NST Part III Experimental and Theoretical Physics Michaelmas 2023

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Particle Physics Major Option

EXAMPLES SHEET QUESTIONS (ALL)

NATURAL UNITS AND HEAVISIDE-LORENTZ UNITS

0. (a) In the units he normally uses, your particle-physics lecturer was $10^{16}/\,\mathrm{GeV}$ tall and had a mass of $4.40\times10^{28}\ \mathrm{GeV}$ when aged $2.11\times10^{33}/\,\mathrm{GeV}$. Calculate his Body Mass Index (BMI) and determine whether he was obese at this point in his life.

   (b) Show that charge can indeed be measured in units of $(\varepsilon_0\hbar c)^{\frac{1}{2}}$. [You may wish to consider dimensional analysis of the Coulomb force law $F=\frac{q_1q_2}{4\pi\varepsilon_0 r^2}$.]

SPECIAL RELATIVITY

1. a) Draw the two leading-order Feynman diagrams for $\mathrm{e^+e^-}\to\mathrm{e^+e^-}$ involving single photon exchange, and write q, the 4-momentum of the exchanged virtual photon, in terms of the 4-momenta of the initial and/or final state particles. By evaluating q^2 in the centre of mass frame, or otherwise, determine whether q is *timelike* ($q^2>0$) or *spacelike* ($q^2<0$) in each case.

   b) The *Mandelstam variables* s,t,u in the scattering process $a+b\to1+2$ are defined in terms of the particle 4-vectors as

$$s=(p_a+p_b)^2,\qquad t=(p_a-p_1)^2,\qquad u=(p_a-p_2)^2\ .$$

   Show that $s+t+u={m_a}^2+{m_b}^2+{m_1}^2+{m_2}^2$.

   c) Show that $\sqrt{s}$ is the total energy of the collision in the centre of mass frame.

   d) At the HERA accelerator in Hamburg, 27.5 GeV electrons are brought into head-on collision with 820 GeV protons. Calculate the centre of mass energy, $\sqrt{s}$, of $\mathrm{e^-p}$ collisions at HERA, and determine the beam energy that would be needed to produce $\mathrm{e^-p}$ collisions with this value of $\sqrt{s}$ using electrons incident on a stationary proton target.

   e) Show that, in the laboratory frame with particle X at rest, the reaction $\nu+X\to\ell+Y$ can only proceed if the incoming neutrino has an energy above a threshold given by

$$E_\nu>\frac{(m_l+m_Y)^2-m_X^2}{2m_X}\ .$$

2. a) For a particle of four-momentum $p^\mu=(E,p_x,p_y,p_z)$, show that the scalar product

$$p^2=E^2-p_x^2-p_y^2-p_z^2$$

is Lorentz invariant by explicitly transforming the four components of p^μ.

b) Use the Lorentz transformations to show that the volume element d^3p/E in momentum space is Lorentz invariant, *i.e.* that

$$\frac{\mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z}{E} = \frac{\mathrm{d}p'_x \mathrm{d}p'_y \mathrm{d}p'_z}{E'} \,.$$

3. In a 2-body decay, $a \to 1 + 2$, show that the three-momentum of the final state particles in the centre of mass frame has magnitude

$$p^* = \frac{1}{2m_a}\sqrt{[m_a^2 - (m_1 + m_2)^2]\,[m_a^2 - (m_1 - m_2)^2]} \,.$$

TWO BODY DECAY

4. According to the hypothesis of SU(3) symmetry (*i.e.* uds flavour independence) of invariant matrix elements, the two-body decay processes $\rho \to \pi\pi$ and $\mathrm{K}^* \to \mathrm{K}\pi$ have invariant matrix elements of the form

$$M_{\mathrm{fi}} = C p_\pi$$

where $C_\rho / C_{K^*} = 2/\sqrt{3}$ and p_π is the final state centre of mass momentum. Show that the predicted ratio of decay rates agrees with experiment to within about 15%.

[Use the result of Question 3 to obtain p_π. Take the π, ρ, K and K^* meson masses to be 139, 770, 494 and 892 MeV respectively. The measured widths are $\Gamma(\rho \to \pi\pi) = 153 \pm 2\,\mathrm{MeV}$ and $\Gamma(\mathrm{K}^* \to \mathrm{K}\pi) = 51.3 \pm 0.8\,\mathrm{MeV}$.]

