin\theta/2 = s$ and $\sin(\pi-\theta)/2 = \cos\theta/2 = c$.

A comparison of Equations (31) and (38) shows that, if we make the replacement $p \to -p$, then $u_\uparrow(p_2)$ is of the same form as $u_\downarrow(p_1)$. Similarly, $u_\downarrow(p_2)$ is then of the same form as $u_\uparrow(p_1)$, apart from an overall normalisation factor of -1.

Similarly, a comparison of Equations (32) and (39) shows that, under $p \to -p$, $u_\uparrow(p_4)$ becomes the same as $u_\downarrow(p_3)$, and $u_\downarrow(p_4)$ becomes the same as $u_\uparrow(p_3)$, apart from overall normalisation factors of -1.

The muon currents can therefore be written down directly using the electron current results, by changing m to M, E_1 to E_2, p to $-p$, $\uparrow$ to $\downarrow$ and $\downarrow$ to $\uparrow$:

$$\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) = 2(E_2c, -ps, -ips, -pc) \tag{40}$$
$$\overline{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) = 2(Ms, 0, 0, 0) \tag{41}$$
$$\overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) = 2(E_2c, -ps, ips, -pc) \tag{42}$$
$$\overline{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) = -2(Ms, 0, 0, 0) \tag{43}$$

d) Some of the currents vanish in the relativistic limit due to helicity conservation. The allowed spin configurations are those for which the helicity of the e^- and the helicity of the μ^- are both preserved in the scattering:

e) In the relativistic limit, we can set $m = M = 0$ and $E_1 = E_2 = E$. The electron currents become

$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2E(c, s, -is, c) \tag{44}$$
$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (0, 0, 0, 0) \tag{45}$$
$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2E(c, s, is, c) \tag{46}$$
$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (0, 0, 0, 0) \tag{47}$$

while the muon curents are:

$$\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) = 2E(c, -s, -is, -c) \tag{48}$$
$$\overline{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) = (0, 0, 0, 0) \tag{49}$$
$$\overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) = 2E(c, -s, is, -c) \tag{50}$$
$$\overline{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) = (0, 0, 0, 0) \tag{51}$$

When the incoming e^- and μ^- are both left-handed (*i.e.* negative helicity) we have $u(p_1) = u_\downarrow(p_1)$ and $u(p_2) = u_\downarrow(p_2)$, and the only non-zero contributions to the electron and muon currents come from Equations (44) and (48). Hence the scalar product of the electron and muon currents is

$$2E(c, s, -is, c) \cdot 2E(c, -s, -is, -c) = 4E^2 \cdot (c^2 + s^2 + c^2 + s^2) = 8E^2$$

and, from Equation (29), the matrix element squared is

$$|M_{\mathrm{LL}}|^2 = \frac{e^4}{(p_1 - p_3)^4} \cdot (8E^2)^2 = \frac{4e^4 s^2}{(p_1 - p_3)^4}$$

where now $s \equiv (p_1 + p_2)^2 = 4E^2$.

The numerator of $|M_{\mathrm{LL}}|^2$ is independent of θ because the incoming left-handed e^- and the incoming left-handed μ^- have oppositely directed spins, and the total spin of the initial state is $S_z = 0$. Hence there is no preferred spatial direction.

f) For M_{RL}, with the incoming e^- right-handed and the μ^- left-handed, we have $u(p_1) = u_\uparrow(p_1)$ and $u(p_2) = u_\downarrow(p_2)$. The only non-zero combination is now given by the scalar product of Equations (46) and (48):

$$\begin{aligned} M_{\mathrm{RL}} \propto 2E(c, s, is, c) \cdot 2E(c, -s, -is, -c) &= 4E^2 \cdot (c^2 + s^2 - s^2 + c^2) \\ &= 8E^2 \cos^2 \theta/2 \\ &= 8E^2 \cdot \tfrac{1}{2}(1 + \cos\theta)\ . \end{aligned}$$

Hence the non-zero matrix elements can be summarised as

$$|M_{\rm RR}|^2 = |M_{\rm LL}|^2 = \frac{e^4}{(p_1-p_3)^4}4s^2$$
$$|M_{\rm LR}|^2 = |M_{\rm RL}|^2 = \frac{e^4}{(p_1-p_3)^4}4s^2\cdot\tfrac{1}{4}(1+\cos\theta)^2$$

where we must have $M_{\rm LL}=M_{\rm RR}$ and $M_{\rm LR}=M_{\rm RL}$ by symmetry of the spin configurations.

g) For unpolarised $e^-\mu^- \to e^-\mu^-$ scattering, sum over the final spins and average over the initial spins to obtain

$$\begin{aligned}\left\langle|M_{\rm fi}|^2\right\rangle &= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\left(|M_{\rm LL}|^2+|M_{\rm RR}|^2+|M_{\rm LR}|^2+|M_{\rm RL}|^2\right)\\ &= \frac{2e^4}{(p_1-p_3)^4}s^2\left[1+\tfrac{1}{4}(1+\cos\theta)^2\right]\end{aligned} \tag{52}$$

With $p_1=(E,0,0,E)$ and $p_3=(E,E\sin\theta,0,E\cos\theta)$, we have

$$(p_1-p_3)^2 = p_1^2+p_3^2-2p_1.p_3 = -2p_1.p_3 = -2E^2(1-\cos\theta)$$

For any $2\to 2$ body elastic scattering process in the centre of mass frame, the differential cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\langle|M_{\rm fi}|^2\rangle$$

Hence:

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e^4}{8\pi^2 s}\cdot\frac{1+\frac{1}{4}(1+\cos\theta)^2}{(1-\cos\theta)^2}} \tag{53}$$

h) With 4-momenta

$$\begin{aligned} p_1 &= (E,0,0,E) & p_3 &= (E,E\sin\theta,0,E\cos\theta)\\ p_2 &= (E,0,0,-E) & p_4 &= (E,-E\sin\theta,0,-E\cos\theta)\end{aligned}$$

the scalar products are

$$\begin{aligned} p_1.p_2 = p_3.p_4 &= 2E^2 = \tfrac{1}{2}s^2\\ p_1.p_4 = p_2.p_3 &= E^2(1+\cos\theta) = \tfrac{1}{4}s(1+\cos\theta)\end{aligned}$$

Hence the spin-averaged matrix element squared of Equation (52) becomes

$$\boxed{\left\langle|M_{\rm fi}|^2\right\rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1.p_2)(p_3.p_4)+(p_1.p_4)(p_2.p_3)\right]}\,.$$

