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Particle Physics Major Option

EXAMPLES SHEET QUESTIONS (ALL)

NATURAL UNITS AND HEAVISIDE-LORENTZ UNITS

1. (a) In the units he normally uses, your particle-physics lecturer was $10^{16}/\text{ GeV}$ tall and had a mass of $4.40 \times 10^{28}\ \text{ GeV}$ when aged $2.11 \times 10^{33}/\text{ GeV}$. Calculate his Body Mass Index (BMI) and determine whether he was obese at this point in his life.

(b) Show that charge can indeed be measured in units of $(\varepsilon_0 \hbar c)^{\frac{1}{2}}$. [You may wish to consider dimensional analysis of the Coulomb force law $F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$.]

SOLUTION

(a) The laborious way of working out the height L and mass of M of the lecturer would be to insert all the right powers of $\hbar$ and c and use $\hbar \approx 1.055 \times 10^{-34}\ Js$ and $c = 3.00 \times\ 10^8\ m/s$. This requires many numbers and lots of use of the calculator. Using this bad way to calculate L we might write something like:

$$
\begin{aligned}
L &= 10^{16}\ \hbar c/\text{ GeV} &(1)\\
&= \frac{(10^{16}) \times (1.055 \times 10^{-34} Js) \times (3.00 \times 10^8\ m/s)}{10^9 \times (1.60 \times 10^{-19} J)} &(2)\\
&= \frac{(10^{16}) \times (1.055 \times 10^{-34}) \times (3.00 \times 10^8)}{10^9 \times (1.60 \times 10^{-19})}\ m &(3)\\
&= \frac{1.055 \times 3.00}{1.60} \times 10^{16-34+8+10}\ m &(4)\\
&= 1.97 \times 10^0\ m &(5)\\
&= 1.97\ m. &(6)
\end{aligned}
$$

Much better would be to use $1 = \hbar c = 197\ \text{ MeV} \cdot \text{ fm}$. This nicer approach would give us:

$$
\begin{aligned}
L &= 10^{16}/\text{ GeV} &(7)\\
&= 10^{16}/\text{ GeV} \times 1 &(8)\\
&= 10^{16}/\text{ GeV} \times (197\ \text{ MeV} \cdot fm) &(9)\\
&= 197 \times 10^{16-9+6-15}\ m &(10)\\
&= 197 \times 10^{-2}\ m &(11)\\
&= 1.97\ m &(12)\\
& &(13)
\end{aligned}
$$

The mass of the lecturer in S.I. units is easier to calculate as $E \sim mc^2$ reminds us that masses are only a factor of c^2 away from energies, and everyone knows c. Therefore

$$\begin{aligned} M &= 4.40 \times 10^{28} \text{ (GeV}/c^2) && (14)\\ &= (4.40 \times 10^{28}) \times (10^9 \times (1.60 \times 10^{-19}\ J))/(3.00 \times 10^8\ m/s)^2 && (15)\\ &= (4.40 * 1.60/9.00) * 10^{28-10-16} \text{ kg} && (16)\\ &= 78 \text{ kg.} && (17)\end{aligned}$$

Hence the BMI (which is mass in kg divided by square of height in metres) is

$$\text{BMI} = 78/(1.97)^2 = 20.1. \qquad (18)$$

According to Wikipedia (`https://en.wikipedia.org/wiki/Body_mass_index`) the WHO defines obsedity as a BMI over 25 if the person is more than 20 years old, so he is not obese given the age supplied (44 years).

(b)

$$\begin{aligned} [q_1 q_2] &= [4pi\varepsilon_0 F r^2] && (19)\\ &= [\varepsilon_0 F L^2] && (20)\\ &= [\varepsilon_0 (FL) L] && (21)\\ &= [\varepsilon_0 E L] && (22)\\ &= [\varepsilon_0 (ET)(L/T)] && (23)\\ &= [\varepsilon_0 \hbar c]. && (24)\end{aligned}$$

SPECIAL RELATIVITY

2. a) Draw the two leading-order Feynman diagrams for $e^+e^- \to e^+e^-$ involving single photon exchange, and write q, the 4-momentum of the exchanged virtual photon, in terms of the 4-momenta of the initial and/or final state particles. By evaluating q^2 in the centre of mass frame, or otherwise, determine whether q is *timelike* ($q^2 > 0$) or *spacelike* ($q^2 < 0$) in each case.

   b) The *Mandelstam variables* s, t, u in the scattering process $a + b \to 1 + 2$ are defined in terms of the particle 4-vectors as

$$s = (p_a + p_b)^2, \qquad t = (p_a - p_1)^2, \qquad u = (p_a - p_2)^2 .$$

   Show that $s + t + u = {m_a}^2 + {m_b}^2 + {m_1}^2 + {m_2}^2$.

   c) Show that $\sqrt{s}$ is the total energy of the collision in the centre of mass frame.

   d) At the HERA accelerator in Hamburg, 27.5 GeV electrons are brought into head-on collision with 820 GeV protons. Calculate the centre of mass energy, $\sqrt{s}$, of e^-p collisions at HERA, and determine the beam energy that would be needed to produce e^-p collisions with this value of $\sqrt{s}$ using electrons incident on a stationary proton target.

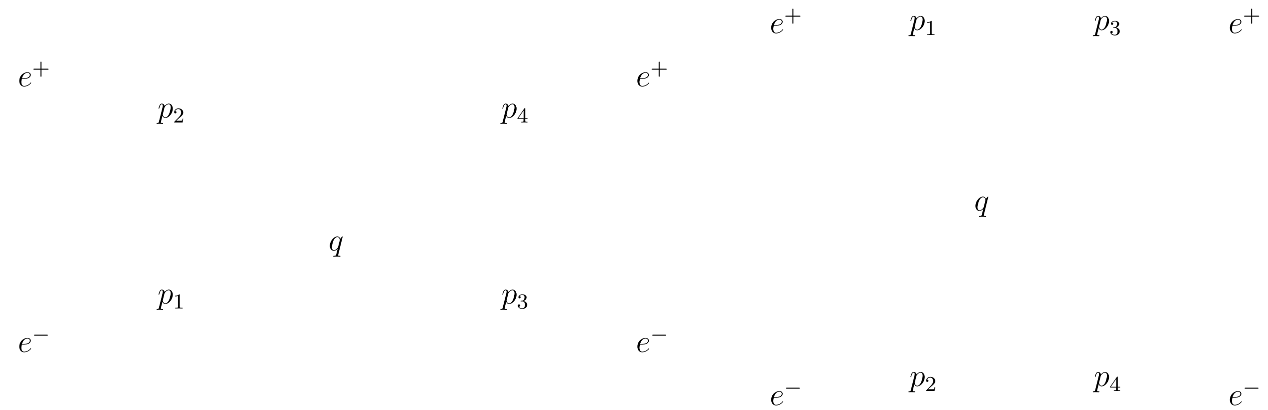
   e) Show that, in the laboratory frame with particle X at rest, the reaction $\nu + X \to \ell + Y$ can only proceed if the incoming neutrino has an energy above a threshold given by

$$E_\nu > \frac{(m_l + m_Y)^2 - m_X^2}{2m_X} .$$

[Aside: when revising at the end of the course you may wish to consider reviewing Question 1 of the January 2017 past Tripos paper for this course as looks more deelply into the connections between Mandelstam variables and the characteristics of different scattering processes.]

SOLUTION

a) The two leading order Feynman diagrams for $e^+e^- \to e^+e^-$ scattering are:



For diagram 1, the 4-momentum of the virtual photon is $q = p_1 + p_2$. In the centre of mass frame, we have $q = p_1 + p_2 = (2E, 0, 0, 0)$, and hence

$$q^2 = 4E^2 > 0 \qquad \Rightarrow \quad q^2 \text{ is timelike.}$$

For diagram 2, $q = p_1 - p_3$. In the centre of mass frame, we have $E_1 = E_3$ (elastic scattering) and hence $q = (0, \boldsymbol{p}_1 - \boldsymbol{p}_3)$. Therefore

$$q^2 = -(\boldsymbol{p}_1 - \boldsymbol{p}_3)^2 < 0 \qquad \Rightarrow \quad q^2 \text{ is spacelike}$$

b) Since $p_a^2 = m_a^2$ *etc.*:

$$\begin{aligned} s + t + u &= (p_a + p_b)^2 + (p_a - p_1)^2 + (p_a - p_2)^2 \\ &= 3p_a^2 + p_b^2 + p_1^2 + p_2^2 + 2p_a.p_b - 2p_a.p_1 - 2p_a.p_2 \\ &= 3m_a^2 + m_b^2 + m_1^2 + m_2^2 + 2p_a.(p_b - p_1 - p_2) \\ &= 3m_a^2 + m_b^2 + m_1^2 + m_2^2 + 2p_a. - p_a \\ &= m_a^2 + m_b^2 + m_1^2 + m_2^2 \end{aligned}$$

where energy-momentum conservation, $p_a + p_b = p_1 + p_2$, has been used in the last line but one.

c) In the centre of mass frame, the 4-momenta of particles a and b can be taken to be $p_a = (E_a, 0, 0, p)$, $p_b = (E_b, 0, 0, -p)$. Hence $p_a + p_b = (E_a + E_b, 0, 0, 0)$ and $s = (p_a + p_b)^2 = (E_a + E_b)^2$. Hence $\sqrt{s} = E_a + E_b$, the total collision energy in the centre of mass frame.

d) HERA: electron and proton masses can be neglected, so 4-momenta are:

$$p_a = (E_a, 0, 0, E_a) \qquad p_b = (E_b, 0, 0, -E_b) \qquad \Rightarrow \qquad p_a + p_b = (E_a + E_b, 0, 0, E_a - E_b)$$

Hence

$$s = (p_a + p_b)^2 = (E_a + E_b)^2 - (E_a - E_b)^2 = 4E_aE_b\ ,$$

which gives

$$\sqrt{s} = 2\sqrt{E_a E_b} = 2\sqrt{27.5\,\mathrm{GeV} * 820\,\mathrm{GeV}} = 300\,\mathrm{GeV}\ .$$

For electrons incident on a stationary proton target:

$$p_a = (E_a, 0, 0, E_a) \qquad p_b = (m_p, 0, 0, 0) \qquad \Rightarrow \qquad p_a + p_b = (E_a + m_p, 0, 0, E_a)\ .$$

Hence

$$s = (p_a + p_b)^2 = (E_a + m_p)^2 - E_a^2 = 2E_a m_p + m_p^2\ ,$$

which gives

$$E_a = \frac{s - m_p^2}{2m_p} = \frac{(300\,\mathrm{GeV})^2 - (0.938\,\mathrm{GeV})^2}{2 \times (0.938\,\mathrm{GeV})} = 47974\,\mathrm{GeV}\ .$$

e) For the scattering process $\nu + \mathrm{X} \to \ell + \mathrm{Y}$ to be kinematically allowed, we must have

$$\sqrt{s} > m_l + m_Y\ . \tag{25}$$

This is easily seen by considering the centre of mass frame: at threshold, the particles ℓ and Y are both produced at rest. Equation (25) involves only Lorentz-invariant quantities, and so can be applied to *any* reference frame. In particular, in the lab frame, with X at rest, we have

$$s = m_X^2 + 2p_\nu \cdot p_X = m_X^2 + 2E_\nu m_X\ .$$

Hence we need

$$m_X^2 + 2E_\nu m_X > (m_l + m_Y)^2$$

which gives a threshold neutrino energy in the lab frame of

$$\boxed{E_\nu > \frac{(m_l + m_Y)^2 - m_X^2}{2m_X}}\ .$$

3. a) For a particle of four-momentum $p^\mu = (E, p_x, p_y, p_z)$, show that the scalar product

$$p^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

is Lorentz invariant by explicitly transforming the four components of p^μ.

b) Use the Lorentz transformations to show that the volume element d^3p/E in momentum space is Lorentz invariant, *i.e.* that

$$\frac{\mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z}{E} = \frac{\mathrm{d}p'_x \mathrm{d}p'_y \mathrm{d}p'_z}{E'}\ .$$

SOLUTION

a) Lorentz transformation (with $c = 1$):

$$E' = \gamma(E - \beta p_x) \qquad p'_y = p_y$$
$$p'_x = \gamma(p_x - \beta E) \qquad p'_z = p_z$$

where $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v/c = v$. Hence

$$\begin{aligned}(p')^2 &= (E')^2 - (p'_x)^2 - (p'_y)^2 - (p'_z)^2 \\ &= \gamma^2(E-\beta p_x)^2 - \gamma^2(p_x - \beta E)^2 - p_y^2 - p_z^2 \\ &= \gamma^2(1-\beta^2)E^2 - \gamma^2(1-\beta^2)p_x^2 - p_y^2 - p_z^2 \\ &= E^2 - p_x^2 - p_y^2 - p_z^2 \\ &= p^2\end{aligned}$$

b) Since $\mathrm{d}p'_y = \mathrm{d}p_y$ and $\mathrm{d}p'_z = \mathrm{d}p_z$ we have

$$\mathrm{d}^3p' = \mathrm{d}p'_x\mathrm{d}p'_y\mathrm{d}p'_z = \frac{\mathrm{d}p'_x}{\mathrm{d}p_x}\cdot \mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z = \frac{\mathrm{d}p'_x}{\mathrm{d}p_x}\mathrm{d}^3p$$

where $p'_x = \gamma(p_x - \beta E)$ and E is to be understood as $E = \sqrt{p_x^2+p_y^2+p_z^2+m^2}$. The derivative is

$$\frac{\mathrm{d}p'_x}{\mathrm{d}p_x} = \frac{\mathrm{d}}{\mathrm{d}p_x}\left[\gamma(p_x-\beta E)\right] = \gamma\left(1-\beta\frac{\mathrm{d}E}{\mathrm{d}p_x}\right) .$$

The components p_y and p_z remain unchanged in the transformation, and so can be treated as constants. Hence

$$\frac{\mathrm{d}E}{\mathrm{d}p_x} = \frac{\mathrm{d}}{\mathrm{d}p_x}\sqrt{p_x^2+p_y^2+p_z^2+m^2} = \frac{p_x}{\sqrt{p_x^2+p_y^2+p_z^2+m^2}} = \frac{p_x}{E} .$$

This gives

$$\frac{\mathrm{d}p'_x}{\mathrm{d}p_x} = \gamma\left(1-\beta\frac{p_x}{E}\right) = \gamma\frac{E-\beta p_x}{E} = \frac{E'}{E} ,$$

and therefore

$$\frac{\mathrm{d}^3p'}{E'} = \frac{1}{E'}\cdot\frac{E'}{E}\mathrm{d}^3p = \frac{\mathrm{d}^3p}{E}$$

4. In a 2-body decay, $a \to 1+2$, show that the three-momentum of the final state particles in the centre of mass frame has magnitude

$$p^* = \frac{1}{2m_a}\sqrt{\left[m_a^2-(m_1+m_2)^2\right]\left[m_a^2-(m_1-m_2)^2\right]} .$$

SOLUTION

Decay $a \to 1+2$: energy conservation gives

$$m_a = E_1 + E_2 = \sqrt{m_1^2 + p^{*\,2}} + \sqrt{m_2^2+p^{*\,2}}$$

Squaring:

$$m_a^2 = E_1^2 + E_2^2 + 2E_1E_2 = m_1^2 + m_2^2 + 2p^{*\,2} + 2\sqrt{(m_1^2+p^{*\,2})(m_2^2+p^{*\,2})}$$

$$\Rightarrow \qquad 2\sqrt{(m_1^2+p^{*\,2})(m_2^2+p^{*\,2})} = m_a^2 - m_1^2 - m_2^2 - 2p^{*\,2} .$$

Squaring again:

$$\Rightarrow \qquad 4(m_1^2 + p^{*\,2})(m_2^2 + p^{*\,2}) = (m_a^2 - m_1^2 - m_2^2 - 2p^{*\,2})^2 \,.$$

Multiplying out and rearranging gives

$$\begin{aligned} 4m_a^2 p^{*\,2} &= (m_a^2 - m_1^2 - m_2^2)^2 - (2m_1m_2)^2 \\ &= (m_a^2 - m_1^2 - m_2^2 - 2m_1m_2)(m_a^2 - m_1^2 - m_2^2 + 2m_1m_2) \\ &= \left[m_a^2 - (m_1+m_2)^2\right]\left[m_a^2 - (m_1-m_2)^2\right] \,. \end{aligned}$$

Hence

$$\boxed{p^* = \frac{1}{2m_a}\sqrt{\left[m_a^2 - (m_1+m_2)^2\right]\left[m_a^2 - (m_1-m_2)^2\right]}} \,.$$

TWO BODY DECAY

5. According to the hypothesis of SU(3) symmetry (*i.e.* uds flavour independence) of invariant matrix elements, the two-body decay processes $\rho \to \pi\pi$ and $\mathrm{K}^* \to \mathrm{K}\pi$ have invariant matrix elements of the form

$$M_{\mathrm{fi}} = Cp_\pi$$

where $C_\rho/C_{K^*} = 2/\sqrt{3}$ and p_π is the final state centre of mass momentum. Show that the predicted ratio of decay rates agrees with experiment to within about 15%.

[Use the result of Question 4 to obtain p_π. Take the π, ρ, K and K^* meson masses to be 139, 770, 494 and 892 MeV respectively. The measured widths are $\Gamma(\rho \to \pi\pi) = 153 \pm 2\,\mathrm{MeV}$ and $\Gamma(\mathrm{K}^* \to \mathrm{K}\pi) = 51.3 \pm 0.8\,\mathrm{MeV}$.]

SOLUTION

a) The matrix element $M_{\mathrm{fi}} = Cp_\pi$ depends only on the centre of mass momentum $p_\pi = p^*$ of the final state particles, not on their directions, *i.e.* the decays are isotropic. For any isotropic two-body decay $a \to 1 + 2$, the decay rate is

$$\Gamma = \frac{p^*}{8\pi m_a^2}|M_{\mathrm{fi}}|^2 = \frac{p^*}{8\pi m_a^2}\cdot(Cp^*)^2 = \frac{C^2 p^{*\,3}}{8\pi m_a^2} \,.$$

From question 3, the centre of mass momentum is given by

$$p^* = \frac{1}{2m_a}\left[(m_a+m_1+m_2)(m_a-m_1+m_2)(m_a+m_1-m_2)(m_a-m_1-m_2)\right]^{1/2} \,.$$

For $\rho \to \pi\pi$, we have $m_a = m_\rho = 770\,\mathrm{MeV}$, $m_1 = m_2 = m_\pi \approx 140\,\mathrm{MeV}$:

$$p^* = \frac{1}{2m_\rho}\sqrt{(m_\rho + 2m_\pi).m_\rho.m_\rho.(m_\rho - 2m_\pi)} = \tfrac{1}{2}\sqrt{m_\rho^2 - 4m_\pi^2} = 359\,\mathrm{MeV}$$

For $\mathrm{K}^* \to \mathrm{K}\pi$, we have $m_a = m_{K^*} = 892\,\mathrm{MeV}$, $m_1 = m_K \approx 494\,\mathrm{MeV}$, $m_2 = m_\pi \approx 140\,\mathrm{MeV}$ giving $p^* = 288\,\mathrm{MeV}$.

$$\Rightarrow \frac{\Gamma(\rho\to\pi\pi)}{\Gamma(\mathrm{K}^*\to\mathrm{K}\pi)} = \frac{C_\rho^2}{C_{\mathrm{K}^*}^2}\cdot\frac{m_{\mathrm{K}^*}^2}{m_\rho^2}\cdot\left(\frac{p_\rho^*}{p_{\mathrm{K}^*}^*}\right)^3 = \left(\frac{2}{\sqrt{3}}\right)^2\cdot\left(\frac{892}{770}\right)^2\cdot\left(\frac{359}{288}\right)^3 = 3.46$$

Data:

$$\Gamma(\rho \to \pi\pi) = 153 \pm 2\,\text{MeV}, \qquad \Gamma(\text{K}^* \to \text{K}\pi) = 51.3 \pm 0.8\,\text{MeV}$$

giving a measured ratio of 2.98 .

6. The π^+ meson decays almost entirely via the two body decay process $\pi^+ \to \mu^+\nu_\mu$, with an invariant matrix element given by

$$|M_{\text{fi}}|^2 = 2G_{\text{F}}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)$$

where $G_{\text{F}} = 1.166 \times 10^{-5}\,\text{GeV}^{-2}$ is the Fermi constant, and f_π is related to the size of the pion wavefunction (the pion being a composite object).

a) Obtain a formula for the $\pi^+ \to \mu^+\nu_\mu$ decay rate. Assuming $f_\pi \sim m_\pi$, calculate the pion lifetime in natural units and in seconds, and compare to measurement.

[$m_\pi = 139.6\,\text{MeV}$, $m_\mu = 105.7\,\text{MeV}$.]

b) By replacing m_μ by m_e, show that the rate of $\pi^+ \to e^+\nu_e$ decay is 1.28×10^{-4} times smaller than the corresponding decay rate to muons. Show also that, on the basis of phase space alone (*i.e.* neglecting the factor $|M_{\text{fi}}|^2$), the decay rate to electrons would be expected to be *greater* than the rate to muons.

SOLUTION

a) From question 3, the momentum of the μ^+ or ν_μ from a $\pi^+ \to \mu^+\nu_\mu$ decay, in the π^+ rest frame, is

$$p^* = \frac{(m_\pi + m_\mu)(m_\pi - m_\mu)}{2m_\pi} = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

and hence the decay rate is

$$\begin{aligned}\Gamma = \frac{p^*}{8\pi m_\pi^2}|M_{\text{fi}}|^2 &= \frac{m_\pi^2 - m_\mu^2}{16\pi m_\pi^3} \cdot 2G_{\text{F}}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2) \\ &= \frac{G_{\text{F}}^2}{8\pi}\frac{m_\mu^2}{m_\pi}(m_\pi^2 - m_\mu^2)^2 \\ &= \frac{(1.166 \times 10^{-5})^2}{8\pi} \cdot \frac{0.105^2}{0.140}(0.140^2 - 0.105^2)^2 \\ &= 3.34 \times 10^{-17}\,\text{GeV}\end{aligned}$$

The pion lifetime is therefore

$$\tau_\pi = \frac{1}{\Gamma} = \frac{1}{3.34 \times 10^{-17}} = 3.0 \times 10^{16}\,\text{GeV}^{-1}$$

which can be converted to SI units using $\hbar = 6.58 \times 10^{-25}\,\text{GeV.s}$:

$$\tau_\pi = (3.0 \times 10^{16}).(6.58 \times 10^{-25}) = 1.97 \times 10^{-8}\,\text{s}$$

b) Ratio of decay rates:

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{m_e^2}{m_\mu^2} \cdot \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = \left(\frac{0.511}{105.6}\right)^2 \cdot \left(\frac{139.6^2 - 0.511^2}{139.6^2 - 105.6^2}\right)^2 = 1.28 \times 10^{-4}$$

On the basis of phase space alone, *i.e.* neglecting the contribution to the decay rate from $|M_{\rm fi}|^2$, we have

$$\Gamma = \frac{p^*}{8\pi m_\pi^2} \propto p^* \, .$$

Hence the ratio of decay rates is just the ratio of the centre of mass momenta appropriate to each decay:

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{p^*(\pi^+ \to e^+\nu_e)}{p^*(\pi^+ \to \mu^+\nu_\mu)} = \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} = 2.34$$

THE DIRAC EQUATION

7. Write down a simplified form of the Dirac equation for a spinor $\psi(t)$ describing a particle of mass m at rest. For the standard Pauli-Dirac representation of the γ matrices, obtain a differential equation for each component ψ_i of the spinor ψ, and hence write down a general solution for the evolution of ψ. Comment on your result and on its relation to the standard plane wave solutions involving $u_1(p)$, $u_2(p)$, $v_1(p)$, $v_2(p)$.

SOLUTION

For a particle of mass m at rest ($\boldsymbol{p} = \boldsymbol{0}$), since $\boldsymbol{p} = -i\boldsymbol{\nabla}$, we have $\partial\psi/\partial x = \partial\psi/\partial y = \partial\psi/\partial z = 0$. Hence $\psi = \psi(t)$ only, and the Dirac equation simplifies to

$$\boxed{i\gamma^0 \frac{\partial \psi}{\partial t} = m\psi} \,.$$

In the Pauli-Dirac representation, this is

$$i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \dot\psi_1 \\ \dot\psi_2 \\ \dot\psi_3 \\ \dot\psi_4 \end{pmatrix} = m \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} ,$$

which gives

$$i\dot\psi_1 = m\psi_1, \qquad i\dot\psi_2 = m\psi_2, \qquad -i\dot\psi_3 = m\psi_3, \qquad -i\dot\psi_4 = m\psi_4 \,.$$

These equations have the solutions

$$\psi_1 = A_1 e^{-imt}, \qquad \psi_2 = A_2 e^{-imt}, \qquad \psi_3 = A_3 e^{+imt}, \qquad \psi_4 = A_4 e^{+imt} ,$$

where the A_i are complex constants. The general solution for ψ is therefore

$$\boxed{\psi = \begin{pmatrix} A_1 e^{-imt} \\ A_2 e^{-imt} \\ A_3 e^{+imt} \\ A_4 e^{+imt} \end{pmatrix}} \,.$$

This can be expressed as a linear combination of the four independent solutions

$$\psi = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \quad N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \quad N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt}, \quad N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt} , \tag{26}$$

where $N = \sqrt{2m}$ to normalise to $2E = 2m$ particles per unit volume.

Thus both positive energy, e^{-imt}, and negative energy, e^{+imt}, solutions unambiguously emerge.

The spinors in Equation (26) can be obtained by setting $E = m$, $p_x = p_y = p_z = 0$ in the standard plane wave solutions $u_1 e^{i(\boldsymbol{p}\cdot\boldsymbol{r} - Et)}$, $u_2 e^{i(\boldsymbol{p}\cdot\boldsymbol{r} - Et)}$, $v_2 e^{-i(\boldsymbol{p}\cdot\boldsymbol{r} - Et)}$, $v_1 e^{-i(\boldsymbol{p}\cdot\boldsymbol{r} - Et)}$, as expected.

8. a) For the standard Pauli-Dirac representation of the γ matrices, and for an arbitrary pair of spinors ψ and ϕ with components ψ_i and ϕ_i, show that the current $\overline{\psi}\gamma^\mu\phi$ is given by

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\
\overline{\psi}\gamma^1\phi &= \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\overline{\psi}\gamma^2\phi &= -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\overline{\psi}\gamma^3\phi &= \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2
\end{aligned}$$

b) For a particle or antiparticle with four-momentum $p^\mu = (E, p_x, p_y, p_z)$, show that

$$\overline{u}_1\gamma^\mu u_1 = \overline{u}_2\gamma^\mu u_2 = \overline{v}_1\gamma^\mu v_1 = \overline{v}_2\gamma^\mu v_2 = 2p^\mu$$

and that

$$\overline{u}_1\gamma^\mu u_2 = \overline{u}_2\gamma^\mu u_1 = \overline{v}_1\gamma^\mu v_2 = \overline{v}_2\gamma^\mu v_1 = 0\ .$$

c) Hence show that the current $j^\mu = \overline{\psi}(p)\gamma^\mu\psi(p)$ corresponding to a general free particle spinor $\psi(p) = u(p)e^{i(\boldsymbol{p}\cdot\boldsymbol{r}-Et)}$ or antiparticle spinor $\psi(p) = v(p)e^{-i(\boldsymbol{p}\cdot\boldsymbol{r}-Et)}$ is given by $j^\mu = 2p^\mu$. Write down the particle density and flux represented by j^μ.

SOLUTION

a) For an arbitrary pair of spinors ψ and ϕ say, with spinor components ψ_i and ϕ_i, standard matrix multiplication gives, for $\mu = 0$,

$$\overline{\psi}\gamma^0\phi = \begin{pmatrix}\psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^*\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4\ .$$

Similarly, for $\mu = 1, 2, 3$, we obtain

$$\overline{\psi}\gamma^1\phi = \begin{pmatrix}\psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^*\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1$$

$$\overline{\psi}\gamma^2\phi = \begin{pmatrix}\psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^*\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & i & 0 & 0\\ -i & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1)$$

$$\overline{\psi}\gamma^3\phi = \begin{pmatrix}\psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^*\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2\ .$$

In summary:

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\
\overline{\psi}\gamma^1\phi &= \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\overline{\psi}\gamma^2\phi &= -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\overline{\psi}\gamma^3\phi &= \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2
\end{aligned}$$

b) For the free particle spinor u_1, the first element of the current 4-vector is

$$\begin{aligned}\overline{u}_1\gamma^0 u_1 &= (E+m)\left[1+\frac{p_z^2}{(E+m)^2}+\frac{(p_x^2+p_y^2)}{(E+m)^2}\right]\\ &= (E+m)\left[1+\frac{p^2}{(E+m)^2}\right]\\ &= \frac{(E+m)^2+p^2}{E+m}=\frac{2E^2+2Em}{E+m}=2E\ ,\end{aligned}$$

where, in the last line, we have made use of the relation $E^2 = p^2 + m^2$.

Repeating this exercise for the remaining terms in the 4-vector current gives, altogether,

$$\overline{u}_1\gamma^0 u_1 = 2E; \qquad \overline{u}_1\gamma^1 u_1 = 2p_x; \qquad \overline{u}_1\gamma^2 u_1 = 2p_y; \qquad \overline{u}_1\gamma^3 u_1 = 2p_z$$

which can be expressed more compactly as

$$\overline{u}_1\gamma^\mu u_1 = (2E, 2p_x, 2p_y, 2p_z) = 2p^\mu\ .$$

Repeating the above exercise for u_2, v_1 and v_2 in place of u_1 gives

$$\overline{u}_1\gamma^\mu u_1 = \overline{u}_2\gamma^\mu u_2 = \overline{v}_1\gamma^\mu v_1 = \overline{v}_2\gamma^\mu v_2 = 2p^\mu\ ,$$

while the cross-terms are easily seen to vanish:

$$\overline{u}_1\gamma^\mu u_2 = \overline{u}_2\gamma^\mu u_1 = \overline{v}_1\gamma^\mu v_2 = \overline{v}_2\gamma^\mu v_1 = 0\ .$$

c) For a particle, with $\psi = u(p)e^{ip.x}$, we have

$$\overline{\psi} = \psi^\dagger\gamma^0 = u(p)^\dagger\gamma^0 e^{-ip.x} = \overline{u}(p)e^{-ip.x}\ ,$$

and hence

$$j^\mu = \overline{\psi}\gamma^\mu\psi = \overline{u}\gamma^\mu u\ .$$

For an antiparticle, we have similarly $j^\mu = \overline{v}\gamma^\mu v$.

A particle spinor $u(p)$ can always be expressed as a linear combination of the basis spinors u_1, u_2:

$$u = \alpha_1 u_1 + \alpha_2 u_2, \qquad |\alpha_1|^2 + |\alpha_2|^2 = 1\ .$$

Hence

$$\overline{u}\gamma^\mu u = |\alpha_1|^2\overline{u}_1\gamma^\mu u_1 + |\alpha_2|^2\overline{u}_2\gamma^\mu u_2 = 2p^\mu\ .$$

Thus

$$\boxed{j^\mu = 2p^\mu}\ .$$

The current 4-vector is $j^\mu = (\rho, \boldsymbol{j})$ so

$$\rho = 2E, \qquad \boldsymbol{j} = 2\boldsymbol{p}\ ,$$

ρ being the particle density and $\boldsymbol{j}$ being the flux.

9. a) For a particle with 4-momentum $p^\mu = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$, show that the spinors $(1+\gamma^5)u_1$ and $(1+\gamma^5)u_2$ are not in general proportional to $u_\uparrow$ but become so in the relativistic limit $E \gg m$.

   b) Define the terms *helicity* and *chirality*. How are chirality and helicity related to the spinors and result described in part (a) ?

   c) What would be the equivalent result to that described in (a) for the corresponding antiparticle spinors $(1+\gamma^5)v_1$ and $(1+\gamma^5)v_2$?

   SOLUTION

   a) For $p^\mu = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$, we have

$$u_\uparrow(p) = \sqrt{E+m}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \\ \frac{p}{E+m}\cos\theta/2 \\ \frac{p}{E+m}e^{i\phi}\sin\theta/2 \end{pmatrix}, \qquad u_\downarrow(p) = \sqrt{E+m}\begin{pmatrix} -\sin\theta/2 \\ e^{i\phi}\cos\theta/2 \\ \frac{p}{E+m}\sin\theta/2 \\ -\frac{p}{E+m}e^{i\phi}\cos\theta/2 \end{pmatrix}$$

But

$$(1+\gamma^5)u_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix}$$
$$= \sqrt{E+m}\begin{pmatrix} 1+p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1+p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix}$$

which, in general, is clearly not proprtional to $u_\uparrow$.

In the limit $E \gg m$, the spinors u_1 and u_2 become

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} \to \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix}$$
$$u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix} \to \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ e^{-i\phi}\sin\theta \\ -\cos\theta \end{pmatrix}.$$

Hence

$$(1+\gamma^5)u_1 \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sqrt{E} \begin{pmatrix} 1 \\ 0 \\ \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} = \sqrt{E} \begin{pmatrix} 1+\cos\theta \\ e^{i\phi}\sin\theta \\ 1+\cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix}$$

$$= 2\sqrt{E}\cos\theta/2 \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \\ \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} = 2\cos\theta/2 \cdot u_\uparrow \quad (27)$$

and similarly:

$$(1+\gamma^5)u_2 \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sqrt{E} \begin{pmatrix} 0 \\ 1 \\ e^{-i\phi}\sin\theta \\ -\cos\theta \end{pmatrix} = \sqrt{E} \begin{pmatrix} e^{-i\phi}\sin\theta \\ 1-\cos\theta \\ e^{-i\phi}\sin\theta \\ 1-\cos\theta \end{pmatrix}$$

$$= 2\sqrt{E}\sin\theta/2 \begin{pmatrix} e^{-i\phi}\cos\theta/2 \\ \sin\theta/2 \\ e^{-i\phi}\cos\theta/2 \\ \sin\theta/2 \end{pmatrix} = 2e^{-i\phi}\sin\theta/2 \cdot u_\uparrow \quad (28)$$

b) The *helicity* operator $h = \mathbf{\Sigma}.\hat{\boldsymbol{p}}$ gives the projection of the particle spin along the direction of motion. A particle or antiparticle with the spin vector aligned along (opposite to) the direction of motion has $h = +1$ ($h = -1$) and is said to be *right-handed* (*left-handed*).