5. The π^+ meson decays almost entirely via the two body decay process $\pi^+ \to \mu^+\nu_\mu$, with an invariant matrix element given by

$$|M_{\mathrm{fi}}|^2 = 2G_{\mathrm{F}}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)$$

where $G_{\mathrm{F}} = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$ is the Fermi constant, and f_π is related to the size of the pion wavefunction (the pion being a composite object).

a) Obtain a formula for the $\pi^+ \to \mu^+\nu_\mu$ decay rate. Assuming $f_\pi \sim m_\pi$, calculate the pion lifetime in natural units and in seconds, and compare to measurement.

[$m_\pi = 139.6\,\mathrm{MeV}$, $m_\mu = 105.7\,\mathrm{MeV}$.]

b) By replacing m_μ by m_e, show that the rate of $\pi^+ \to \mathrm{e}^+\nu_{\mathrm{e}}$ decay is 1.28×10^{-4} times smaller than the corresponding decay rate to muons. Show also that, on the basis of phase space alone (*i.e.* neglecting the factor $|M_{\mathrm{fi}}|^2$), the decay rate to electrons would be expected to be *greater* than the rate to muons.

THE DIRAC EQUATION

6. Write down a simplified form of the Dirac equation for a spinor $\psi(t)$ describing a particle of mass m at rest. For the standard Pauli-Dirac representation of the γ matrices, obtain a differential equation for each component ψ_i of the spinor ψ, and hence write down a general solution for the evolution of ψ. Comment on your result and on its relation to the standard plane wave solutions involving $u_1(p)$, $u_2(p)$, $v_1(p)$, $v_2(p)$.

7. a) For the standard Pauli-Dirac representation of the γ matrices, and for an arbitrary pair of spinors ψ and ϕ with components ψ_i and ϕ_i, show that the current $\overline{\psi}\gamma^\mu\phi$ is given by

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\
\overline{\psi}\gamma^1\phi &= \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\overline{\psi}\gamma^2\phi &= -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\overline{\psi}\gamma^3\phi &= \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2
\end{aligned}$$

b) For a particle or antiparticle with four-momentum $p^\mu = (E, p_x, p_y, p_z)$, show that

$$\overline{u}_1\gamma^\mu u_1 = \overline{u}_2\gamma^\mu u_2 = \overline{v}_1\gamma^\mu v_1 = \overline{v}_2\gamma^\mu v_2 = 2p^\mu$$

and that

$$\overline{u}_1\gamma^\mu u_2 = \overline{u}_2\gamma^\mu u_1 = \overline{v}_1\gamma^\mu v_2 = \overline{v}_2\gamma^\mu v_1 = 0\ .$$

c) Hence show that the current $j^\mu = \overline{\psi}(p)\gamma^\mu\psi(p)$ corresponding to a general free particle spinor $\psi(p) = u(p)e^{i(\boldsymbol{p}\boldsymbol{.}\boldsymbol{r}-Et)}$ or antiparticle spinor $\psi(p) = v(p)e^{-i(\boldsymbol{p}\boldsymbol{.}\boldsymbol{r}-Et)}$ is given by $j^\mu = 2p^\mu$. Write down the particle density and flux represented by j^μ.

8. a) For a particle with 4-momentum $p^\mu = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$, show that the spinors $(1+\gamma^5)u_1$ and $(1+\gamma^5)u_2$ are not in general proportional to $u_\uparrow$ but become so in the relativistic limit $E \gg m$.

   b) Define the terms *helicity* and *chirality*. How are chirality and helicity related to the spinors and result described in part (a) ?

   c) What would be the equivalent result to that described in (a) for the corresponding antiparticle spinors $(1+\gamma^5)v_1$ and $(1+\gamma^5)v_2$?

9. a) Without resorting to an explicit representation of the Dirac gamma matrices, show that the matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ has the following properties:

$$(\gamma^5)^2 = 1, \qquad \gamma^{5\dagger} = \gamma^5, \qquad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5\ .$$

b) Show that the adjoint spinors $\overline{\psi_\mathrm{L}}$ and $\overline{\psi_\mathrm{R}}$ corresponding to the left-handed and right-handed components $\psi_\mathrm{L} \equiv \frac{1}{2}(1-\gamma^5)\psi$ and $\psi_\mathrm{R} \equiv \frac{1}{2}(1+\gamma^5)\psi$ are:

$$\begin{aligned}
\overline{\psi_\mathrm{L}} &= \overline{\psi}\tfrac{1}{2}(1+\gamma^5) \\
\overline{\psi_\mathrm{R}} &= \overline{\psi}\tfrac{1}{2}(1-\gamma^5)\ .
\end{aligned}$$

c) Show that $\overline{\psi_\mathrm{L}}\gamma^\mu\psi_\mathrm{R} = \overline{\psi_\mathrm{R}}\gamma^\mu\psi_\mathrm{L} = 0$, and that the current $\overline{\psi}\gamma^\mu\psi$ can be decomposed as

$$\overline{\psi}\gamma^\mu\psi = \overline{\psi_\mathrm{L}}\gamma^\mu\psi_\mathrm{L} + \overline{\psi_\mathrm{R}}\gamma^\mu\psi_\mathrm{R}\ .$$