It was shown in Handout 3 that the Lorentz-invariant cross section $\mathrm{d}\sigma/\mathrm{d}t = \mathrm{d}\sigma/\mathrm{d}q^2$ is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{1}{64\pi s(p_{\rm i}^*)^2}|M_{\rm fi}|^2$$

where $p_{\rm i}^*$ is the centre of mass momentum of either of the initial state particles. At high energies when masses are negligible (as here), we have $p_{\rm i}^* = E$ and hence $4(p_{\rm i}^*)^2 = 4E^2 = s$. Hence

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{1}{16\pi s^2}|M_{\rm fi}|^2 = \frac{1}{16\pi s^2} \cdot \frac{8e^4}{q^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3)\right] \ .$$

In terms of s and q^2, the scalar products are

$$s = (p_1 + p_2)^2 = 2p_1.p_2 = 2p_3.p_4$$
$$q^2 = (p_1 - p_3)^2 = -2p_1.p_3$$
$$p_1.p_4 = p_1.(p_1 + p_2 - p_3) = p_1.p_2 - p_1.p_3 = \tfrac{1}{2}s + \tfrac{1}{2}q^2$$
$$p_2.p_3 = p_1.p_4$$

Hence

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{1}{16\pi s^2} \cdot \frac{8e^4}{q^4} \left[(\tfrac{1}{2}s)(\tfrac{1}{2}s) + (\tfrac{1}{2}s + \tfrac{1}{2}q^2)(\tfrac{1}{2}s + \tfrac{1}{2}q^2)\right] \ .$$

Using $e^2 = 4\pi\alpha$, this can be written as

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{2\pi\alpha^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s}\right)^2\right]} \ .$$

Alternatively, start from Equation (53) and use

$$q^2 = (p_1 - p_3)^2 = -2p_1.p_3 = -\tfrac{1}{2}s(1 - \cos\theta)$$

to transform the cross section directly:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \left|\frac{\mathrm{d}\cos\theta}{\mathrm{d}q^2}\right| \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{2}{s} \cdot \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}$$
$$1 - \cos\theta = \frac{2Q^2}{s}$$
$$1 + \cos\theta = 2\left(1 - \frac{Q^2}{s}\right)$$

i) The Lorentz invariant matrix element for a given spin configuration is

$$M_{ijkl} = -\frac{e^2}{(p_1 - p_3)^2}\left[\overline{u}_k(p_3)\gamma^\mu u_i(p_1)\right]\left[\overline{u}_l(p_4)\gamma_\nu u_j(p_2)\right]$$

where $i, j, k, l = \uparrow$ or $\downarrow$ (or $= 1, 2$) specifies the spin state of each of the incoming and outgoing particles in the collision. For unpolarised $e^-\mu^- \to e^-\mu^-$ scattering, sum over the final spins and average over the initial e^- and μ^- spins to obtain

$$\begin{aligned}
\left\langle |M_{\rm fi}|^2 \right\rangle &= \frac{1}{2} \cdot \frac{1}{2} \cdot \sum_{i,j,k,l=1}^{2} |M_{ijkl}|^2 \\
&= \frac{1}{4}\frac{e^4}{(p_1 - p_3)^4} \sum_{i,j,k,l=1}^{2} \left[\overline{u}_k(p_3)\gamma^\mu u_i(p_1)\right]\left[\overline{u}_k(p_3)\gamma^\nu u_i(p_1)\right]^* \left[\overline{u}_l(p_4)\gamma_\mu u_j(p_2)\right]\left[\overline{u}_l(p_4)\gamma_\nu u_j(p_2)\right]^* \\
&= \frac{1}{4}\frac{e^4}{(p_1 - p_3)^4} L^{\mu\nu} W_{\mu\nu}
\end{aligned}$$

where the electron and muon tensors $L^{\mu\nu}$ and $W^{\mu\nu}$ are given by

$$L^{\mu\nu} \equiv \sum_{i,k=1}^{2} \left[\overline{u}_k(p_3)\gamma^\mu u_i(p_1)\right] \left[\overline{u}_k(p_3)\gamma^\nu u_i(p_1)\right]^*$$
$$W_{\mu\nu} \equiv \sum_{j,l=1}^{2} \left[\overline{u}_l(p_4)\gamma_\mu u_j(p_2)\right] \left[\overline{u}_l(p_4)\gamma_\nu u_j(p_2)\right]^*$$

j) Writing out the sum over spins explicitly, the electron tensor $L^{\mu\nu}$ is given by

$$\begin{aligned} L^{\mu\nu} &= [\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)]\,[\overline{u}_\downarrow(p_3)\gamma^\nu u_\downarrow(p_1)]^* + [\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1)]\,[\overline{u}_\uparrow(p_3)\gamma^\nu u_\downarrow(p_1)]^* \\ &\quad + [\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1)]\,[\overline{u}_\uparrow(p_3)\gamma^\nu u_\uparrow(p_1)]^* + [\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1)]\,[\overline{u}_\downarrow(p_3)\gamma^\nu u_\uparrow(p_1)]^* \ . \end{aligned}$$

Substituting the electron currents given in Equations (34)-(37), and using matrix notation, the sum is

$$\begin{aligned} \begin{pmatrix} L^{00} & L^{01} & L^{02} & L^{03} \\ L^{10} & L^{11} & L^{12} & L^{13} \\ L^{20} & L^{21} & L^{22} & L^{23} \\ L^{30} & L^{31} & L^{32} & L^{33} \end{pmatrix} &= 4 \begin{pmatrix} E_1 c \\ ps \\ -ips \\ pc \end{pmatrix} \begin{pmatrix} E_1 c & ps & ips & pc \end{pmatrix} + 4 \begin{pmatrix} ms \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} ms & 0 & 0 & 0 \end{pmatrix} \\ &\quad + 4 \begin{pmatrix} E_1 c \\ ps \\ ips \\ pc \end{pmatrix} \begin{pmatrix} E_1 c & ps & -ips & pc \end{pmatrix} + 4 \begin{pmatrix} ms \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} ms & 0 & 0 & 0 \end{pmatrix} \\ &= 8 \begin{pmatrix} E_1^2 c^2 + m^2 s^2 & E_1 psc & 0 & E_1 pc^2 \\ E_1 psc & p^2 s^2 & 0 & p^2 sc \\ 0 & 0 & p^2 s^2 & 0 \\ E_1 pc^2 & p^2 sc & 0 & p^2 c^2 \end{pmatrix} \end{aligned} \tag{54}$$

Now consider

$$L^{\mu\nu} = 4 \left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu} \left(m^2 - p_1.p_3 \right) \right] \ .$$