Any (particle or antiparticle) spinor ψ can be expressed as the sum of its *left-handed* and *right-handed chiral components*

$$\psi = \psi_\mathrm{L} + \psi_\mathrm{R}; \qquad \psi_\mathrm{L} \equiv \tfrac{1}{2}(1-\gamma^5)\psi \qquad \psi_\mathrm{R} \equiv \tfrac{1}{2}(1+\gamma^5)\psi\ .$$

In the extreme relativistic limit ($E \gg m$), the left-handed and right-handed chiral components are also eigenstates of the helicity operator:

For a *particle*:	ψ_L has helicity -1	ψ_R has helicity $+1$
For an *antiparticle*:	ψ_L has helicity $+1$	ψ_R has helicity -1

The results in part a) show that, in the relativistic limit, and *only* in the relativistic limit, the right-handed chiral components $(1+\gamma^5)u_1$ and $(1+\gamma^5)u_2$ are both proportional to $u_\uparrow$, *i.e.* are both positive helicity eigenstates. Since any particle spinor u can be expressed as a linear combination of u_1 and u_2, this result holds quite generally *i.e.* in the relativistic limit, the right-handed chiral component $(1+\gamma^5)u$ becomes a right-handed helicity eigenstate for any particle spinor u.

c) For antiparticles, the right-handed chiral component $\frac{1}{2}(1+\gamma^5)\psi$ becomes a left-handed helicity eigenstate in the relativistic limit. Hence $(1+\gamma^5)v_1$ and $(1+\gamma^5)v_2$ will both become proportional to $v_\downarrow$ in the relativsitic limit.

10. a) Without resorting to an explicit representation of the Dirac gamma matrices, show that the matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ has the following properties:

$$(\gamma^5)^2 = 1, \qquad \gamma^{5\dagger} = \gamma^5, \qquad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5 \ .$$

b) Show that the adjoint spinors $\overline{\psi_{\rm L}}$ and $\overline{\psi_{\rm R}}$ corresponding to the left-handed and right-handed components $\psi_{\rm L} \equiv \frac{1}{2}(1-\gamma^5)\psi$ and $\psi_{\rm R} \equiv \frac{1}{2}(1+\gamma^5)\psi$ are:

$$\begin{aligned} \overline{\psi_{\rm L}} &= \overline{\psi}\tfrac{1}{2}(1+\gamma^5) \\ \overline{\psi_{\rm R}} &= \overline{\psi}\tfrac{1}{2}(1-\gamma^5) \ . \end{aligned}$$

c) Show that $\overline{\phi_{\rm L}}\gamma^\mu\psi_{\rm R} = \overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm L} = 0$, and that the current $\overline{\phi}\gamma^\mu\psi$ can be decomposed as

$$\overline{\phi}\gamma^\mu\psi = \overline{\phi_{\rm L}}\gamma^\mu\psi_{\rm L} + \overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm R} \ .$$

SOLUTION

a) Repeatedly use the fact that the γ matrices anticommute and satisfy $(\gamma^0)^2 = 1,\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$:

$$\begin{aligned} (\gamma^5)^2 = (i\gamma^0\gamma^1\gamma^2\gamma^3)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 && \\ &= \ \ \gamma^0\gamma^1\gamma^2\gamma^0\gamma^3\gamma^1\gamma^2\gamma^3 && \text{since} \quad \gamma^3\gamma^0 = -\gamma^0\gamma^3 \\ &= -\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 && \text{since} \quad \gamma^2\gamma^0 = -\gamma^0\gamma^2 \\ &= \ \ \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 && \text{since} \quad \gamma^1\gamma^0 = -\gamma^0\gamma^1 \\ &= \ \ \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 && \text{since} \quad (\gamma^0)^2 = 1 \\ &= -\gamma^1\gamma^2\gamma^1\gamma^3\gamma^2\gamma^3 && \text{since} \quad \gamma^3\gamma^1 = -\gamma^1\gamma^3 \\ &= \ \ \gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3 && \text{since} \quad \gamma^2\gamma^1 = -\gamma^1\gamma^2 \\ &= -\gamma^2\gamma^3\gamma^2\gamma^3 && \text{since} \quad (\gamma^1)^2 = -1 \\ &= \ \ \gamma^3\gamma^2\gamma^2\gamma^3 && \text{since} \quad \gamma^2\gamma^3 = -\gamma^3\gamma^2 \\ &= -\gamma^3\gamma^3 && \text{since} \quad (\gamma^2)^2 = -1 \\ &= \ \ 1 && \end{aligned}$$

Using $\gamma^{0\dagger} = \gamma^0$, $\gamma^{1\dagger} = -\gamma^1$, $\gamma^{2\dagger} = -\gamma^2$, $\gamma^{3\dagger} = -\gamma^3$:

$$\begin{aligned} \gamma^{5\dagger} = (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger = -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} = i\gamma^3\gamma^2\gamma^1\gamma^0 & \\ = -i\gamma^2\gamma^1\gamma^0\gamma^3 = -i\gamma^1\gamma^0\gamma^2\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 & \end{aligned}$$

Consider $\gamma^5\gamma^2$ for example:

$$\gamma^5\gamma^2 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = i\gamma^0\gamma^2\gamma^1\gamma^2\gamma^3 = -i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^2\gamma^5$$

and similarly: $\gamma^5\gamma^0 = -\gamma^0\gamma^5,\ \gamma^5\gamma^1 = -\gamma^1\gamma^5,\ \gamma^5\gamma^3 = -\gamma^3\gamma^5$ giving altogether $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$.

b) An adjoint spinor is defined as $\overline{\psi} \equiv \psi^\dagger\gamma^0$, so that

$$\begin{aligned} \overline{\psi_{\rm L}} = \psi_L^\dagger\gamma^0 &= \left[\tfrac{1}{2}(1-\gamma^5)\psi\right]^\dagger\gamma^0 && \\ &= \psi^\dagger\tfrac{1}{2}(1-\gamma^5)\gamma^0 && \text{since} \quad \gamma^{5\dagger} = \gamma^5 \\ &= \psi^\dagger\gamma^0\tfrac{1}{2}(1+\gamma^5) && \text{since} \quad \gamma^0\gamma^5 = -\gamma^5\gamma^0 \\ &= \overline{\psi}\tfrac{1}{2}(1+\gamma^5) && \end{aligned}$$

and similarly:

$$\overline{\psi_{\rm R}} = \overline{\psi}\tfrac{1}{2}(1-\gamma^5)\ .$$

c) Separate the spinor ψ into its left- and right-handed components via

$$\psi = \tfrac{1}{2}(1-\gamma^5)\psi + \tfrac{1}{2}(1+\gamma^5)\psi = \psi_{\rm L} + \psi_{\rm R}$$

For the adjoint spinor:

$$\overline{\psi} = \psi^\dagger\gamma^0 = (\psi_{\rm L}+\psi_{\rm R})^\dagger\gamma^0 = \psi_{\rm L}^\dagger\gamma^0 + \psi_{\rm R}^\dagger\gamma^0 = \overline{\psi_{\rm L}} + \overline{\psi_{\rm R}}$$

Hence

$$\begin{aligned}\overline{\phi}\gamma^\mu\psi &= \left[\,\overline{\phi_{\rm L}} + \overline{\phi_{\rm R}}\,\right]\gamma^\mu\left[\psi_{\rm L}+\psi_{\rm R}\right] \\ &= \overline{\phi_{\rm L}}\gamma^\mu\psi_{\rm L} + \overline{\phi_{\rm L}}\gamma^\mu\psi_{\rm R} + \overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm L} + \overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm R}\end{aligned}$$

But

$$\begin{aligned}\overline{\phi_{\rm L}}\gamma^\mu\psi_{\rm R} &= \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\cdot\gamma^\mu\cdot\tfrac{1}{2}(1+\gamma^5)\psi \\ &= \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\cdot\tfrac{1}{2}(1-\gamma^5)\gamma^\mu\psi \\ &= 0\end{aligned}$$

since $(1+\gamma^5)(1-\gamma^5) = 1-(\gamma^5)^2 = 0$. Similarly: $\overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm L} = 0$ giving

$$\overline{\phi}\gamma^\mu\psi = \overline{\phi_{\rm L}}\gamma^\mu\psi_{\rm L} + \overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm R}$$

as required. Alternatively, show directly that

$$\begin{aligned}\overline{\phi_{\rm L}}\gamma^\mu\psi_{\rm L} &= \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\cdot\gamma^\mu\cdot\tfrac{1}{2}(1-\gamma^5)\psi \\ &= \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\cdot\tfrac{1}{2}(1+\gamma^5)\gamma^\mu\psi \\ &= \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\gamma^\mu\psi\end{aligned}$$

and similarly

$$\overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm R} = \overline{\phi}\tfrac{1}{2}(1-\gamma^5)\gamma^\mu\psi\ ,$$

again giving

$$\overline{\phi_{\rm L}}\gamma^\mu\psi_{\rm L} + \overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm R} = \overline{\phi}\tfrac{1}{2}(1+\gamma^5)\gamma^\mu\psi + \overline{\phi}\tfrac{1}{2}(1-\gamma^5)\gamma^\mu\psi = \overline{\phi}\gamma^\mu\psi\ .$$

Thus, for interactions between spin $\frac{1}{2}$ particles (or antiparticles) and photons in QED, the left-handed chiral component of a spinor couples only to another left-handed chiral component $(\overline{\phi_{\rm L}}\gamma^\mu\psi_{\rm L})$ and the right-handed chiral component couples only to another right-handed chiral component $(\overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm R})$. There is no coupling between the left-handed and right-handed chiral components: $(\overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm L} = 0, \overline{\phi_{\rm R}}\gamma^\mu\psi_{\rm L} = 0)$.

At high energies, the left-handed and right-handed chiral components become helicity eigenstates with definite helicity and we have *helicity conservation* in QED: the *particle* helicity is preserved at a QED vertex.

ELECTRON-MUON ELASTIC SCATTERING

11. a) Show that the general matrix element for $e^-\mu^- \to e^-\mu^-$ scattering via single photon exchange is

$$M_{\rm fi} = -\frac{e^2}{(p_1-p_3)^2} g_{\mu\nu}\left[\overline{u}(p_3)\gamma^\mu u(p_1)\right]\left[\overline{u}(p_4)\gamma^\nu u(p_2)\right]$$

where p_1 and p_3 are the initial and final e^- four-momenta and p_2 and p_4 are the initial and final μ^- four-momenta.

b) Show that, for scattering in the centre of mass frame with incoming and outgoing e^- four-momenta $p_1^\mu = (E_1, 0, 0, p)$ and $p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta)$, the electron current for the various possible electron spin combinations is

$$\begin{aligned}
\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(E_1 c, ps, -ips, pc)\\
\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(ms, 0, 0, 0)\\
\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= 2(E_1 c, ps, ips, pc)\\
\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) &= -2(ms, 0, 0, 0)
\end{aligned}$$

where m is the electron mass and $s \equiv \sin\theta/2$, $c \equiv \cos\theta/2$.

c) Write down the incoming and outgoing muon 4-momenta p_2 and p_4, and the helicity eigenstate spinors $u_\uparrow(p_2)$, $u_\downarrow(p_2)$, $u_\uparrow(p_4)$ and $u_\downarrow(p_4)$. [Take the muon mass to be M and the muon energy to be E_2]. By comparing the forms of the muon and electron spinors, explain how the muon currents

$$\begin{aligned}
\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(E_2 c, -ps, -ips, -pc)\\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(Ms, 0, 0, 0)\\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) &= 2(E_2 c, -ps, ips, -pc)\\
\overline{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) &= -2(Ms, 0, 0, 0)
\end{aligned}$$

can be written down (up to overall factors of ± 1) without any further calculation.

d) Explain why some of the above currents vanish in the relativistic limit where the electron mass and muon mass can be neglected. Sketch the spin configurations which are allowed in this limit.

e) Show that, in the relativistic limit, the matrix element squared $|M_{\rm LL}|^2$ for the case where the incoming e^- and incoming μ^- are both left-handed is given by

$$|M_{\rm LL}|^2 = \frac{4e^4 s^2}{(p_1-p_3)^4}$$

where $s = (p_1+p_2)^2$. Why is the numerator of $|M_{\rm LL}|^2$ independent of θ ?

f) Find a similar expression for the matrix element $|M_{\rm RL}|^2$ for a right-handed incoming e^- and a left-handed incoming μ^-, and explain why $|M_{\rm RL}|^2$ vanishes when $\theta = \pi$. Write down the corresponding results for $|M_{\rm RR}|^2$ and $|M_{\rm LR}|^2$.

g) Show that, in the relativistic limit, the differential cross section for unpolarised $e^-\mu^- \to e^-\mu^-$ scattering in the centre of mass frame is

$$\frac{{\rm d}\sigma}{{\rm d}\Omega} = \frac{2\alpha^2}{s}\cdot\frac{1+\frac{1}{4}(1+\cos\theta)^2}{(1-\cos\theta)^2}\,.$$

h) Show that the spin-averaged matrix element squared (in this ultra-relativistic limit) can be expressed in Lorentz-invariant form as

$$\left\langle |M_{\rm fi}|^2 \right\rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) \right] ,$$

and that a Lorentz invariant form for the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{2\pi\alpha^2}{q^4} \left[1 + \left(1 + \frac{q^2}{s} \right)^2 \right]$$

where $q^2 = (p_1 - p_3)^2$.

The remainder of this question involves the derivation of a general expression for $\langle |M_{\rm fi}|^2 \rangle$ for the case of finite electron and muon masses, and is optional:

i) Show that the spin-averaged matrix element squared for unpolarised $e^-\mu^- \to e^-\mu^-$ scattering can be written in the form

$$\left\langle |M_{\rm fi}|^2 \right\rangle = \frac{1}{4} \sum_{\rm spins} |M_{\rm fi}|^2 = \frac{1}{4} \frac{e^4}{(p_1 - p_3)^4} L^{\mu\nu} W_{\mu\nu}$$

where the electron and muon tensors $L^{\mu\nu}$ and $W^{\mu\nu}$ are given by

$$L^{\mu\nu} \equiv \sum_{\rm spins} \left[\overline{u}(p_3)\gamma^\mu u(p_1) \right] \left[\overline{u}(p_3)\gamma^\nu u(p_1) \right]^*$$

$$W_{\mu\nu} \equiv \sum_{\rm spins} \left[\overline{u}(p_4)\gamma_\mu u(p_2) \right] \left[\overline{u}(p_4)\gamma_\nu u(p_2) \right]^*$$

j) Using the electron currents from part b) above, show that the components of the electron tensor $L^{\mu\nu}$ are

$$\begin{pmatrix} L^{00} & L^{01} & L^{02} & L^{03} \\ L^{10} & L^{11} & L^{12} & L^{13} \\ L^{20} & L^{21} & L^{22} & L^{23} \\ L^{30} & L^{31} & L^{32} & L^{33} \end{pmatrix} = 8 \begin{pmatrix} E_1^2 c^2 + m^2 s^2 & E_1 psc & 0 & E_1 pc^2 \\ E_1 psc & p^2 s^2 & 0 & p^2 sc \\ 0 & 0 & p^2 s^2 & 0 \\ E_1 pc^2 & p^2 sc & 0 & p^2 c^2 \end{pmatrix} ,$$

and hence verify that $L^{\mu\nu}$ has the Lorentz invariant form

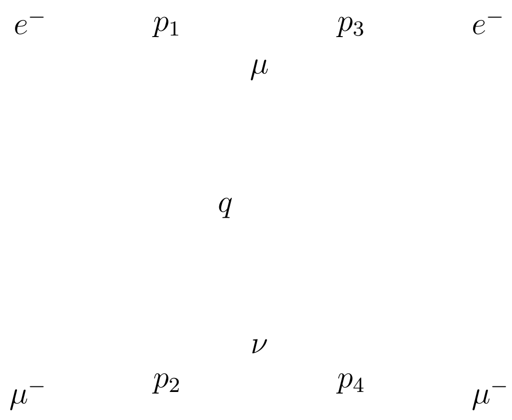
$$L^{\mu\nu} = 4 \left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu} \left(m^2 - p_1.p_3 \right) \right] .$$

k) Write down the corresponding expression for $W^{\mu\nu}$ and hence show that

$$\langle |M_{\rm fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_2.p_4)m^2 + 2m^2 M^2 \right]$$

SOLUTION

a) The QED process $e^-\mu^- \to e^-\mu^-$ involves a single Feynman diagram at leading order:



Applying the Feynman rules gives

$$-iM_{\rm fi} = [\overline{u}(p_3)\cdot -ie\gamma^\mu \cdot u(p_1)]\cdot \frac{-ig_{\mu\nu}}{(p_1-p_3)^2}\cdot [\overline{u}(p_4)\cdot -ie\gamma^\nu u(p_2)]$$

and hence

$$\boxed{M_{\rm fi} = -\frac{e^2}{(p_1-p_3)^2} g_{\mu\nu}\,[\overline{u}(p_3)\gamma^\mu u(p_1)]\,[\overline{u}(p_4)\gamma^\nu u(p_2)]} \tag{29}$$

b) For a particle of mass m with four-momentum $p^\mu = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$, the helicity eigenstate spinors are

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix}\cos\theta/2\\ e^{i\phi}\sin\theta/2\\ p/(E+m)\cos\theta/2\\ p/(E+m)e^{i\phi}\sin\theta/2\end{pmatrix};\qquad u_\downarrow = \sqrt{E+m}\begin{pmatrix}-\sin\theta/2\\ e^{i\phi}\cos\theta/2\\ p/(E+m)\sin\theta/2\\ -p/(E+m)e^{i\phi}\cos\theta/2\end{pmatrix} \tag{30}$$

For the incoming electron, with $p_1 = (E_1, 0, 0, p)$, the two possible spinors are:

$$u_\uparrow(p_1) = \sqrt{E_1+m}\begin{pmatrix}1\\0\\p/(E_1+m)\\0\end{pmatrix};\quad u_\downarrow(p_1) = \sqrt{E_1+m}\begin{pmatrix}0\\1\\0\\-p/(E_1+m)\end{pmatrix} \tag{31}$$

For the outgoing electron, with $p_3 = (E_1, p\sin\theta, 0, p\cos\theta)$, the spinors are:

$$u_\uparrow(p_3) = \sqrt{E_1+m}\begin{pmatrix}c\\s\\p/(E_1+m)\cdot c\\p/(E_1+m)\cdot s\end{pmatrix};\quad u_\downarrow(p_3) = \sqrt{E_1+m}\begin{pmatrix}-s\\c\\p/(E_1+m)\cdot s\\-p/(E_1+m)\cdot c\end{pmatrix} \tag{32}$$

where $c \equiv \cos\theta/2$ and $s \equiv \sin\theta/2$. Noting that the spinors are real, matrix multiplication gives

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi_1\phi_1 + \psi_2\phi_2 + \psi_3\phi_3 + \psi_4\phi_4\\
\overline{\psi}\gamma^1\phi &= \psi_1\phi_4 + \psi_2\phi_3 + \psi_3\phi_2 + \psi_4\phi_1\\
\overline{\psi}\gamma^2\phi &= -i(\psi_1\phi_4 - \psi_2\phi_3 + \psi_3\phi_2 - \psi_4\phi_1)\\
\overline{\psi}\gamma^3\phi &= \psi_1\phi_3 - \psi_2\phi_4 + \psi_3\phi_1 - \psi_4\phi_2
\end{aligned}$$

Start with $\overline{u}_\downarrow(p_3)$ and $u_\downarrow(p_1)$:

$$\overline{u}_\downarrow(p_3)\gamma^0 u_\downarrow(p_1) = (E_1+m)\left[c + \frac{p^2}{(E_1+m)^2}c\right] = \frac{(E_1+m)^2+p^2}{(E_1+m)}c = \frac{2E_1^2+2mE_1}{(E_1+m)}c = 2E_1c$$

where we have used $m^2+p^2=E_1^2$ in the last-but-one step. Similarly, for γ^1, γ^2, γ^3 we have

$$\overline{u}_\downarrow(p_3)\gamma^1 u_\downarrow(p_1) = (E_1+m)\left[\frac{p}{E_1+m}s + \frac{p}{E_1+m}s\right] = 2ps$$

$$\overline{u}_\downarrow(p_3)\gamma^2 u_\downarrow(p_1) = (E_1+m)\left[\frac{-ip}{E_1+m}s - \frac{ip}{E_1+m}s\right] = -2ips$$

$$\overline{u}_\downarrow(p_3)\gamma^3 u_\downarrow(p_1) = (E_1+m)\left[\frac{p}{E_1+m}c + \frac{p}{E_1+m}c\right] = 2pc$$

In summary

$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (2E_1c, 2ps, -2ips, 2pc) \tag{33}$$

Similarly for the other possible spin configurations, giving overall:

$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2(E_1c, ps, -ips, pc) \tag{34}$$
$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2(ms, 0, 0, 0) \tag{35}$$
$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2(E_1c, ps, ips, pc) \tag{36}$$
$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = -2(ms, 0, 0, 0) \tag{37}$$

c) For the incoming μ^-, with four-momentum $p_2 = (E_2, 0, 0, -p)$ and $E_2 = \sqrt{p^2+M^2}$, the helicity eigenstate spinors can be obtained from Equation (30) by setting $\theta = \pi$ and $\phi = 0$:

$$u_\uparrow(p_2) = \sqrt{E_2+M}\begin{pmatrix} 0 \\ 1 \\ 0 \\ p/(E_2+M) \end{pmatrix}; \qquad u_\downarrow(p_2) = \sqrt{E_2+M}\begin{pmatrix} -1 \\ 0 \\ p/(E_2+M) \\ 0 \end{pmatrix} \tag{38}$$

For the outgoing μ^-, with 4-momentum $p_4 = (E_2, -p\sin\theta, 0, -p\cos\theta)$, the helicity eigenstate spinors can be obtained from Equation (30) by setting $\theta \to \pi - \theta$ and $\phi = \pi$:

$$u_\uparrow(p_4) = \sqrt{E_2+M}\begin{pmatrix} s \\ -c \\ p/(E_2+M)\cdot s \\ p/(E_2+M)\cdot -c \end{pmatrix}; \qquad u_\downarrow(p_4) = \sqrt{E_2+M}\begin{pmatrix} -c \\ -s \\ p/(E_2+M)\cdot c \\ -p/(E_2+M)\cdot -s \end{pmatrix} \tag{39}$$

using $\cos(\pi-\theta)/2 = \sin\theta/2 = s$ and $\sin(\pi-\theta)/2 = \cos\theta/2 = c$.

A comparison of Equations (31) and (38) shows that, if we make the replacement $p \to -p$, then $u_\uparrow(p_2)$ is of the same form as $u_\downarrow(p_1)$. Similarly, $u_\downarrow(p_2)$ is then of the same form as $u_\uparrow(p_1)$, apart from an overall normalisation factor of -1.

Similarly, a comparison of Equations (32) and (39) shows that, under $p \to -p$, $u_\uparrow(p_4)$ becomes the same as $u_\downarrow(p_3)$, and $u_\downarrow(p_4)$ becomes the same as $u_\uparrow(p_3)$, apart from overall normalisation factors of -1.

The muon currents can therefore be written down directly using the electron current results, by changing m to M, E_1 to E_2, p to $-p$, $\uparrow$ to $\downarrow$ and $\downarrow$ to $\uparrow$:

$$\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) = 2(E_2 c, -ps, -ips, -pc) \tag{40}$$
$$\overline{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) = 2(Ms, 0, 0, 0) \tag{41}$$
$$\overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) = 2(E_2 c, -ps, ips, -pc) \tag{42}$$
$$\overline{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) = -2(Ms, 0, 0, 0) \tag{43}$$

d) Some of the currents vanish in the relativistic limit due to helicity conservation. The allowed spin configurations are those for which the helicity of the e^- and the helicity of the μ^- are both preserved in the scattering:

e) In the relativistic limit, we can set $m = M = 0$ and $E_1 = E_2 = E$. The electron currents become

$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2E(c, s, -is, c) \tag{44}$$
$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (0, 0, 0, 0) \tag{45}$$
$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2E(c, s, is, c) \tag{46}$$
$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (0, 0, 0, 0) \tag{47}$$

while the muon curents are:

$$\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) = 2E(c, -s, -is, -c) \tag{48}$$
$$\overline{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) = (0, 0, 0, 0) \tag{49}$$
$$\overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) = 2E(c, -s, is, -c) \tag{50}$$
$$\overline{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) = (0, 0, 0, 0) \tag{51}$$

When the incoming e^- and μ^- are both left-handed (*i.e.* negative helicity) we have $u(p_1) = u_\downarrow(p_1)$ and $u(p_2) = u_\downarrow(p_2)$, and the only non-zero contributions to the electron and muon currents come from Equations (44) and (48). Hence the scalar product of the electron and muon currents is

$$2E(c, s, -is, c) \cdot 2E(c, -s, -is, -c) = 4E^2 \cdot (c^2 + s^2 + c^2 + s^2) = 8E^2$$

and, from Equation (29), the matrix element squared is

$$|M_{\mathrm{LL}}|^2 = \frac{e^4}{(p_1 - p_3)^4} \cdot (8E^2)^2 = \frac{4e^4 s^2}{(p_1 - p_3)^4}$$

where now $s \equiv (p_1 + p_2)^2 = 4E^2$.

The numerator of $|M_{\mathrm{LL}}|^2$ is independent of θ because the incoming left-handed e^- and the incoming left-handed μ^- have oppositely directed spins, and the total spin of the initial state is $S_z = 0$. Hence there is no preferred spatial direction.

f) For M_{RL}, with the incoming e^- right-handed and the μ^- left-handed, we have $u(p_1) = u_\uparrow(p_1)$ and $u(p_2) = u_\downarrow(p_2)$. The only non-zero combination is now given by the scalar product of Equations (46) and (48):

$$\begin{aligned} M_{\mathrm{RL}} \propto 2E(c, s, is, c) \cdot 2E(c, -s, -is, -c) &= 4E^2 \cdot (c^2 + s^2 - s^2 + c^2) \\ &= 8E^2 \cos^2 \theta/2 \\ &= 8E^2 \cdot \tfrac{1}{2}(1 + \cos\theta)\ . \end{aligned}$$

Hence the non-zero matrix elements can be summarised as

$$|M_{\rm RR}|^2 = |M_{\rm LL}|^2 = \frac{e^4}{(p_1-p_3)^4}4s^2$$
$$|M_{\rm LR}|^2 = |M_{\rm RL}|^2 = \frac{e^4}{(p_1-p_3)^4}4s^2\cdot\tfrac{1}{4}(1+\cos\theta)^2$$

where we must have $M_{\rm LL}=M_{\rm RR}$ and $M_{\rm LR}=M_{\rm RL}$ by symmetry of the spin configurations.

g) For unpolarised $e^-\mu^- \to e^-\mu^-$ scattering, sum over the final spins and average over the initial spins to obtain

$$\begin{aligned}\left\langle|M_{\rm fi}|^2\right\rangle &= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\left(|M_{\rm LL}|^2+|M_{\rm RR}|^2+|M_{\rm LR}|^2+|M_{\rm RL}|^2\right)\\ &= \frac{2e^4}{(p_1-p_3)^4}s^2\left[1+\tfrac{1}{4}(1+\cos\theta)^2\right]\end{aligned} \tag{52}$$

With $p_1=(E,0,0,E)$ and $p_3=(E,E\sin\theta,0,E\cos\theta)$, we have

$$(p_1-p_3)^2 = p_1^2+p_3^2-2p_1.p_3 = -2p_1.p_3 = -2E^2(1-\cos\theta)$$

For any $2\to 2$ body elastic scattering process in the centre of mass frame, the differential cross section is given by

$$\frac{{\rm d}\sigma}{{\rm d}\Omega} = \frac{1}{64\pi^2 s}\langle|M_{\rm fi}|^2\rangle$$

Hence:

$$\boxed{\frac{{\rm d}\sigma}{{\rm d}\Omega} = \frac{e^4}{8\pi^2 s}\cdot\frac{1+\frac{1}{4}(1+\cos\theta)^2}{(1-\cos\theta)^2}} \tag{53}$$

h) With 4-momenta

$$\begin{aligned} p_1 &= (E,0,0,E) & p_3 &= (E,E\sin\theta,0,E\cos\theta)\\ p_2 &= (E,0,0,-E) & p_4 &= (E,-E\sin\theta,0,-E\cos\theta)\end{aligned}$$

the scalar products are

$$\begin{aligned} p_1.p_2 &= p_3.p_4 = 2E^2 = \tfrac{1}{2}s^2\\ p_1.p_4 &= p_2.p_3 = E^2(1+\cos\theta) = \tfrac{1}{4}s(1+\cos\theta)\end{aligned}$$

Hence the spin-averaged matrix element squared of Equation (52) becomes

$$\boxed{\left\langle|M_{\rm fi}|^2\right\rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1.p_2)(p_3.p_4)+(p_1.p_4)(p_2.p_3)\right]}\,.$$

It was shown in Handout 3 that the Lorentz-invariant cross section ${\rm d}\sigma/{\rm d}t = {\rm d}\sigma/{\rm d}q^2$ is given by

$$\frac{{\rm d}\sigma}{{\rm d}q^2} = \frac{1}{64\pi s(p_{\rm i}^*)^2}|M_{\rm fi}|^2$$

where p_{i}^* is the centre of mass momentum of either of the initial state particles. At high energies when masses are negligible (as here), we have $p_{\mathrm{i}}^* = E$ and hence $4(p_{\mathrm{i}}^*)^2 = 4E^2 = s$. Hence

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{1}{16\pi s^2}|M_{\mathrm{fi}}|^2 = \frac{1}{16\pi s^2} \cdot \frac{8e^4}{q^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3)\right] \; .$$

In terms of s and q^2, the scalar products are

$$s = (p_1 + p_2)^2 = 2p_1.p_2 = 2p_3.p_4$$
$$q^2 = (p_1 - p_3)^2 = -2p_1.p_3$$
$$p_1.p_4 = p_1.(p_1 + p_2 - p_3) = p_1.p_2 - p_1.p_3 = \tfrac{1}{2}s + \tfrac{1}{2}q^2$$
$$p_2.p_3 = p_1.p_4$$

Hence

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{1}{16\pi s^2} \cdot \frac{8e^4}{q^4} \left[(\tfrac{1}{2}s)(\tfrac{1}{2}s) + (\tfrac{1}{2}s + \tfrac{1}{2}q^2)(\tfrac{1}{2}s + \tfrac{1}{2}q^2)\right] \; .$$

Using $e^2 = 4\pi\alpha$, this can be written as

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{2\pi\alpha^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s}\right)^2\right]} \; .$$

Alternatively, start from Equation (53) and use

$$q^2 = (p_1 - p_3)^2 = -2p_1.p_3 = -\tfrac{1}{2}s(1 - \cos\theta)$$

to transform the cross section directly:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \left|\frac{\mathrm{d}\cos\theta}{\mathrm{d}q^2}\right| \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{2}{s} \cdot \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}$$
$$1 - \cos\theta = \frac{2Q^2}{s}$$
$$1 + \cos\theta = 2\left(1 - \frac{Q^2}{s}\right)$$

i) The Lorentz invariant matrix element for a given spin configuration is

$$M_{ijkl} = -\frac{e^2}{(p_1 - p_3)^2}\left[\overline{u}_k(p_3)\gamma^\mu u_i(p_1)\right]\left[\overline{u}_l(p_4)\gamma_\nu u_j(p_2)\right]$$

where $i, j, k, l = \uparrow$ or $\downarrow$ (or $= 1, 2$) specifies the spin state of each of the incoming and outgoing particles in the collision. For unpolarised $e^-\mu^- \to e^-\mu^-$ scattering, sum over the final spins and average over the initial e^- and μ^- spins to obtain

$$\begin{aligned}
\left\langle |M_{\mathrm{fi}}|^2 \right\rangle &= \frac{1}{2} \cdot \frac{1}{2} \cdot \sum_{i,j,k,l=1}^{2} |M_{ijkl}|^2 \\
&= \frac{1}{4}\frac{e^4}{(p_1 - p_3)^4} \sum_{i,j,k,l=1}^{2} \left[\overline{u}_k(p_3)\gamma^\mu u_i(p_1)\right]\left[\overline{u}_k(p_3)\gamma^\nu u_i(p_1)\right]^* \left[\overline{u}_l(p_4)\gamma_\mu u_j(p_2)\right]\left[\overline{u}_l(p_4)\gamma_\nu u_j(p_2)\right]^* \\
&= \frac{1}{4}\frac{e^4}{(p_1 - p_3)^4} L^{\mu\nu} W_{\mu\nu}
\end{aligned}$$

where the electron and muon tensors $L^{\mu\nu}$ and $W^{\mu\nu}$ are given by

$$L^{\mu\nu} \equiv \sum_{i,k=1}^{2} \left[\overline{u}_k(p_3)\gamma^\mu u_i(p_1)\right]\left[\overline{u}_k(p_3)\gamma^\nu u_i(p_1)\right]^*$$
$$W_{\mu\nu} \equiv \sum_{j,l=1}^{2} \left[\overline{u}_l(p_4)\gamma_\mu u_j(p_2)\right]\left[\overline{u}_l(p_4)\gamma_\nu u_j(p_2)\right]^*$$

j) Writing out the sum over spins explicitly, the electron tensor $L^{\mu\nu}$ is given by

$$\begin{aligned} L^{\mu\nu} = &\left[\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)\right]\left[\overline{u}_\downarrow(p_3)\gamma^\nu u_\downarrow(p_1)\right]^* + \left[\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1)\right]\left[\overline{u}_\uparrow(p_3)\gamma^\nu u_\downarrow(p_1)\right]^* \\ &+ \left[\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1)\right]\left[\overline{u}_\uparrow(p_3)\gamma^\nu u_\uparrow(p_1)\right]^* + \left[\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1)\right]\left[\overline{u}_\downarrow(p_3)\gamma^\nu u_\uparrow(p_1)\right]^* \ . \end{aligned}$$

Substituting the electron currents given in Equations (34)-(37), and using matrix notation, the sum is

$$\begin{aligned} \begin{pmatrix} L^{00} & L^{01} & L^{02} & L^{03} \\ L^{10} & L^{11} & L^{12} & L^{13} \\ L^{20} & L^{21} & L^{22} & L^{23} \\ L^{30} & L^{31} & L^{32} & L^{33} \end{pmatrix} &= 4\begin{pmatrix} E_1c \\ ps \\ -ips \\ pc \end{pmatrix}\begin{pmatrix} E_1c & ps & ips & pc \end{pmatrix} + 4\begin{pmatrix} ms \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} ms & 0 & 0 & 0 \end{pmatrix} \\ &+ 4\begin{pmatrix} E_1c \\ ps \\ ips \\ pc \end{pmatrix}\begin{pmatrix} E_1c & ps & -ips & pc \end{pmatrix} + 4\begin{pmatrix} ms \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} ms & 0 & 0 & 0 \end{pmatrix} \\ &= 8\begin{pmatrix} E_1^2c^2 + m^2s^2 & E_1psc & 0 & E_1pc^2 \\ E_1psc & p^2s^2 & 0 & p^2sc \\ 0 & 0 & p^2s^2 & 0 \\ E_1pc^2 & p^2sc & 0 & p^2c^2 \end{pmatrix} \end{aligned} \tag{54}$$

Now consider

$$L^{\mu\nu} = 4\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}\left(m^2 - p_1.p_3\right)\right] \ .$$

In matrix notation, this is

$$\begin{aligned} L^{\mu\nu} = &4\begin{pmatrix} p_1^0 \\ p_1^1 \\ p_1^2 \\ p_1^3 \end{pmatrix}\begin{pmatrix} p_3^0 & p_3^1 & p_3^2 & p_3^3 \end{pmatrix} + 4\begin{pmatrix} p_3^0 \\ p_3^1 \\ p_3^2 \\ p_3^3 \end{pmatrix}\begin{pmatrix} p_1^0 & p_1^1 & p_1^2 & p_1^3 \end{pmatrix} \\ &+ 4\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot (m^2 - p_1.p_3) \end{aligned}$$

With $p_1 = (E_1, 0, 0, p)$ and $p_3 = (E_1, p\sin\theta, 0, p\cos\theta)$, we have

$$m^2 - p_1.p_3 = m^2 - (E_1^2 - p^2\cos\theta) = p^2(\cos\theta - 1) = -2p^2s^2 \ ,$$

where we have used $E_1^2 = p^2 + m^2$ and $1 - \cos\theta = 2\sin^2\theta/2 = 2s^2$. Hence

$$
\begin{aligned}
L^{\mu\nu} &= 4 \begin{pmatrix} E_1 \\ 0 \\ 0 \\ p \end{pmatrix} \begin{pmatrix} E_1 & p\sin\theta & 0 & p\cos\theta \end{pmatrix} + 4 \begin{pmatrix} E_1 \\ p\sin\theta \\ 0 \\ p\cos\theta \end{pmatrix} \begin{pmatrix} E_1 & 0 & 0 & p \end{pmatrix} \\
&+ 4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot -2p^2 s^2 \\
&= 4 \begin{pmatrix} 2E_1^2 - 2p^2 s^2 & E_1 p\sin\theta & 0 & E_1 p(1+\cos\theta) \\ E_1 p\sin\theta & 2p^2 s^2 & 0 & p^2\sin\theta \\ 0 & 0 & 2p^2 s^2 & 0 \\ E_1 p(1+\cos\theta) & p^2\sin\theta & 0 & 2p^2\cos\theta + 2p^2 s^2 \end{pmatrix}
\end{aligned}
$$

Using the relations $\sin\theta = 2\sin\theta/2\cos\theta/2 = 2sc$, $1+\cos\theta = 2\cos^2\theta/2 = 2c^2$ and $E_1^2 = p^2 + m^2$, this is readily seen to be equal to Equation (54).

k) The muon tensor $W^{\mu\nu}$ can be written down immediately as

$$
W_{\mu\nu} = 4\left[p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + g_{\mu\nu}\left(M^2 - p_2.p_4\right)\right] \ .
$$

Hence

$$
\begin{aligned}
\langle |M_{\mathrm{fi}}|^2 \rangle = \frac{4e^4}{(p_1 - p_3)^4} &\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}\left(m^2 - p_1.p_3\right)\right] \\
&\times \left[p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + g_{\mu\nu}\left(M^2 - p_2.p_4\right)\right]
\end{aligned} \tag{55}
$$

giving finally

$$
\boxed{\langle |M_{\mathrm{fi}}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_2.p_4)m^2 + 2m^2M^2\right]}
$$

12. a) The elastic form factors for the proton are well described by the form

$$G(q^2) = \frac{G(0)}{(1+|q^2|/0.71)^2}$$

with q^2 in GeV^2. Show that an exponential charge distribution in the proton

$$\rho(\boldsymbol{r}) = \rho_0 e^{-\lambda r}$$

leads to this form for $G(q^2)$ (insofar as $|q^2| = |\boldsymbol{q}^2|$), and calculate λ.

b) Show that, for any spherically symmetric charge distribution, the mean square radius is given by

$$\langle r^2 \rangle = -\frac{6}{G(0)} \left[\frac{\mathrm{d}G(q^2)}{\mathrm{d}|q^2|} \right]_{q^2=0}$$

and estimate the r.m.s. charge radius of the proton.

c) The pion form factor may be determined in πe^- scattering. Use the following data to estimate the r.m.s. charge radius of the pion.