ELECTRON-MUON ELASTIC SCATTERING

10. a) Show that the matrix element for $\mathrm{e}^-\mu^- \to \mathrm{e}^-\mu^-$ scattering via single photon exchange is

$$M_{\mathrm{fi}} = -\frac{e^2}{(p_1-p_3)^2} g_{\mu\nu} \left[\overline{u}(p_3)\gamma^\mu u(p_1)\right] \left[\overline{u}(p_4)\gamma^\nu u(p_2)\right]$$

where p_1 and p_3 are the initial and final e^- four-momenta and p_2 and p_4 are the initial and final μ^- four-momenta.

b) Show that, for scattering in the centre of mass frame with incoming and outgoing e^- four-momenta $p_1^\mu = (E_1, 0, 0, p)$ and $p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta)$, the electron current for the various possible electron spin combinations is

$$\begin{aligned}
\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(E_1 c, ps, -ips, pc)\\
\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(ms, 0, 0, 0)\\
\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= 2(E_1 c, ps, ips, pc)\\
\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) &= -2(ms, 0, 0, 0)
\end{aligned}$$

where m is the electron mass and $s \equiv \sin\theta/2$, $c \equiv \cos\theta/2$.

c) Write down the incoming and outgoing muon 4-momenta p_2 and p_4, and the helicity eigenstate spinors $u_\uparrow(p_2)$, $u_\downarrow(p_2)$, $u_\uparrow(p_4)$ and $u_\downarrow(p_4)$. [Take the muon mass to be M and the muon energy to be E_2]. By comparing the forms of the muon and electron spinors, explain how the muon currents

$$\begin{aligned}
\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(E_2 c, -ps, -ips, -pc)\\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(Ms, 0, 0, 0)\\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) &= 2(E_2 c, -ps, ips, -pc)\\
\overline{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) &= -2(Ms, 0, 0, 0)
\end{aligned}$$

can be written down (up to overall factors of ± 1) without any further calculation.

d) Explain why some of the above currents vanish in the relativistic limit where the electron mass and muon mass can be neglected. Sketch the spin configurations which are allowed in this limit.

e) Show that, in the relativistic limit, the matrix element squared $|M_{\mathrm{LL}}|^2$ for the case where the incoming e^- and incoming μ^- are both left-handed is given by

$$|M_{\mathrm{LL}}|^2 = \frac{4e^4 s^2}{(p_1-p_3)^4}$$

where $s = (p_1+p_2)^2$. Why is the numerator of $|M_{\mathrm{LL}}|^2$ independent of θ ?

f) Find a similar expression for the matrix element $|M_{\mathrm{RL}}|^2$ for a right-handed incoming e^- and a left-handed incoming μ^-, and explain why $|M_{\mathrm{RL}}|^2$ vanishes when $\theta = \pi$. Write down the corresponding results for $|M_{\mathrm{RR}}|^2$ and $|M_{\mathrm{LR}}|^2$.

g) Show that, in the relativistic limit, the differential cross section for unpolarised $\mathrm{e}^-\mu^- \to \mathrm{e}^-\mu^-$ scattering in the centre of mass frame is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\alpha^2}{s} \cdot \frac{1+\frac{1}{4}(1+\cos\theta)^2}{(1-\cos\theta)^2}\,.$$

h) Show that the spin-averaged matrix element squared can be expressed in Lorentz-invariant form as

$$\langle |M_{\mathrm{fi}}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) \right] ,$$

and that a Lorentz invariant form for the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{2\pi\alpha^2}{q^4} \left[1 + \left(1 + \frac{q^2}{s} \right)^2 \right]$$

where $q^2 = (p_1 - p_3)^2$.