In matrix notation, this is

$$\begin{aligned} L^{\mu\nu} &= 4 \begin{pmatrix} p_1^0 \\ p_1^1 \\ p_1^2 \\ p_1^3 \end{pmatrix} \begin{pmatrix} p_3^0 & p_3^1 & p_3^2 & p_3^3 \end{pmatrix} + 4 \begin{pmatrix} p_3^0 \\ p_3^1 \\ p_3^2 \\ p_3^3 \end{pmatrix} \begin{pmatrix} p_1^0 & p_1^1 & p_1^2 & p_1^3 \end{pmatrix} \\ &\quad + 4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot (m^2 - p_1.p_3) \end{aligned}$$

With $p_1 = (E_1, 0, 0, p)$ and $p_3 = (E_1, p\sin\theta, 0, p\cos\theta)$, we have

$$m^2 - p_1.p_3 = m^2 - (E_1^2 - p^2\cos\theta) = p^2(\cos\theta - 1) = -2p^2 s^2 \ ,$$

where we have used $E_1^2 = p^2 + m^2$ and $1 - \cos\theta = 2\sin^2\theta/2 = 2s^2$. Hence

$$
\begin{aligned}
L^{\mu\nu} &= 4 \begin{pmatrix} E_1 \\ 0 \\ 0 \\ p \end{pmatrix} \begin{pmatrix} E_1 & p\sin\theta & 0 & p\cos\theta \end{pmatrix} + 4 \begin{pmatrix} E_1 \\ p\sin\theta \\ 0 \\ p\cos\theta \end{pmatrix} \begin{pmatrix} E_1 & 0 & 0 & p \end{pmatrix} \\
&+ 4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot -2p^2s^2 \\
&= 4 \begin{pmatrix} 2E_1^2 - 2p^2s^2 & E_1 p\sin\theta & 0 & E_1 p(1+\cos\theta) \\ E_1 p\sin\theta & 2p^2s^2 & 0 & p^2\sin\theta \\ 0 & 0 & 2p^2s^2 & 0 \\ E_1 p(1+\cos\theta) & p^2\sin\theta & 0 & 2p^2\cos\theta + 2p^2s^2 \end{pmatrix}
\end{aligned}
$$

Using the relations $\sin\theta = 2\sin\theta/2\cos\theta/2 = 2sc$, $1+\cos\theta = 2\cos^2\theta/2 = 2c^2$ and $E_1^2 = p^2 + m^2$, this is readily seen to be equal to Equation (54).

k) The muon tensor $W^{\mu\nu}$ can be written down immediately as

$$W_{\mu\nu} = 4\left[p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + g_{\mu\nu}\left(M^2 - p_2.p_4\right)\right] \ .$$

Hence

$$
\begin{aligned}
\left\langle |M_{\rm fi}|^2 \right\rangle = \frac{4e^4}{(p_1 - p_3)^4} &\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}\left(m^2 - p_1.p_3\right)\right] \\
&\times \left[p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + g_{\mu\nu}\left(M^2 - p_2.p_4\right)\right]
\end{aligned}
\tag{55}
$$

giving finally

$$\boxed{\left\langle |M_{\rm fi}|^2 \right\rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_2.p_4)m^2 + 2m^2M^2\right]}$$

12. a) The elastic form factors for the proton are well described by the form

$$G(q^2) = \frac{G(0)}{(1 + |q^2|/0.71)^2}$$

with q^2 in GeV^2. Show that an exponential charge distribution in the proton

$$\rho(\boldsymbol{r}) = \rho_0 e^{-\lambda r}$$

leads to this form for $G(q^2)$ (insofar as $|q^2| = |\boldsymbol{q}^2|$), and calculate λ.

b) Show that, for any spherically symmetric charge distribution, the mean square radius is given by

$$\langle r^2 \rangle = -\frac{6}{G(0)} \left[\frac{\mathrm{d}G(q^2)}{\mathrm{d}|q^2|} \right]_{q^2=0}$$

and estimate the r.m.s. charge radius of the proton.

c) The pion form factor may be determined in πe^- scattering. Use the following data to estimate the r.m.s. charge radius of the pion.

$\|q^2\|$ (GeV^2)	$G_E^2(q^2)$
0.015	0.944 ± 0.007
0.042	0.849 ± 0.009
0.074	0.777 ± 0.016
0.101	0.680 ± 0.017
0.137	0.646 ± 0.027
0.173	0.534 ± 0.030
0.203	0.529 ± 0.040
0.223	0.487 ± 0.049

SOLUTION

a) For elastic scattering, there is no energy transfer to the target particle and the 4-momentum transfer q is of the form $q^\mu = (0, \boldsymbol{q})$. Hence $|q^2| = |\boldsymbol{q}|^2$, and the form factor is given by the Fourier transform of the charge distribution:

$$G(q^2) = G(\boldsymbol{q}^2) = \int e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \rho(\boldsymbol{r}) \mathrm{d}^3 r \tag{56}$$

For a spherically symmetric charge distribution, and choosing the constant vector $\boldsymbol{q}$ to lie along the $+z$ axis:

$$\begin{aligned}
G(q^2) &= \int_0^{2\pi} \int_{-1}^{+1} \int_0^{\infty} e^{iqr\cos\theta} \rho(r) r^2 \mathrm{d}r \mathrm{d}\cos\theta \mathrm{d}\phi \\
&= 2\pi \int_0^{\infty} \rho(r) r^2 \cdot \int_{-1}^{+1} e^{iqr\cos\theta} \mathrm{d}\cos\theta \cdot \mathrm{d}r \\
&= 2\pi \int_0^{\infty} \rho(r) r^2 \cdot \left[\frac{e^{iqr\cos\theta}}{iqr} \right]_{-1}^{+1} \cdot \mathrm{d}r \\
&= \frac{4\pi}{q} \int_0^{\infty} \rho(r) r \sin(qr) \mathrm{d}r
\end{aligned}$$

For the exponential charge distribution $\rho(r) = \rho_0 e^{-\lambda r}$:

$$G(q^2) = \frac{4\pi\rho_0}{q}\int_0^\infty re^{-\lambda r}\sin(qr)\mathrm{d}r$$
$$= \frac{4\pi\rho_0}{q}\frac{1}{2i}\int_0^\infty r\left[e^{-\lambda r+iqr} - e^{-\lambda r-iqr}\right]\mathrm{d}r$$

Integration by parts gives

$$\int_0^\infty re^{-\alpha r}\mathrm{d}r = \frac{1}{\alpha^2}$$

for any constant α, so that

$$G(q^2) = \frac{2\pi\rho_0}{iq}\left[\frac{1}{(\lambda-iq)^2} - \frac{1}{(\lambda+iq)^2}\right] = \frac{8\pi\lambda\rho_0}{(\lambda^2+q^2)^2}\ .$$