$\lvert q^2\rvert$ (GeV^2)	$G_E^2(q^2)$
0.015	0.944 ± 0.007
0.042	0.849 ± 0.009
0.074	0.777 ± 0.016
0.101	0.680 ± 0.017
0.137	0.646 ± 0.027
0.173	0.534 ± 0.030
0.203	0.529 ± 0.040
0.223	0.487 ± 0.049

SOLUTION

a) For elastic scattering, there is no energy transfer to the target particle and the 4-momentum transfer q is of the form $q^\mu = (0, \boldsymbol{q})$. Hence $|q^2| = |\boldsymbol{q}|^2$, and the form factor is given by the Fourier transform of the charge distribution:

$$G(q^2) = G(\boldsymbol{q}^2) = \int e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \rho(\boldsymbol{r}) \mathrm{d}^3 r \tag{56}$$

For a spherically symmetric charge distribution, and choosing the constant vector $\boldsymbol{q}$ to lie along the $+z$ axis:

$$\begin{aligned} G(q^2) &= \int_0^{2\pi} \int_{-1}^{+1} \int_0^{\infty} e^{iqr\cos\theta} \rho(r) r^2 \mathrm{d}r \mathrm{d}\cos\theta \mathrm{d}\phi \\ &= 2\pi \int_0^{\infty} \rho(r) r^2 \cdot \int_{-1}^{+1} e^{iqr\cos\theta} \mathrm{d}\cos\theta \cdot \mathrm{d}r \\ &= 2\pi \int_0^{\infty} \rho(r) r^2 \cdot \left[\frac{e^{iqr\cos\theta}}{iqr} \right]_{-1}^{+1} \cdot \mathrm{d}r \\ &= \frac{4\pi}{q} \int_0^{\infty} \rho(r) r \sin(qr) \mathrm{d}r \end{aligned}$$

For the exponential charge distribution $\rho(r) = \rho_0 e^{-\lambda r}$:

$$\begin{aligned} G(q^2) &= \frac{4\pi\rho_0}{q} \int_0^\infty r e^{-\lambda r} \sin(qr) \mathrm{d}r \\ &= \frac{4\pi\rho_0}{q} \frac{1}{2i} \int_0^\infty r \left[e^{-\lambda r + iqr} - e^{-\lambda r - iqr} \right] \mathrm{d}r \end{aligned}$$

Integration by parts gives

$$\int_0^\infty r e^{-\alpha r} \mathrm{d}r = \frac{1}{\alpha^2}$$

for any constant α, so that

$$G(q^2) = \frac{2\pi\rho_0}{iq} \left[\frac{1}{(\lambda - iq)^2} - \frac{1}{(\lambda + iq)^2} \right] = \frac{8\pi\lambda\rho_0}{(\lambda^2 + q^2)^2} \ .$$

Thus the form factor is of the required ("dipole") form:

$$G(q^2) = \frac{G(0)}{(1 + |q^2|/0.71)^2}$$

with $G(0) = 8\pi\rho_0/\lambda^3$ and

$$\boxed{\lambda = \sqrt{0.71\,\mathrm{GeV}^2} = 0.84\,\mathrm{GeV}}$$

Note that, from equation (56), $G(0)$ is just the total charge of the target particle:

$$G(0) = \int \rho(\boldsymbol{r}) \mathrm{d}^3 r = Q \ .$$

For an exponential charge distribution, it is easy to check that

$$G(0) = \int_0^\infty \rho_0 e^{-\lambda r} \cdot 4\pi r^2 \mathrm{d}r = 4\pi\rho_0 \int_0^\infty r^2 e^{-\lambda r} \mathrm{d}r = 4\pi\rho_0 \cdot \frac{2}{\lambda^3} \ ,$$

consistent with the expression above. It is conventional and convenient to express the charge density ρ in units of $+e$ so that, for a proton target, $G(0) = 1$. This corresponds to choosing the normalisation constant ρ_0 to be $\rho_0 = \lambda^3/8\pi$.

b) A Taylor expansion gives

$$G(q^2) = \int e^{i\boldsymbol{q}.\boldsymbol{r}} \rho(\boldsymbol{r}) \mathrm{d}^3 r = \int \left(1 + i\boldsymbol{q}.\boldsymbol{r} - \tfrac{1}{2}(\boldsymbol{q}.\boldsymbol{r})^2 + \cdots \right) \rho(\boldsymbol{r}) \mathrm{d}^3 r$$

But $G(0) = 1$ and

$$\int (\boldsymbol{q}.\boldsymbol{r}) \rho(\boldsymbol{r}) \mathrm{d}^3 r = 0 \qquad \text{since the integrand is an odd function of } \boldsymbol{r}$$

so that

$$G(q^2) = 1 - \int \tfrac{1}{2}(\boldsymbol{q}.\boldsymbol{r})^2 \rho(\boldsymbol{r}) \mathrm{d}^3 r + \cdots \ .$$

But the Taylor expansion can also be written as

$$G(q^2) = G(0) + q^2 \frac{\mathrm{d}G}{\mathrm{d}q^2}\bigg|_{q^2=0} + \cdots$$

so that

$$q^2 \frac{\mathrm{d}G}{\mathrm{d}q^2}\bigg|_{q^2=0} = -\int \tfrac{1}{2}(\boldsymbol{q}.\boldsymbol{r})^2 \rho(\boldsymbol{r}) \mathrm{d}^3 r \,.$$

For a spherically symmetric charge distribution, and choosing $\boldsymbol{q}$ to lie along the $+z$-axis, this becomes

$$\begin{aligned}
q^2 \frac{\mathrm{d}G}{\mathrm{d}q^2}\bigg|_{q^2=0} &= -\int_0^{2\pi}\int_{-1}^{+1}\int_0^{\infty} \tfrac{1}{2}\cdot q^2 r^2 \cos^2\theta \cdot \rho(r)\, r^2 \mathrm{d}r \mathrm{d}\cos\theta \mathrm{d}\phi \\
\Rightarrow \qquad \frac{\mathrm{d}G}{\mathrm{d}q^2}\bigg|_{q^2=0} &= -\int_0^{2\pi}\int_{-1}^{+1}\int_0^{\infty} \tfrac{1}{2} r^4 \cos^2\theta \rho(r)\, \mathrm{d}r \mathrm{d}\cos\theta \mathrm{d}\phi \\
&= -\tfrac{2}{3}\pi \int_0^{\infty} r^4 \rho(r) \mathrm{d}r
\end{aligned}$$

But the mean square radius of the charge distribution is, by definition,

$$\langle r^2 \rangle = \frac{1}{G(0)}\int r^2 \rho(\boldsymbol{r}) \mathrm{d}^3 r = \frac{1}{G(0)}\int_0^{\infty} r^2 \rho(r)\, 4\pi r^2 \mathrm{d}r = \frac{1}{G(0)} 4\pi \int_0^{\infty} r^4 \rho(r)\, \mathrm{d}r$$

and hence

$$\boxed{\langle r^2 \rangle = -\frac{6}{G(0)} \frac{\mathrm{d}G(q^2)}{\mathrm{d}|q^2|}\bigg|_{q^2=0}}$$

For the particular case of an exponential charge distribution, we have

$$G(q^2) = \frac{G(0)}{(1 + |q^2|/\lambda^2)^2}$$

and differentiation gives

$$\frac{\mathrm{d}G(q^2)}{\mathrm{d}q^2} = G(0) \cdot -2\left(1 + \frac{|q^2|}{\lambda^2}\right)^{-3} \cdot \frac{1}{\lambda^2} \qquad \Rightarrow \qquad \frac{\mathrm{d}G}{\mathrm{d}q^2}\bigg|_{q^2=0} = \frac{-2G(0)}{\lambda^2}$$

$$\Rightarrow \qquad \langle r^2 \rangle = -6 \cdot \frac{-2G(0)}{\lambda^2} = \frac{12}{\lambda^2} \,.$$

Hence the rms charge radius is

$$\sqrt{\langle r^2 \rangle} = \frac{\sqrt{12}}{\lambda} = \frac{\sqrt{12}}{0.84\,\text{GeV}} \times 0.197\,\text{GeV.fm} = 0.81\,\text{fm}$$

where $\hbar c = 0.197$ GeV.fm has been used to convert from natural units to SI units.

c) From a plot of $G_E(q^2)$ versus $|q^2|$, the slope at $q^2 = 0$ can be estimated to be

$$\left.\frac{\mathrm{d}G(q^2)}{\mathrm{d}|q^2|}\right|_{q^2=0} \approx -1.9\,\mathrm{GeV}^{-2} \,.$$

$$\Rightarrow \qquad \sqrt{\langle r^2 \rangle} \approx \sqrt{-6 \times -1.9} = 3.38\,\mathrm{GeV}^{-1} = 3.38\,\mathrm{GeV}^{-1} \times (0.197\,\mathrm{GeV.fm}) = 0.67\,\mathrm{fm}$$

In fact, the "dipole" form $G(q^2) = G(0)/(1 + |q^2|/\lambda^2)^2$ provides a good description of the pion form factor data. The dashed curve in the figure (drawn by eye rather than fitted) shows the function

$$G_E(q^2) = \frac{1}{1 + |q^2|/(1.05\,\mathrm{GeV}^2)} \,,$$

so that $\lambda^2 \approx 1.05\,\mathrm{GeV}^2$. The dotted line shows the tangent to this curve at $q^2 = 0$, with slope

$$\left.\frac{\mathrm{d}G}{\mathrm{d}q^2}\right|_{q^2=0} = \frac{-2G(0)}{\lambda^2} = \frac{-2}{1.05\,\mathrm{GeV}^2} = -1.90\,\mathrm{GeV}^{-2} \,.$$

DEEP-INELASTIC SCATTERING

13. The figure below shows a deep-inelastic scattering event $e^+\mathrm{p} \to e^+\mathrm{X}$ recorded by the H1 experiment at the HERA collider. The positron beam, of energy $E_1 = 27.5\,\mathrm{GeV}$, enters from the left and the proton beam, of energy $E_2 = 820\,\mathrm{GeV}$, enters from the right. The energy of the outgoing positron is measured to be $E_3 = 31\,\mathrm{GeV}$. The picture is to scale, so angles may be read off the diagram if required.

a) Show that the Bjorken scaling variable x is given by

$$x = \frac{E_3}{E_2}\left[\frac{1-\cos\theta}{2-(E_3/E_1)(1+\cos\theta)}\right]$$

where θ is the angle through which the positron has scattered.

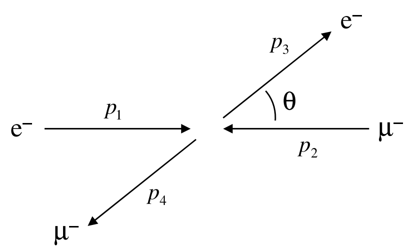
b) Estimate the values of Q^2, x and y for this event.

c) Estimate the invariant mass M_X of the final state hadronic system.

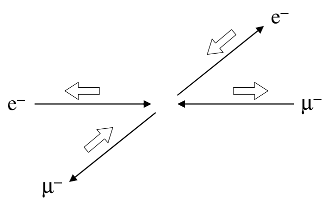
d) Draw quark level diagrams to illustrate the possible origins of this event. Using the plot overleaf of the parton distribution functions $xu_\mathrm{V}(x)$, $xd_\mathrm{V}(x)$, $x\overline{u}(x)$ and $x\overline{d}(x)$, estimate the relative probabilities of the various possible quark-level processes for the event. Note that the Q^2 in the plot overleaf need not be exactly the same as the Q^2 in this event – Bjorken scaling requires only that it be similar. So do not worry about any relatively small differences between the two Q^2 scales.

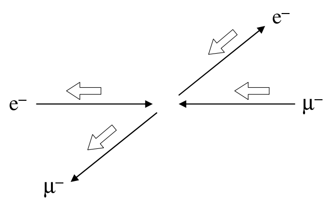
[Neglect contributions from the heavier quarks s, c, b, t.]

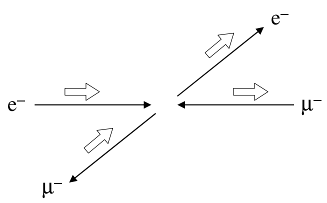
e) Estimate the relative contributions of the F_1 and F_2 terms to the deep-inelastic cross section for the x and Q^2 values corresponding to this event.

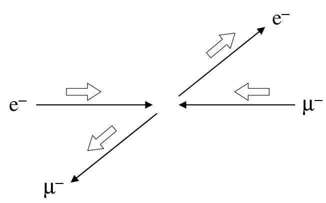


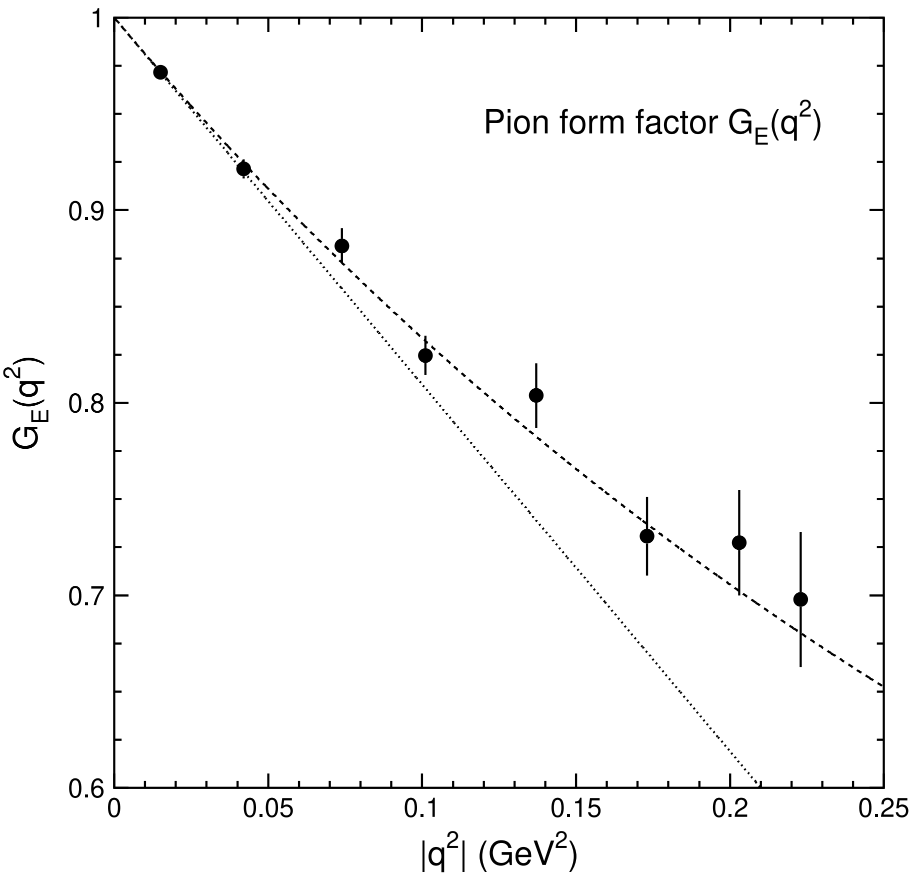
e⁻
p_1
p_3
e⁻
θ
μ⁻
p_2
p_4
μ⁻

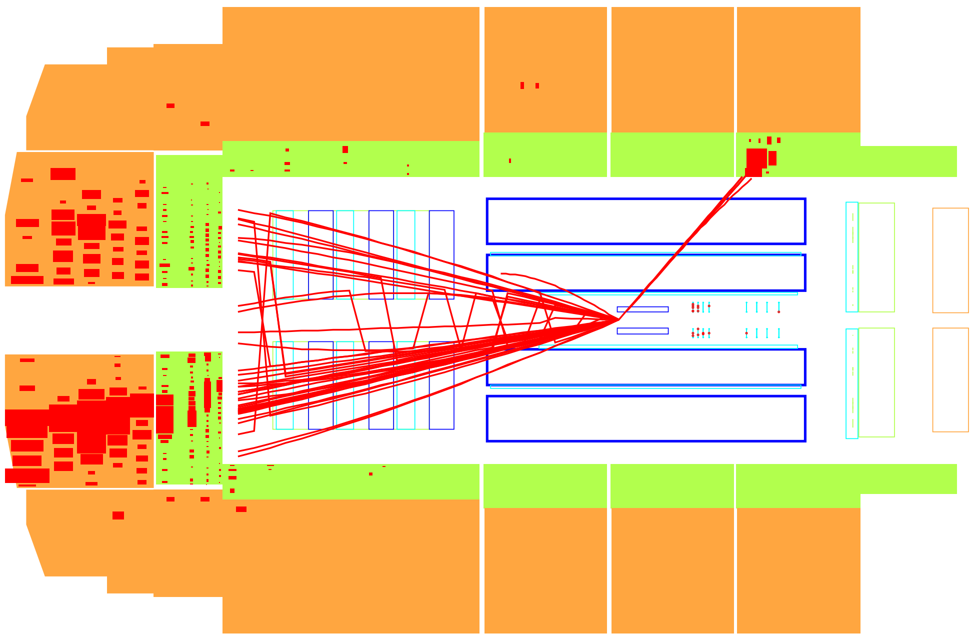

e⁻
e⁻
μ⁻
μ⁻

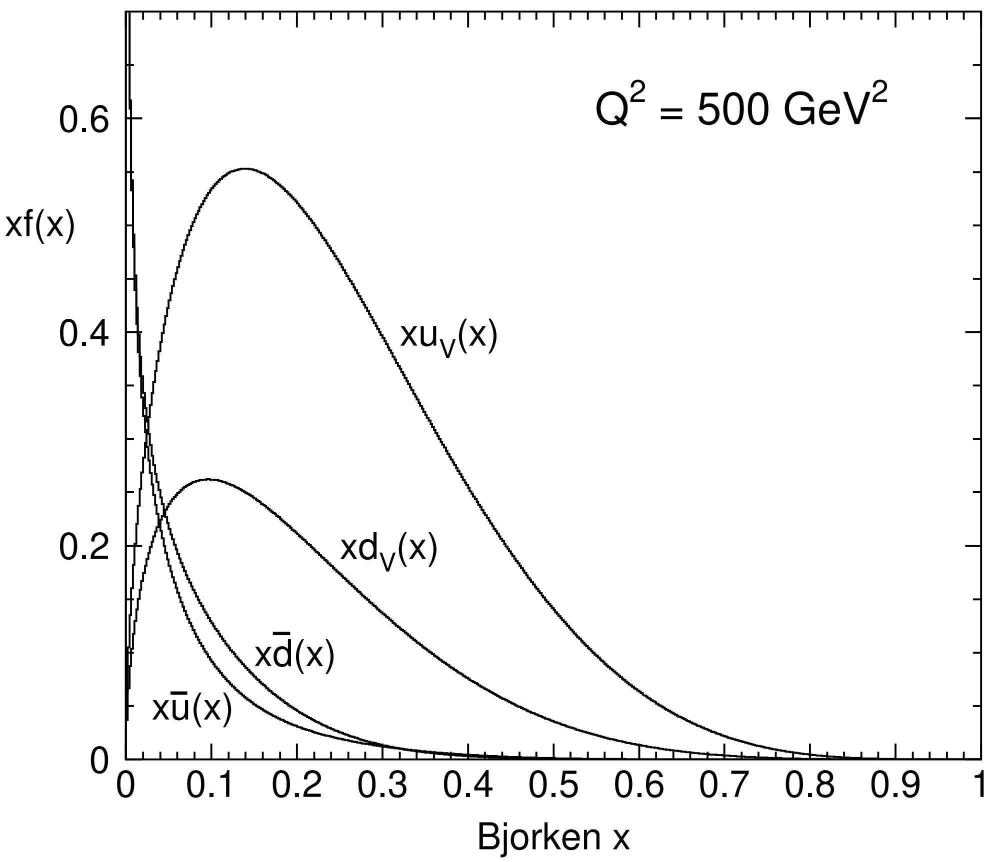

e⁻
e⁻
μ⁻
μ⁻


e⁻
e⁻
μ⁻
μ⁻


e⁻
e⁻
μ⁻
μ⁻


Pion form factor $G_E(q^2)$
$G_E(q^2)$
$|q^2|$ (GeV^2)
1
0.9
0.8
0.7
0.6
0
0.05
0.1
0.15
0.2
0.25





Q^2 = 500 GeV^2
xf(x)
0.6
0.4
0.2
0
xu$_V$(x)
xd$_V$(x)
x$\bar{d}$(x)
x$\bar{u}$(x)
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Bjorken x

SOLUTION

a) For $e^+\mathrm{p} \to e^+\mathrm{X}$ at HERA, choose four-momenta to be:

$$p_1 = (E_1, 0, 0, E_1)\,, \qquad p_2 = (E_2, 0, 0, -E_2)\,, \qquad p_3 = (E_3, E_3 \sin\theta, 0, E_3\cos\theta)\ .$$

Then

$$q^2 = -2p_1.p_3 = -2E_1E_3(1-\cos\theta)$$
$$p_2.q = p_2.p_1 - p_2.p_3 = 2E_1E_2 - E_2E_3(1+\cos\theta)$$

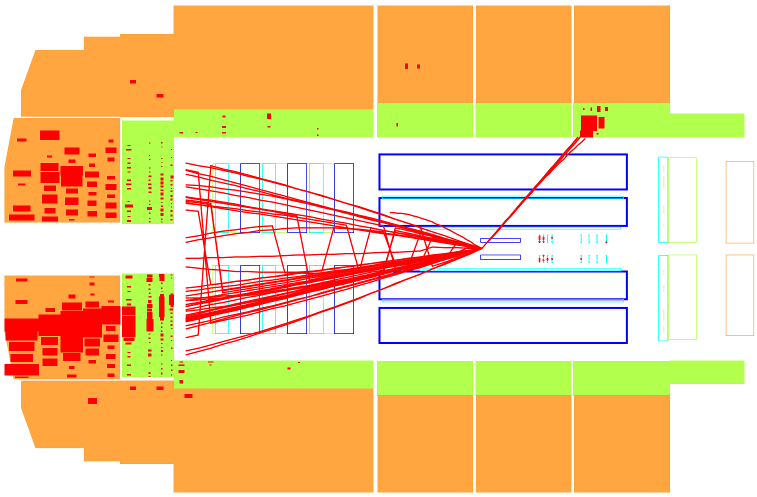
The Bjorken scaling variable x is defined as

$$x \equiv \frac{-q^2}{2p_2.q}$$

Hence

$$\boxed{x = \frac{E_3}{E_2}\left[\frac{1-\cos\theta}{2-(E_3/E_1)(1+\cos\theta)}\right]}\ .$$

b) For the particular event shown, we can estimate the e^+ scattering angle to be $\theta \approx 50^\circ$. We are given



$E_1 = 27.5\,\mathrm{GeV}$, $E_2 = 820\,\mathrm{GeV}$, $E_3 = 31\,\mathrm{GeV}$. Hence

$$x = \frac{31}{820}\left[\frac{1-\cos 50^\circ}{2-(31/27.5)(1+\cos 50^\circ)}\right] = 0.091\ .$$

$$Q^2 = 2E_1E_3(1-\cos\theta) = 2\times 27.5\times 31\times(1-\cos 50^\circ) = 609\,\mathrm{GeV}^2$$

$$y = \frac{p_2.q}{p_2.p_1} = 1 - \frac{p_2.p_3}{p_2.p_1} = 1 - \frac{E_3(1+\cos\theta)}{2E_1} = 1 - \frac{31\times(1+\cos 50^\circ)}{2\times 27.5} = 0.074$$

c) The final state hadronic system has four-momentum $p_4 = p_2 + q$. Hence its invariant mass M_X is given by

$$M_\mathrm{X}^2 = (p_2+q)^2 = M^2 + 2p_2.q - Q^2 = M^2 + \frac{Q^2}{x} - Q^2\ .$$

Hence

$$M_{\mathrm{X}} = \sqrt{(0.938)^2 + \frac{609}{0.091} - 609} = 78.0\,\mathrm{GeV}\ .$$

d) At quark level, the possible origins of the event are $e^+\mathrm{u} \to e^+\mathrm{u}$, $e^+\mathrm{d} \to e^+\mathrm{d}$, $e^+\overline{\mathrm{u}} \to e^+\overline{\mathrm{u}}$, $e^+\overline{\mathrm{d}} \to e^+\overline{\mathrm{d}}$.

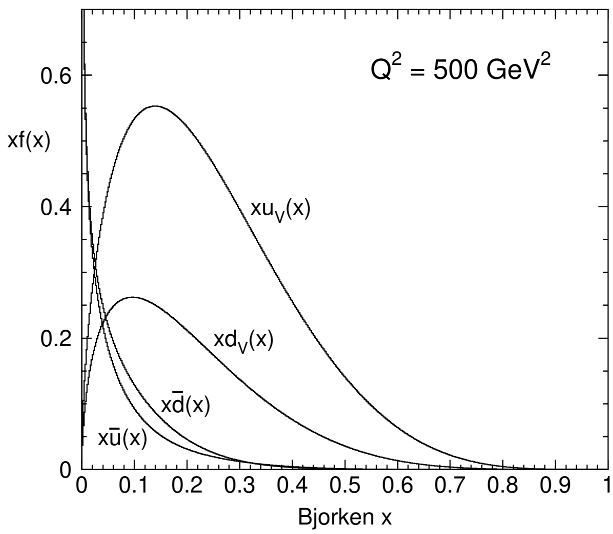
The parton model prediction for the $e^+\mathrm{p}$ cross section is

$$\frac{\mathrm{d}^2\sigma^{\mathrm{ep}}}{\mathrm{d}x\mathrm{d}Q^2} = \frac{2\pi\alpha^2}{Q^4}\left[1+(1-y)^2\right]\left[\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\overline{u}(x) + \frac{1}{9}\overline{d}(x)\right]\ .$$

Hence the relative probability for these processes is

$$\mathrm{u} : \mathrm{d} : \overline{\mathrm{u}} : \overline{\mathrm{d}} = \frac{4}{9}u(x) : \frac{1}{9}d(x) : \frac{4}{9}\overline{u}(x) : \frac{1}{9}\overline{d}(x)\ .$$

From the plot, for $x \approx 0.09$, we can estimate



$$xu_{\mathrm{V}}(x) \approx 0.52, \qquad xd_{\mathrm{V}}(x) \approx 0.26, \qquad x\overline{u}(x) \approx 0.10, \qquad x\overline{d}(x) \approx 0.14\ .$$

Remembering that

$$u(x) = u_{\mathrm{V}}(x) + u_{\mathrm{S}}(x) = u_{\mathrm{V}}(x) + \overline{u}(x)$$
$$d(x) = d_{\mathrm{V}}(x) + d_{\mathrm{S}}(x) = d_{\mathrm{V}}(x) + \overline{d}(x)\ ,$$

we obtain the estimates

$$u(x) \approx (0.52 + 0.10)/0.09 = 6.89$$
$$d(x) \approx (0.26 + 0.14)/0.09 = 4.44$$
$$\overline{u}(x) \approx 0.10/0.09 = 1.11$$
$$\overline{d}(x) \approx 0.14/0.09 = 1.56\ .$$

Including the factors of 4/9 or 1/9, the relative probabilities are therefore

$$\boxed{\mathrm{u} : \mathrm{d} : \bar{\mathrm{u}} : \bar{\mathrm{d}} \approx 3.06 : 0.494 : 0.494 : 0.173 = 0.73 : 0.12 : 0.12 : 0.04}\ .$$

e) The deep-inelastic $e^+\mathrm{p}$ cross section is

$$\frac{\mathrm{d}^2\sigma^{\mathrm{ep}}}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2^{\mathrm{ep}}}{x} + \frac{1}{2}y^2\frac{2xF_1^{\mathrm{ep}}}{x}\right]$$

Therefore, assuming the Callan-Gross relation $F_2^{\mathrm{ep}} = 2xF_1^{\mathrm{ep}}$, the F_2 and F_1 terms contribute to the cross section in the ratio

$$F_2 : F_1 = (1-y) : \tfrac{1}{2}y^2 = 1 - 0.075 : \tfrac{1}{2}(0.075)^2 \approx 1 : 0.0028\ .$$

In other words, the cross section is dominated by the F_2 term, with the F_1 term contributing only about 0.3% of events.

14. a) Show that the lab frame differential cross section $\mathrm{d}^2\sigma/\mathrm{d}E_3\mathrm{d}\Omega$ for deep-inelastic scattering is related to the Lorentz invariant differential cross section $\mathrm{d}^2\sigma/\mathrm{d}\nu\mathrm{d}Q^2$ via

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}Q^2} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2}$$

where E_1 and E_3 are the energies of the incoming and outgoing lepton, $\nu = E_1 - E_3$, and $Q^2 = -q^2 = -(p_1 - p_3)^2$. [When you do this, make sure you understand that differential cross sections transform as Jacobians, not as partial derivatives!]

Show further that

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2} = \frac{2Mx^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}$$

where M is the mass of the target nucleon and $x = Q^2/2M\nu$.

b) Show that

$$\frac{2Mx^2}{Q^2}\cdot\frac{y^2}{2} = \frac{1}{M}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}$$

and that

$$1 - y - \frac{M^2x^2y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2}\,.$$

c) Show that the Lorentz invariant cross section for deep-inelastic electromagnetic scattering,

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)\frac{F_2}{x} + \frac{y^2}{2}\frac{2xF_1}{x}\right]$$

becomes

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{F_2}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1}{M}\sin^2\frac{\theta}{2}\right]$$

in the lab frame.

d) An experiment consists of an electron beam of maximum energy $20\,\mathrm{GeV}$ and a variable angle spectrometer which can detect scattered electrons with energies greater than $2\,\mathrm{GeV}$. Find the range of values of θ over which deep-inelastic scattering events can be studied for $x = 0.2$ and $Q^2 = 2\,\mathrm{GeV}^2$.

[You may find it helpful to determine $E_1 - E_3$ (fixed), and E_1E_3 in terms of θ, and then sketch the various constraints on E_1 and E_3 on a 2D plot of E_3 against E_1.]

e) Outline a possible experimental strategy for measuring $F_1(x, Q^2)$ and $F_2(x, Q^2)$ for the above values of x and Q^2.