The remainder of this question involves the derivation of a general expression for $\langle |M_{\mathrm{fi}}|^2 \rangle$ for the case of finite electron and muon masses, and is optional:

i) Show that the spin-averaged matrix element squared for unpolarised $\mathrm{e}^-\mu^- \to \mathrm{e}^-\mu^-$ scattering can be written in the form

$$\langle |M_{\mathrm{fi}}|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |M_{\mathrm{fi}}|^2 = \frac{1}{4} \frac{e^4}{(p_1 - p_3)^4} L^{\mu\nu} W_{\mu\nu}$$

where the electron and muon tensors $L^{\mu\nu}$ and $W^{\mu\nu}$ are given by

$$L^{\mu\nu} \equiv \sum_{\text{spins}} \left[\overline{u}(p_3)\gamma^\mu u(p_1) \right] \left[\overline{u}(p_3)\gamma^\nu u(p_1) \right]^*$$

$$W_{\mu\nu} \equiv \sum_{\text{spins}} \left[\overline{u}(p_4)\gamma_\mu u(p_2) \right] \left[\overline{u}(p_4)\gamma_\nu u(p_2) \right]^*$$

j) Using the electron currents from part b) above, show that the components of the electron tensor $L^{\mu\nu}$ are

$$\begin{pmatrix} L^{00} & L^{01} & L^{02} & L^{03} \\ L^{10} & L^{11} & L^{12} & L^{13} \\ L^{20} & L^{21} & L^{22} & L^{23} \\ L^{30} & L^{31} & L^{32} & L^{33} \end{pmatrix} = 8 \begin{pmatrix} E_1^2 c^2 + m^2 s^2 & E_1 psc & 0 & E_1 pc^2 \\ E_1 psc & p^2 s^2 & 0 & p^2 sc \\ 0 & 0 & p^2 s^2 & 0 \\ E_1 pc^2 & p^2 sc & 0 & p^2 c^2 \end{pmatrix} ,$$

and hence verify that $L^{\mu\nu}$ has the Lorentz invariant form

$$L^{\mu\nu} = 4 \left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu} \left(m^2 - p_1.p_3 \right) \right] .$$

k) Write down the corresponding expression for $W^{\mu\nu}$ and hence show that

$$\langle |M_{\mathrm{fi}}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_2.p_4)m^2 + 2m^2M^2 \right]$$

11. a) The elastic form factors for the proton are well described by the form

$$G(q^2) = \frac{G(0)}{(1 + |q^2|/0.71)^2}$$

with q^2 in GeV^2. Show that an exponential charge distribution in the proton

$$\rho(\boldsymbol{r}) = \rho_0 e^{-\lambda r}$$

leads to this form for $G(q^2)$ (insofar as $|q^2| = |\boldsymbol{q}^2|$), and calculate λ.

b) Show that, for any spherically symmetric charge distribution, the mean square radius is given by

$$\langle r^2 \rangle = -\frac{6}{G(0)} \left[\frac{\mathrm{d}G(q^2)}{\mathrm{d}|q^2|} \right]_{q^2=0}$$

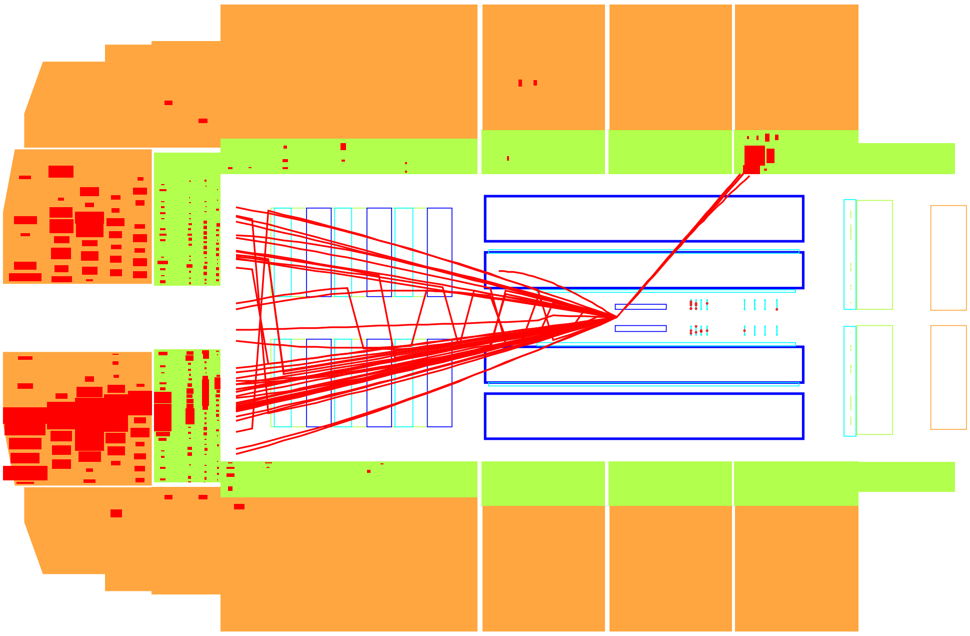
and estimate the r.m.s. charge radius of the proton.