Thus the form factor is of the required ("dipole") form:

$$G(q^2) = \frac{G(0)}{(1+|q^2|/0.71)^2}$$

with $G(0) = 8\pi\rho_0/\lambda^3$ and

$$\boxed{\lambda = \sqrt{0.71\,\mathrm{GeV}^2} = 0.84\,\mathrm{GeV}}$$

Note that, from equation (56), $G(0)$ is just the total charge of the target particle:

$$G(0) = \int \rho(\boldsymbol{r})\mathrm{d}^3r = Q\ .$$

For an exponential charge distribution, it is easy to check that

$$G(0) = \int_0^\infty \rho_0 e^{-\lambda r}\cdot 4\pi r^2\mathrm{d}r = 4\pi\rho_0\int_0^\infty r^2 e^{-\lambda r}\mathrm{d}r = 4\pi\rho_0\cdot\frac{2}{\lambda^3}\ ,$$

consistent with the expression above. It is conventional and convenient to express the charge density ρ in units of $+e$ so that, for a proton target, $G(0) = 1$. This corresponds to choosing the normalisation constant ρ_0 to be $\rho_0 = \lambda^3/8\pi$.

b) A Taylor expansion gives

$$G(q^2) = \int e^{i\boldsymbol{q}.\boldsymbol{r}}\rho(\boldsymbol{r})\mathrm{d}^3r = \int\left(1 + i\boldsymbol{q}.\boldsymbol{r} - \tfrac{1}{2}(\boldsymbol{q}.\boldsymbol{r})^2 + \cdots\right)\rho(\boldsymbol{r})\mathrm{d}^3r$$

But $G(0) = 1$ and

$$\int(\boldsymbol{q}.\boldsymbol{r})\rho(\boldsymbol{r})\mathrm{d}^3r = 0 \qquad \text{since the integrand is an odd function of } \boldsymbol{r}$$

so that

$$G(q^2) = 1 - \int \tfrac{1}{2}(\boldsymbol{q}.\boldsymbol{r})^2\rho(\boldsymbol{r})\mathrm{d}^3r + \cdots\ .$$

But the Taylor expansion can also be written as

$$G(q^2) = G(0) + q^2 \left.\frac{\mathrm{d}G}{\mathrm{d}q^2}\right|_{q^2=0} + \cdots$$

so that

$$q^2 \left.\frac{\mathrm{d}G}{\mathrm{d}q^2}\right|_{q^2=0} = -\int \tfrac{1}{2}(\boldsymbol{q}.\boldsymbol{r})^2 \rho(\boldsymbol{r}) \mathrm{d}^3 r \, .$$

For a spherically symmetric charge distribution, and choosing $\boldsymbol{q}$ to lie along the $+z$-axis, this becomes

$$\begin{aligned}
q^2 \left.\frac{\mathrm{d}G}{\mathrm{d}q^2}\right|_{q^2=0} &= -\int_0^{2\pi}\int_{-1}^{+1}\int_0^{\infty} \tfrac{1}{2} \cdot q^2 r^2 \cos^2\theta \cdot \rho(r)\, r^2 \mathrm{d}r \mathrm{d}\cos\theta \mathrm{d}\phi \\
\Rightarrow \qquad \left.\frac{\mathrm{d}G}{\mathrm{d}q^2}\right|_{q^2=0} &= -\int_0^{2\pi}\int_{-1}^{+1}\int_0^{\infty} \tfrac{1}{2} r^4 \cos^2\theta \rho(r)\, \mathrm{d}r \mathrm{d}\cos\theta \mathrm{d}\phi \\
&= -\tfrac{2}{3}\pi \int_0^{\infty} r^4 \rho(r) \mathrm{d}r
\end{aligned}$$

But the mean square radius of the charge distribution is, by definition,

$$\langle r^2 \rangle = \frac{1}{G(0)} \int r^2 \rho(\boldsymbol{r}) \mathrm{d}^3 r = \frac{1}{G(0)} \int_0^{\infty} r^2 \rho(r)\, 4\pi r^2 \mathrm{d}r = \frac{1}{G(0)} 4\pi \int_0^{\infty} r^4 \rho(r)\, \mathrm{d}r$$

and hence

$$\boxed{\langle r^2 \rangle = -\frac{6}{G(0)} \left.\frac{\mathrm{d}G(q^2)}{\mathrm{d}|q^2|}\right|_{q^2=0}}$$

For the particular case of an exponential charge distribution, we have

$$G(q^2) = \frac{G(0)}{(1 + |q^2|/\lambda^2)^2}$$

and differentiation gives

$$\frac{\mathrm{d}G(q^2)}{\mathrm{d}q^2} = G(0) \cdot -2 \left(1 + \frac{|q^2|}{\lambda^2}\right)^{-3} \cdot \frac{1}{\lambda^2} \qquad \Rightarrow \qquad \left.\frac{\mathrm{d}G}{\mathrm{d}q^2}\right|_{q^2=0} = \frac{-2G(0)}{\lambda^2}$$

$$\Rightarrow \qquad \langle r^2 \rangle = -6 \cdot \frac{-2G(0)}{\lambda^2} = \frac{12}{\lambda^2} \, .$$

Hence the rms charge radius is

$$\sqrt{\langle r^2 \rangle} = \frac{\sqrt{12}}{\lambda} = \frac{\sqrt{12}}{0.84\,\mathrm{GeV}} \times 0.197\,\mathrm{GeV.fm} = 0.81\,\mathrm{fm}$$

where $\hbar c = 0.197\,\mathrm{GeV.fm}$ has been used to convert from natural units to SI units.

c) From a plot of $G_E(q^2)$ versus $|q^2|$, the slope at $q^2 = 0$ can be estimated to be

$$\left.\frac{\mathrm{d}G(q^2)}{\mathrm{d}|q^2|}\right|_{q^2=0} \approx -1.9\,\mathrm{GeV}^{-2} .$$

$$\Rightarrow \qquad \sqrt{\langle r^2 \rangle} \approx \sqrt{-6 \times -1.9} = 3.38\,\mathrm{GeV}^{-1} = 3.38\,\mathrm{GeV}^{-1} \times (0.197\,\mathrm{GeV.fm}) = 0.67\,\mathrm{fm}$$

In fact, the "dipole" form $G(q^2) = G(0)/(1 + |q^2|/\lambda^2)^2$ provides a good description of the pion form factor data. The dashed curve in the figure (drawn by eye rather than fitted) shows the function

$$G_E(q^2) = \frac{1}{1 + |q^2|/(1.05\,\mathrm{GeV}^2)} ,$$

so that $\lambda^2 \approx 1.05\,\mathrm{GeV}^2$. The dotted line shows the tangent to this curve at $q^2 = 0$, with slope

$$\left.\frac{\mathrm{d}G}{\mathrm{d}q^2}\right|_{q^2=0} = \frac{-2G(0)}{\lambda^2} = \frac{-2}{1.05\,\mathrm{GeV}^2} = -1.90\,\mathrm{GeV}^{-2} .$$