SOLUTION

a) Changing variables from $d\Omega = 2\pi d\cos\theta$ to

$$Q^2 = -q^2 = 2E_1E_3(1 - \cos\theta)$$

gives

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\cos\theta} = \frac{1}{2\pi}\left|\frac{\mathrm{d}Q^2}{\mathrm{d}\cos\theta}\right|\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}Q^2} = \frac{1}{2\pi}2E_1E_3\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}Q^2}$$

and hence

$$\boxed{\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}Q^2} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2}} \tag{57}$$

To change variables from ν to x, use

$$x = \frac{Q^2}{2M\nu} \qquad \Rightarrow \qquad \nu = \frac{Q^2}{2Mx}$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \left|\frac{\mathrm{d}\nu}{\mathrm{d}x}\right| \frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2}$$

which gives directly

$$\boxed{\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2} = \frac{2Mx^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}} \tag{58}$$

b) Since

$$Q^2 = 4E_1E_3\sin^2\theta/2$$

and

$$y = \frac{\nu}{E_1}$$

we have

$$\frac{E_3}{E_1}\sin^2\theta/2 = \frac{Q^2}{4E_1^2} = \frac{Q^2y^2}{4\nu^2}\ .$$

Using $\nu = Q^2/2Mx$, we then obtain

$$\boxed{\frac{2Mx^2}{Q^2}\cdot\tfrac{1}{2}y^2 = \frac{1}{M}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}} \tag{59}$$

Hence

$$\boxed{1 - y - \frac{M^2x^2y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2}}\ . \tag{60}$$

c) The Lorentz invariant cross section for deep-inelastic electromagnetic scattering is

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)\frac{F_2}{x} + \frac{y^2}{2}\frac{2xF_1}{x}\right]$$

Combining Equations (57) and (58), we have

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2} = \frac{E_1E_3}{\pi}\frac{2Mx^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}$$

The F_2 term contains the combination of factors

$$\frac{2Mx^2}{Q^2}\left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)\frac{1}{x} = \frac{2Mx}{Q^2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2} = \frac{1}{\nu}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}\ ,$$

where we have used Equation (60). Using Equation (59) for the F_1 term, we then obtain

$$\frac{d^2\sigma}{dE_3 d\Omega} = \frac{E_1 E_3}{\pi}\frac{4\pi\alpha^2}{Q^4}\left[\left(\frac{1}{\nu}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}\right)F_2 + \left(\frac{1}{M}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}\right)2F_1\right]$$

Since $Q^2 = 4E_1 E_3 \sin^2\theta/2$, we finally obtain

$$\boxed{\frac{d^2\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{F_2}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1}{M}\sin^2\frac{\theta}{2}\right]}$$

d) Given $x = 0.2$ and $Q^2 = 2\,\mathrm{GeV}^2$, the electron energies E_1 and E_3 are fixed via

$$E_1 - E_3 = \frac{Q^2}{2Mx} = \frac{2\,\mathrm{GeV}^2}{2\times(0.938\,\mathrm{GeV})\times 0.2} = 5.33\,\mathrm{GeV} \tag{61}$$

and

$$E_1 E_3 = \frac{Q^2}{4\sin^2\theta/2}\ . \tag{62}$$

The experimental constraints $E_1 < 20\,\mathrm{GeV}$ and $E_3 > 2\,\mathrm{GeV}$ then lead to constraints on the angle θ. To obtain these, it may help to think in terms of a graphical solution of Equations (61) and (62) on a plot of E_3 versus E_1. Equation (61) corresponds to a straight line running at 45° while Equation (62) gives an infinite set of hyperbolae, each hyperbola corresponding to a different possible value of θ.

The minimum possible value of θ corresponds to taking the maximum possible beam energy $E_1 = 20\,\mathrm{GeV}$:

$$\sin^2\theta/2 = \frac{Q^2}{4E_1E_3} = \frac{2}{4\times 20\times(20-5.33)} = 1.70\times 10^{-3}$$

which gives

$$\theta_{\text{min}} = 4.73^\circ\ .$$

The maximum possible value of θ is determined by the minimum detectable scattered electron energy of $E_3 = 2\,\mathrm{GeV}$:

$$\sin^2\theta/2 = \frac{Q^2}{4E_1E_3} = \frac{2}{4\times(2+5.33)\times 2} = 0.034$$

which gives

$$\theta_{\text{max}} = 21.3^\circ\ .$$

Strategy: choose several values of θ between about 5° and 20°, measure reduced cross section at each value of θ and plot versus $\tan^2\theta/2$. Should give a straight line (note ν is fixed) with slope $2F_1/M$ and intercept F_2/ν:

$$\frac{d^2\sigma}{dE_3 d\Omega}\Bigg/\frac{\alpha^2\cos^2\theta/2}{4E_1^2\sin^4\theta/2} = \left[\frac{F_2}{\nu} + \frac{2F_1}{M}\tan^2\frac{\theta}{2}\right]$$

Each θ setting requires a different beam energy given by solving the quadratic equation

$$E_1(E_1 - 5.33) = \frac{Q^2}{4\sin^2\theta/2}$$

This gives

$$2E_1 = 5.33 + \sqrt{(5.33)^2 + \frac{Q^2}{\sin^2 \theta/2}}$$

Gives $E_1 = 19.1\,\mathrm{GeV}$ for $\theta = 5^\circ$ and $E_1 = 7.5\,\mathrm{GeV}$ for $\theta = 20^\circ$.

Note that $y = (E_1 - E_3)/E_1$ varies between 0.28 and 0.71 so get healthy contribution from F_1.

HADRONS AND QCD

15. [This is lifted from the 2016 Tripos Paper]

Suppose there exists a 'Bogus' universe in which the laws of physics are the same as in ours, except in one respect: quantum chromodynamics in the 'Bogus' universe is based on an $SU(2)$ colour symmetry having only two colours ('red' and 'green') rather than the three colour $SU(3)$ symmetry of our own.

(a) Determine which 'Bogus mesons' and 'Bogus baryons' (or their nearest equivalents) could exist by constructing any important colour, flavour and spin wave-functions. Categorise the expected 'Bogus' hadrons by type (meson/baryon), spin, and the multiplets they inhabit. Compare 'Bogus' hadron structure to that in our own universe, highlighting the main similarities and differences. [*Above you need only consider light quarks types: u, d and s.*]

(b) The change from $SU(3)$ colour to $SU(2)$ colour could affect more than the basic hadron structure considered above. It could have consequences in other areas of particle physics and even further afield. Discuss any such expected differences between the 'Bogus' universe and our own.

[Aside: if you want practice of multiplying $SU(3)$ multiplets together, consider looking at part (h) of Question 2 in the January 2025 past Tripos paper for this course. A worked answer to it is also provided on the course website.]

SOLUTION

(a) **Bogus mesons**

A key fact is that the $SU(2)$ colour theory will require a

$$\frac{1}{\sqrt{2}}\left(r\bar{r} + g\bar{g}\right)$$

equivalent of the $SU(3)$

$$\frac{1}{\sqrt{3}}\left(r\bar{r} + g\bar{g} + b\bar{b}\right)$$

colour-anticolour singlet thereby permitting mesons to exist for most of the same reasons they can in the real universe. A poor answer would omit this altogether. A medium answer would mention it without proof merely appealing to its plausibility and connection to colour confinement hypothesis. A good answer might demonstrate that this really is a singlet by consideration of the action of properly defined ladder operators on it, etc. It might even go on to question whether the colour confinement hypothesis would still be important in the bogus universe. Answers will hopefully consider the potential spin wavefunctions of the 'real' mesons, noting those in the bogus universe could be identical.

A good answer would hopefully re-capitulate the flavour part of the notes (that coveres the meson nonets) noting that, as in 'real'-space, the bogus universe allows any spin combinations with any flavour combinations since the lack of any identical fermions in the mesons leads no need to have antisymmetry of the overall wavefunction.

The spetra of excited mesonic states would presumably differ in the real universe from that in the bogus, as the different colout potential would space excitations differently.

Bogus baryons Here the key fact is that the three colour singlet of $SU(3)$

$$\frac{1}{\sqrt{6}}(rbg - rbg + gbr - brb + brg - bgr)$$

is replaced in the bogus universe by the

$$\frac{1}{\sqrt{2}}(rg - gr)$$

two-colour singlet of $SU(2)$, meaning that the colour confinement hypothesis (if still needed!) would permit two-quark baryons and forbid three-quark baryons. Again, a poor answer would neglect to mention this at all. A medium answer would just state it. A good answer would argue the case clearly.

The disappearance of one colour would not change the approximate (u,d)-isopin $SU(2)$ flavour or (u, d, s)-isospin $SU(3)$ flavour symmetries available to nature – but the need for only two quark states would require us now to consider only the $3 \otimes 3 = 6 \oplus \bar{3}$ not the $3 \otimes 3 \otimes 3 = 10 + 8 + 8 + 1$ version of before. A good answer would work out that the 6 is symmetric in the two quark flavours, while the $\bar{3}$ is antisymmetric.

What flavour/spin/colour combinations would be allowed? Assuming the lowest angular momentum states would have $L = 0$ making them even parity, and given that the colour singlet is already antisymmetric, we'd need flavour $\times$ spin to be symmetric. We would need to combine the 6 with a symmetric $S = 1$ spin-triplet, or the antisymmetric $\bar{3}$ with an antisymmetric $S = 0$ spin-singlet.

The bogus (u, d, s)-baryons would therefore be expected to come in an $S = 1$ hextet and an $S = 0$ triplet of di-quark states.

Note that the charges of these bogus baryons would be non-integer: the lightest three (uu, ud, dd) having charges $\frac{4}{3}$, $\frac{1}{3}$ and $-\frac{2}{3}$ respectively.

(b) Here is a non-exhaustive list of potential answers:

- Mesons play very little role in the day-to-day life of organisms on present-day earth, as they can usually decay (via q qbar annihilation) to other things, and so life on earth is based on the more stable bosons. Changes to the mesonic structure the measons might be expected to be less important in the current universe, though presumably they would make considerable differences to some parts of the big-bang/cosmological models around the transition from radiation to matter domination.

- The change in baryon structure, however (removal of the proton!!) would have very profound implications for chemistry. With the lightest baryons now being fractionally charged, atoms as we know them would cease to exist. Indeed the whole periodic table is based on assembling elements from two nucleon types (proton and neutron) and would have to change to a system based on three nucleons ... so elements would be in trouble too.

- The bogus universe would only have $2^3 - 1 = 3$ gluons, not the $3^3 - 1 = 9$ gluons in the real universe.

- The rate of running of α_s will change due to fewer gluons/colours.
- The linear term in effective colour potential between two quarks would probably be different (less?) as a result of fewer quarks, possibly making jets less jetty.
- The possiblilty of $q\bar{q}q\bar{q}$ states would be present in both Bogus and Real universes. But whereas the real universe forbids $qqqq$ and allows $qqqq\bar{q}$ states, the Bogus would allow $qqqq$ and forbid $qqqq\bar{q}$ due to the change in which contains a colour singleton.
- Colour factors would change leading to, say, some hadron-hadron cross sections to get enhanced or reduced.
- A good answer that has not already considered this point in an earlier part (a) or (b) might andvance some ideas on why/whether the colour confinement hypothesis would hold for $SU(2)$-based colour.

WEAK INTERACTIONS

16. Following on from Question 10, show that, for a free particle spinor ψ:

$$\overline{\psi_{\rm L}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\psi_{\rm R} = \overline{\psi_{\rm R}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\psi_{\rm L} = \overline{\psi_{\rm R}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\psi_{\rm R} = 0$$

$$\overline{\psi_{\rm L}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\psi_{\rm L} = \overline{\psi}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\psi$$

where $\psi_{\rm L} \equiv \frac{1}{2}(1-\gamma^5)\psi$ and $\psi_{\rm R} \equiv \frac{1}{2}(1+\gamma^5)\psi$. Explain the relevance of these results to the weak interactions. What are the equivalent results for currents of the form $\overline{\psi}\gamma^\mu \frac{1}{2}(1+\gamma^5)\psi$?

SOLUTION

From Question 7, we have $(\gamma^5)^2 = 1$ and hence

$$(1-\gamma^5)(1+\gamma^5) = 0 \qquad (1-\gamma^5)^2 = 2(1-\gamma^5) \qquad (1+\gamma^5)^2 = 2(1+\gamma^5)$$

Also from question 6, $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$, and so

$$(1+\gamma^5)\gamma^\mu = \gamma^\mu(1-\gamma^5) \qquad (1-\gamma^5)\gamma^\mu = \gamma^\mu(1+\gamma^5)$$

Also from question 7:

$$\overline{\psi_{\rm L}} = \overline{\psi}\tfrac{1}{2}(1+\gamma^5) \qquad \overline{\psi_{\rm R}} = \overline{\psi}\tfrac{1}{2}(1-\gamma^5)$$

Hence

$$\overline{\psi_{\rm L}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\psi_{\rm R} = \overline{\psi}\tfrac{1}{2}(1+\gamma^5)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\tfrac{1}{2}(1+\gamma^5)\psi = 0$$

$$\overline{\psi_{\rm R}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\psi_{\rm L} = \overline{\psi}\tfrac{1}{2}(1-\gamma^5)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\tfrac{1}{2}(1-\gamma^5)\psi = \overline{\psi}\gamma^\mu \tfrac{1}{2}(1+\gamma^5)\tfrac{1}{2}(1-\gamma^5)\psi = 0$$

$$\overline{\psi_{\rm R}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\psi_{\rm R} = \overline{\psi}\tfrac{1}{2}(1-\gamma^5)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\tfrac{1}{2}(1+\gamma^5)\psi = 0$$

$$\begin{aligned}\overline{\psi_{\rm L}}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\psi_{\rm L} &= \overline{\psi}\tfrac{1}{2}(1+\gamma^5)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\tfrac{1}{2}(1-\gamma^5)\psi \\ &= \overline{\psi}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\tfrac{1}{2}(1-\gamma^5)\psi = \overline{\psi}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\psi\end{aligned}$$

The $V - A$ interaction of a charged $W^{\pm}$ boson with a quark or lepton gives rise to currents of the form $\overline{\psi}\gamma^\mu \frac{1}{2}(1-\gamma^5)\psi$ in expressions for the matrix element $M_{\rm fi}$. The results above show that only the left-handed chiral component $\psi_{\rm L} \equiv \frac{1}{2}(1-\gamma^5)\psi$ of all the particles or antiparticles involved produce non-zero matrix elements in charged current weak interactions.

For currents of the form $\overline{\psi}\gamma^\mu \frac{1}{2}(1+\gamma^5)\psi$, the corresponding results are:

$$\overline{\psi_{\rm R}}\gamma^\mu \tfrac{1}{2}(1+\gamma^5)\psi_{\rm L} = \overline{\psi_{\rm L}}\gamma^\mu \tfrac{1}{2}(1+\gamma^5)\psi_{\rm R} = \overline{\psi_{\rm L}}\gamma^\mu \tfrac{1}{2}(1+\gamma^5)\psi_{\rm L} = 0$$
$$\overline{\psi_{\rm R}}\gamma^\mu \tfrac{1}{2}(1+\gamma^5)\psi_{\rm R} = \overline{\psi}\gamma^\mu \tfrac{1}{2}(1+\gamma^5)\psi$$

Thus only the *right-handed* chiral component $\psi_{\rm R} \equiv \frac{1}{2}(1+\gamma^5)\psi$ now gives non-zero currents.

Interactions of the Z^0 boson with quarks or leptons give rise to currents of *both* of the above forms: $\overline{\psi}\gamma^\mu \frac{1}{2}(1-\gamma^5)\psi$ *and* $\overline{\psi}\gamma^\mu \frac{1}{2}(1+\gamma^5)\psi$, with relative strengths determined by the left- and right-handed coupling constants c_L and c_R. The former involve purely the *left-handed* chiral components, the latter purely the *right-handed* chiral components of the particles or antiparticles involved.

17. a) In Question 6, the decay rate for $\pi^- \to e^-\overline{\nu}_e$ was found to be 1.28×10^{-4} times that for $\pi^- \to \mu^-\overline{\nu}_\mu$, whereas, on the basis of phase space alone, one would expect a higher decay rate to electrons. Explain why the weak interaction gives such a small decay rate to electrons.

b) The Lorentz invariant matrix element for $\pi^- \to \mu^-\overline{\nu}_\mu$ decay is

$$M_{\rm fi} = \frac{g_{\rm W}^2}{4m_{\rm W}^2} g_{\mu\nu} f_\pi p_1^\mu \overline{u}(p_3)\gamma^\nu \tfrac{1}{2}(1-\gamma^5)v(p_4)$$

where p_1, p_3 and p_4 are the 4-momenta of the π^-, μ^- and $\overline{\nu}_\mu$, respectively, and f_π is a constant which must be determined experimentally. Verify that this matrix element follows from the Feynman rules, with the quark current $\overline{u}\gamma^\mu(1-\gamma^5)v$ taken to be of the form $-f_\pi p_1^\mu$.

[The free particle spinors u, v cannot be used for quarks and antiquarks in a hadronic bound state; a quark current of the form given can be shown to be the most general possibility.]

c) Show that (as in Question 6) the Lorentz-invariant matrix element squared is

$$|M_{\rm fi}|^2 = 2G_{\rm F}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2) \ .$$

[Use the spinors u_1, u_2, v_1, v_2 for this calculation rather than the spinors $u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow$. Work in the π^- rest frame, and choose the 4-momenta of the μ^- and $\overline{\nu}_\mu$ to be $p_3 = (E, 0, 0, p)$ and $p_4 = (p, 0, 0, -p)$, with $E = \sqrt{p^2 + m_\mu^2}$.]

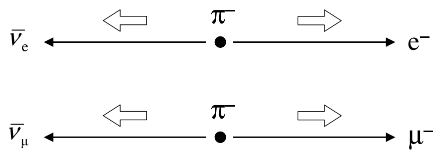
d) Show that the square of the *non-invariant* matrix element $T_{\rm fi}$ is proportional to $1 - \beta$:

$$|T_{\rm fi}|^2 = \frac{G_{\rm F}^2}{2} f_\pi^2 m_\pi (1-\beta)$$

where β is the velocity of the μ^-.

SOLUTION

a) The antineutrino from the $\pi^- \to e^-\overline{\nu}_e$ or $\pi^- \to \mu^-\overline{\nu}_\mu$ decay always has positive helicity. Therefore, to conserve angular momentum (the π^- has spin zero), the e^- or μ^- must also have positive helicity: The W boson couples only to the left-handed chiral component $\psi_L = \frac{1}{2}(1-\gamma^5)\psi$. In the relativistic



limit, this implies that the W boson couples only to negative helicity particles or positive helicity antiparticles. Since $m_e \ll m_\pi$, the e^- is highly relativistic, $\beta \approx 1$. In this limit, a positive helicity e^- cannot couple to the W boson, and the decay $\pi^- \to e^-\overline{\nu}_e$ is therefore completely suppressed.

The μ^- is much heavier than the electron ($m_\mu/m_\pi \approx 0.76$) and so is produced with a value of β appreciably less than 1 ($\beta \approx 0.73$: see below). Since the μ^- is not ultra-relativistic, its left-handed chiral component contains an appreciable mixture of both the left-handed and right-handed helicity eigenstates. Therefore, there is an appreciable probability that the μ^- can be emitted with positive helicity, as required in the $\pi^- \to \mu^-\overline{\nu}_\mu$ decay.

b) From the Feynman rules:

$$-iM_{\rm fi} = -i\frac{g_W}{\sqrt{2}} \cdot \tfrac{1}{2} i f_\pi p_1^\mu \cdot \frac{-ig_{\mu\nu}}{q^2 - m_W^2} \cdot \overline{u}(p_3) \cdot -i\frac{g_W}{\sqrt{2}}\gamma^\nu \tfrac{1}{2}(1-\gamma^5) \cdot v(p_4) \tag{63}$$

where the factor $\overline{u}\gamma^\mu(1-\gamma^5)v$ that would have appeared for free quarks and antiquarks has been replaced by $if_\pi p_1^\mu$. For pion decay, we have $q^2 = m_\pi^2 \ll m_W^2$, giving

$$\boxed{M_{\rm fi} = \frac{g_W^2}{4m_W^2} g_{\mu\nu} f_\pi p_1^\mu \overline{u}(p_3)\gamma^\nu \tfrac{1}{2}(1-\gamma^5)v(p_4)} \,.$$

c) The μ^- 4-momentum is $p_3 = (E, 0, 0, p)$ with $E^2 = p^2 + m_\mu^2$. The possible μ^- spinors are:

$$u_1(p_3) = \sqrt{E + m_\mu} \begin{pmatrix} 1 \\ 0 \\ p/(E + m_\mu) \\ 0 \end{pmatrix}, \qquad u_2(p_3) = \sqrt{E + m_\mu} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -p/(E + m_\mu) \end{pmatrix}$$

with corresponding adjoint spinors

$$\bar{u}_1(p_3) = \sqrt{E + m_\mu}\ (1, 0, -p/(E + m_\mu), 0)\,, \qquad \bar{u}_2(p_3) = \sqrt{E + m_\mu}\ (0, 1, 0, p/(E + m_\mu))$$

The $\overline{\nu}_\mu$ 4-momentum is $p_4 = (p, 0, 0, -p)$, and the $\overline{\nu}_\mu$ spinors are therefore

$$v_1(p_4) = \sqrt{p} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad v_2(p_4) = \sqrt{p} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

In the π^- rest frame we have $p_1 = (m_\pi, 0, 0, 0)$. Hence only the $\mu = \nu = 0$ term in the sum in the expression for $M_{\rm fi}$ is non-zero:

$$M_{\rm fi} = \frac{g_{\rm W}^2}{4m_{\rm W}^2} f_\pi p_1^0 \overline{u}(p_3)\gamma^0 \tfrac{1}{2}(1 - \gamma^5) v(p_4)$$

$$= \frac{g_{\rm W}^2}{4m_{\rm W}^2} f_\pi m_\pi \overline{u}(p_3)\gamma^0 \tfrac{1}{2}(1 - \gamma^5) v(p_4)$$

But

$$\tfrac{1}{2}(1 - \gamma^5) v_1(p_4) = \tfrac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \sqrt{p} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\tfrac{1}{2}(1 - \gamma^5) v_2(p_4) = \tfrac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \sqrt{p} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \tfrac{1}{2}\sqrt{p} \begin{pmatrix} -2 \\ 0 \\ 2 \\ 0 \end{pmatrix} = v_2(p_4)$$

Thus only the spinor $v_2(p_4)$ gives a non-zero contribution. This is as expected; for an antiparticle travelling in the $-z$ direction, v_2 is the positive helicity eigenstate, and antineutrinos always have positive helicity.

Premultiplying by γ^0 gives

$$\gamma^0 \tfrac{1}{2}(1 - \gamma^5) v_2(p_4) = \gamma^0 v_2(p_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sqrt{p} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{p} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

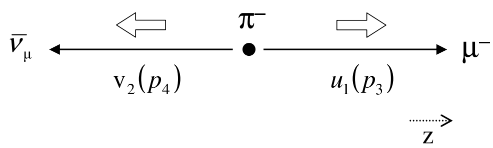
Premultiplying in turn by $\overline{u}_1(p_3)$ and $\overline{u}_2(p_3)$ gives

$$\overline{u}_1(p_3)\gamma^0\tfrac{1}{2}(1-\gamma^5)v_2(p_4) = \sqrt{E+m_\mu}\sqrt{p}\left(-1+\frac{p}{E+m_\mu}\right) \tag{64}$$

$$\overline{u}_2(p_3)\gamma^0\tfrac{1}{2}(1-\gamma^5)v_2(p_4) = 0\ .$$

Thus only the spinor $u_1(p_3)$ gives a non-zero contribution. This was anticipated in part (a) above; the μ^- is expected to have positive helicity, and for a particle travelling in the $+z$ direction the spinor u_1 is the positive helicity eigenstate. In summary, the only non-zero combination of spinors is as shown in the figure overleaf, and, from equations (63) and (64), the matrix element for this case is

$$M_{\rm fi} = \frac{g_{\rm W}^2}{4m_{\rm W}^2} f_\pi m_\pi \sqrt{E+m_\mu}\sqrt{p}\left(-1+\frac{p}{E+m_\mu}\right)\ .$$



To find p (the centre of mass momentum), use energy conservation $m_\pi = E + p$:

$$m_\pi^2 = (E+p)^2 = E^2 + p^2 + 2Ep = 2p^2 + m_\mu^2 + 2p\sqrt{p^2+m_\mu^2}$$

$$\Rightarrow \qquad 4p^2(p^2+m_\mu^2) = \left(m_\pi^2 - m_\mu^2 - 2p^2\right)^2$$

$$\Rightarrow \qquad 4p^2m_\mu^2 = \left(m_\pi^2 - m_\mu^2\right)^2 - 4p^2\left(m_\pi^2 - m_\mu^2\right)$$

$$\Rightarrow \qquad p = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

(or use the result derived in question 3). Hence

$$E + m_\mu = m_\pi - p + m_\mu = m_\pi - \frac{m_\pi^2 - m_\mu^2}{2m_\pi} + m_\mu = \frac{(m_\pi + m_\mu)^2}{2m_\pi}$$

$$\Rightarrow \qquad -1 + \frac{p}{E + m_\mu} = -1 + \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \cdot \frac{2m_\pi}{(m_\pi + m_\mu)^2} = -1 + \frac{m_\pi - m_\mu}{m_\pi + m_\mu} = \frac{-2m_\mu}{m_\pi + m_\mu} \, .$$

Hence

$$\begin{aligned} M_{\rm fi} &= \frac{g_{\rm W}^2}{4m_{\rm W}^2} f_\pi m_\pi \sqrt{E + m_\mu}\sqrt{p}\left(-1 + \frac{p}{E + m_\mu}\right) \\ &= \frac{g_{\rm W}^2}{4m_{\rm W}^2} f_\pi m_\pi \cdot \frac{m_\pi + m_\mu}{\sqrt{2m_\pi}} \cdot \sqrt{\frac{m_\pi^2 - m_\mu^2}{2m_\pi}} \cdot \frac{-2m_\mu}{m_\pi + m_\mu} \\ &= -\left(\frac{g_{\rm W}}{2m_{\rm W}}\right)^2 f_\pi m_\mu \sqrt{m_\pi^2 - m_\mu^2} \end{aligned}$$

Using the relation

$$\frac{G_{\rm F}}{\sqrt{2}} = \frac{g_{\rm W}^2}{8m_{\rm W}^2} \, ,$$

we finally obtain

$$\boxed{\left\langle |M_{\rm fi}|^2 \right\rangle = 2G_{\rm F}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)}$$

d) The non-invariant matrix element squared is obtained by extracting a factor of $2E$ for every initial state and final state particle:

$$|M_{\rm fi}|^2 = 2E_\pi \cdot 2E_\mu \cdot 2E_\nu \cdot |T_{\rm fi}|^2 = 2m_\pi \cdot 2E \cdot 2p \cdot |T_{\rm fi}|^2$$

But

$$E = m_\pi - p = m_\pi - \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}$$

Hence

$$\begin{aligned} |T_{\rm fi}|^2 = \frac{1}{8m_\pi} \cdot \frac{1}{E} \cdot \frac{1}{p} \cdot |M_{\rm fi}|^2 &= \frac{1}{8m_\pi} \cdot \frac{1}{E} \cdot \frac{1}{p} \cdot 2G_{\rm F}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2) \\ &= \frac{1}{8m_\pi} \cdot \frac{2m_\pi}{m_\pi^2 + m_\mu^2} \cdot \frac{2m_\pi}{m_\pi^2 - m_\mu^2} \cdot 2G_{\rm F}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2) \\ &= \frac{m_\pi}{m_\pi^2 + m_\mu^2} \cdot G_{\rm F}^2 f_\pi^2 m_\mu^2 \end{aligned}$$

But

$$1 - \beta = 1 - \frac{p}{E} = 1 - \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} = \frac{2m_\mu^2}{m_\pi^2 + m_\mu^2}$$

giving finally

$$\boxed{|T_{\rm fi}|^2 = \frac{G_{\rm F}^2}{2} f_\pi^2 m_\pi (1 - \beta)}$$

DEEP INELASTIC SCATTERING

18. Find the maximum possible value of Q^2 in deep-inelastic neutrino scattering for a neutrino beam energy of 400 GeV, and compare with m_{W}^2.

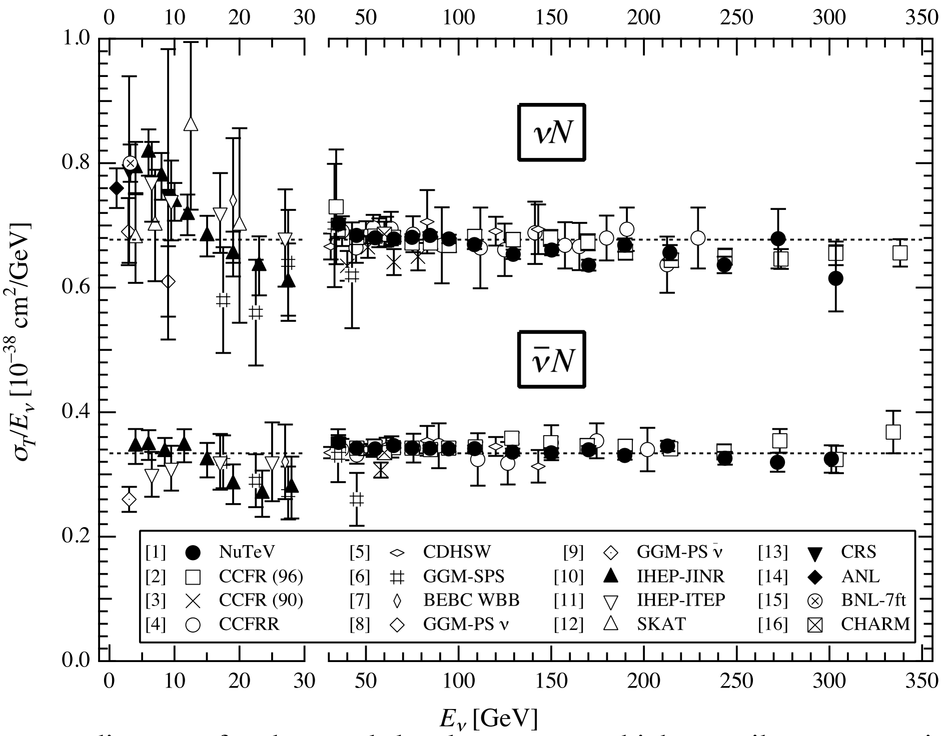
SOLUTION

In terms of the lab frame neutrino beam energy E_1, we have $Q^2 = 2ME_1xy$. Since $0 < x < 1$ and $0 < y < 1$, the maximum value of Q^2 is

$$(Q^2)_{max} = 2ME_1 = 2 \times (0.938\,\mathrm{GeV}) \times (400\,\mathrm{GeV}) = 750.4\,\mathrm{GeV}^2$$

This compares with $m_{\mathrm{W}}^2 = (80.4\,\mathrm{GeV})^2 = 6460\,\mathrm{GeV}^2$, justifying the approximation $q^2 \ll m_{\mathrm{W}}^2$ for current neutrino experiments.

19. The figure below shows the measured total cross sections $\sigma(\nu_\mu + \mathrm{N} \to \mu^- + \text{hadrons})/E_\nu$ and $\sigma(\overline{\nu}_\mu + \mathrm{N} \to \mu^- + \text{hadrons})/E_{\overline{\nu}}$ for charged-current neutrino and antineutrino scattering, averaged over proton and neutron targets.



a) Draw Feynman diagrams for the quark-level processes which contribute to neutrino-nucleon and antineutrino-nucleon scattering. (Neglect the s, c, b and t quark flavours).

b) Show that the parton model predicts total cross sections of the form

$$\sigma^{\nu N} \equiv \tfrac{1}{2}\left(\sigma^{\nu \mathrm{p}} + \sigma^{\nu \mathrm{n}}\right) = \frac{G_{\mathrm{F}}^2 s}{2\pi}\left[f_{\mathrm{q}} + \tfrac{1}{3} f_{\overline{\mathrm{q}}}\right]$$

$$\sigma^{\overline{\nu} N} \equiv \tfrac{1}{2}\left(\sigma^{\overline{\nu} \mathrm{p}} + \sigma^{\overline{\nu} \mathrm{n}}\right) = \frac{G_{\mathrm{F}}^2 s}{2\pi}\left[\tfrac{1}{3} f_{\mathrm{q}} + f_{\overline{\mathrm{q}}}\right]$$

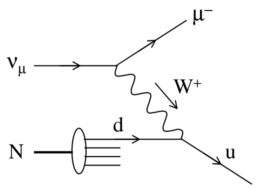
where s is the neutrino-nucleon centre of mass energy squared, and $f_{\rm q} = f_{\rm u} + f_{\rm d}$ and $f_{\overline{\rm q}} = f_{\overline{\rm u}} + f_{\overline{\rm d}}$ are the average momentum fractions carried by $\rm u$ and $\rm d$ quarks and antiquarks.

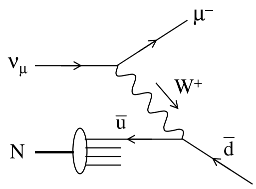
c) Estimate the average fractions of the nucleon momentum carried by quarks, antiquarks and gluons.

[Take $G_{\rm F} = 1.166 \times 10^{-5}\,{\rm GeV}^{-2}$.]

SOLUTION

a) For neutrino-nucleon scattering, the possible quark-level processes are $\nu_\mu + {\rm d} \to \mu^- + {\rm u}$ and $\nu_\mu + \overline{\rm u} \to \mu^- + \overline{\rm d}$:





In the second case, the initial $\overline{\rm u}$ in the nucleon must belong to the quark-antiquark sea, having been produced via ${\rm g} \to {\rm u}\overline{\rm u}$ for example. In the first case, the initial $\rm d$ could be either a valence quark or a sea quark.

For antineutrino-nucleon scattering, the possible quark level processes are $\overline{\nu}_\mu + {\rm u} \to \mu^+ + {\rm d}$ and $\overline{\nu}_\mu + \overline{\rm d} \to \mu^+ + \overline{\rm u}$:

b) For $\nu{\rm p}$ scattering, the differential cross section in the quark-parton model was derived in the lectures:

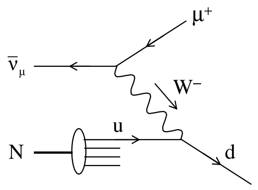
$$\frac{{\rm d}^2\sigma^{\nu{\rm p}}}{{\rm d}x{\rm d}y} = \frac{G_{\rm F}^2 x s}{\pi}\left[d(x) + (1-y)^2\bar{u}(x)\right]$$

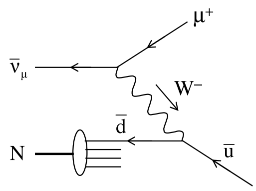
The total cross section is therefore

$$\sigma^{\nu{\rm p}} = \int_0^1\int_0^1 \frac{{\rm d}^2\sigma^{\nu{\rm p}}}{{\rm d}x{\rm d}y}{\rm d}x{\rm d}y = \frac{G_{\rm F}^2 s}{\pi}\int_0^1\left[xd(x) + \tfrac{1}{3}x\bar{u}(x)\right]{\rm d}x = \frac{G_{\rm F}^2 s}{\pi}\left[f_{\rm d} + \tfrac{1}{3}f_{\overline{\rm u}}\right]$$

where

$$f_d \equiv \int_0^1 xd(x){\rm d}x \qquad \text{and} \qquad f_{\bar{u}} \equiv \int_0^1 x\bar{u}(x){\rm d}x$$





are the fractions of the proton's momentum carried by d quark and $\bar{u}$ antiquark constituents, respectively.