c) The pion form factor may be determined in πe^- scattering. Use the following data to estimate the r.m.s. charge radius of the pion.

$\|q^2\|$ (GeV^2)	$G_E^2(q^2)$
0.015	0.944 ± 0.007
0.042	0.849 ± 0.009
0.074	0.777 ± 0.016
0.101	0.680 ± 0.017
0.137	0.646 ± 0.027
0.173	0.534 ± 0.030
0.203	0.529 ± 0.040
0.223	0.487 ± 0.049

DEEP-INELASTIC SCATTERING

12. The figure below shows a deep-inelastic scattering event $e^+p \to e^+X$ recorded by the H1 experiment at the HERA collider. The positron beam, of energy $E_1 = 27.5\,\mathrm{GeV}$, enters from the left and the proton beam, of energy $E_2 = 820\,\mathrm{GeV}$, enters from the right. The energy of the outgoing positron is measured to be $E_3 = 31\,\mathrm{GeV}$. The picture is to scale, so angles may be read off the diagram if required.



a) Show that the Bjorken scaling variable x is given by

$$x = \frac{E_3}{E_2}\left[\frac{1-\cos\theta}{2-(E_3/E_1)(1+\cos\theta)}\right]$$

where θ is the angle through which the positron has scattered.

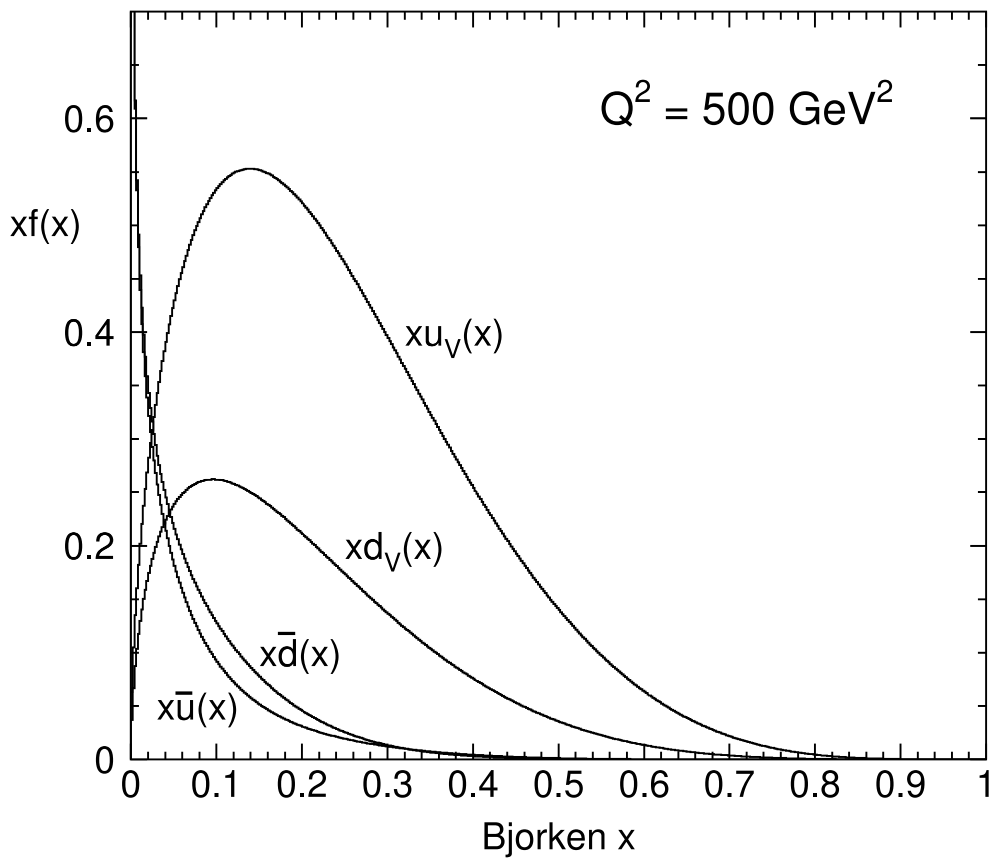
b) Estimate the values of Q^2, x and y for this event.