DEEP-INELASTIC SCATTERING

13. The figure below shows a deep-inelastic scattering event $e^+\mathrm{p} \to e^+\mathrm{X}$ recorded by the H1 experiment at the HERA collider. The positron beam, of energy $E_1 = 27.5\,\mathrm{GeV}$, enters from the left and the proton beam, of energy $E_2 = 820\,\mathrm{GeV}$, enters from the right. The energy of the outgoing positron is measured to be $E_3 = 31\,\mathrm{GeV}$. The picture is to scale, so angles may be read off the diagram if required.

a) Show that the Bjorken scaling variable x is given by

$$x = \frac{E_3}{E_2}\left[\frac{1-\cos\theta}{2-(E_3/E_1)(1+\cos\theta)}\right]$$

where θ is the angle through which the positron has scattered.

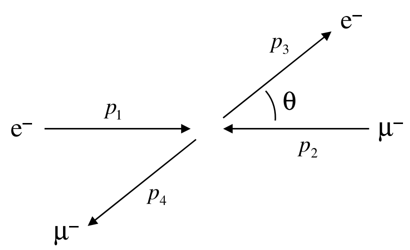
b) Estimate the values of Q^2, x and y for this event.

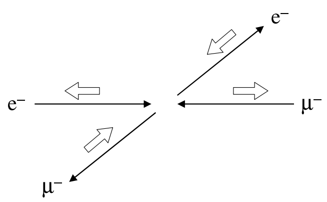
c) Estimate the invariant mass M_X of the final state hadronic system.

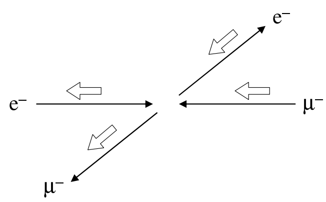
d) Draw quark level diagrams to illustrate the possible origins of this event. Using the plot overleaf of the parton distribution functions $xu_\mathrm{V}(x)$, $xd_\mathrm{V}(x)$, $x\overline{u}(x)$ and $x\overline{d}(x)$, estimate the relative probabilities of the various possible quark-level processes for the event. Note that the Q^2 in the plot overleaf need not be exactly the same as the Q^2 in this event – Bjorken scaling requires only that it be similar. So do not worry about any relatively small differences between the two Q^2 scales.

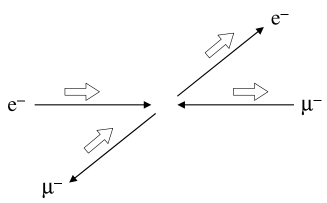
[Neglect contributions from the heavier quarks s, c, b, t.]

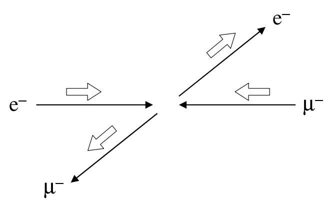
e) Estimate the relative contributions of the F_1 and F_2 terms to the deep-inelastic cross section for the x and Q^2 values corresponding to this event.