For $\overline{\nu}$p scattering, we have $\overline{\nu}_\mu + \mathrm{u} \to \mu^+ + \mathrm{d}$ and $\overline{\nu}_\mu + \overline{\mathrm{d}} \to \mu^+ + \overline{\mathrm{u}}$:

$$\frac{\mathrm{d}^2\sigma^{\overline{\nu}\mathrm{p}}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2 xs}{\pi}\left[(1-y)^2 u(x) + \overline{d}(x)\right] \ .$$

For scattering from a neutron target, we have

$$\frac{\mathrm{d}^2\sigma^{\nu\mathrm{n}}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2 xs}{\pi}\left[d^\mathrm{n}(x) + (1-y)^2\overline{u}^\mathrm{n}(x)\right] = \frac{G_\mathrm{F}^2 xs}{\pi}\left[u(x) + (1-y)^2\overline{d}(x)\right]$$

$$\frac{\mathrm{d}^2\sigma^{\overline{\nu}\mathrm{n}}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2 xs}{\pi}\left[(1-y)^2 u^\mathrm{n}(x) + \overline{d}^\mathrm{n}(x)\right] = \frac{G_\mathrm{F}^2 xs}{\pi}\left[(1-y)^2 d(x) + \overline{u}(x)\right]$$

using $d^\mathrm{n}(x) = u^\mathrm{p}(x) = u(x)$ etc. Altogether then, the total cross sections are as follows:

$$\sigma^{\nu\mathrm{p}} = \frac{G_\mathrm{F}^2 s}{\pi}\left[f_d + \tfrac{1}{3}f_{\bar{u}}\right] \qquad \sigma^{\overline{\nu}\mathrm{p}} = \frac{G_\mathrm{F}^2 s}{\pi}\left[\tfrac{1}{3}f_u + f_{\bar{d}}\right]$$

$$\sigma^{\nu\mathrm{n}} = \frac{G_\mathrm{F}^2 s}{\pi}\left[f_u + \tfrac{1}{3}f_{\bar{d}}\right] \qquad \sigma^{\overline{\nu}\mathrm{n}} = \frac{G_\mathrm{F}^2 s}{\pi}\left[\tfrac{1}{3}f_d + f_{\bar{u}}\right]$$

Averaged over proton and neutron targets:

$$\sigma^{\nu N} = \tfrac{1}{2}\left(\sigma^{\nu\mathrm{p}} + \sigma^{\nu\mathrm{n}}\right) = \frac{G_\mathrm{F}^2 s}{2\pi}\left[f_d + f_u + \tfrac{1}{3}f_{\bar{d}} + \tfrac{1}{3}f_{\bar{u}}\right]$$

$$\sigma^{\overline{\nu} N} = \tfrac{1}{2}\left(\sigma^{\overline{\nu}\mathrm{p}} + \sigma^{\overline{\nu}\mathrm{n}}\right) = \frac{G_\mathrm{F}^2 s}{2\pi}\left[\tfrac{1}{3}f_d + \tfrac{1}{3}f_u + f_{\bar{d}} + f_{\bar{u}}\right]$$

c) In the lab frame, we have $s = (p_\nu + p_N)^2$ with $p_\nu = (E_\nu, 0, 0, E_\nu)$ and $p_N = (M, 0, 0, 0)$:

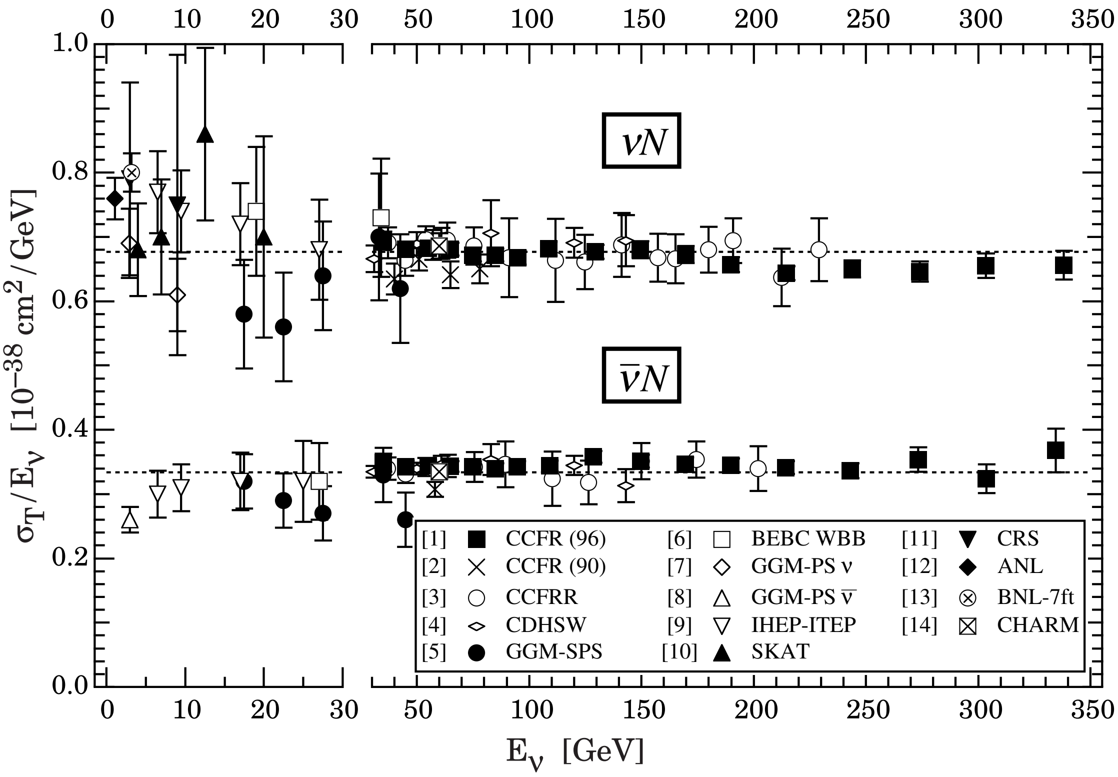
$$s = M^2 + 2p_\nu.p_N = M^2 + 2ME_\nu \approx 2ME_\nu \ ,$$

for $E_\nu \gg M$. Hence

$$\frac{\sigma^{\nu N}}{E_\nu} = \frac{G_F^2 M}{\pi}\left[f_q + \tfrac{1}{3} f_{\bar{q}}\right]$$
$$\frac{\sigma^{\bar{\nu} N}}{E_\nu} = \frac{G_F^2 M}{\pi}\left[\tfrac{1}{3} f_q + f_{\bar{q}}\right]$$

where $f_q = f_u + f_d$ and $f_{\bar{q}} = f_{\bar{u}} + f_{\bar{d}}$ are the momentum fractions for quarks and antiquarks, respectively.

Thus, at high energy, $\sigma^{\nu N}/E_\nu$ and $\sigma^{\bar{\nu} N}/E_\nu$ are expected to be constant, as seen in the figure.



Comparing with the measured values gives ($1\,\mathrm{cm}^2 = 10^{26}\,\mathrm{fm}^2$)

$$f_q + \tfrac{1}{3} f_{\bar{q}} = \frac{\pi}{G_F^2 M}\frac{\sigma^{\nu N}}{E_\nu} \approx \frac{\pi \times (0.68 \times 10^{-38}\,\mathrm{cm}^2\,\mathrm{GeV}^{-1})}{(1.166 \times 10^{-5}\,\mathrm{GeV}^{-2})^2 \times (0.938\,\mathrm{GeV})}\frac{1}{(0.197\,\mathrm{GeV\,fm})^2} \approx 0.43$$
$$\tfrac{1}{3} f_q + f_{\bar{q}} = \frac{\pi}{G_F^2 M}\frac{\sigma^{\bar{\nu} N}}{E_\nu} \approx \frac{\pi \times (0.33 \times 10^{-38}\,\mathrm{cm}^2\,\mathrm{GeV}^{-1})}{(1.166 \times 10^{-5}\,\mathrm{GeV}^{-2})^2 \times (0.938\,\mathrm{GeV})}\frac{1}{(0.197\,\mathrm{GeV\,fm})^2} \approx 0.21$$

Hence

$$f_q = \frac{3}{8}\,(3 \times 0.43 - 0.21) = 0.41$$
$$f_{\bar{q}} = \frac{3}{8}\,(3 \times 0.21 - 0.43) = 0.08$$

with the remaining momentum, $f_g \approx 0.50$, being carried by gluons.

20. The figure below shows measurements of the cross section $\mathrm{d}\sigma/\mathrm{d}Q^2$ from the H1 experiment at HERA for the neutral current (NC) processes $e^-\mathrm{p} \to e^-\mathrm{X}$ and $e^+\mathrm{p} \to e^+\mathrm{X}$, and the charged current (CC) processes $e^-\mathrm{p} \to \nu_e\mathrm{X}$ and $e^+\mathrm{p} \to \overline{\nu}_e\mathrm{X}$, with unpolarised incoming e^+ or e^- and proton beams:
a) Draw Feynman diagrams for the quark-level processes which contribute to CC $e^-\mathrm{p} \to \nu_e\mathrm{X}$ and $e^+\mathrm{p} \to \overline{\nu}_e\mathrm{X}$ scattering. (Neglect the s, c, b and t quark flavours).

b) The HERA data extends to values of $Q^2 > m_\mathrm{W}^2$. Starting from the parton model cross sections $\mathrm{d}^2\sigma/\mathrm{d}x\mathrm{d}y$ for (anti)neutrino-nucleon scattering derived in the lectures for $Q^2 \ll m_\mathrm{W}^2$, explain why the CC cross sections can be written down directly as

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}(e^+\mathrm{p} \to \overline{\nu}_e\mathrm{X}) = \frac{G_\mathrm{F}^2 m_\mathrm{W}^4}{2\pi x(Q^2 + m_\mathrm{W}^2)^2} x\left[\overline{u}(x) + (1-y)^2 d(x)\right]$$

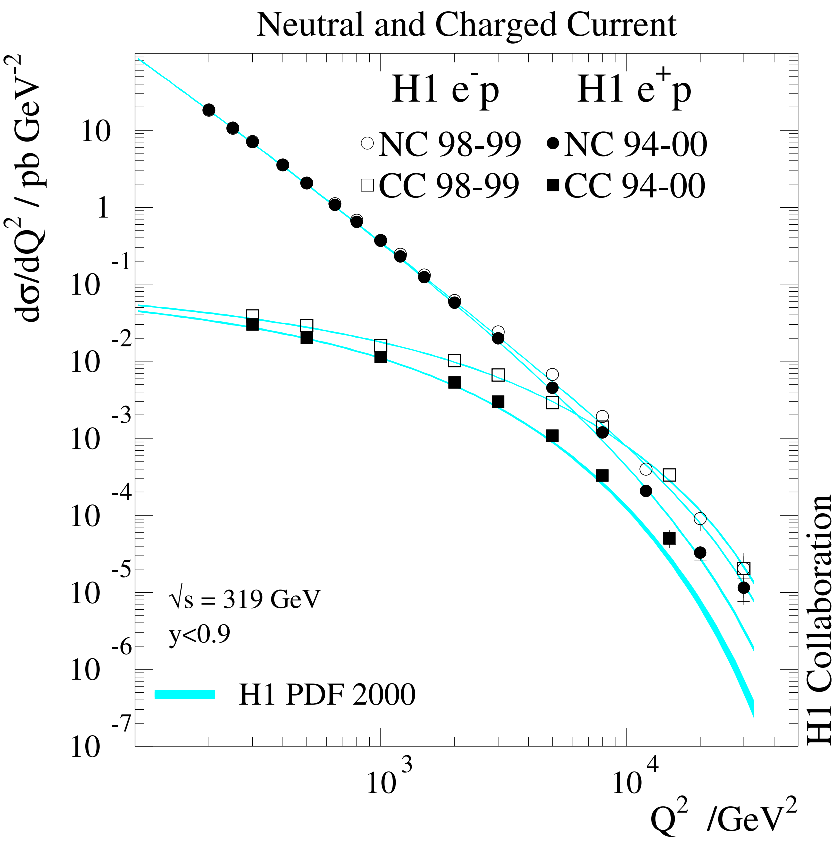
$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}(e^-\mathrm{p} \to \nu_e\mathrm{X}) = \frac{G_\mathrm{F}^2 m_\mathrm{W}^4}{2\pi x(Q^2 + m_\mathrm{W}^2)^2} x\left[u(x) + (1-y)^2 \overline{d}(x)\right]$$

c) Explain why the $e^-\mathrm{p}$ CC cross section is always higher than the $e^+\mathrm{p}$ CC cross section.

d) Explain why the CC cross sections become approximately constant as Q^2 decreases, while the NC cross sections grow indefinitely large. Account approximately for the observed slope of the NC cross sections at low values of Q^2.

e) Explain why the NC cross sections become similar in magnitude to the CC cross sections at high values of $Q^2 \sim m_\mathrm{Z}^2$.

f) (optional) Explain why the two NC cross sections are equal at low Q^2, but differ at high Q^2.



SOLUTION

Measurements at HERA of the NC processes $e^-\mathrm{p} \to e^-\mathrm{X}$ and $e^+\mathrm{p} \to e^+\mathrm{X}$, and the CC processes $e^-\mathrm{p} \to \nu_e\mathrm{X}$ and $e^+\mathrm{p} \to \overline{\nu}_e\mathrm{X}$, for unpolarised e^+, e^- and proton beams:

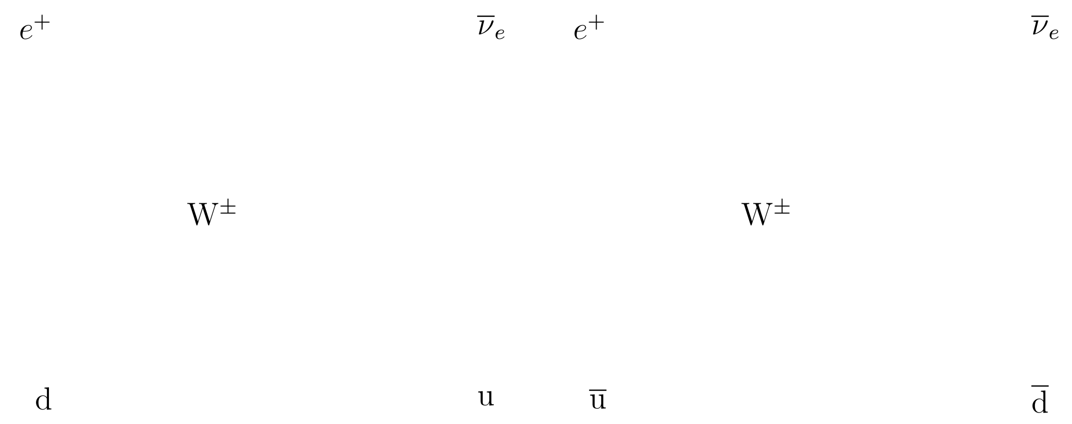
a) For the CC process $e^-\mathrm{p} \to \nu_e\mathrm{X}$, the quark-level processes are $e^-\mathrm{u} \to \nu_e\mathrm{d}$ and $e^-\overline{\mathrm{d}} \to \nu_e\overline{\mathrm{u}}$:

e^- ν_e e^- ν_e

$\mathrm{W}^\pm$ $\mathrm{W}^\pm$

u d $\overline{\mathrm{d}}$ $\overline{\mathrm{u}}$

For $e^+\mathrm{p} \to \overline{\nu}_e\mathrm{X}$ scattering, the quark-level processes are $e^+\mathrm{d} \to \overline{\nu}_e\mathrm{u}$ and $e^+\overline{\mathrm{u}} \to \overline{\nu}_e\overline{\mathrm{d}}$:



b) For νp scattering, the differential cross section in the quark-parton model for $Q^2 \gg m_W^2$ was derived in the lectures:

$$\frac{d^2\sigma^{\nu p}}{dx dy} = \frac{G_F^2 x s}{\pi} \left[d(x) + (1-y)^2 \overline{u}(x) \right] .$$

Since $Q^2 = sxy$ (for $s \gg M^2$), we have

$$\frac{d^2\sigma}{dx dQ^2} = \frac{dy}{dQ^2} \frac{d^2\sigma}{dx dy} = \frac{1}{sx} \frac{d^2\sigma}{dx dy} ,$$

and hence

$$\frac{d^2\sigma^{\nu p}}{dx dQ^2} = \frac{G_F^2}{\pi} \left[d(x) + (1-y)^2 \overline{u}(x) \right] .$$

The $W^\pm$ propagator contributes a factor $-ig_{\mu\nu}/(q^2 - m_W^2) = ig_{\mu\nu}/(Q^2 + m_W^2)$ to the matrix element M_{fi}. For $Q^2 \ll m_W^2$, this becomes $ig_{\mu\nu}/m_W^2$. Hence, relaxing the approximation $Q^2 \ll m_W^2$ gives an extra factor $m_W^2/(Q^2 + m_W^2)$ in the matrix element, or $m_W^4/(Q^2 + m_W^2)^2$ in the cross section:

$$\frac{d^2\sigma^{\nu p}}{dx dQ^2} = \frac{G_F^2}{\pi} \frac{m_W^4}{(Q^2 + m_W^2)^2} \left[d(x) + (1-y)^2 \overline{u}(x) \right] .$$

For νp scattering, it was only necessary to average over the two possible spin states of the proton, since the incoming neutrino is always in a unique spin state. For unpolarised $e^\pm p$ scattering, it is necessary to average over the two possible $e^\pm$ spin states and over the two possible proton spin states, so that, relative to (anti)neutrino scattering, an extra factor of $\frac{1}{2}$ is needed.

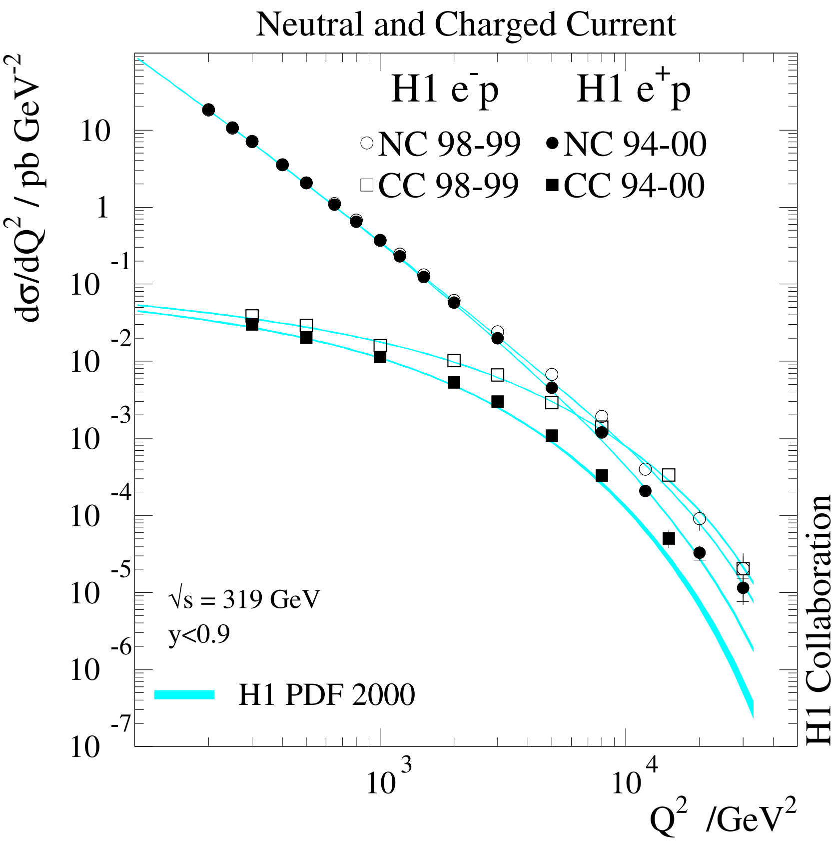
For $e^- p \to \nu_e X$, summing over $e^- u \to \nu_e d$ and $e^- \overline{d} \to \nu_e \overline{u}$, with their appropriate y distributions, and including the extra factor of one-half gives

$$\frac{d^2\sigma}{dx dQ^2}(e^- p \to \nu_e X) = \frac{G_F^2 m_W^4}{2\pi x (Q^2 + m_W^2)^2} x \left[u(x) + (1-y)^2 \overline{d}(x) \right]$$

Similarly, for $e^+ p \to \overline{\nu}_e X$ scattering, summing over $e^+ d \to \overline{\nu}_e u$ and $e^+ \overline{u} \to \overline{\nu}_e \overline{d}$ gives

$$\frac{d^2\sigma}{dx dQ^2}(e^+ p \to \overline{\nu}_e X) = \frac{G_F^2 m_W^4}{2\pi x (Q^2 + m_W^2)^2} x \left[\overline{u}(x) + (1-y)^2 d(x) \right]$$

c) The $e^- p$ CC cross section is higher than the $e^+ p$ CC cross section because $u(x)$ is larger than $d(x)$ (by about a factor of two). In addition, the $d(x)$ contribution to $e^+ p$ is suppressed by the extra factor $(1-y)^2$.



d) At low Q^2, the propagator factor $m_{\mathrm{W}}^4/(Q^2+m_{\mathrm{W}}^2)^2$ in the CC cross sections tends to a constant (unity), whereas the photon propagator factor $1/Q^4$ in the NC cross sections grows without limit.

From the plot, the cross section falls from approximately $90\,\mathrm{pb\,GeV}^{-2}$ at $Q^2 = 100\,\mathrm{GeV}^2$ to approximately $0.33\,\mathrm{pb\,GeV}^{-2}$ at $Q^2 = 1000\,\mathrm{GeV}^2$. Parameterising the cross section as $\mathrm{d}\sigma/\mathrm{d}Q^2 \propto 1/(Q^2)^n$, we can estimate

$$n \approx -\frac{\Delta(\log_{10}(\mathrm{d}\sigma/\mathrm{d}Q^2)}{\Delta(\log_{10}Q^2)} \approx \frac{\log_{10}90 - \log_{10}0.4}{3-2} \approx 2.35$$

so that we have $\mathrm{d}\sigma/\mathrm{d}Q^2 \approx 1/(Q^2)^{2.35} \approx 1/Q^{4.7}$, reasonably close to $1/Q^4$.

e) At low Q^2, the two NC processes $e^-\mathrm{p} \to e^-\mathrm{X}$ and $e^+\mathrm{p} \to e^+\mathrm{X}$ are dominated at leading order by single photon exchange and the leading-order cross sections are equal.

At high Q^2, there is a significant contribution also from the weak interactions, via Z^0 exchange. The $e^+\mathrm{p}$ and $e^-\mathrm{p}$ cross sections differ because the contribution from F_3 changes sign, similarly to the sign change for the F_3 contributions to neutrino and antineutrino scattering. Hence, for $Q^2 \sim m_{\mathrm{Z}}^2$, the $e^+\mathrm{p}$ and $e^-\mathrm{p}$ NC cross sections differ, and become similar in magnitude to the CC cross sections.

NEUTRINO OSCILLATIONS

21. In the Daya Bay experiment (`arXiv:1203.1669` and `arXiv:1310.6732`) electron antineutrinos from six nuclear reactors were observed in six detectors in three experimental halls, some $\approx 0.5\,\mathrm{km}$ and some $\approx 1.5\,\mathrm{km}$ distant from the reactors. The nuclear reactors emit electron antineutrinos of mean energy $E \approx 3\,\mathrm{MeV}$, and the detectors can resolve their energy to within a few percent.

a) Show that neutrino oscillations associated with the (solar) mass-squared difference $|\Delta m^2_{12}| \approx 7 \times 10^{-5}\,\mathrm{eV}^2$ can be neglected for the Daya Bay experiment, and that

$$P(\overline{\nu}_e \to \overline{\nu}_e) \approx 1 - \sin^2 2\theta_{13} \sin^2 \Delta_{23}$$

where

$$\Delta_{23} \equiv \frac{\Delta m^2_{23} L}{4E} \; .$$

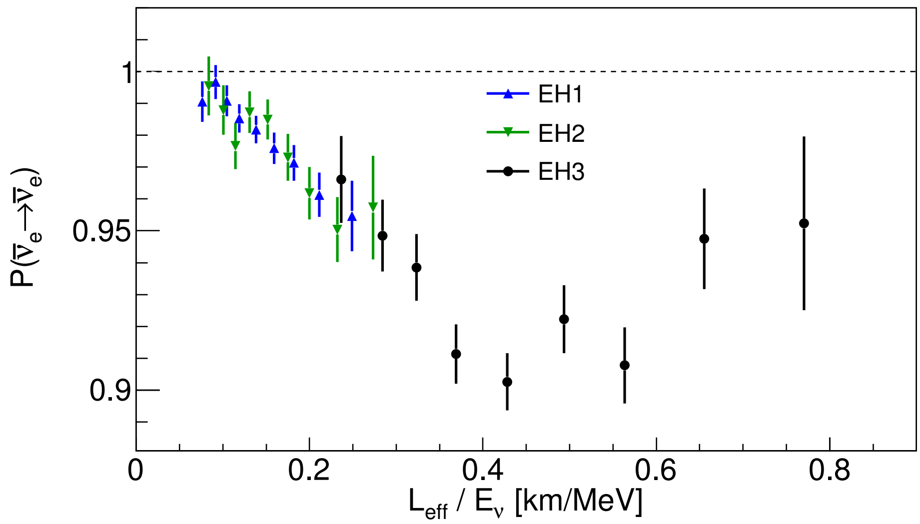
b) In the limit $|\Delta m^2_{23}| \gg (E/L)$, explain why a given measurement, P, of the survival probability $P(\overline{\nu}_e \to \overline{\nu}_e)$ determines the neutrino mixing to be $\sin^2 2\theta_{13} = 2(1-P)$.

c) In the limit $|\Delta m^2_{23}| \ll (E/L)$, show that a given measurement, P, of the survival probability $P(\overline{\nu}_e \to \overline{\nu}_e)$ determines the neutrino mixing to be $\sin^2 2\theta_{13} \propto 1/(\Delta m^2_{23})^2$, with constant of proportionality $(1-P)(4E/L)^2$.

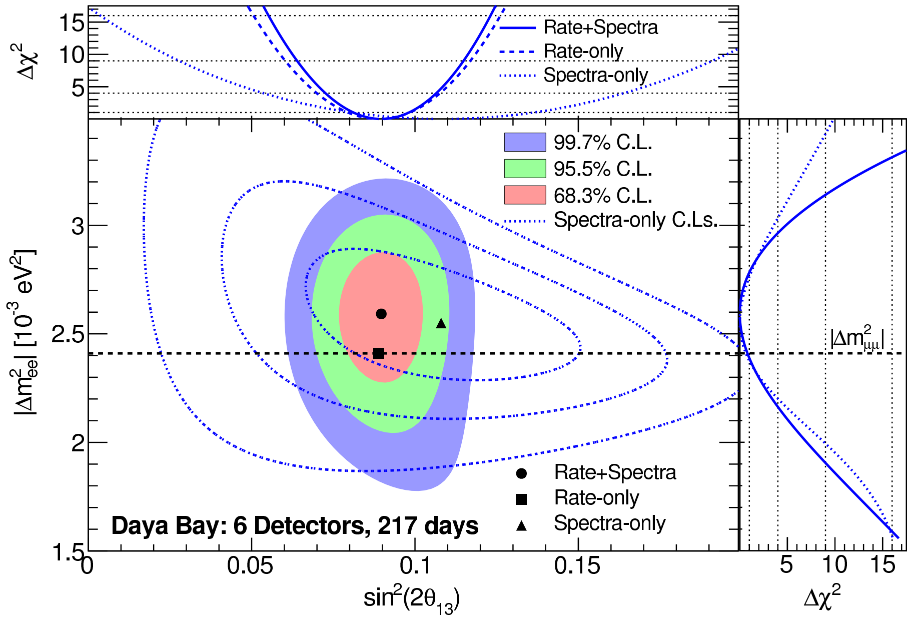
d) The third experimental hall is a (weighted) distance of $1.63\,\mathrm{km}$ from the reactor complex. A detector here sees a fractional deficit in the number of electron antineutrinos of 0.071 ± 0.010, compared to that expected from the neutrino fluxes of the reactors. Place a lower bound on the value of $\sin^2 2\theta_{13}$.

The deficit is observed to monotonically decrease for neutrinos of energy greater than $4\,\mathrm{MeV}$ average. What bound does this place on Δm^2_{23}?

e) The plot below shows the ratio of the number of observed to number of expected electron antineutrinos, as a function of the effective detector-reactor distance L_{eff} over the observed neutrino energies E_ν. It comprises data from all the detectors in the three experimental halls. Estimate values for $\sin^2 2\theta_{13}$ and Δm^2_{23}.



f) Sketch your results of parts (d) and (e) on a plot of the values of $\sin^2 2\theta_{13}$ and Δm^2_{23}, as fitted to the data by the Daya Bay collaboration.



SOLUTION

a) For $E = 3\,\mathrm{MeV}$, the wavelength associated with the solar mass-squared difference $|\Delta m^2_{12}| \approx 7 \times 10^{-5}\,\mathrm{eV}^2$ is

$$\lambda_{12} = \frac{4\pi E}{\Delta m^2_{12}} = \frac{4\pi \times 3\,\mathrm{MeV}}{7 \times 10^{-5}\,\mathrm{eV}^2} \times (0.197\,\mathrm{GeV\,fm}) = 105\,\mathrm{km}\,.$$

For the Daya Bay experiment, located $L \approx 1\,\mathrm{km}$ from the reactor core, oscillations due to Δm^2_{12} therefore do not have time to develop appreciably and can safely be neglected.

It was shown in Handout 12 that, neglecting CP violation (i.e. assuming the PMNS matrix is real) and using $|\Delta m^2_{23}| \approx |\Delta m^2_{13}|$, a general expression for the survival probability $P(\nu_e \to \nu_e)$ is

$$P(\nu_e \to \nu_e) \approx 1 - 4U^2_{e1}U^2_{e2}\sin^2\Delta_{12} - 4\left(1 - U^2_{e3}\right)U^2_{e3}\sin^2\Delta_{23}\,.$$

where

$$\Delta_{12} \equiv \frac{\Delta m^2_{12} L}{4E}\,, \qquad \Delta_{23} \equiv \frac{\Delta m^2_{23} L}{4E}\,.$$

The result is the same for $P(\overline{\nu}_e \to \overline{\nu}_e)$, as the PMNS matrix is assumed to be real. Neglecting the term involving Δm^2_{12}, and using $U_{e3} = \sin\theta_{13} = s_{13}$, we obtain

$$\begin{aligned} P(\overline{\nu}_e \to \overline{\nu}_e) &\approx 1 - 4\left(1 - U^2_{e3}\right)U^2_{e3}\sin^2\Delta_{23} \\ &= 1 - 4\left(1 - s^2_{13}\right)s^2_{13}\sin^2\Delta_{23} \\ &= 1 - \sin^2 2\theta_{13}\sin^2\Delta_{23}\,. \end{aligned} \tag{65}$$

b) As $|\Delta m^2_{23}| \to \infty$, the oscillation wavelength

$$\lambda_{23} = \frac{4\pi E}{\Delta m^2_{23}} \ll L$$

becomes much smaller than the dimensions of the detector and the experiment is no longer sensitive to individual oscillations but only to an *average* over many oscillations. The factor $\sin^2 \Delta_{23}$ in Equation (65) should therefore be replaced by its average value of one-half:

$$P(\nu_e \to \nu_e) = 1 - \tfrac{1}{2}\sin^2 2\theta_{13}\ .$$

A measured value P of the oscillation probability $P(\nu_e \to \nu_e)$ therefore determines the mixing to be $\sin^2 2\theta_{13} = 2(1-P)$.

c) If $\Delta m^2_{23} \ll (E/L)$, then $\Delta_{23} \equiv (\Delta m^2_{23} L)/4E \ll 1$ and $\sin^2 \Delta_{23} \approx (\Delta_{23})^2$. Hence

$$P(\nu_e \to \nu_e) \approx 1 - \sin^2 2\theta_{13} (\Delta_{23})^2 = 1 - \sin^2 2\theta_{13} \left(\frac{\Delta m^2_{23} L}{4E}\right)^2\ .$$

This can be rearranged to give

$$\sin^2 2\theta_{13} = (1-P)\left(\frac{4E}{L}\right)^2 \frac{1}{(\Delta m^2_{23})^2} \tag{66}$$

i.e. $\sin^2 2\theta_{13} \propto 1/(\Delta m^2_{23})^2$ with constant of proportionality $(1-P)(4E/L)^2$.

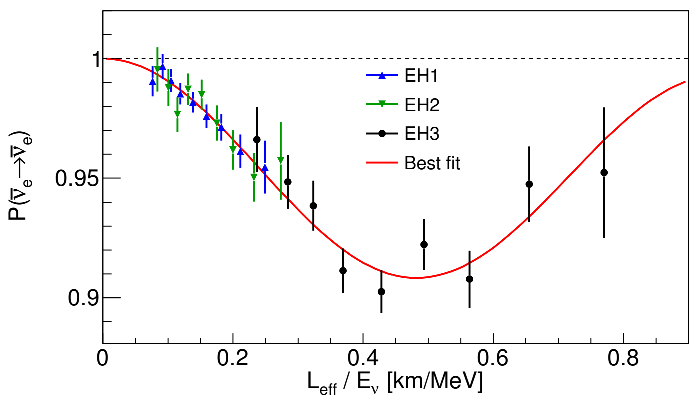
d) The smallest compatible value of $\sin^2 2\theta_{13}$ would occur if the third experimental hall is positioned at a minimum of the survival probability, viz., $\sin^2 \frac{\Delta m^2_{23} L}{4E} = 1$.

Then $\sin^2 2\theta_{13} = 1 - P = 0.071 \pm 0.010$, or $\sin^2 2\theta_{13} > 0.071 - 2 \times 0.010$ at approx. 97.5% confidence.

If neutrinos above 4 MeV show increasingly less chance of oscillating with increasing energy, then $\sin^2 \frac{\Delta m^2_{23} L}{4E} < 1$ for all $E > 4\,\text{MeV}$, and hence, for $E = 4\,\text{MeV}$:

$$\left|\frac{1.267 \Delta m^2_{23}(\text{eV}^2) L(\text{m})}{E(\text{MeV})}\right| < \frac{1}{2}\pi$$
$$|\Delta m^2_{23}| < 3.0 \times 10^{-3}\,\text{eV}^2$$

e) Here is the plot with the "best fit" line provided by the Daya Bay collaboration.



The minimum is around $(480\text{m}/\,\text{MeV}, 0.908) = (\frac{1}{2}\pi\frac{1}{1.267|\Delta m^2_{23}(\text{eV}^2)|}, 1 - \sin^2 2\theta_{13})$, whence $\sin^2 2\theta_{13} = 0.09$, $|\Delta m^2_{23}| = 2.6 \times 10^{-3}\,\text{eV}^2$.