c) Estimate the invariant mass M_X of the final state hadronic system.

d) Draw quark level diagrams to illustrate the possible origins of this event. Using the plot overleaf of the parton distribution functions $xu_V(x)$, $xd_V(x)$, $x\overline{u}(x)$ and $x\overline{d}(x)$, estimate the relative probabilities of the various possible quark-level processes for the event. Note that the Q^2 in the plot overleaf need not be exactly the same as the Q^2 in this event – Bjorken scaling requires only that it be similar. So do not worry about any relatively small differences between the two Q^2 scales.

[Neglect contributions from the heavier quarks s, c, b, t.]

e) Estimate the relative contributions of the F_1 and F_2 terms to the deep-inelastic cross section for the x and Q^2 values corresponding to this event.



$Q^2 = 500\ GeV^2$
xf(x)
$xu_V(x)$
$xd_V(x)$
$x\bar{d}(x)$
$x\bar{u}(x)$
0
0.2
0.4
0.6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Bjorken x

13. a) Show that the lab frame differential cross section $\mathrm{d}^2\sigma/\mathrm{d}E_3\mathrm{d}\Omega$ for deep-inelastic scattering is related to the Lorentz invariant differential cross section $\mathrm{d}^2\sigma/\mathrm{d}\nu\mathrm{d}Q^2$ via

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}Q^2} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2}$$

where E_1 and E_3 are the energies of the incoming and outgoing lepton, $\nu = E_1 - E_3$, and $Q^2 = -q^2 = -(p_1 - p_3)^2$. [When you do this, make sure you understand that differential cross sections transform as Jacobians, not as partial derivatives!]

Show further that

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2} = \frac{2Mx^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}$$

where M is the mass of the target nucleon and $x = Q^2/2M\nu$.

b) Show that

$$\frac{2Mx^2}{Q^2}\cdot\frac{y^2}{2} = \frac{1}{M}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}$$

and that

$$1 - y - \frac{M^2x^2y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2}\,.$$

c) Show that the Lorentz invariant cross section for deep-inelastic electromagnetic scattering,

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)\frac{F_2}{x} + \frac{y^2}{2}\frac{2xF_1}{x}\right]$$

becomes

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{F_2}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1}{M}\sin^2\frac{\theta}{2}\right]$$

in the lab frame.

d) An experiment consists of an electron beam of maximum energy $20\,\mathrm{GeV}$ and a variable angle spectrometer which can detect scattered electrons with energies greater than $2\,\mathrm{GeV}$. Find the range of values of θ over which deep-inelastic scattering events can be studied for $x = 0.2$ and $Q^2 = 2\,\mathrm{GeV}^2$.

[You may find it helpful to determine $E_1 - E_3$ (fixed), and E_1E_3 in terms of θ, and then sketch the various constraints on E_1 and E_3 on a 2D plot of E_3 against E_1.]

e) Outline a possible experimental strategy for measuring $F_1(x, Q^2)$ and $F_2(x, Q^2)$ for the above values of x and Q^2.

HADRONS AND QCD

14. [This is lifted from the 2016 Tripos Paper]

Suppose there exists a 'Bogus' universe in which the laws of physics are the same as in ours, except in one respect: quantum chromodynamics in the 'Bogus' universe is based on an $SU(2)$ colour symmetry having only two colours ('red' and 'green') rather than the three colour $SU(3)$ symmetry of our own.

(a) Determine which 'Bogus mesons' and 'Bogus baryons' (or their nearest equivalents) could exist by constructing any important colour, flavour and spin wave-functions. Categorise the expected 'Bogus' hadrons by type (meson/baryon), spin, and the multiplets they inhabit. Compare 'Bogus' hadron structure to that in our own universe, highlighting the main similarities and differences. [*Above you need only consider light quarks types:* u, d *and* s.]

(b) The change from $SU(3)$ colour to $SU(2)$ colour could affect more than the basic hadron structure considered above. It could have consequences in other areas of particle physics and even further afield. Discuss any such expected differences between the 'Bogus' universe and our own.

15. [This question has been deleted.]