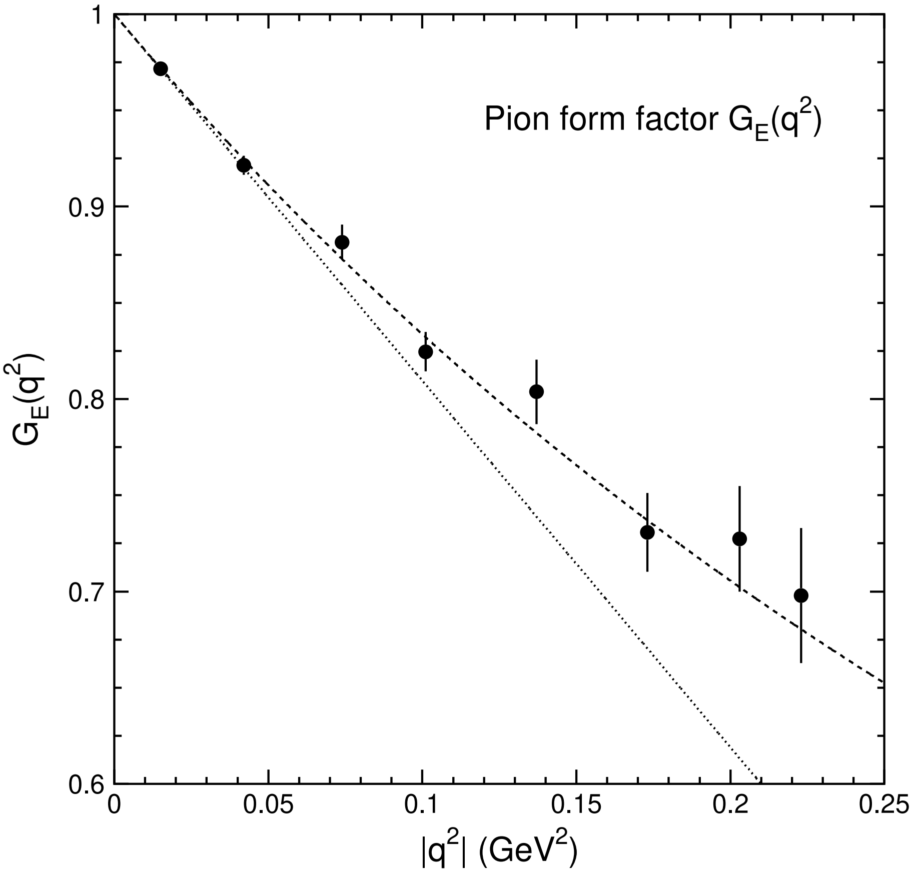

e⁻
p1
p3
e⁻
θ
μ⁻
p2
p4
μ⁻

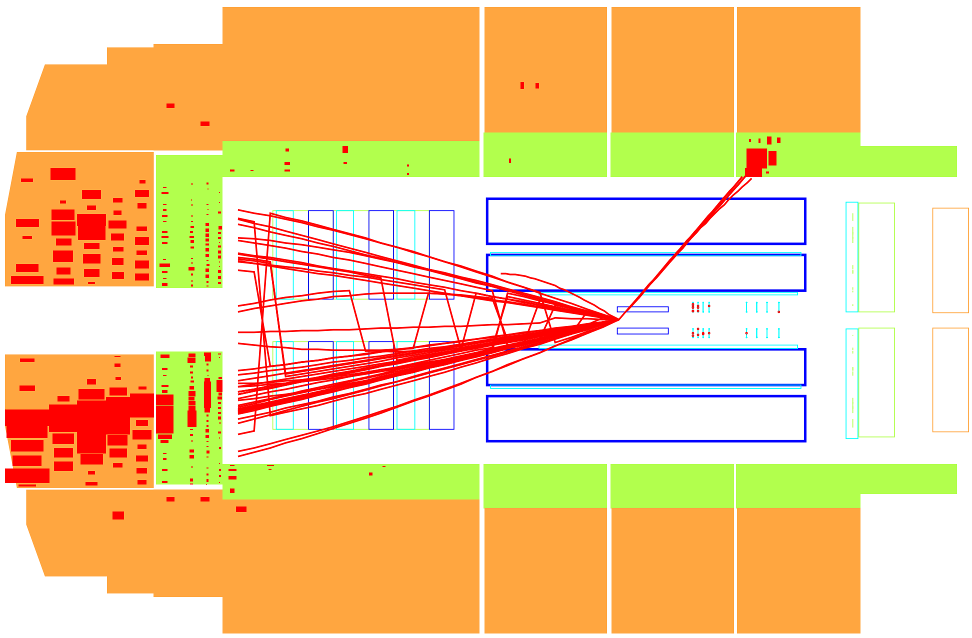

e⁻
e⁻
μ⁻
μ⁻

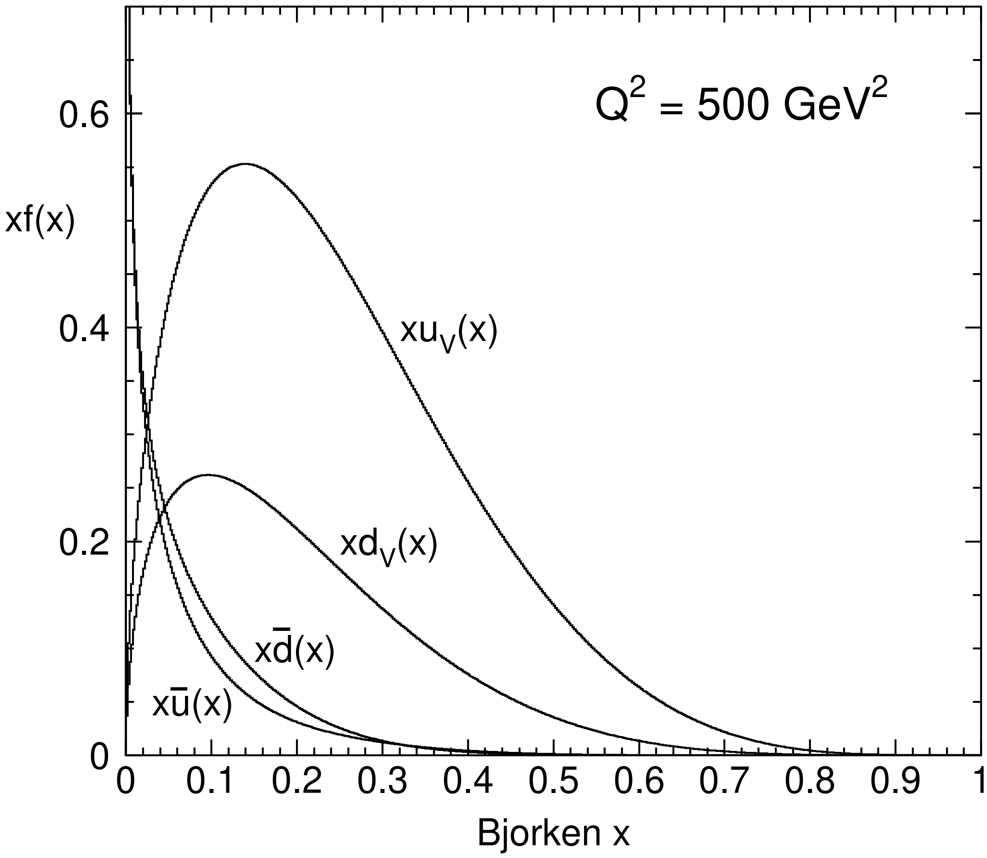

e⁻
e⁻
μ⁻
μ⁻


e⁻
e⁻
μ⁻
μ⁻


e⁻
e⁻
μ⁻
μ⁻


Pion form factor $G_E(q^2)$
$G_E(q^2)$
1
0.9
0.8
0.7
0.6
0
0.05
0.1
0.15
0.2
0.25
$|q^2|$ (GeV^2)





$Q^2 = 500\ \mathrm{GeV}^2$
xf(x)
$xu_V(x)$
$xd_V(x)$
$x\bar{d}(x)$
$x\bar{u}(x)$
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Bjorken x

SOLUTION

a) For $e^+\mathrm{p} \to e^+\mathrm{X}$ at HERA, choose four-momenta to be:

$$p_1 = (E_1, 0, 0, E_1)\,, \qquad p_2 = (E_2, 0, 0, -E_2)\,, \qquad p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta)\ .$$

Then

$$q^2 = -2p_1.p_3 = -2E_1E_3(1-\cos\theta)$$
$$p_2.q = p_2.p_1 - p_2.p_3 = 2E_1E_2 - E_2E_3(1+\cos\theta)$$

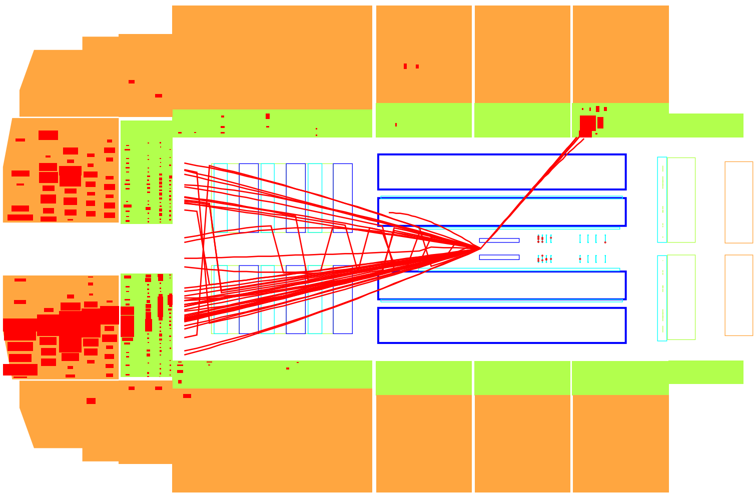
The Bjorken scaling variable x is defined as

$$x \equiv \frac{-q^2}{2p_2.q}$$

Hence

$$\boxed{x = \frac{E_3}{E_2}\left[\frac{1-\cos\theta}{2-(E_3/E_1)(1+\cos\theta)}\right]}\ .$$

b) For the particular event shown, we can estimate the e^+ scattering angle to be $\theta \approx 50^\circ$. We are given