22. a) It was shown in the lectures (see Equation (14) of Handout 12) that a general expression for the probability that an initial ν_e oscillates into a ν_μ is

$$P(\nu_e \to \nu_\mu) = 2\sum_{i<j} \mathrm{Re}\left(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j}\left[e^{-i(E_i-E_j)t} - 1\right]\right) \ .$$

Show that

$$P(\nu_e \to \nu_\mu) = -4\sum_{i<j} \mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin^2\Delta_{ij} + 2\sum_{i<j} \mathrm{Im}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin 2\Delta_{ij}$$

where

$$\Delta_{ij} \equiv \frac{(m_i^2 - m_j^2)L}{4E} \equiv \frac{\Delta m^2_{ij}L}{4E} \ .$$

b) Use the unitarity of the PMNS matrix to show that

$$\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3}) = -\mathrm{Im}(U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3}) = -\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2}) \equiv -J, \text{ say} \ .$$

c) Hence show that

$$P(\nu_e \to \nu_\mu) = -4\sum_{i<j} \mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin^2\Delta_{ij} + 8J\sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23}$$

[You may wish to use the trigonometric identity

$$\sin A + \sin B - \sin(A+B) = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{A+B}{2} \ .]$$

d) The standard parameterisation of the PMNS matrix is

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. Show that, in this parameterisation,

$$J = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\sin\delta$$

and find the maximum possible value of $|J|$ given the present experimental knowledge of the mixing angles θ_{12}, θ_{23} and θ_{13}.

e) The conversion probabilities for antineutrinos are obtained by replacing U by U^*. Show that

$$P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu) = 16J\sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23} \ .$$

f) It is proposed to build a "neutrino factory" to search for evidence of CP violation in neutrino oscillations; $P(\nu_e \to \nu_\mu) \neq P(\overline{\nu}_e \to \overline{\nu}_\mu)$. A neutrino factory would produce an intense beam of neutrinos with typical energy $10\,\mathrm{GeV}$. Roughly how far away should a neutrino detector be positioned to optimise the chances of observing CP violation, and how large an effect might be expected ?

SOLUTION

a) It was shown in the lectures (see Equation (6) of Handout 12) that

$$\begin{aligned} P(\nu_e \to \nu_\mu) = &\ 2\mathrm{Re}(U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2}\left[e^{-i(E_1-E_2)t}-1\right]) \\ &+ 2\mathrm{Re}(U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3}\left[e^{-i(E_1-E_3)t}-1\right]) \\ &+ 2\mathrm{Re}(U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3}\left[e^{-i(E_2-E_3)t}-1\right]) \end{aligned}$$

or equivalently

$$P(\nu_e \to \nu_\mu) = 2\sum_{i<j}\mathrm{Re}\left(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j}\left[e^{-i(E_i-E_j)t}-1\right]\right) ,$$

where e^{-iE_it} is being used as a shorthand for $e^{ip_ix-iE_it}$. For any pair of complex numbers $z_1 = x_1+iy_1$ and $z_2 = x_2 + iy_2$, we have

$$\mathrm{Re}(z_1z_2) = \mathrm{Re}\left[(x_1+iy_1)(x_2+iy_2)\right] = x_1x_2 - y_1y_2 = \mathrm{Re}(z_1)\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\mathrm{Im}(z_2)$$

and hence each term in the sum is

$$\begin{aligned} &\mathrm{Re}\left(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j}\left[e^{-i(E_i-E_j)t}-1\right]\right) \\ =&\mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\mathrm{Re}\left[e^{-i(E_i-E_j)t}-1\right] - \mathrm{Im}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\mathrm{Im}\left[e^{-i(E_i-E_j)t}-1\right] \\ =&\mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\left[\cos(E_i-E_j)t-1\right] + \mathrm{Im}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin(E_i-E_j)t \\ =&-2\mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin^2(E_i-E_j)t + \mathrm{Im}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin(E_i-E_j)t \\ =&-2\mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin^2 2\Delta_{ij} + \mathrm{Im}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin 2\Delta_{ij} , \end{aligned}$$

where

$$\Delta_{ij} \equiv \frac{(m_i^2-m_j^2)L}{4E} \equiv \frac{\Delta m^2_{ij}L}{4E}$$

and we have used the approximation

$$p_ix - E_it \approx (p_i - E_i)x \approx -\frac{m_i^2L}{2E} .$$

Hence

$$\boxed{P(\nu_e \to \nu_\mu) = -4\sum_{i<j}\mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin^2\Delta_{ij} + 2\sum_{i<j}\mathrm{Im}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin 2\Delta_{ij}} . \qquad (67)$$

b) Unitarity gives

$$\begin{aligned} U^*_{e1}U_{\mu 1} + U^*_{e2}U_{\mu 2} + U^*_{e3}U_{\mu 3} = 0 \\ U_{e1}U^*_{\mu 1} + U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3} = 0 \end{aligned}$$

so that

$$\begin{aligned} \mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3}) &= \mathrm{Im}\left[U_{e1}U^*_{\mu 1}(-U^*_{e1}U_{\mu 1} - U^*_{e2}U_{\mu 2})\right] = -\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2}) \\ \mathrm{Im}(U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3}) &= \mathrm{Im}\left[(-U_{e1}U^*_{\mu 1} - U_{e3}U^*_{\mu 3})U^*_{e3}U_{\mu 3}\right] = -\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3}) \end{aligned}$$

In summary:

$$\boxed{\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3}) = -\mathrm{Im}(U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3}) = -\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2}) \equiv -J} .$$

c) The last term in Equation (67) involves

$$\begin{aligned}
P_I &\equiv 2\sum_{i<j} \mathrm{Im}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin 2\Delta_{ij} \\
&= 2\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2})\sin 2\Delta_{12} + 2\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3})\sin 2\Delta_{13} + 2\mathrm{Im}(U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3})\sin 2\Delta_{23} \\
&= 2J(\sin 2\Delta_{12} - \sin 2\Delta_{13} + \sin 2\Delta_{23})
\end{aligned}$$

But the Δ_{ij} are related via

$$\Delta_{12} + \Delta_{23} = \Delta_{13}$$

so

$$P_I = 2J\left[\sin 2\Delta_{12} - \sin 2(\Delta_{12}+\Delta_{23}) + \sin 2\Delta_{23}\right]$$

Using the trigonometric identity

$$\sin A + \sin B - \sin(A+B) = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{A+B}{2}$$

this becomes

$$P_I = 8J\sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23}$$

In summary

$$\boxed{P(\nu_e\to\nu_\mu) = -4\sum_{i<j}\mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j})\sin^2\Delta_{ij} + 8J\sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23}}\ . \qquad (68)$$

d) We have

$$J \equiv \mathrm{Im}\left(U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2}\right)$$

Since $U_{e1} = c_{12}c_{13}$ and $U_{e2} = s_{12}c_{13}$ are real, and since

$$\mathrm{Im}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Im}(z_2) + \mathrm{Im}(z_1)\mathrm{Re}(z_2)$$

we have

$$J = U_{e1}U_{e2}\left[\mathrm{Re}(U_{\mu 1})\mathrm{Im}(U_{\mu 2}) - \mathrm{Im}(U_{\mu 1})\mathrm{Re}(U_{\mu 2})\right]\ .$$

But $U_{\mu 1} = -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta}$ and $U_{\mu 2} = c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta}$ so that

$$\begin{aligned}
J &= c_{12}c_{13}^2 s_{12}\left[(s_{12}c_{23} + c_{12}s_{23}s_{13}\cos\delta)s_{12}s_{23}s_{13}\sin\delta + c_{12}s_{23}s_{13}\sin\delta(c_{12}c_{23} - s_{12}s_{23}s_{13}\cos\delta)\right] \\
&= c_{12}c_{13}^2 s_{12}\left[s_{12}^2 c_{23}s_{23}s_{13}\sin\delta + c_{12}^2 c_{23}s_{23}s_{13}\sin\delta\right] \\
&= c_{13}^2 c_{12}s_{12}c_{23}s_{23}s_{13}\sin\delta\ .
\end{aligned}$$

Using $\sin 2\theta_{12} = 2\sin\theta_{12}\cos\theta_{12} = 2s_{12}c_{12}$ *etc.* this can also be written

$$\boxed{J = c_{13}^2 s_{13}s_{12}c_{12}s_{23}c_{23}\sin\delta = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\sin\delta}\ .$$

Experimentally, we have

$$c_{12} \approx 0.85;\quad s_{12}\approx 0.53;\quad c_{23}\approx s_{23}\approx \frac{1}{\sqrt{2}};\quad \sin^2\theta_{13} < 0.065\ .$$

Taking θ_{13} as large as possible ($s_{13} = 0.25$, $c_{13} = 0.97$) and $\sin\delta = 1$ gives

$$J_{max} = c_{13}^2 s_{13} s_{12} c_{12} s_{23} c_{23} = (0.97)^2 (0.25)(0.53)(0.85)\frac{1}{2} = 0.053$$

e) The conversion probabilities for antineutrinos are obtained by replacing U by U^* in Equation (68):

$$P(\overline{\nu}_e \to \overline{\nu}_\mu) = -4\sum_{i<j} \text{Re}(U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^*) \sin^2\Delta_{ij} + 8\bar{J}\sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23}$$

where

$$\bar{J} \equiv \text{Im}\left(U_{e1}^* U_{\mu 1} U_{e2} U_{\mu 2}^*\right) \; .$$

But

$$\begin{aligned} \text{Re}(U_{e1}^* U_{\mu 1} U_{e2} U_{\mu 2}^*) &= \quad \text{Re}(U_{e1} U_{\mu 1}^* U_{e2}^* U_{\mu 2}) \\ \text{Im}(U_{e1}^* U_{\mu 1} U_{e2} U_{\mu 2}^*) &= -\text{Im}(U_{e1} U_{\mu 1}^* U_{e2}^* U_{\mu 2}) \end{aligned}$$

so $\bar{J} = -J$ and hence

$$\boxed{P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu) = 16J\sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23}}$$

Therefore, unless U is purely real, we have CP violation:

$$P(\nu_e \to \nu_\mu) \neq P(\overline{\nu}_e \to \overline{\nu}_\mu)$$

As an aside, it is easy to show in similar fashion that, unless U is purely real we also have

$$P(\nu_e \to \nu_\mu) \neq P(\nu_\mu \to \nu_e)$$

which is T violation. However, even if U is complex, we always have

$$P(\nu_e \to \nu_\mu) = P(\overline{\nu}_\mu \to \overline{\nu}_e)$$

and therefore CPT is always conserved.

f) Since $|\Delta m_{13}^2| \approx |\Delta m_{23}^2|$, we have $|\Delta_{13}| \approx |\Delta_{23}|$ and hence

$$\Delta P \equiv P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu) \approx 16J\sin\Delta_{12}\sin^2\Delta_{13} \; .$$

For a neutrino energy $E = 10\,\text{GeV}$, the wavelengths associated with the Δ_{12} and Δ_{13} terms are

$$\begin{aligned} \lambda_{12} &= \frac{4\pi E}{\Delta m_{12}^2} = \frac{4\pi \times 10\,\text{GeV}}{7\times 10^{-5}\,\text{eV}^2} \times (0.197\,\text{GeV fm}) = 350,000\,\text{km} \\ \lambda_{13} &= \frac{4\pi E}{\Delta m_{13}^2} = \frac{4\pi \times 10\,\text{GeV}}{2\times 10^{-3}\,\text{eV}^2} \times (0.197\,\text{GeV fm}) = 10,000\,\text{km} \; . \end{aligned}$$

To get a potentially measurable CP violating effect needs ΔP as large as possible. The $\sin^2\Delta_{13}$ term is maximised at $L = 5,000\,\text{km}$, $L = 15,000\,\text{km}$ *etc.* , while the long-wavelength $\sin\Delta_{12}$ term grows approximately linearly with distance but doesn't reach a maximum until $L \approx 175,000\,\mathrm{km}$. For an Earth-bound experiment, $L \approx 5,000\,\mathrm{km}$ is about the optimum length.

With $\sin^2 \Delta_{13} \approx 1$, the largest CP violating effect which can be expected is

$$\begin{aligned}
\Delta P &\approx 16 J_{max} \sin \Delta_{12} = 16 J_{max} \sin\left(\frac{\Delta m_{12}^2 L}{4E}\right) \\
&= 16 \times 0.053 \times \sin\left(\frac{7 \times 10^{-5}\,\mathrm{eV}^2 \times 5000\,\mathrm{km}}{4 \times 10\,\mathrm{GeV}} \frac{1}{0.197\,\mathrm{GeV.\,fm}}\right) \\
&= 16 \times 0.053 \times \sin(0.044) \\
&= 0.038\ .
\end{aligned}$$

Notice that a measurement of the sign of ΔP would also give the sign of $\Delta m_{12}^2 = m_1^2 - m_2^2$, and would therefore distinguish between the two cases $m_1 > m_2$ and $m_1 < m_2$.

CP VIOLATION AND THE CKM MATRIX

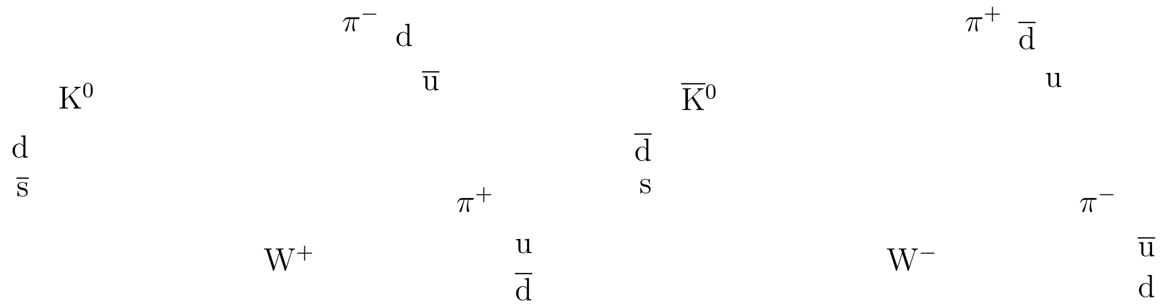
23. a) Draw Feynman diagrams for the decays $\mathrm{K}^0 \to \pi^+\pi^-$ and $\overline{\mathrm{K}}^0 \to \pi^+\pi^-$, and for the decays $\mathrm{K}^0 \to \pi^0\pi^0$ and $\overline{\mathrm{K}}^0 \to \pi^0\pi^0$.

b) Draw Feynman diagrams for the decays $\mathrm{K}^0 \to \pi^- e^+ \nu_e$ and $\overline{\mathrm{K}}^0 \to \pi^+ e^- \overline{\nu}_e$, and explain why the decays $\overline{\mathrm{K}}^0 \to \pi^- e^+ \nu_e$ and $\mathrm{K}^0 \to \pi^+ e^- \overline{\nu}_e$ cannot occur.

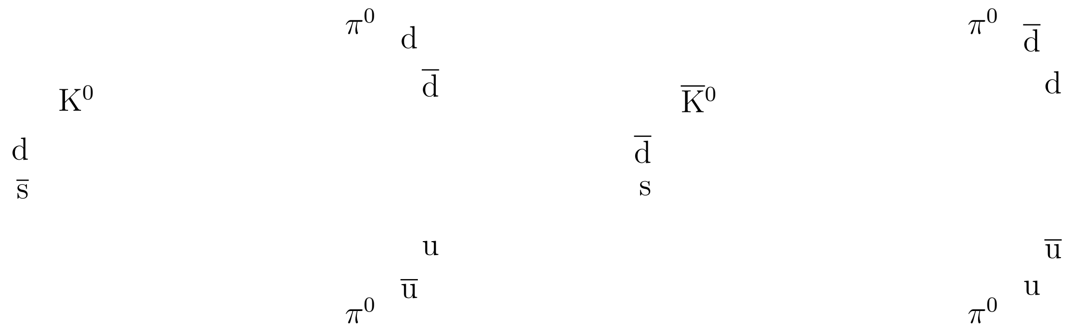
c) How does the decay rate for each of the above decays depend on the Cabibbo angle θ_C ?

SOLUTION

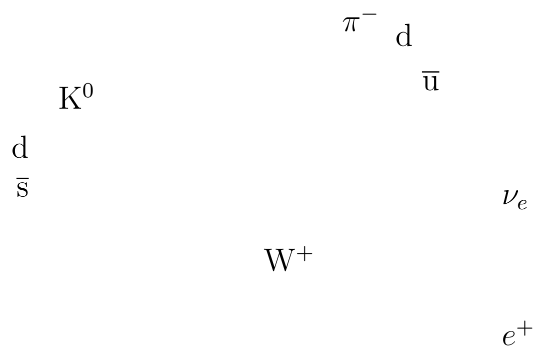
a) The leading-order Feynman diagrams for the decays $\mathrm{K}^0 \to \pi^+\pi^-$ and $\overline{\mathrm{K}}^0 \to \pi^+\pi^-$ involve the subprocesses $\overline{\mathrm{s}} \to \overline{\mathrm{u}}\mathrm{u}\overline{\mathrm{d}}$ and $\mathrm{s} \to \mathrm{u}\overline{\mathrm{u}}\mathrm{d}$, with a virtual W^+ or W^- boson, respectively:

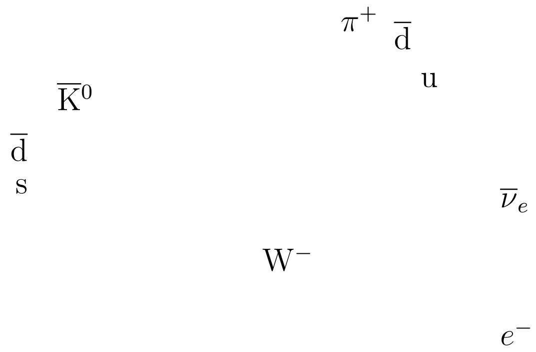


The same subprocesses are involved in the decays $\mathrm{K}^0 \to \pi^0\pi^0$ and $\overline{\mathrm{K}}^0 \to \pi^0\pi^0$, but with a different decay topology:



b) The leading-order Feynman diagrams for the decays $\mathrm{K}^0 \to \pi^- e^+ \nu_e$ and $\overline{\mathrm{K}}^0 \to \pi^+ e^- \overline{\nu}_e$ involve the subprocesses $\overline{\mathrm{s}} \to \overline{\mathrm{u}} e^+ \nu_e$ and $\mathrm{s} \to \mathrm{u} e^- \overline{\nu}_e$:





The decays $\overline{\mathrm{K}}^0 \to \pi^- e^+ \nu_e$ and $\mathrm{K}^0 \to \pi^+ e^- \overline{\nu}_e$ cannot occur because they would need $\mathrm{s} \to \overline{\mathrm{u}} e^+ \nu_e$ and $\overline{\mathrm{s}} \to \mathrm{u} e^- \overline{\nu}_e$, which are forbidden by charge conservation.

Hence, decays to final states containing an e^+ directly measure the K^0 component of the beam while decays to e^- directly measure the $\overline{\mathrm{K}}^0$ component.

c) The decays $\mathrm{K}^0 \to \pi^+\pi^-$ and $\mathrm{K}^0 \to \pi^0\pi^0$ both involve the quark level processes $\overline{\mathrm{s}} \to \overline{\mathrm{u}}$+virtual W^+ and $\mathrm{W}^+ \to \mathrm{u}\overline{\mathrm{d}}$. The first of these vertices gives a factor $\sin\theta_{\mathrm{C}}$ in the matrix element M_{fi}, while the second gives a factor $\cos\theta_{\mathrm{C}}$. Overall therefore, the decay rate is proportional to $\sin^2\theta_{\mathrm{C}}\cos^2\theta_{\mathrm{C}}$.

For $\overline{\mathrm{K}}^0 \to \pi^+\pi^-$ and $\overline{\mathrm{K}}^0 \to \pi^0\pi^0$, the quark level processes are $\mathrm{s} \to \mathrm{u}$ + virtual W^- and $\mathrm{W}^- \to \overline{\mathrm{u}}\mathrm{d}$, which again gives a decay rate proportional to $\sin^2\theta_{\mathrm{C}}\cos^2\theta_{\mathrm{C}}$.

For the decays $\mathrm{K}^0 \to \pi^- e^+ \nu_e$ and $\overline{\mathrm{K}}^0 \to \pi^+ e^- \overline{\nu}_e$, the vertices $\overline{\mathrm{s}} \to \overline{\mathrm{u}}$ + virtual W^+ and $\mathrm{s} \to \mathrm{u}$ + virtual W^- both give a factor $\sin\theta_{\mathrm{C}}$ in the matrix element. The decay rates are therefore both proportional to $\sin^2\theta_{\mathrm{C}}$.

24. In the CPLEAR experiment at CERN, neutral kaons are produced in low energy proton-antiproton collisions via the channels $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\overline{\mathrm{K}}^0$ and $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\mathrm{K}^0$. The strangeness of the initial $\overline{\mathrm{K}}^0$ or K^0 is tagged by the charge of the accompanying K^+ or K^-, and the K^0 or $\overline{\mathrm{K}}^0$ is subsequently detected via decays into the semileptonic final states $\pi^- e^+ \nu_e$ and $\pi^+ e^- \overline{\nu}_e$.

a) Draw Feynman diagrams for the reactions $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\overline{\mathrm{K}}^0$ and $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\mathrm{K}^0$, and explain why the reactions $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\mathrm{K}^0$ and $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\overline{\mathrm{K}}^0$ cannot occur.

b) Show that, for a system which is initially in a pure K^0 state, the decay rates R_+ and R_- to the semileptonic final states $\pi^- e^+ \nu_e$ and $\pi^+ e^- \overline{\nu}_e$ depend on the proper decay time t as

$$R_+ \equiv \Gamma(\mathrm{K}^0_{t=0} \to \pi^- e^+ \nu_e) = N_{\pi e\nu}\tfrac{1}{4}\left[e^{-\Gamma_{\mathrm{S}}t} + e^{-\Gamma_{\mathrm{L}}t} + 2e^{-(\Gamma_{\mathrm{S}}+\Gamma_{\mathrm{L}})t/2}\cos\Delta m t\right]$$
$$R_- \equiv \Gamma(\mathrm{K}^0_{t=0} \to \pi^+ e^- \overline{\nu}_e) \approx N_{\pi e\nu}\tfrac{1}{4}\left[1 - 4\mathrm{Re}\epsilon\right]\left[e^{-\Gamma_{\mathrm{S}}t} + e^{-\Gamma_{\mathrm{L}}t} - 2e^{-(\Gamma_{\mathrm{S}}+\Gamma_{\mathrm{L}})t/2}\cos\Delta m t\right]$$

where $\Gamma_S = 1/\tau_S$, $\Gamma_L = 1/\tau_L$, $\Delta m = m_{\mathrm{L}} - m_{\mathrm{S}}$, ϵ is the CP violation parameter, and $N_{\pi e\nu}$ is an overall normalisation constant. Show that the corresponding expressions for a system which is initially in a pure $\overline{\mathrm{K}}^0$ state are

$$\bar{R}_+ \equiv \Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^- e^+ \nu_e) \approx N_{\pi e\nu}\tfrac{1}{4}\left[1 + 4\mathrm{Re}\epsilon\right]\left[e^{-\Gamma_{\mathrm{S}}t} + e^{-\Gamma_{\mathrm{L}}t} - 2e^{-(\Gamma_{\mathrm{S}}+\Gamma_{\mathrm{L}})t/2}\cos\Delta m t\right]$$
$$\bar{R}_- \equiv \Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^+ e^- \overline{\nu}_e) = N_{\pi e\nu}\tfrac{1}{4}\left[e^{-\Gamma_{\mathrm{S}}t} + e^{-\Gamma_{\mathrm{L}}t} + 2e^{-(\Gamma_{\mathrm{S}}+\Gamma_{\mathrm{L}})t/2}\cos\Delta m t\right] .$$

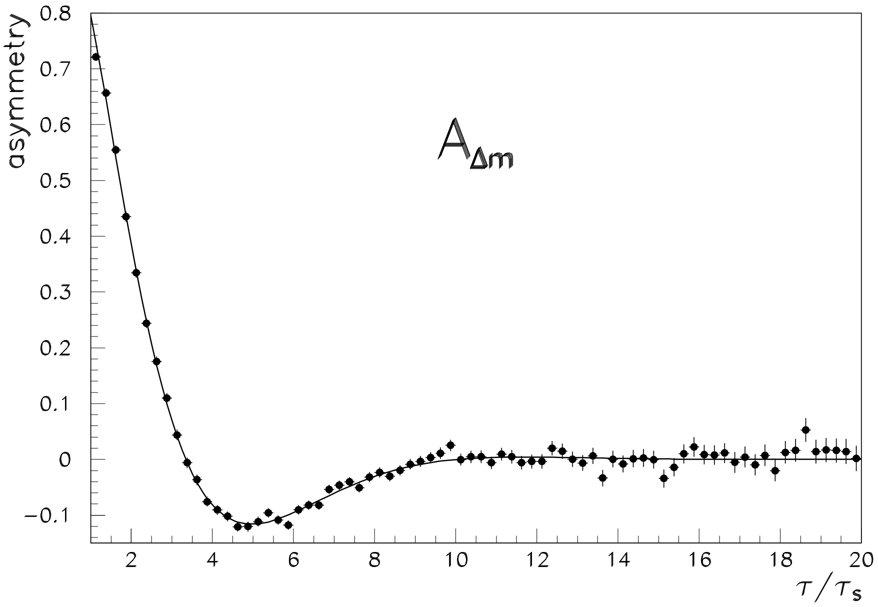
c) The figure overleaf shows a measurement from the CPLEAR experiment of the asymmetry

$$A_{\Delta m} \equiv \frac{(R_+ + \overline{R}_-) - (\overline{R}_+ + R_-)}{(R_+ + \overline{R}_-) + (\overline{R}_+ + R_-)}$$

as a function of the proper decay time $\tau = t$ (plotted in units of the $\mathrm{K_S}$ lifetime $\tau_S = 0.9 \times 10^{-10}$ s). Show that $A_{\Delta m}$ is given by

$$A_{\Delta m} = \frac{2\cos(\Delta m t)\, e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}}{e^{-\Gamma_\mathrm{S} t} + e^{-\Gamma_\mathrm{L} t}}$$

and obtain an estimate of the mass difference Δm.



d) Show that the time-reversal asymmetry

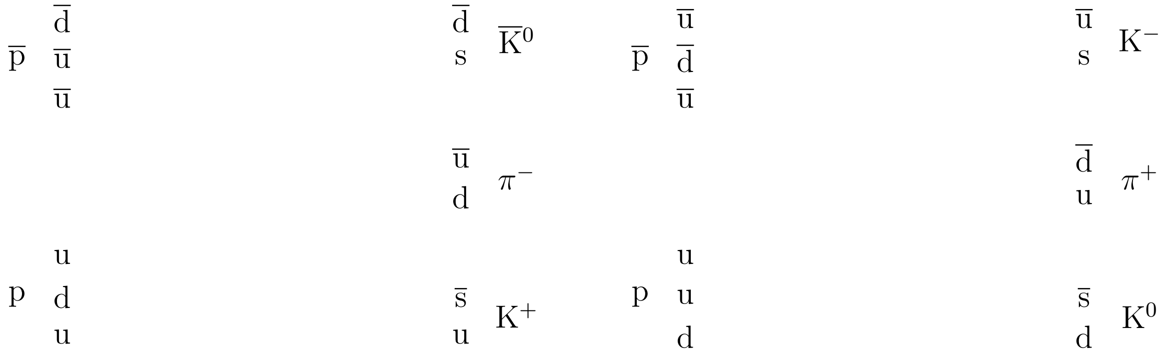
$$A_T \equiv \frac{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \mathrm{K}^0) - \Gamma(\mathrm{K}^0_{t=0} \to \overline{\mathrm{K}}^0)}{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \mathrm{K}^0) + \Gamma(\mathrm{K}^0_{t=0} \to \overline{\mathrm{K}}^0)}$$

is independent of the decay time t and that

$$A_T \approx 4\mathrm{Re}(\epsilon) = 4|\epsilon|\cos\phi\,.$$

SOLUTION

a) Feynman diagrams for $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\overline{\mathrm{K}}^0$ and $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\mathrm{K}^0$:



These are strong interaction processes involving $u\bar{u} \to g \to s\bar{s}$ at the quark level. The reactions $\bar{p}p \to K^+\pi^-K^0$ and $\bar{p}p \to K^-\pi^+\overline{K}^0$ cannot occur through the strong interactions because the final states contain $\bar{s}\bar{s}$ or ss rather than $s\bar{s}$, which would not conserve strangeness.

b) The K^0 can be expressed in terms of the states K_L and K_S as

$$\left|K^0\right\rangle = \sqrt{\frac{1+|\epsilon|^2}{2}}\frac{1}{1+\epsilon}\left(|K_L\rangle + |K_S\rangle\right)$$

The states K_L and K_S have well defined masses and lifetimes and evolve with time as

$$\begin{aligned} |K_L(t)\rangle &= |K_L\rangle\,\theta_L(t) = |K_L\rangle\,e^{-im_Lt-\Gamma_Lt/2} \\ |K_S(t)\rangle &= |K_S\rangle\,\theta_S(t) = |K_S\rangle\,e^{-im_St-\Gamma_St/2}\ . \end{aligned}$$

Hence the initial K^0 state evolves with time as

$$\left|K^0(t)\right\rangle = \sqrt{\frac{1+|\epsilon|^2}{2}}\frac{1}{1+\epsilon}\left(|K_L\rangle\,\theta_L + |K_S\rangle\,\theta_S\right)$$

Now express this time evolution in terms of the eigenstates K^0 and $\overline{K}^0$:

$$\begin{aligned} \left|K^0(t)\right\rangle &= \sqrt{\frac{1+|\epsilon|^2}{2}}\frac{1}{1+\epsilon}\times \\ &\times\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\epsilon|^2}}\left(\left[(1+\epsilon)\left|K^0\right\rangle + (1-\epsilon)\left|\overline{K}^0\right\rangle\right]\theta_L + \left[(1+\epsilon)\left|K^0\right\rangle - (1-\epsilon)\left|\overline{K}^0\right\rangle\right]\theta_S\right) \\ &= \frac{1}{2}(\theta_L+\theta_S)\left|K^0\right\rangle + \frac{1}{2}\frac{1-\epsilon}{1+\epsilon}(\theta_L-\theta_S)\left|\overline{K}^0\right\rangle \end{aligned}$$

Hence

$$\begin{aligned} \Gamma(K^0_{t=0}\to K^0) &\propto \tfrac{1}{4}\left|\theta_L+\theta_S\right|^2 \\ \Gamma(K^0_{t=0}\to \overline{K}^0) &\propto \tfrac{1}{4}\left|\frac{1-\epsilon}{1+\epsilon}\right|^2\left|\theta_L-\theta_S\right|^2 \end{aligned}$$

But

$$\left|\frac{1-\epsilon}{1+\epsilon}\right|^2 = \frac{(1-\epsilon^*)(1-\epsilon)}{(1+\epsilon^*)(1+\epsilon)} \approx \frac{1-2\mathrm{Re}\epsilon}{1+2\mathrm{Re}\epsilon} \approx 1-4\mathrm{Re}\epsilon$$

Since e^+ can only come from K^0 and e^- can only come from $\overline{\mathrm{K}}^0$, we have

$$R_+ \equiv \Gamma(\mathrm{K}^0_{t=0} \to \pi^- e^+ \nu_e) = N_{\pi e\nu} \tfrac{1}{4} \left[e^{-\Gamma_\mathrm{S} t} + e^{-\Gamma_\mathrm{L} t} + 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2} \cos \Delta m t \right] \tag{69}$$

$$R_- \equiv \Gamma(\mathrm{K}^0_{t=0} \to \pi^+ e^- \overline{\nu}_e) = N_{\pi e\nu} \tfrac{1}{4} \left[1 - 4\mathrm{Re}\epsilon \right] \left[e^{-\Gamma_\mathrm{S} t} + e^{-\Gamma_\mathrm{L} t} - 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2} \cos \Delta m t \right] \tag{70}$$

Similarly, an initially pure $\overline{\mathrm{K}}^0$ becomes

$$\begin{aligned} \left|\overline{\mathrm{K}}^0\right\rangle &= \sqrt{\frac{1+|\epsilon|^2}{2}} \frac{1}{1-\epsilon} \left(|\mathrm{K_L}\rangle \, \theta_\mathrm{L} - |\mathrm{K_S}\rangle \, \theta_\mathrm{S} \right) \\ &= \frac{1}{2} \frac{1+\epsilon}{1-\epsilon} (\theta_\mathrm{L} - \theta_\mathrm{S}) \left|\mathrm{K}^0\right\rangle + \frac{1}{2} (\theta_\mathrm{L} + \theta_\mathrm{S}) \left|\overline{\mathrm{K}}^0\right\rangle \end{aligned}$$

and we find

$$\begin{aligned} \Gamma(\overline{\mathrm{K}}^0_{t=0} \to \overline{\mathrm{K}}^0) &= \tfrac{1}{4} \left|\theta_\mathrm{L} + \theta_\mathrm{S}\right|^2 \\ \Gamma(\overline{\mathrm{K}}^0_{t=0} \to \mathrm{K}^0) &= \tfrac{1}{4} \left| \frac{1+\epsilon}{1-\epsilon} \right|^2 \left|\theta_\mathrm{L} - \theta_\mathrm{S}\right|^2 \end{aligned}$$

and

$$\bar{R}_+ \equiv \Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^- e^+ \nu_e) = N_{\pi e\nu} \tfrac{1}{4} \left[1 + 4\mathrm{Re}\epsilon \right] \left[e^{-\Gamma_\mathrm{S} t} + e^{-\Gamma_\mathrm{L} t} - 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2} \cos \Delta m t \right] \tag{71}$$

$$\bar{R}_- \equiv \Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^+ e^- \overline{\nu}_e) = N_{\pi e\nu} \tfrac{1}{4} \left[e^{-\Gamma_\mathrm{S} t} + e^{-\Gamma_\mathrm{L} t} + 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2} \cos \Delta m t \right] \tag{72}$$

c) The asymmetry $A_{\Delta m}$ is defined as

$$A_{\Delta m} \equiv \frac{(R_+ + \bar{R}_-) - (R_- + \bar{R}_+)}{(R_+ + \bar{R}_-) + (R_- + \bar{R}_+)} \, .$$

From Equations (69)-(72), we have

$$\begin{aligned} R_+ + \bar{R}_- &= N_{\pi e\nu} \tfrac{1}{2} \left[e^{-\Gamma_\mathrm{S} t} + e^{-\Gamma_\mathrm{L} t} + 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2} \cos \Delta m t \right] \\ R_- + \bar{R}_+ &= N_{\pi e\nu} \tfrac{1}{2} \left[e^{-\Gamma_\mathrm{S} t} + e^{-\Gamma_\mathrm{L} t} - 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2} \cos \Delta m t \right] \end{aligned}$$

and hence

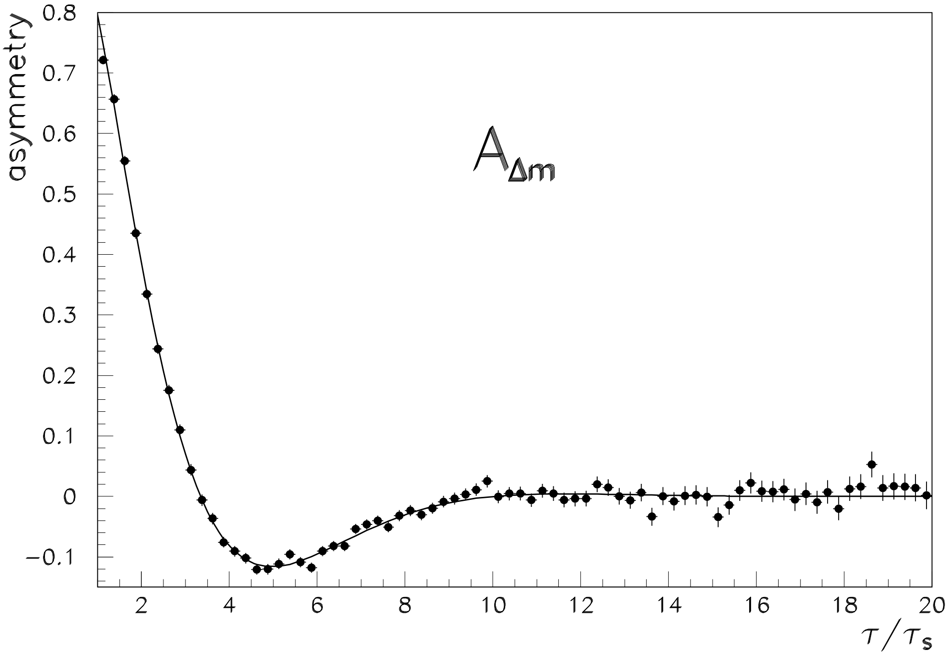
$$A_{\Delta m} = \frac{2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2} \cos \Delta m t}{e^{-\Gamma_\mathrm{S} t} + e^{-\Gamma_\mathrm{L} t}}$$

The value of Δm can be estimated from the plot by considering the time for which the asymmetry A first becomes zero: $\tau/\tau_S \approx 3.3$. At this point, we have $\Delta m.\tau = \pi/2$, and hence

$$\begin{aligned} \Delta m = \frac{\pi}{2\tau} &\approx \frac{\pi \times (6.58 \times 10^{-25}\,\mathrm{GeV.s})}{2 \times 3.3 \times (0.9 \times 10^{-10}\,\mathrm{s})} \\ &= 3.5 \times 10^{-15}\,\mathrm{GeV} \end{aligned}$$

(using $\hbar = 6.58 \times 10^{-25}$ GeV.s and $\tau_S = 0.9 \times 10^{-10}$ s).

d) From Equations (70) and (71), we see that, in the presence of CP violation, the rate for the transition $\mathrm{K}^0 \to \overline{\mathrm{K}}^0$ is no longer equal to the rate for $\overline{\mathrm{K}}^0 \to \mathrm{K}^0$. Thus we have T violation: the laws of physics are not invariant under time reversal, $t \to -t$, at the microscopic level.