WEAK INTERACTIONS

16. Following on from Question 9, show that, for a free particle spinor ψ:

$$\overline{\psi_{\rm L}}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi_{\rm R} = \overline{\psi_{\rm R}}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi_{\rm L} = \overline{\psi_{\rm R}}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi_{\rm R} = 0$$

$$\overline{\psi_{\rm L}}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi_{\rm L} = \overline{\psi}\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\psi$$

where $\psi_{\rm L} \equiv \frac{1}{2}(1-\gamma^5)\psi$ and $\psi_{\rm R} \equiv \frac{1}{2}(1+\gamma^5)\psi$. Explain the relevance of these results to the weak interactions. What are the equivalent results for currents of the form $\overline{\psi}\gamma^\mu\frac{1}{2}(1+\gamma^5)\psi$?

17. a) In Question 5, the decay rate for $\pi^- \to {\rm e}^-\overline{\nu}_{\rm e}$ was found to be 1.28×10^{-4} times that for $\pi^- \to \mu^-\overline{\nu}_\mu$, whereas, on the basis of phase space alone, one would expect a higher decay rate to electrons. Explain why the weak interaction gives such a small decay rate to electrons.

b) The Lorentz invariant matrix element for $\pi^- \to \mu^-\overline{\nu}_\mu$ decay is

$$M_{\rm fi} = \frac{g_{\rm W}^2}{4m_{\rm W}^2} g_{\mu\nu} f_\pi p_1^\mu \overline{u}(p_3)\gamma^\nu\tfrac{1}{2}(1-\gamma^5)v(p_4)$$

where p_1, p_3 and p_4 are the 4-momenta of the π^-, μ^- and $\overline{\nu}_\mu$, respectively, and f_π is a constant which must be determined experimentally. Verify that this matrix element follows from the Feynman rules, with the quark current $\overline{u}\gamma^\mu(1-\gamma^5)v$ taken to be of the form $-f_\pi p_1^\mu$.

[The free particle spinors u, v cannot be used for quarks and antiquarks in a hadronic bound state; a quark current of the form given can be shown to be the most general possibility.]

c) Show that (as in Question 5) the Lorentz-invariant matrix element squared is

$$|M_{\rm fi}|^2 = 2G_{\rm F}^2 f_\pi^2 m_\mu^2(m_\pi^2 - m_\mu^2)\ .$$

[Use the spinors u_1, u_2, v_1, v_2 for this calculation rather than the spinors $u_\uparrow$, $u_\downarrow$, $v_\uparrow$, $v_\downarrow$. Work in the π^- rest frame, and choose the 4-momenta of the μ^- and $\overline{\nu}_\mu$ to be $p_3 = (E,0,0,p)$ and $p_4 = (p,0,0,-p)$, with $E = \sqrt{p^2+m_\mu^2}$.]

d) Show that the square of the *non-invariant* matrix element $T_{\rm fi}$ is proportional to $1-\beta$:

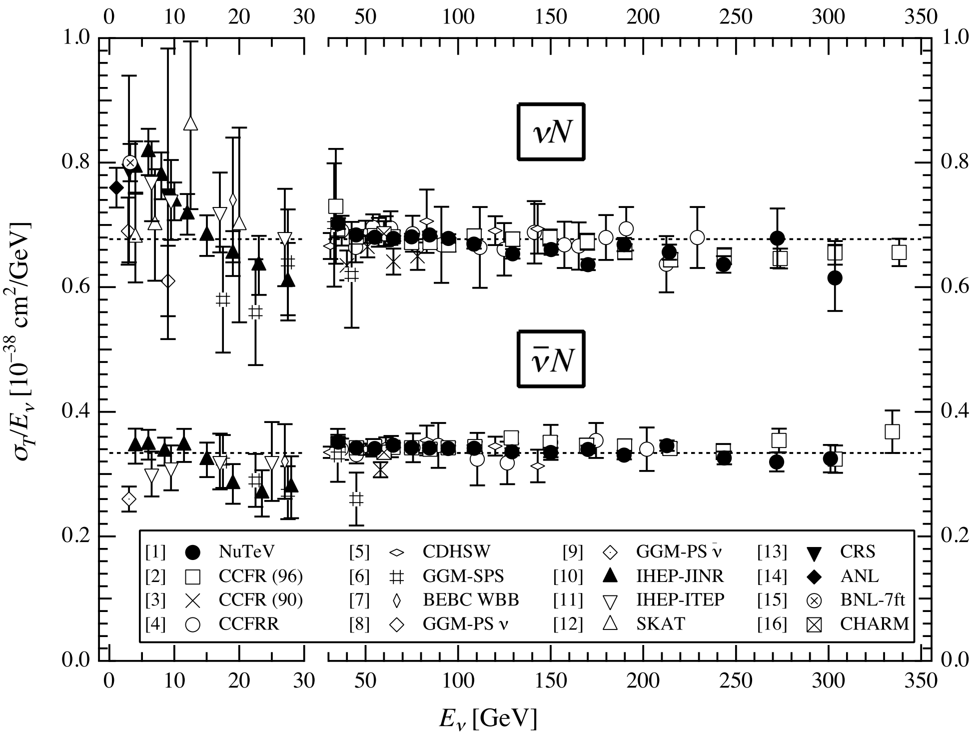
$$|T_{\rm fi}|^2 = \frac{G_{\rm F}^2}{2} f_\pi^2 m_\pi\,(1-\beta)$$