$E_1 = 27.5\,\mathrm{GeV}$, $E_2 = 820\,\mathrm{GeV}$, $E_3 = 31\,\mathrm{GeV}$. Hence

$$x = \frac{31}{820}\left[\frac{1-\cos 50^\circ}{2-(31/27.5)(1+\cos 50^\circ)}\right] = 0.091\ .$$

$$Q^2 = 2E_1E_3(1-\cos\theta) = 2\times 27.5\times 31\times(1-\cos 50^\circ) = 609\,\mathrm{GeV}^2$$

$$y = \frac{p_2.q}{p_2.p_1} = 1-\frac{p_2.p_3}{p_2.p_1} = 1-\frac{E_3(1+\cos\theta)}{2E_1} = 1-\frac{31\times(1+\cos 50^\circ)}{2\times 27.5} = 0.074$$

c) The final state hadronic system has four-momentum $p_4 = p_2 + q$. Hence its invariant mass M_X is given by

$$M_\mathrm{X}^2 = (p_2+q)^2 = M^2 + 2p_2.q - Q^2 = M^2 + \frac{Q^2}{x} - Q^2\ .$$

Hence

$$M_{\mathrm{X}} = \sqrt{(0.938)^2 + \frac{609}{0.091} - 609} = 78.0\,\mathrm{GeV}\ .$$

d) At quark level, the possible origins of the event are $e^+\mathrm{u} \to e^+\mathrm{u}$, $e^+\mathrm{d} \to e^+\mathrm{d}$, $e^+\overline{\mathrm{u}} \to e^+\overline{\mathrm{u}}$, $e^+\overline{\mathrm{d}} \to e^+\overline{\mathrm{d}}$.

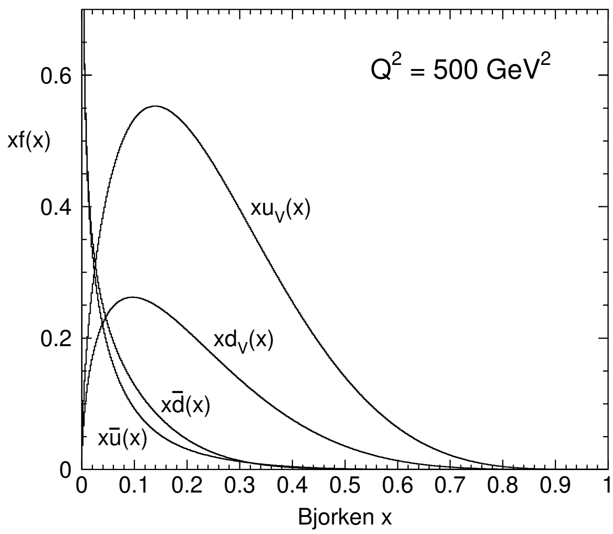
The parton model prediction for the $e^+\mathrm{p}$ cross section is

$$\frac{\mathrm{d}^2\sigma^{\mathrm{ep}}}{\mathrm{d}x\mathrm{d}Q^2} = \frac{2\pi\alpha^2}{Q^4}\left[1+(1-y)^2\right]\left[\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\overline{u}(x) + \frac{1}{9}\overline{d}(x)\right]\ .$$

Hence the relative probability for these processes is

$$\mathrm{u} : \mathrm{d} : \overline{\mathrm{u}} : \overline{\mathrm{d}} = \frac{4}{9}u(x) : \frac{1}{9}d(x) : \frac{4}{9}\overline{u}(x) : \frac{1}{9}\overline{d}(x)\ .$$

From the plot, for $x \approx 0.09$, we can estimate



$$xu_{\mathrm{V}}(x) \approx 0.52, \qquad xd_{\mathrm{V}}(x) \approx 0.26, \qquad x\overline{u}(x) \approx 0.10, \qquad x\overline{d}(x) \approx 0.14\ .$$

Remembering that

$$u(x) = u_{\mathrm{V}}(x) + u_{\mathrm{S}}(x) = u_{\mathrm{V}}(x) + \overline{u}(x)$$
$$d(x) = d_{\mathrm{V}}(x) + d_{\mathrm{S}}(x) = d_{\mathrm{V}}(x) + \overline{d}(x)\ ,$$

we obtain the estimates

$$u(x) \approx (0.52 + 0.10)/0.09 = 6.89$$
$$d(x) \approx (0.26 + 0.14)/0.09 = 4.44$$
$$\overline{u}(x) \approx 0.10/0.09 = 1.11$$
$$\overline{d}(x) \approx 0.14/0.09 = 1.56\ .$$

Including the factors of 4/9 or 1/9, the relative probabilities are therefore

$$\boxed{\mathrm{u} : \mathrm{d} : \bar{\mathrm{u}} : \bar{\mathrm{d}} \approx 3.06 : 0.494 : 0.494 : 0.173 = 0.73 : 0.12 : 0.12 : 0.04}\,.$$

e) The deep-inelastic $e^+\mathrm{p}$ cross section is

$$\frac{\mathrm{d}^2\sigma^{\mathrm{ep}}}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2^{\mathrm{ep}}}{x} + \frac{1}{2}y^2\frac{2xF_1^{\mathrm{ep}}}{x}\right]$$

Therefore, assuming the Callan-Gross relation $F_2^{\mathrm{ep}} = 2xF_1^{\mathrm{ep}}$, the F_2 and F_1 terms contribute to the cross section in the ratio

$$F_2 : F_1 = (1-y) : \tfrac{1}{2}y^2 = 1 - 0.075 : \tfrac{1}{2}(0.075)^2 \approx 1 : 0.0028\,.$$

In other words, the cross section is dominated by the F_2 term, with the F_1 term contributing only about 0.3% of events.

14. a) Show that the lab frame differential cross section $\mathrm{d}^2\sigma/\mathrm{d}E_3\mathrm{d}\Omega$ for deep-inelastic scattering is related to the Lorentz invariant differential cross section $\mathrm{d}^2\sigma/\mathrm{d}\nu\mathrm{d}Q^2$ via

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}Q^2} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2}$$

where E_1 and E_3 are the energies of the incoming and outgoing lepton, $\nu = E_1 - E_3$, and $Q^2 = -q^2 = -(p_1 - p_3)^2$. [When you do this, make sure you understand that differential cross sections transform as Jacobians, not as partial derivatives!]