This is quantified by defining the time-reversal asymmetry A_T (Kabir test)

$$A_T \equiv \frac{\Gamma(\overline{\mathrm{K}}^0 \to \mathrm{K}^0) - \Gamma(\mathrm{K}^0 \to \overline{\mathrm{K}}^0)}{\Gamma(\overline{\mathrm{K}}^0 \to \mathrm{K}^0) + \Gamma(\mathrm{K}^0 \to \overline{\mathrm{K}}^0)}$$

In terms of measurable semi-leptonic decay rates, this is

$$A_T = \frac{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^- e^+ \nu_e) - \Gamma(\mathrm{K}^0_{t=0} \to \pi^+ e^- \overline{\nu}_e)}{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^- e^+ \nu_e) + \Gamma(\mathrm{K}^0_{t=0} \to \pi^+ e^- \overline{\nu}_e)}$$

From Equations (70) and (71), the time-dependent terms cancel in the ratio leaving an asymmetry A_T which is constant, independent of time:

$$A_T \approx \frac{(1 + 4\mathrm{Re}\epsilon) - (1 - 4\mathrm{Re}\epsilon)}{(1 + 4\mathrm{Re}\epsilon) + (1 - 4\mathrm{Re}\epsilon)}$$

So approximately:

$$\boxed{A_T \approx 4\mathrm{Re}(\epsilon) = 4|\epsilon|\cos\phi}$$

THE Z BOSON

25. Consider the decay of the Z^0 to a fermion-antifermion pair, $Z^0 \to f\bar{f}$, where the fermion couples to the Z^0 with vector and axial vector coupling constants c_V and c_A:

a) Use the Feynman rules to show that the matrix element for the decay $Z^0 \to f\bar{f}$ can be written in the form

$$\begin{aligned} M_{fi} &= c_L \cdot g_Z \epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_4) + c_R \cdot g_Z \epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1+\gamma^5)v(p_4) \\ &\equiv c_L \cdot M_L + c_R \cdot M_R \end{aligned}$$

where p_1 is the Z^0 4-momentum, p_3 and p_4 are the 4-momenta of the fermion and antifermion, and $c_L = \frac{1}{2}(c_V + c_A)$, $c_R = \frac{1}{2}(c_V - c_A)$.

b) Assuming the fermion mass can be neglected, draw diagrams illustrating the spin configurations which result in non-zero values of M_L and M_R.

c) Use the results of the calculation of the $W^- \to e^-\overline{\nu}_e$ decay rate in the lectures to show that, for unpolarised Z^0's,

$$\langle |M_{fi}|^2 \rangle = \tfrac{2}{3} g_Z^2 m_Z^2 (c_L^2 + c_R^2)$$

and hence that the decay rate is

$$\Gamma(Z^0 \to f\bar{f}) = \frac{g_Z^2 m_Z}{48\pi}(c_V^2 + c_A^2)\ .$$

SOLUTION

a) The leading-order Feynman diagram for the decay $Z^0 \to f\bar{f}$ is

$\bar{f}$

p_4

Z^0 μ

p_1

p_3

f

The Feynman rules determine the invariant matrix element M_{fi} for the decay to be

$$\begin{aligned} -iM_{fi} &= \epsilon_\mu(p_1) \cdot \overline{u}(p_3) \cdot -i\frac{g_Z}{2}\gamma^\mu(c_V - c_A\gamma^5) \cdot v(p_4) \\ \Rightarrow \qquad M_{fi} &= \tfrac{1}{2} g_Z \epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu(c_V - c_A\gamma^5)v(p_4) \end{aligned}$$

where p_3 and p_4 are the 4-momenta of the fermion and antifermion, and $\epsilon_\mu(p_1)$ is the polarisation vector of the Z^0, with 4-momentum p_1.

The left-handed and right-handed couplings c_L and c_R are given by

$$\begin{aligned} c_V &= c_L + c_R & \qquad c_L &= \tfrac{1}{2}(c_V + c_A) \\ c_A &= c_L - c_R & \qquad c_R &= \tfrac{1}{2}(c_V - c_A) \end{aligned}$$

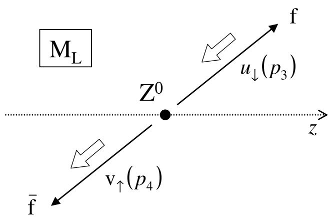
Then:

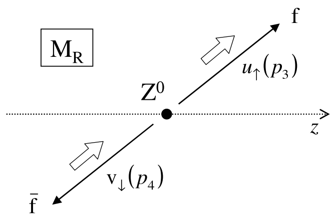
$$\tfrac{1}{2}(c_{\mathrm{V}} - c_{\mathrm{A}}\gamma^5) = c_L \cdot \tfrac{1}{2}(1-\gamma^5) + c_R \cdot \tfrac{1}{2}(1+\gamma^5)$$

and the expression for the matrix element becomes

$$\begin{aligned} M_{\mathrm{fi}} &= c_L \cdot g_{\mathrm{Z}}\epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_4) + c_R \cdot g_{\mathrm{Z}}\epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1+\gamma^5)v(p_4) \\ &\equiv c_L \cdot M_{\mathrm{L}} + c_R \cdot M_{\mathrm{R}} \end{aligned}$$

b) The spin configurations corresponding to each term are:





The M_{L} term, containing the factor $1-\gamma^5$, corresponds to a $V-A$ interaction, and only left-handed chiral components contribute. Neglecting the fermion and antifermion masses, this is equivalent to saying that the fermion must have negative helicity and the antifermion positive helicity.

The M_{R} term contains the factor $1+\gamma^5$, corresponding to a $V+A$ interaction, and only right-handed chiral components contribute (see question 13). The fermion has positive helicity and the antifermion negative helicity.

c) The term

$$M_{\mathrm{L}} = g_{\mathrm{Z}}\epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_4)$$

is identical to the matrix element

$$M_{\mathrm{fi}}(\mathrm{W}^- \to e^-\overline{\nu}_e) = \frac{g_{\mathrm{W}}}{\sqrt{2}}\epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_4)$$

for $\mathrm{W}^- \to e^-\overline{\nu}_e$ decay evaluated in handout 13, except that $g_{\mathrm{W}}/\sqrt{2}$ is replaced by g_{Z}. For unpolarised W decays, it was shown that

$$\langle |M_{\mathrm{fi}}|^2 \rangle = \tfrac{1}{3} g_{\mathrm{W}}^2 m_{\mathrm{W}}^2 \,.$$

Therefore, the contribution of M_L to the matrix element squared for Z^0 decays is

$$\langle|M_\mathrm{fi}|^2\rangle = \tfrac{1}{3}(\sqrt{2}g_\mathrm{Z})^2 m_\mathrm{Z}^2 = \tfrac{2}{3}g_\mathrm{Z}^2 m_\mathrm{Z}^2$$

By symmetry of the spin configurations corresponding to M_L and M_R, the values of $\langle|M_\mathrm{fi}|^2\rangle$ from M_L and M_R must be equal. Therefore the overall result is

$$\boxed{\langle|M_\mathrm{fi}|^2\rangle = \tfrac{2}{3}g_\mathrm{Z}^2 m_\mathrm{Z}^2 (c_L^2 + c_R^2)}$$

The decay rate is

$$\Gamma = \frac{p^*}{8\pi m_\mathrm{Z}^2}\,\langle|M_\mathrm{fi}|^2\rangle$$

where $p^* = m_\mathrm{Z}/2$ is the centre of mass momentum of either final state particle. Hence we obtain finally

$$\boxed{\Gamma(\mathrm{Z}^0 \to \mathrm{f}\bar{\mathrm{f}}) = \frac{g_\mathrm{Z}^2 m_\mathrm{Z}}{24\pi}(c_L^2 + c_R^2) = \frac{g_\mathrm{Z}^2 m_\mathrm{Z}}{48\pi}(c_\mathrm{V}^2 + c_\mathrm{A}^2)}$$

where we have used

$$c_L^2 + c_R^2 = \tfrac{1}{4}(c_\mathrm{V} + c_\mathrm{A})^2 + \tfrac{1}{4}(c_\mathrm{V} - c_\mathrm{A})^2 = \tfrac{1}{2}(c_\mathrm{V}^2 + c_\mathrm{A}^2)$$

26. a) Use the result of question 25 to compute the total width of the Z^0, and compare to experiment. [Take $\sin^2\theta_W = 0.23$, and remember that quarks have three colour states].

b) What will be the value of

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

at the peak of the Z^0 resonance ?

c) Calculate the cross section for $e^+e^- \to Z^0$ at the resonance peak, and show that the cross-section for $e^+e^- \to \mu^+\mu^-$ is increased by a factor of ≈ 200 relative to the QED cross section.

d) The width $\Gamma(Z^0 \to b\bar{b})$ has been measured at LEP to be 0.378 GeV. Show that the weak isospin of the b quark is compatible with a value of -0.5. Explain why this result effectively guaranteed the existence of the top quark, even before it was directly discovered.

[$G_F = 1.166 \times 10^{-5}\,\text{GeV}^{-2}$.]

SOLUTION

a) First add up all the factors of $c_V^2 + c_A^2$ for all possible Z^0 final states. In the Standard Model we have

$$c_V = I_W^3 - 2Q\sin^2\theta_W, \qquad c_A = I_W^3$$

where I_W^3 is the third component of weak isospin and Q is the particle charge in units of $|e|$. Assuming $\sin^2\theta_W = 0.23$, and remembering a colour factor of 3 for the quark-antiquark final states, this gives:

particles	c_V	c_A	$c_V^2 + c_A^2$
e^-, μ^-, τ^-	$-\frac{1}{2} + 2\sin^2\theta_W = -0.04$	$-\frac{1}{2}$	3×0.2516
ν_e, ν_μ, ν_τ	$+\frac{1}{2}$	$+\frac{1}{2}$	3×0.5
u, c	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_W = 0.193$	$+\frac{1}{2}$	$3 \times 2 \times 0.2874$
d, s, b	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W = -0.347$	$-\frac{1}{2}$	$3 \times 3 \times 0.3702$

which gives a total of $\sum(c_V^2 + c_A^2) = 7.311$. Hence

$$\begin{aligned}\Gamma_Z = \frac{g_Z^2 m_Z}{48\pi}\sum(c_V^2 + c_A^2) &= \frac{e^2}{\sin^2\theta_W(1-\sin^2\theta_W)} \cdot \frac{m_Z}{48\pi}\sum(c_V^2 + c_A^2) \\ &= \frac{\alpha}{\sin^2\theta_W(1-\sin^2\theta_W)} \cdot \frac{m_Z}{12}\sum(c_V^2 + c_A^2) \\ &= \frac{(1/137)}{0.23 \times (1-0.23)} \times \frac{91.2\,\text{GeV}}{12} \times 7.311 = 2.29\,\text{GeV}\end{aligned}$$

[The experimental value, $\Gamma_Z = 2.49\,\text{GeV}$, is larger than this because of higher order corrections.]

b) At the peak of the Z^0 resonance, from the Breit-Wigner formula,

$$\begin{aligned}R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} &= \frac{\Gamma(e^+e^- \to \text{hadrons})}{\Gamma(e^+e^- \to \mu^+\mu^-)} \\ &= \frac{(c_V^2 + c_A^2)_{\text{hadrons}}}{(c_V^2 + c_A^2)_{\mu^+\mu^-}} = \frac{3 \times 2 \times 0.2874 + 3 \times 3 \times 0.3702}{0.2516} = 20.1\end{aligned}$$

c) The Breit-Wigner formula gives the total Z^0 cross section on the peak of the resonance as:

$$\sigma(e^+e^- \to \mathrm{Z}^0 \to \text{anything}) = \frac{12\pi}{m_\mathrm{Z}^2}\frac{\Gamma(\mathrm{Z}^0 \to e^+e^-)}{\Gamma_Z} = \frac{12\pi}{m_\mathrm{Z}^2}\mathrm{BR}(\mathrm{Z}^0 \to e^+e^-)$$

The $\mathrm{Z}^0 \to e^+e^-$ branching ratio is

$$\mathrm{BR}(\mathrm{Z}^0 \to e^+e^-) = \frac{0.2516}{7.311} = 3.44\%$$

so

$$\begin{aligned}\sigma(e^+e^- \to \mathrm{Z}^0 \to \text{anything}) &= \frac{12\pi}{(91.2\,\mathrm{GeV})^2} \times 0.0344 = 1.56\times 10^{-4}\mathrm{GeV}^{-2} \\ &= 1.56\times 10^{-4}\mathrm{GeV}^{-2} \times (0.197\,\mathrm{GeV.fm})^2 = 0.061\,\mathrm{fm}^2 = 61\,\mathrm{nb}\end{aligned}$$

The QED cross section is

$$\sigma_{QED} = \sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$$

On the Z^0 resonance peak, with $s = m_\mathrm{Z}^2$, we have

$$\begin{aligned}\frac{\sigma(e^+e^- \to \mathrm{Z}^0 \to \mu^+\mu^-)}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)} &= \frac{(12\pi/m_\mathrm{Z}^2)\mathrm{BR}(\mathrm{Z}^0 \to e^+e^-)\mathrm{BR}(\mathrm{Z}^0 \to \mu^+\mu^-)}{4\pi\alpha^2/3m_\mathrm{Z}^2} \\ &= \frac{9}{\alpha^2}\cdot\left[\mathrm{BR}(\mathrm{Z}^0 \to e^+e^-)\right]^2 = 9\times(137)^2\times(0.0344)^2 = 200.0\end{aligned}$$

d) From Question 25, and including a factor 3 for colour, the $\mathrm{Z}^0 \to \mathrm{b\overline{b}}$ width is

$$\Gamma(\mathrm{Z}^0 \to \mathrm{b\overline{b}}) = 3\frac{g_\mathrm{Z}^2 m_\mathrm{Z}}{48\pi}\left[(c_\mathrm{V}^b)^2 + (c_\mathrm{A}^b)^2\right] = \frac{g_\mathrm{Z}^2 m_\mathrm{Z}}{16\pi}\left[(c_\mathrm{V}^b)^2 + (c_\mathrm{A}^b)^2\right]$$

where the coupling g_Z is given by

$$g_\mathrm{Z}^2 = \frac{e^2}{\sin^2\theta_W(1-\sin^2\theta_W)} = \frac{4\pi\alpha}{\sin^2\theta_W(1-\sin^2\theta_W)}$$

Hence

$$\begin{aligned}(c_\mathrm{V}^b)^2 + (c_\mathrm{A}^b)^2 &= \frac{16\pi}{g_\mathrm{Z}^2 m_\mathrm{Z}}\cdot\Gamma(\mathrm{Z}^0 \to \mathrm{b\overline{b}}) \\ &= \frac{16\pi\Gamma(\mathrm{Z}^0 \to \mathrm{b\overline{b}})}{m_\mathrm{Z}}\cdot\frac{\sin^2\theta_W(1-\sin^2\theta_W)}{4\pi\alpha} \\ &= \frac{4\times 0.378\,\mathrm{GeV}}{91.2\,\mathrm{GeV}}\times 0.23\times(1-0.23)\times 137 = 0.402\end{aligned}$$

But $c_\mathrm{V} = I_W^3 - 2Q\sin^2\theta_W$ and $c_\mathrm{A} = I_W^3$, where $Q_b = -\frac{1}{3}$ for the b quark. Hence

$$(c_\mathrm{V}^b)^2 + (c_\mathrm{A}^b)^2 = (I_W^3 + \tfrac{2}{3}\times 0.23)^2 + (I_W^3)^2 = 0.402\,.$$

Solving the quadratic equation for I_W^3 gives $I_W^3 = 0.36$ or $I_W^3 = -0.52$, which clearly suggests $I_W^3 = -\frac{1}{2}$ for the b quark.

Since $I_W^3 = -\frac{1}{2}$ for the b quark, it must be a member of a weak isospin doublet, and there must be a partner state with $I_W^3 = +\frac{1}{2}$. Thus the existence of the top quark could be inferred long before it was directly discovered.

27. a) It was shown in the lectures that the centre of mass frame differential cross section $\mathrm{d}\sigma_{\mathrm{LR}}/\mathrm{d}\cos\theta$ for the process $e^+e^- \to \mathrm{f}\bar{\mathrm{f}}$ on the peak of the Z^0 resonance, for the case that the incoming electron is left-handed and the outgoing fermion is right-handed, is given by

$$\frac{\mathrm{d}\sigma_{\mathrm{LR}}}{\mathrm{d}\cos\theta} \propto (c_{\mathrm{L}}^{e})^2 (c_{\mathrm{R}}^{\mathrm{f}})^2 (1-\cos\theta)^2 \ .$$

Show that the corresponding forward and backward cross sections $\sigma_{\mathrm{LR}}^{\mathrm{F}}$ and $\sigma_{\mathrm{LR}}^{\mathrm{B}}$ are given by

$$\sigma_{\mathrm{LR}}^{\mathrm{F}} \propto (c_{\mathrm{L}}^{e})^2 (c_{\mathrm{R}}^{\mathrm{f}})^2, \qquad \sigma_{\mathrm{LR}}^{\mathrm{B}} \propto 7 (c_{\mathrm{L}}^{e})^2 (c_{\mathrm{R}}^{\mathrm{f}})^2 \ ,$$

and write down similar expressions for the cross sections $\sigma_{\mathrm{RL}}^{\mathrm{F}}$, $\sigma_{\mathrm{RL}}^{\mathrm{B}}$, $\sigma_{\mathrm{LL}}^{\mathrm{F}}$, $\sigma_{\mathrm{LL}}^{\mathrm{B}}$, $\sigma_{\mathrm{RR}}^{\mathrm{F}}$, $\sigma_{\mathrm{RR}}^{\mathrm{B}}$.

b) The asymmetry $A_{\mathrm{LR}}^{\mathrm{FB}}$ is defined as

$$A_{\mathrm{LR}}^{\mathrm{FB}} \equiv \frac{(\sigma_{\mathrm{L}}^{\mathrm{F}} - \sigma_{\mathrm{L}}^{\mathrm{B}}) - (\sigma_{\mathrm{R}}^{\mathrm{F}} - \sigma_{\mathrm{R}}^{\mathrm{B}})}{(\sigma_{\mathrm{L}}^{\mathrm{F}} + \sigma_{\mathrm{L}}^{\mathrm{B}}) + (\sigma_{\mathrm{R}}^{\mathrm{F}} + \sigma_{\mathrm{R}}^{\mathrm{B}})}$$

where $\sigma_{\mathrm{L}} \equiv \sigma_{\mathrm{LL}} + \sigma_{\mathrm{LR}}$ and $\sigma_{\mathrm{R}} \equiv \sigma_{\mathrm{RL}} + \sigma_{\mathrm{RR}}$ are the total cross sections for left-handed and right-handed incoming electrons, respectively. Show that

$$A_{\mathrm{LR}}^{\mathrm{FB}} = \frac{3}{4} \frac{(c_{\mathrm{L}}^{\mathrm{f}})^2 - (c_{\mathrm{R}}^{\mathrm{f}})^2}{(c_{\mathrm{L}}^{\mathrm{f}})^2 + (c_{\mathrm{R}}^{\mathrm{f}})^2} \equiv \frac{3}{4} A_{\mathrm{f}} \ ,$$

and compare with the similar predictions for the asymmetries A_{LR} and A_{FB}.

c) Using a polarised electron beam, the SLD experiment has recently measured $A_{\mathrm{LR}}^{\mathrm{FB}}$ for the process $e^+e^- \to \mathrm{c}\bar{\mathrm{c}}$, and obtained the result $A_{\mathrm{c}} = 0.6712 \pm 0.0274$. Determine the corresponding value of $\sin^2\theta_{\mathrm{W}}$ and (optionally) its error.

SOLUTION

a) The differential cross section $\mathrm{d}\sigma_{\mathrm{LR}}/\mathrm{d}\cos\theta$ for the process $e^+e^- \to \mathrm{f}\bar{\mathrm{f}}$ for the case that the incoming electron is left-handed and the outgoing fermion is right-handed is given by

$$\frac{\mathrm{d}\sigma_{\mathrm{LR}}}{\mathrm{d}\cos\theta} \propto (c_{\mathrm{L}}^{e})^2 (c_{\mathrm{R}}^{\mathrm{f}})^2 (1-\cos\theta)^2 \ .$$

The forward and backward cross sections are defined by

$$\sigma_{\mathrm{F}} \equiv \int_0^1 \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\,\mathrm{d}\cos\theta, \qquad \sigma_{\mathrm{B}} \equiv \int_{-1}^0 \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\,\mathrm{d}\cos\theta \ .$$

Using the integrals

$$\int_0^1 (1-\cos\theta)^2\,\mathrm{d}\cos\theta = \frac{1}{3} \qquad \int_{-1}^0 (1-\cos\theta)^2\,\mathrm{d}\cos\theta = \frac{7}{3} \ ,$$

Therefore the forward and backward cross sections $\sigma^{\rm F}_{\rm LR}$ and $\sigma^{\rm B}_{\rm LR}$ are in a ratio 7:1

$$\boxed{\sigma^{\rm F}_{\rm LR} \propto (c^e_{\rm L})^2 (c^{\rm f}_{\rm R})^2, \qquad \sigma^{\rm B}_{\rm LR} \propto 7 (c^e_{\rm L})^2 (c^{\rm f}_{\rm R})^2} \; .$$

Similarly, we have

$$\begin{aligned}
\frac{{\rm d}\sigma_{\rm RL}}{{\rm d}\cos\theta} &\propto (c^e_{\rm R})^2 (c^{\rm f}_{\rm L})^2 (1-\cos\theta)^2 \\
\frac{{\rm d}\sigma_{\rm LL}}{{\rm d}\cos\theta} &\propto (c^e_{\rm L})^2 (c^{\rm f}_{\rm L})^2 (1+\cos\theta)^2 \\
\frac{{\rm d}\sigma_{\rm RR}}{{\rm d}\cos\theta} &\propto (c^e_{\rm R})^2 (c^{\rm f}_{\rm R})^2 (1+\cos\theta)^2
\end{aligned}$$

and hence

$$\begin{aligned}
\sigma^{\rm F}_{\rm RL} &\propto (c^e_{\rm R})^2 (c^{\rm f}_{\rm L})^2, & \sigma^{\rm B}_{\rm RL} &\propto 7 (c^e_{\rm R})^2 (c^{\rm f}_{\rm L})^2 \\
\sigma^{\rm F}_{\rm LL} &\propto 7 (c^e_{\rm L})^2 (c^{\rm f}_{\rm L})^2, & \sigma^{\rm B}_{\rm LL} &\propto (c^e_{\rm L})^2 (c^{\rm f}_{\rm L})^2 \\
\sigma^{\rm F}_{\rm RR} &\propto 7 (c^e_{\rm R})^2 (c^{\rm f}_{\rm R})^2, & \sigma^{\rm B}_{\rm RR} &\propto (c^e_{\rm R})^2 (c^{\rm f}_{\rm R})^2
\end{aligned}$$

b) The asymmetry $A^{\rm FB}_{\rm LR}$ is defined as

$$A^{\rm FB}_{\rm LR} \equiv \frac{(\sigma^{\rm F}_{\rm L} - \sigma^{\rm B}_{\rm L}) - (\sigma^{\rm F}_{\rm R} - \sigma^{\rm B}_{\rm R})}{(\sigma^{\rm F}_{\rm L} + \sigma^{\rm B}_{\rm L}) + (\sigma^{\rm F}_{\rm R} + \sigma^{\rm B}_{\rm R})}$$

where $\sigma_{\rm L} \equiv \sigma_{\rm LL} + \sigma_{\rm LR}$ and $\sigma_{\rm R} \equiv \sigma_{\rm RL} + \sigma_{\rm RR}$ are the total cross sections for left-handed and right-handed incoming electrons, respectively. Therefore

$$\begin{aligned}
\sigma^{\rm F}_{\rm L} = \sigma^{\rm F}_{\rm LL} + \sigma^{\rm F}_{\rm LR} &\propto 7(c^e_{\rm L})^2(c^{\rm f}_{\rm L})^2 + (c^e_{\rm L})^2(c^{\rm f}_{\rm R})^2 = (c^e_{\rm L})^2 \left[7(c^{\rm f}_{\rm L})^2 + (c^{\rm f}_{\rm R})^2\right] \\
\sigma^{\rm B}_{\rm L} = \sigma^{\rm B}_{\rm LL} + \sigma^{\rm B}_{\rm LR} &\propto (c^e_{\rm L})^2(c^{\rm f}_{\rm L})^2 + 7(c^e_{\rm L})^2(c^{\rm f}_{\rm R})^2 = (c^e_{\rm L})^2 \left[(c^{\rm f}_{\rm L})^2 + 7(c^{\rm f}_{\rm R})^2\right] \\
\sigma^{\rm F}_{\rm R} = \sigma^{\rm F}_{\rm RL} + \sigma^{\rm F}_{\rm RR} &\propto (c^e_{\rm R})^2(c^{\rm f}_{\rm L})^2 + 7(c^e_{\rm R})^2(c^{\rm f}_{\rm R})^2 = (c^e_{\rm R})^2 \left[(c^{\rm f}_{\rm L})^2 + 7(c^{\rm f}_{\rm R})^2\right] \\
\sigma^{\rm B}_{\rm R} = \sigma^{\rm B}_{\rm RL} + \sigma^{\rm B}_{\rm RR} &\propto 7(c^e_{\rm R})^2(c^{\rm f}_{\rm L})^2 + (c^e_{\rm R})^2(c^{\rm f}_{\rm R})^2 = (c^e_{\rm R})^2 \left[7(c^{\rm f}_{\rm L})^2 + (c^{\rm f}_{\rm R})^2\right]
\end{aligned}$$

Hence

$$A^{\rm FB}_{\rm LR} = \frac{(c^e_{\rm L})^2\left[6(c^\mu_{\rm L})^2 - 6(c^\mu_{\rm R})^2\right] + (c^e_{\rm R})^2\left[6(c^\mu_{\rm L})^2 - 6(c^\mu_{\rm R})^2\right]}{(c^e_{\rm L})^2\left[8(c^\mu_{\rm L})^2 + 8(c^\mu_{\rm R})^2\right] + (c^e_{\rm R})^2\left[8(c^\mu_{\rm L})^2 + 8(c^\mu_{\rm R})^2\right]}$$

Hence

$$\boxed{A^{\rm FB}_{\rm LR} = \frac{3}{4}\frac{(c^{\rm f}_{\rm L})^2 - (c^{\rm f}_{\rm R})^2}{(c^{\rm f}_{\rm L})^2 + (c^{\rm f}_{\rm R})^2} \equiv \frac{3}{4} A_{\rm f}} \; .$$

For comparison, the expressions for the asymmetries $A_{\rm LR}$ and $A_{\rm FB}$ derived in lectures are given by

$$A_{\rm LR} = A_e, \qquad A_{\rm FB} = \frac{3}{4} A_e A_f \; .$$

c) We have

$$A_c = \frac{(c^c_{\rm L})^2 - (c^c_{\rm R})^2}{(c^c_{\rm L})^2 + (c^c_{\rm R})^2}$$

where

$$c_{\rm L}^c = \tfrac{1}{2} - \tfrac{2}{3}\sin^2\theta_{\rm W}, \qquad c_{\rm R}^c = -\tfrac{2}{3}\sin^2\theta_{\rm W}\ .$$

Writing $x \equiv \sin^2\theta_{\rm W}$, we have

$$A_c = \frac{(\frac{1}{2} - \frac{2}{3}x)^2 - (-\frac{2}{3}x)^2}{(\frac{1}{2} - \frac{2}{3}x)^2 + (-\frac{2}{3}x)^2} = \frac{(\frac{1}{4} - \frac{2}{3}x)}{(\frac{1}{4} - \frac{2}{3}x + \frac{8}{9}x^2)}\ .$$

This can be rearranged to give the quadratic equation

$$32A_c x^2 + 24(1 - A_c)x + 9(A_c - 1) = 0\ .$$

For the central measured value of $A_c = 0.6712$, this quadratic equation can be solved to give $x = \sin^2\theta_{\rm W} = 0.2305$.

To estimate the error on $\sin^2\theta_{\rm W}$, consider the upper end of the SLD measurement $A_c = 0.6712 + 0.0274 = 0.6986$. Solving the quadratic equation for this value of A_c gives $x = \sin^2\theta_{\rm W} = 0.2223$, which is $\delta(\sin^2\theta_{\rm W}) = -0.0082$ below the central value of 0.2305 .

Similarly, solving the quadratic equation for the lower end of the SLD measurement $A_c = 0.6712 - 0.0274 = 0.6438$ gives $x = \sin^2\theta_{\rm W} = 0.2382$, which is $\delta(\sin^2\theta_{\rm W}) = +0.0077$ above the central value of 0.2305 .

Overall therefore, we can estimate

$$\boxed{\sin^2\theta_{\rm W} = 0.230 \pm 0.008}\ .$$

THE TOP QUARK

28. a) The top quark decays into final states containing 1) two quarks and an antiquark, or 2) a quark, a lepton and an antilepton. List the possible final states of each type and draw the generic leading order Feynman diagram for these decays. Explain why the total top quark decay rate is dominated by the rate for the decay $\mathrm{t} \to \mathrm{W}^+\mathrm{b}$ into a real W^+ boson and b quark.

b) Use the Feynman rules to show that the matrix element for the decay $\mathrm{t} \to \mathrm{W}^+\mathrm{b}$ is given by

$$M_{\mathrm{fi}} = \frac{g_{\mathrm{W}}}{\sqrt{2}}\epsilon^*_\mu(p_4)\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_1)$$

where p_1 is the 4-momentum of the top quark and p_3 and p_4 are the 4-momenta of the b quark and W^+, respectively.

c) Consider the decay $\mathrm{t} \to \mathrm{W}^+\mathrm{b}$ in the top quark rest frame, with the b quark travelling in the $+z$ direction. Neglect the b quark mass. Draw diagrams illustrating the two spin configurations which are allowed in this case. Show that, when the top quark spin points in the $+z$ direction, the matrix element M_{fi} is given by

$$M_\uparrow = -g_{\mathrm{W}}\sqrt{2m_{\mathrm{t}}p^*}$$

where $p^* = (m_{\mathrm{t}}^2 - m_{\mathrm{W}}^2)/2m_{\mathrm{t}}$ is the magnitude of the three-momenta of the W^+ and the b quark. Show that when the top quark spin points in the $-z$ direction, the matrix element becomes

$$M_\downarrow = -g_{\mathrm{W}}\frac{m_{\mathrm{t}}}{m_{\mathrm{W}}}\sqrt{m_{\mathrm{t}}p^*}\ .$$

d) Explain why the decay of an unpolarised sample of top quarks must be isotropic, and show that the total decay rate in this case is

$$\Gamma = \frac{G_{\mathrm{F}}m_{\mathrm{t}}^3}{8\pi\sqrt{2}}\left(1-\frac{m_{\mathrm{W}}^2}{m_{\mathrm{t}}^2}\right)^2\left(1+\frac{2m_{\mathrm{W}}^2}{m_{\mathrm{t}}^2}\right)\ .$$

e) Calculate the top quark lifetime. Use the uncertainty principle to estimate a typical hadronisation timescale and comment on the result.

SOLUTION

a) The possible top quark decays into two quarks and an antiquark are

$$\begin{aligned}
&\mathrm{t} \to \mathrm{du\overline{d}},\ \mathrm{du\overline{s}},\ \mathrm{du\overline{b}},\ \mathrm{dc\overline{d}},\ \mathrm{dc\overline{s}},\ \mathrm{dc\overline{b}}\\
&\mathrm{t} \to \mathrm{su\overline{d}},\ \mathrm{su\overline{s}},\ \mathrm{su\overline{b}},\ \mathrm{sc\overline{d}},\ \mathrm{sc\overline{s}},\ \mathrm{sc\overline{b}}\\
&\mathrm{t} \to \mathrm{bu\overline{d}},\ \mathrm{bu\overline{s}},\ \mathrm{bu\overline{b}},\ \mathrm{bc\overline{d}},\ \mathrm{bc\overline{s}},\ \mathrm{bc\overline{b}}\ .
\end{aligned}$$

The possible decays into a quark, a lepton and an antilepton are

$$\begin{aligned}
&\mathrm{t} \to \mathrm{d}e^+\nu_e,\ \mathrm{d}\mu^+\nu_\mu,\ \mathrm{d}\tau^+\nu_\tau\\
&\mathrm{t} \to \mathrm{s}e^+\nu_e,\ \mathrm{s}\mu^+\nu_\mu,\ \mathrm{s}\tau^+\nu_\tau\\
&\mathrm{t} \to \mathrm{b}e^+\nu_e,\ \mathrm{b}\mu^+\nu_\mu,\ \mathrm{b}\tau^+\nu_\tau
\end{aligned}$$

The leading order Feynman diagram for all these decays contains $\mathrm{t} \to \mathrm{d} + \mathrm{W}^+$, $\mathrm{t} \to \mathrm{s} + \mathrm{W}^+$ or $\mathrm{t} \to \mathrm{b} + \mathrm{W}^+$, followed by $\mathrm{W}^+ \to \mathrm{q\overline{q}}$ or $\mathrm{W}^+ \to \ell^+\nu_\ell$:

b, s, d

t

u, c ν_e ν_μ ν_τ

W^+

$\overline{d}, \overline{s}, \overline{b}$ e^+ μ^+ τ^+

Because $V_{\rm tb} \approx 1$ is much bigger than $V_{\rm td}$ or $V_{\rm ts}$, the decays containing $\rm t \to b + W^+$ completely dominate. In addition, the W^+ propagator factor is proportional to $1/(q^2 - m_{\rm W}^2)$, which is a maximum when $q^2 \approx m_{\rm W}^2$, *i.e.* when the W^+ boson is real rather than virtual.

Hence the total top quark decay rate is dominated by the rate for the decay $\rm t \to W^+ b$ into a real W^+ boson and $\rm b$ quark.

b) The Feynman diagram for $\rm t \to W^+ b$ decay is

p_3 b

p_1

t

p_4 W^+

The Feynman rules give a factor $\epsilon_\mu^*(p_4)$ for the outgoing real W^+ boson:

$$-iM_{\rm fi} = \overline{u}(p_3) \cdot \frac{-ig_{\rm W}}{\sqrt{2}} \gamma^\mu \tfrac{1}{2}(1 - \gamma^5) \cdot u(p_1) \cdot \epsilon_\mu^*(p_4)$$

where p_1 is the 4-momentum of the top quark and p_3 and p_4 are the 4-momenta of the $\rm b$ quark and W^+, respectively. Hence

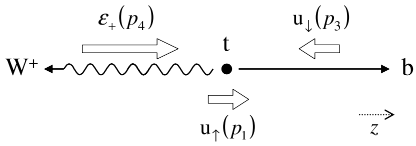
$$\boxed{M_{\rm fi} = \frac{g_{\rm W}}{\sqrt{2}} \epsilon_\mu^*(p_4) \overline{u}(p_3) \gamma^\mu \tfrac{1}{2}(1 - \gamma^5) u(p_1)}\ .$$

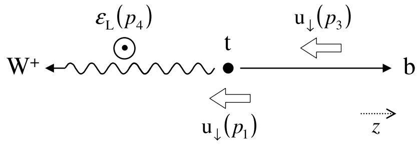
c) Consider the decay $\rm t \to W^+ b$ in the top quark rest frame, with the $\rm b$ quark travelling in the $+z$ direction, and neglect the $\rm b$ quark mass.

Since we have a $V - A$ interaction, in the massless limit the $\rm b$ quark must be left-handed, and must therefore have its spin pointing in the $-z$ direction. If the top quark spin points along $+z$, then, to conserve angular momentum, the W^+ spin must also point along $+z$. Alternatively, if the top quark spin points along $-z$, then the W^+ spin must be longitudinal, *i.e.* $S_z = 0$. In summary, the two allowed spin configurations are:

Since the $\rm b$ quark is left-handed, we have $u(p_3) = u_\downarrow(p_3)$. Since $\frac{1}{2}(1 - \gamma^5) u_\downarrow(p_3) = u_\downarrow(p_3)$, the matrix element then becomes

$$M_{\rm fi} = \frac{g_{\rm W}}{\sqrt{2}} \epsilon_\mu^*(p_4) \overline{u}_\downarrow(p_3) \gamma^\mu u(p_1)\ .$$





The four-momenta of the top quark, b quark and W^+ boson can be taken to be

$$p_1 = (m_\mathrm{t}, 0, 0, 0)\,, \qquad p_3 = (p^*, 0, 0, p^*)\,, \qquad p_4 = (E, 0, 0, -p^*)$$

where $E = \sqrt{(p^*)^2 + m_\mathrm{W}^2}$ is the energy of the W^+ boson and the centre of mass momentum p^* is the magnitude of the W^+ and b quark 3-momenta.

The two possible spin states for the initial t quark, and the final state b quark spinor are then

$$u_\uparrow(p_1) = \sqrt{2m_\mathrm{t}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad u_\downarrow(p_1) = \sqrt{2m_\mathrm{t}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad u_\downarrow(p_3) = \sqrt{p^*}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}.$$

For the case $u(p_1) = u_\uparrow(p_1)$, standard matrix multiplication gives the current as

$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = \sqrt{2m_\mathrm{t}p^*}\,(0, -1, -i, 0)\;,$$

while for the case $u(p_1) = u_\downarrow(p_1)$ the current becomes

$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = \sqrt{2m_\mathrm{t}p^*}\,(1, 0, 0, 1)\;.$$

The three possible spin states for the W^+ are

$$\epsilon_+^\mu(p_4) = \frac{-1}{\sqrt{2}}\,(0, 1, i, 0)\,, \qquad \epsilon_-^\mu(p_4) = \frac{1}{\sqrt{2}}\,(0, 1, -i, 0)\,, \qquad \epsilon_L^\mu(p_4) = \frac{1}{m_\mathrm{W}}\,(-p^*, 0, 0, E)\;.$$

For the case $u(p_1) = u_\uparrow(p_1)$, the scalar products $\epsilon^*.j$ are

$$\epsilon_+^*.j = \frac{-1}{\sqrt{2}}(0,1,-i,0) \cdot \sqrt{2m_\mathrm{t}p^*}\,(0,-1,-i,0) = -2\sqrt{m_\mathrm{t}p^*}$$
$$\epsilon_-^*.j = \frac{1}{\sqrt{2}}(0,1,i,0) \cdot \sqrt{2m_\mathrm{t}p^*}\,(0,-1,-i,0) = 0$$
$$\epsilon_L^*.j = \frac{1}{m_\mathrm{W}}(-p^*,0,0,E) \cdot \sqrt{2m_\mathrm{t}p^*}\,(0,-1,-i,0) = 0$$

Thus, as anticipated above, when the top quark spin points in the $+z$ direction, the matrix element is non-zero only when the W^+ spin also points in the $+z$ direction.

The matrix element M_fi for this case is given by

$$\boxed{M_\uparrow = -g_\mathrm{W}\sqrt{2m_\mathrm{t}p^*}}\ .$$

The centre of mass momentum p^* can be found using the general result derived in Question 3, or by eliminating the energy E between the two equations $E = \sqrt{(p^*)^2 + m_\mathrm{W}^2}$ and $m_\mathrm{t} = E + p^*$ (energy conservation). Either method gives

$$p^* = \frac{m_\mathrm{t}^2 - m_\mathrm{W}^2}{2m_\mathrm{t}}\ .$$

For the case $u(p_1) = u_\downarrow(p_1)$ when the top quark spin points in the $-z$ direction, the scalar products are

$$\epsilon_+^*.j = \frac{-1}{\sqrt{2}}(0,1,-i,0) \cdot \sqrt{2m_\mathrm{t}p^*}\,(1,0,0,1) = 0$$
$$\epsilon_-^*.j = \frac{1}{\sqrt{2}}(0,1,i,0) \cdot \sqrt{2m_\mathrm{t}p^*}\,(1,0,0,1) = 0$$
$$\epsilon_L^*.j = \frac{1}{m_\mathrm{W}}(-p^*,0,0,E) \cdot \sqrt{2m_\mathrm{t}p^*}\,(1,0,0,1) = -\frac{\sqrt{2m_\mathrm{t}p^*}}{m_\mathrm{W}}(E+p^*)$$

Thus, as anticipated above, the matrix element is non-zero only when the W^+ spin is longitudinal.

Energy conservation in the decay gives

$$m_\mathrm{t} = E + p^*\ .$$

Hence the matrix element for this case can be written

$$\boxed{M_\downarrow = -g_\mathrm{W}\frac{m_\mathrm{t}}{m_\mathrm{W}}\sqrt{m_\mathrm{t}p^*}}\ .$$

d) The decay of an unpolarised sample of top quarks must be isotropic because there is no preferred spatial direction in the initial state.

The spin-averaged matrix element squared is

$$\left\langle |M_\mathrm{fi}|^2 \right\rangle = \tfrac{1}{2}\left(|M_\uparrow|^2 + |M_\downarrow|^2\right) = \frac{1}{2}g_\mathrm{W}^2 m_\mathrm{t}p^*\left[2 + \frac{m_\mathrm{t}^2}{m_\mathrm{W}^2}\right]$$

The total top quark decay rate is

$$\Gamma = \frac{p^*}{8\pi m_\mathrm{t}^2}\left\langle|M_\mathrm{fi}|^2\right\rangle = \frac{g_\mathrm{W}^2(p^*)^2}{16\pi m_\mathrm{t}}\left[2+\frac{m_\mathrm{t}^2}{m_\mathrm{W}^2}\right]$$
$$= \frac{g_\mathrm{W}^2}{16\pi m_\mathrm{t}}\left(\frac{m_\mathrm{t}^2-m_\mathrm{W}^2}{2m_\mathrm{t}}\right)^2\left[2+\frac{m_\mathrm{t}^2}{m_\mathrm{W}^2}\right]$$
$$= \frac{g_\mathrm{W}^2 m_\mathrm{t}^3}{64\pi m_\mathrm{W}^2}\left(1-\frac{m_\mathrm{W}^2}{m_\mathrm{t}^2}\right)^2\left(1+\frac{2m_\mathrm{W}^2}{m_\mathrm{t}^2}\right) .$$

In terms of G_F, using

$$\frac{G_\mathrm{F}}{\sqrt{2}} = \frac{g_\mathrm{W}^2}{8m_\mathrm{W}^2} ,$$

the total decay rate is

$$\boxed{\Gamma = \frac{G_\mathrm{F} m_\mathrm{t}^3}{8\pi\sqrt{2}}\left(1-\frac{m_\mathrm{W}^2}{m_\mathrm{t}^2}\right)^2\left(1+\frac{2m_\mathrm{W}^2}{m_\mathrm{t}^2}\right)} .$$

e) With $G_\mathrm{F} = 1.166\times10^{-5}\,\mathrm{GeV}^{-2}$, $m_\mathrm{t} = 178\,\mathrm{GeV}$ and $m_\mathrm{W} = 80.4\,\mathrm{GeV}$, the top quark decay rate is

$$\Gamma = \frac{(1.166\times10^{-5})\times(178)^3}{8\pi\sqrt{2}}\left(1-\frac{(80.4)^2}{(178)^2}\right)^2\left(1+\frac{2(80.4)^2}{(178)^2}\right) = 1.65\,\mathrm{GeV} .$$

The top quark lifetime is then

$$\tau = \frac{6.582\times10^{-25}\,\mathrm{GeV.\,s}}{1.65\,\mathrm{GeV}} = 4.0\times10^{-25}\,\mathrm{s} .$$

Taking a typical energy involved in hadronisation to be the pion mass, the timescale can be crudely estimated to be

$$\tau_\mathrm{had} \sim \frac{6.6\times10^{-25}\,\mathrm{GeV.\,s}}{0.135\,\mathrm{GeV}} \sim 10^{-23}\,\mathrm{s} .$$

Thus the top quark lifetime is much less than the time it takes for quarks to hadronise. The top quark therefore decays before it can form a hadron - hadrons containing t quarks are not expected to exist.

THE HIGGS BOSON

29. a) Use the Feynman rules to show that the matrix element for the decay $\mathrm{H} \to \mathrm{W^+W^-}$ is

$$M_{\mathrm{fi}} = -g_{\mathrm{W}} m_{\mathrm{W}} g_{\mu\nu} \epsilon^{\mu}(p_2)^* \epsilon^{\nu}(p_3)^*$$

where p_2 and p_3 are the 4-momenta of the W^- and W^+, respectively.

b) Show that $M_{\mathrm{fi}} = -g_{\mathrm{W}} m_{\mathrm{W}}$ when both W bosons are left-handed or both are right-handed, that $M_{\mathrm{fi}} = (g_{\mathrm{W}}/m_{\mathrm{W}})(\frac{1}{2}m_{\mathrm{H}}^2 - m_{\mathrm{W}}^2)$ when both W bosons are longitudinally polarised, and that $M_{\mathrm{fi}} = 0$ for the six remaining combinations of W boson spin states.

c) Show that the $\mathrm{H} \to \mathrm{W^+W^-}$ decay rate is

$$\Gamma(\mathrm{H} \to \mathrm{W^+W^-}) = \frac{G_{\mathrm{F}} m_{\mathrm{H}}^3}{8\pi\sqrt{2}} \sqrt{1 - 4\lambda^2}\left(1 - 4\lambda^2 + 12\lambda^4\right)$$

where $\lambda = m_{\mathrm{W}}/m_{\mathrm{H}}$.

d) For $\mathrm{H} \to \mathrm{Z^0Z^0}$ decays, an extra factor of $\frac{1}{2}$ is required to account for the fact that the final state contains two identical particles. Show that

$$\Gamma(\mathrm{H} \to \mathrm{Z^0Z^0}) = \tfrac{1}{2}\Gamma(\mathrm{H} \to \mathrm{W^+W^-})|_{(m_{\mathrm{W}} \to m_{\mathrm{Z}})}\,.$$

e) For $\mathrm{H} \to \mathrm{f\bar{f}}$ decays into a fermion-antifermion pair, the decay rate is

$$\Gamma(\mathrm{H} \to \mathrm{f\bar{f}}) = N_c \frac{G_{\mathrm{F}}}{\sqrt{2}} \frac{m_{\mathrm{f}}^2 m_{\mathrm{H}}}{4\pi} \left(1 - \frac{4m_{\mathrm{f}}^2}{m_{\mathrm{H}}^2}\right)^{3/2}$$

where N_c is the number of colour degrees of freedom of the fermion f of mass m_{f} [See Tripos paper, Jan 2002, for a derivation of this result]. Compute the $\mathrm{H} \to \mathrm{W^+W^-}$, $\mathrm{H} \to \mathrm{Z^0Z^0}$ and $\mathrm{H} \to \mathrm{t\bar{t}}$ branching ratios and the total Higgs width Γ for a Higgs mass of 500 GeV. [Note that the decay rates into $\mathrm{f\bar{f}}$ final states other than $\mathrm{H} \to \mathrm{t\bar{t}}$ are negligibly small since $m_{\mathrm{f}} \ll m_{\mathrm{t}}$.]

SOLUTION

a) The leading-order Feynman diagram for the decay $\mathrm{H} \to \mathrm{W^+W^-}$ is

W^+

p_3

H

p_1

p_2

W^-

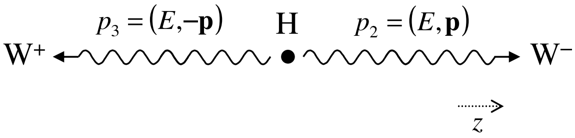
The Feynman rules give a factor $ig_{\mathrm{W}} m_{\mathrm{W}} g_{\mu\nu}$ for the HWW vertex, a factor $\epsilon^{\mu}(p_2)^*$ for the outgoing W^- boson, and a factor $\epsilon^{\nu}(p_3)^*$ for the outgoing W^+ boson. The product of these factors determines $-iM_{\mathrm{fi}}$:

$$-iM_{\mathrm{fi}} = ig_{\mathrm{W}} m_{\mathrm{W}} g_{\mu\nu} \cdot \epsilon^{\mu}(p_2)^* \cdot \epsilon^{\nu}(p_3)^*$$

and hence the matrix element is

$$\boxed{M_{\rm fi} = -g_{\rm W} m_{\rm W} g_{\mu\nu} \epsilon^{\mu}(p_2)^* \epsilon^{\nu}(p_3)^*}\,.$$

b) Take the W^- and W^+ 4-momenta to be $p_2 = (E, 0, 0, p)$ and $p_3 = (E, 0, 0, -p)$, with $E^2 = p^2 + m_{\rm W}^2$ and $E = m_{\rm H}/2$:



The three possible polarisation 4-vectors for the W^- are:

$$\begin{aligned}
\epsilon^{\mu}_{+}(p_2) &= -\frac{1}{\sqrt{2}}(0, 1, i, 0) \qquad h = +1 \\
\epsilon^{\mu}_{-}(p_2) &= \frac{1}{\sqrt{2}}(0, 1, -i, 0) \qquad h = -1 \\
\epsilon^{\mu}_{L}(p_2) &= \frac{1}{m_{\rm W}}(p, 0, 0, E) \qquad h = 0
\end{aligned}$$

while the three possible polarisation 4-vectors for the W^+ are:

$$\begin{aligned}
\epsilon_+^\nu(p_3) &= -\frac{1}{\sqrt{2}}(0,1,i,0) \qquad && h=-1 \\
\epsilon_-^\nu(p_3) &= \frac{1}{\sqrt{2}}(0,1,-i,0) \qquad && h=+1 \\
\epsilon_L^\nu(p_3) &= \frac{1}{m_\mathrm{W}}(-p,0,0,E) \qquad && h=0
\end{aligned}$$

Therefore, of the nine possible 4-vector scalar products of the form $\epsilon(p_2).\epsilon(p_3)$, only three are non-zero:

$$\begin{aligned}
\epsilon_+^\mu(p_2).\epsilon_-^\nu(p_3) &= -\frac{1}{\sqrt{2}}(0,1,i,0).\frac{1}{\sqrt{2}}(0,1,-i,0) \quad = +1 \\
\epsilon_-^\mu(p_2).\epsilon_+^\nu(p_3) &= \frac{1}{\sqrt{2}}(0,1,-i,0).-\frac{1}{\sqrt{2}}(0,1,i,0) \quad = +1 \\
\epsilon_L^\mu(p_2).\epsilon_L^\nu(p_3) &= \frac{1}{m_\mathrm{W}}(p,0,0,E).\frac{1}{m_\mathrm{W}}(-p,0,0,E) = \frac{1}{m_\mathrm{W}^2}(-p^2-E^2)
\end{aligned}$$

Thus the matrix element is non-zero only if both W bosons have the same helicity. This is to be expected: the Higgs boson has spin zero, and these are therefore the only possibilities consistent with conservation of angular momentum.

Since $E^2 = p^2 + m_\mathrm{W}^2$ and $E = m_\mathrm{H}/2$, we have

$$p = \sqrt{E^2 - m_\mathrm{W}^2} = \sqrt{\tfrac{1}{4}m_\mathrm{H}^2 - m_\mathrm{W}^2}$$

and hence

$$p^2 + E^2 = \tfrac{1}{4}m_\mathrm{H}^2 - m_\mathrm{W}^2 + \tfrac{1}{4}m_\mathrm{H}^2 = \tfrac{1}{2}m_\mathrm{H}^2 - m_\mathrm{W}^2\,.$$

Therefore the non-zero matrix elements are:

$$\begin{aligned}
\mathrm{H}\to\mathrm{W}_+\mathrm{W}_+ : &\qquad M_\mathrm{fi} = -g_\mathrm{W}m_\mathrm{W} \\
\mathrm{H}\to\mathrm{W}_-\mathrm{W}_- : &\qquad M_\mathrm{fi} = -g_\mathrm{W}m_\mathrm{W} \\
\mathrm{H}\to\mathrm{W}_L\mathrm{W}_L : &\qquad M_\mathrm{fi} = -g_\mathrm{W}m_\mathrm{W}\cdot -\frac{1}{m_\mathrm{W}^2}(p^2+E^2) = \frac{g_\mathrm{W}}{m_\mathrm{W}}(\tfrac{1}{2}m_\mathrm{H}^2 - m_\mathrm{W}^2)
\end{aligned}$$

where W_+, W_-, W_L denotes a W with helicity $h = +1, -1, 0$ respectively.

c) For an isotropic two-body decay, the decay rate is

$$\Gamma = \frac{p^*}{8\pi m_\mathrm{H}^2}|M_\mathrm{fi}|^2$$

where

$$p^* = p = \sqrt{\tfrac{1}{4}m_\mathrm{H}^2 - m_\mathrm{W}^2} = \frac{m_\mathrm{H}}{2}\sqrt{1 - \frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}}\,.$$

Hence, for the case where both W's are transversely polarised, we have

$$\begin{aligned}\Gamma(\mathrm{H} \to \mathrm{W}_+\mathrm{W}_+) = \Gamma(\mathrm{H} \to \mathrm{W}_-\mathrm{W}_-) &= \frac{p^*}{8\pi m_\mathrm{H}^2} \cdot g_\mathrm{W}^2 m_\mathrm{W}^2 \\ &= \frac{g_\mathrm{W}^2}{8\pi}\frac{m_\mathrm{W}^2}{2m_\mathrm{H}}\sqrt{1-\frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}}\end{aligned}$$

while if the W's are longitudinally polarised we have

$$\begin{aligned}\Gamma(\mathrm{H} \to \mathrm{W}_L\mathrm{W}_L) &= \frac{p^*}{8\pi m_\mathrm{H}^2} \cdot \frac{g_\mathrm{W}^2}{m_\mathrm{W}^2}(p^2+E^2)^2 \\ &= \frac{1}{8\pi m_\mathrm{H}^2}\frac{m_\mathrm{H}}{2}\sqrt{1-\frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}}\frac{g_\mathrm{W}^2}{m_\mathrm{W}^2}\left(\tfrac{1}{2}m_\mathrm{H}^2 - m_\mathrm{W}^2\right)^2 \\ &= \frac{g_\mathrm{W}^2}{64\pi}\frac{m_\mathrm{H}^3}{m_\mathrm{W}^2}\sqrt{1-\frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}}\left(1-\frac{2m_\mathrm{W}^2}{m_\mathrm{H}^2}\right)^2\end{aligned}$$

The total decay rate is obtained by summing over all possible final state spins:

$$\Gamma(\mathrm{H} \to \mathrm{W}^+\mathrm{W}^-) = \Gamma(\mathrm{H} \to \mathrm{W}_+\mathrm{W}_+) + \Gamma(\mathrm{H} \to \mathrm{W}_-\mathrm{W}_-) + \Gamma(\mathrm{H} \to \mathrm{W}_L\mathrm{W}_L) \tag{73}$$

$$= 2 \times \frac{g_\mathrm{W}^2}{8\pi}\frac{m_\mathrm{W}^2}{2m_\mathrm{H}}\sqrt{1-\frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}} + \frac{g_\mathrm{W}^2}{64\pi}\frac{m_\mathrm{H}^3}{m_\mathrm{W}^2}\sqrt{1-\frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}}\left(1-\frac{2m_\mathrm{W}^2}{m_\mathrm{H}^2}\right)^2 \tag{74}$$

$$= \frac{g_\mathrm{W}^2}{64\pi}\frac{m_\mathrm{H}^3}{m_\mathrm{W}^2}\sqrt{1-\frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}}\left(\frac{8m_\mathrm{W}^4}{m_\mathrm{H}^4} + 1 - \frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2} + \frac{4m_\mathrm{W}^4}{m_\mathrm{H}^4}\right) \tag{75}$$

$$= \frac{g_\mathrm{W}^2}{64\pi}\frac{m_\mathrm{H}^3}{m_\mathrm{W}^2}\sqrt{1-\frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}}\left(1 - \frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2} + \frac{12m_\mathrm{W}^4}{m_\mathrm{H}^4}\right) \tag{76}$$

Using

$$\frac{G_F}{\sqrt{2}} = \frac{g_\mathrm{W}^2}{8m_\mathrm{W}^2} \tag{77}$$

we finally obtain

$$\Gamma(\mathrm{H} \to \mathrm{W}^+\mathrm{W}^-) = \frac{G_\mathrm{F} m_\mathrm{H}^3}{8\pi\sqrt{2}}\sqrt{1-\frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}}\left(1 - \frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2} + \frac{12m_\mathrm{W}^4}{m_\mathrm{H}^4}\right) \tag{78}$$

d) For the decay $\mathrm{H} \to \mathrm{Z}^0\mathrm{Z}^0$, the Feynman rules give a vertex factor $ig_\mathrm{Z} m_\mathrm{Z} g_{\mu\nu}$ in place of $ig_\mathrm{W} m_\mathrm{W} g_{\mu\nu}$. Thus the $\mathrm{H} \to \mathrm{Z}^0\mathrm{Z}^0$ decay rate is given by Equation (76) with m_W replaced by m_Z, g_W replaced by g_Z, and with an extra factor of $\frac{1}{2}$ included to take into account the fact that the $\mathrm{H} \to \mathrm{Z}^0\mathrm{Z}^0$ decay contains two identical particles in the final state:

$$\Gamma(\mathrm{H} \to \mathrm{Z}^0\mathrm{Z}^0) = \frac{1}{2} \cdot \frac{g_\mathrm{Z}^2}{64\pi}\frac{m_\mathrm{H}^3}{m_\mathrm{Z}^2}\sqrt{1-\frac{4m_\mathrm{Z}^2}{m_\mathrm{H}^2}}\left(1 - \frac{4m_\mathrm{Z}^2}{m_\mathrm{H}^2} + \frac{12m_\mathrm{Z}^4}{m_\mathrm{H}^4}\right) .$$

The relations $m_{\rm W} = m_{\rm Z}\cos\theta_W$ and $g_{\rm W} = g_{\rm Z}\cos\theta_W$ give $g_{\rm W}^2/m_{\rm W}^2 = g_{\rm Z}^2/m_{\rm Z}^2$. Hence

$$\Gamma(\mathrm{H}\to \mathrm{Z}^0\mathrm{Z}^0) = \frac{1}{2}\cdot\frac{g_{\rm W}^2}{64\pi}\frac{m_{\rm H}^3}{m_{\rm W}^2}\sqrt{1-\frac{4m_{\rm Z}^2}{m_{\rm H}^2}}\left(1-\frac{4m_{\rm Z}^2}{m_{\rm H}^2}+\frac{12m_{\rm Z}^4}{m_{\rm H}^4}\right) .$$

Using Equation (77) to convert from $g_{\rm W}$ to $G_{\rm F}$, we then obtain

$$\Gamma(\mathrm{H}\to \mathrm{Z}^0\mathrm{Z}^0) = \frac{1}{2}\cdot\frac{G_{\rm F}m_{\rm H}^3}{8\pi\sqrt{2}}\sqrt{1-\frac{4m_{\rm Z}^2}{m_{\rm H}^2}}\left(1-\frac{4m_{\rm Z}^2}{m_{\rm H}^2}+\frac{12m_{\rm Z}^4}{m_{\rm H}^4}\right) .$$

A comparison with Equation (78) then shows immediately that

$$\boxed{\Gamma(\mathrm{H}\to \mathrm{Z}^0\mathrm{Z}^0) = \tfrac{1}{2}\Gamma(\mathrm{H}\to \mathrm{W}^+\mathrm{W}^-)|_{(m_{\rm W}\to m_{\rm Z})}} .$$

e) The decay rate into a fermion-antifermion pair is given by

$$\Gamma(\mathrm{H}\to \mathrm{f}\bar{\mathrm{f}}) = N_c\frac{G_{\rm F}}{\sqrt{2}}\frac{m_f^2 m_{\rm H}}{4\pi}\left(1-\frac{4m_f^2}{m_{\rm H}^2}\right)^{3/2} .$$

For the decay $\mathrm{H}\to \mathrm{t}\bar{\mathrm{t}}$, with $N_c = 3$, $m_{\rm H} = 500\,$GeV, $m_t = 175\,$GeV and $G_{\rm F} = 1.166\times10^{-5}\,$GeV^{-2} we have

$$\begin{aligned}\Gamma(\mathrm{H}\to \mathrm{t}\bar{\mathrm{t}}) &= 3\times\frac{1.166\times10^{-5}}{\sqrt{2}}\times\frac{175^2\times500}{4\pi}\left(1-\frac{4\times175^2}{500^2}\right)^{3/2}\ \mathrm{GeV}\\ &= 11.0\,\mathrm{GeV} .\end{aligned}$$

For the decays $\mathrm{H}\to \mathrm{W}^+\mathrm{W}^-$ and $\mathrm{H}\to \mathrm{Z}^0\mathrm{Z}^0$, with $m_{\rm W} = 80.4\,$GeV and $m_{\rm Z} = 91.2\,$GeV, we have $m_{\rm W}^2/m_{\rm H}^2 = (80.4/500)^2 = 0.0259$ and $m_{\rm Z}^2/m_{\rm H}^2 = (91.2/500)^2 = 0.0333$, giving

$$\begin{aligned}\Gamma(\mathrm{H}\to \mathrm{W}^+\mathrm{W}^-) &= \frac{1.166\times10^{-5}\times500^3}{8\pi\sqrt{2}}\sqrt{1-4\times0.0259}\left(1-4\times0.0259+12\times0.0259^2\right)\\ &= 35.1\,\mathrm{GeV}\end{aligned}$$

$$\begin{aligned}\Gamma(\mathrm{H}\to \mathrm{Z}^0\mathrm{Z}^0) &= \frac{1.166\times10^{-5}\times500^3}{16\pi\sqrt{2}}\sqrt{1-4\times0.0333}\left(1-4\times0.0333+12\times0.0333^2\right)\\ &= 16.8\,\mathrm{GeV}\end{aligned}$$

The total width of a Higgs boson of mass 500 GeV is therefore

$$\Gamma = 11.0+35.1+16.8 = 62.9\,\mathrm{GeV}$$

and the branching ratios are

$$\begin{aligned}\mathrm{BR}(\mathrm{H}\to \mathrm{W}^+\mathrm{W}^-) &= 35.1/62.9 = 55.8\%\\ \mathrm{BR}(\mathrm{H}\to \mathrm{Z}^0\mathrm{Z}^0) &= 16.8/62.9 = 26.7\%\\ \mathrm{BR}(\mathrm{H}\to \mathrm{t}\bar{\mathrm{t}}) &= 11.0/62.9 = 17.5\% .\end{aligned}$$

NUMERICAL ANSWERS

1. a) $L = 1.98\ m$, $M = 79$ kg. BMI $= 20.1$
2. d) $\sqrt{s} = 300\,$GeV; $E = 48000\,$GeV
3.
4.
5. $\Gamma(\rho \to \pi\pi)/\Gamma(\mathrm{K}^* \to \mathrm{K}\pi) = 3.46$; expt $= 2.98$
6. a) $\tau_\pi = 3.0 \times 10^{16}\,$GeV^{-1} = $1.97 \times 10^{-8}\,$s; expt $= 2.6 \times 10^{-8}\,$s

   b) from phase space alone: $\Gamma(\pi^+ \to e^+\nu_e)/\Gamma(\pi^+ \to \mu^+\nu_\mu) = 2.34$
7.
8.
9.
10.
11.
12. a) $\lambda = 0.84\,$GeV; b) 0.81 fm; c) $\approx 0.68\,$fm
13. b) $x \approx 0.09$, $Q^2 \approx 610\,\mathrm{GeV}^2$, $y \approx 0.075$; c) $M_\mathrm{X} \approx 78\,\mathrm{GeV}$

    d) relative probabilities that scattering is from u, d, $\overline{\mathrm{u}}$, $\overline{\mathrm{d}}$ are

    $$\mathrm{u} : \mathrm{d} : \overline{\mathrm{u}} : \overline{\mathrm{d}} \approx 0.73 : 0.12 : 0.12 : 0.04\ .$$

    e) the F_1 term contributes only $\approx 0.3\%$ of events.
14. d) $4.7^\circ < \theta < 21.3^\circ$
15.
16.
17.
18. $(Q^2)_{\mathrm{max}} \approx 750\,\mathrm{GeV}^2$
19. $f_\mathrm{q} \approx 0.41$, $f_{\overline{\mathrm{q}}} \approx 0.08$, $f_\mathrm{g} \approx 0.51$
20.
21. d) $\sin^2\theta_{13} > 0.051$ at 97.5% C.L., $|\Delta m^2_{23}| < 3.0 \times 10^{-3}\,\mathrm{eV}^2$; e) $\sin^2\theta_{13} = 0.09$, $|\Delta m^2_{23}| = 2.6 \times 10^{-3}\,\mathrm{eV}^2$

22. d) $|J|_{\max} = 0.053$; f) about $5000\,\mathrm{km}$, $|\Delta P|_{\max} \approx 0.04$

23.

24.

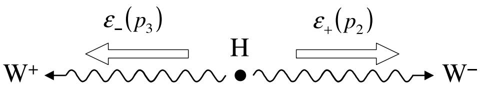
25.

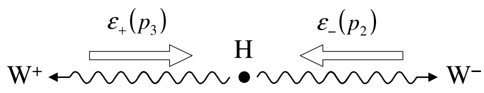
26. a) $\Gamma_Z = 2.3\,$GeV; b) $R = 20.1$; c) 61 nb

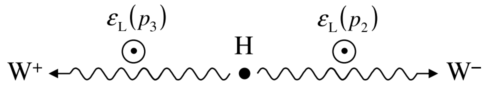
27. c) $\sin^2\theta_{\mathrm{W}} \approx 0.230 \pm 0.008$

28. e) $\tau \approx 4.0 \times 10^{-25}\,\mathrm{s}$, $\tau_{\mathrm{had}} \sim \times 10^{-23}\,\mathrm{s}$

29. e) $\mathrm{BR}(\mathrm{H} \to \mathrm{W}^+\mathrm{W}^-) = 55.8\%$, $\mathrm{BR}(\mathrm{H} \to \mathrm{Z}^0\mathrm{Z}^0) = 26.7\%$, $\mathrm{BR}(\mathrm{H} \to \mathrm{t}\bar{\mathrm{t}}) = 17.5\%$;
$\Gamma = 62.9\,\mathrm{GeV}$


$\varepsilon_-(p_3)$
$\varepsilon_+(p_2)$
H
W⁺
W⁻


$\varepsilon_+(p_3)$
$\varepsilon_-(p_2)$
H
W⁺
W⁻


$\varepsilon_L(p_3)$
$\varepsilon_L(p_2)$
H
W⁺
W⁻