Show further that

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2} = \frac{2Mx^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}$$

where M is the mass of the target nucleon and $x = Q^2/2M\nu$.

b) Show that

$$\frac{2Mx^2}{Q^2}\cdot\frac{y^2}{2} = \frac{1}{M}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}$$

and that

$$1 - y - \frac{M^2x^2y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2}\,.$$

c) Show that the Lorentz invariant cross section for deep-inelastic electromagnetic scattering,

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)\frac{F_2}{x} + \frac{y^2}{2}\frac{2xF_1}{x}\right]$$

becomes

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{F_2}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1}{M}\sin^2\frac{\theta}{2}\right]$$

in the lab frame.

d) An experiment consists of an electron beam of maximum energy $20\,\mathrm{GeV}$ and a variable angle spectrometer which can detect scattered electrons with energies greater than $2\,\mathrm{GeV}$. Find the range of values of θ over which deep-inelastic scattering events can be studied for $x = 0.2$ and $Q^2 = 2\,\mathrm{GeV}^2$.

[You may find it helpful to determine $E_1 - E_3$ (fixed), and E_1E_3 in terms of θ, and then sketch the various constraints on E_1 and E_3 on a 2D plot of E_3 against E_1.]

e) Outline a possible experimental strategy for measuring $F_1(x, Q^2)$ and $F_2(x, Q^2)$ for the above values of x and Q^2.

SOLUTION

a) Changing variables from $d\Omega = 2\pi d\cos\theta$ to

$$Q^2 = -q^2 = 2E_1E_3(1 - \cos\theta)$$

gives

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\cos\theta} = \frac{1}{2\pi}\left|\frac{\mathrm{d}Q^2}{\mathrm{d}\cos\theta}\right|\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}Q^2} = \frac{1}{2\pi}2E_1E_3\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}Q^2}$$

and hence

$$\boxed{\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}Q^2} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2}} \tag{57}$$

To change variables from ν to x, use

$$x = \frac{Q^2}{2M\nu} \qquad \Rightarrow \qquad \nu = \frac{Q^2}{2Mx}$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \left|\frac{\mathrm{d}\nu}{\mathrm{d}x}\right| \frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2}$$

which gives directly

$$\boxed{\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2} = \frac{2Mx^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}} \tag{58}$$

b) Since

$$Q^2 = 4E_1E_3\sin^2\theta/2$$

and

$$y = \frac{\nu}{E_1}$$

we have

$$\frac{E_3}{E_1}\sin^2\theta/2 = \frac{Q^2}{4E_1^2} = \frac{Q^2y^2}{4\nu^2}\ .$$

Using $\nu = Q^2/2Mx$, we then obtain

$$\boxed{\frac{2Mx^2}{Q^2}\cdot\tfrac{1}{2}y^2 = \frac{1}{M}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}} \tag{59}$$

Hence

$$\boxed{1 - y - \frac{M^2x^2y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2}}\ . \tag{60}$$

c) The Lorentz invariant cross section for deep-inelastic electromagnetic scattering is

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)\frac{F_2}{x} + \frac{y^2}{2}\frac{2xF_1}{x}\right]$$

Combining Equations (57) and (58), we have

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2} = \frac{E_1E_3}{\pi}\frac{2Mx^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}$$

The F_2 term contains the combination of factors

$$\frac{2Mx^2}{Q^2}\left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)\frac{1}{x} = \frac{2Mx}{Q^2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2} = \frac{1}{\nu}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}\ ,$$

where we have used Equation (60). Using Equation (59) for the F_1 term, we then obtain

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{4\pi\alpha^2}{Q^4}\left[\left(\frac{1}{\nu}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}\right)F_2 + \left(\frac{1}{M}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}\right)2F_1\right]$$

Since $Q^2 = 4E_1E_3\sin^2\theta/2$, we finally obtain

$$\boxed{\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{F_2}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1}{M}\sin^2\frac{\theta}{2}\right]}$$

d) Given $x = 0.2$ and $Q^2 = 2\,\mathrm{GeV}^2$, the electron energies E_1 and E_3 are fixed via

$$E_1 - E_3 = \frac{Q^2}{2Mx} = \frac{2\,\mathrm{GeV}^2}{2\times(0.938\,\mathrm{GeV})\times 0.2} = 5.33\,\mathrm{GeV} \tag{61}$$

and

$$E_1E_3 = \frac{Q^2}{4\sin^2\theta/2}\ . \tag{62}$$

The experimental constraints $E_1 < 20\,\mathrm{GeV}$ and $E_3 > 2\,\mathrm{GeV}$ then lead to constraints on the angle θ. To obtain these, it may help to think in terms of a graphical solution of Equations (61) and (62) on a plot of E_3 versus E_1. Equation (61) corresponds to a straight line running at 45° while Equation (62) gives an infinite set of hyperbolae, each hyperbola corresponding to a different possible value of θ.

The minimum possible value of θ corresponds to taking the maximum possible beam energy $E_1 = 20\,\mathrm{GeV}$:

$$\sin^2\theta/2 = \frac{Q^2}{4E_1E_3} = \frac{2}{4\times 20\times(20-5.33)} = 1.70\times 10^{-3}$$

which gives

$$\theta_{\text{min}} = 4.73^\circ\ .$$

The maximum possible value of θ is determined by the minimum detectable scattered electron energy of $E_3 = 2\,\mathrm{GeV}$:

$$\sin^2\theta/2 = \frac{Q^2}{4E_1E_3} = \frac{2}{4\times(2+5.33)\times 2} = 0.034$$

which gives

$$\theta_{\text{max}} = 21.3^\circ\ .$$

Strategy: choose several values of θ between about 5° and 20°, measure reduced cross section at each value of θ and plot versus $\tan^2\theta/2$. Should give a straight line (note ν is fixed) with slope $2F_1/M$ and intercept F_2/ν:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega}\bigg/\frac{\alpha^2\cos^2\theta/2}{4E_1^2\sin^4\theta/2} = \left[\frac{F_2}{\nu} + \frac{2F_1}{M}\tan^2\frac{\theta}{2}\right]$$

Each θ setting requires a different beam energy given by solving the quadratic equation

$$E_1(E_1 - 5.33) = \frac{Q^2}{4\sin^2\theta/2}$$

This gives

$$2E_1 = 5.33 + \sqrt{(5.33)^2 + \frac{Q^2}{\sin^2 \theta/2}}$$

Gives $E_1 = 19.1\,\mathrm{GeV}$ for $\theta = 5°$ and $E_1 = 7.5\,\mathrm{GeV}$ for $\theta = 20°$.

Note that $y = (E_1 - E_3)/E_1$ varies between 0.28 and 0.71 so get healthy contribution from F_1.