where β is the velocity of the μ^-.

DEEP INELASTIC SCATTERING

18. Find the maximum possible value of Q^2 in deep-inelastic neutrino scattering for a neutrino beam energy of $400\,\mathrm{GeV}$, and compare with m_{W}^2.

19. The figure below shows the measured total cross sections $\sigma(\nu_\mu + \mathrm{N} \to \mu^- + \text{hadrons})/E_\nu$ and $\sigma(\overline{\nu}_\mu + \mathrm{N} \to \mu^- + \text{hadrons})/E_{\overline{\nu}}$ for charged-current neutrino and antineutrino scattering, averaged over proton and neutron targets.



a) Draw Feynman diagrams for the quark-level processes which contribute to neutrino-nucleon and antineutrino-nucleon scattering. (Neglect the s, c, b and t quark flavours).

b) Show that the parton model predicts total cross sections of the form

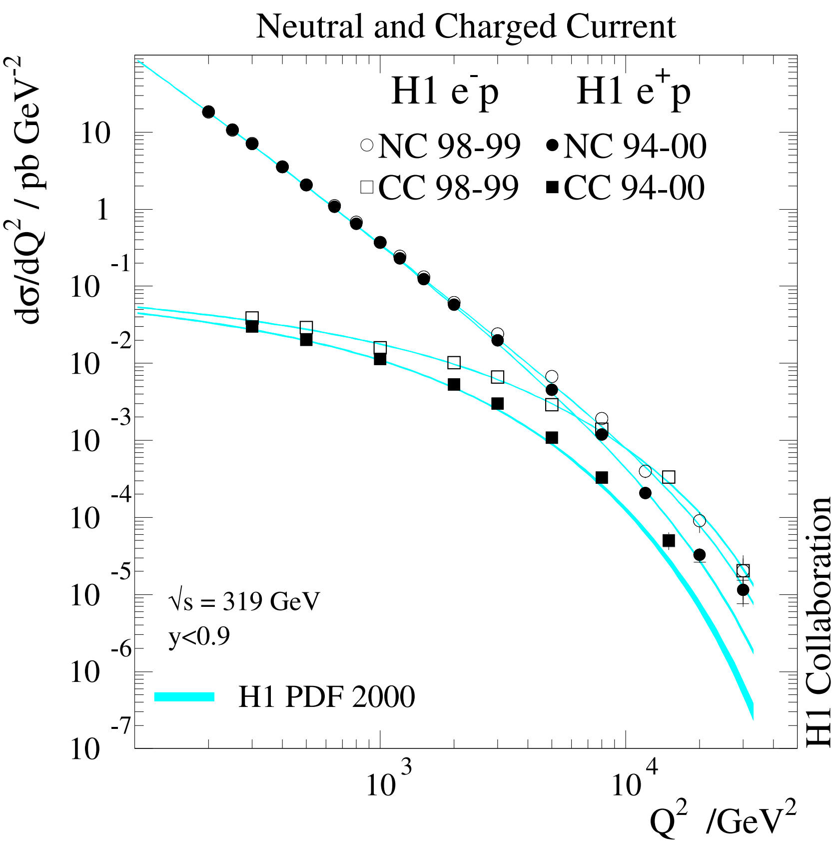
$$\sigma^{\nu N} \equiv \tfrac{1}{2}\left(\sigma^{\nu \mathrm{p}} + \sigma^{\nu \mathrm{n}}\right) = \frac{G_{\mathrm{F}}^2 s}{2\pi}\left[f_{\mathrm{q}} + \tfrac{1}{3} f_{\overline{\mathrm{q}}}\right]$$
$$\sigma^{\overline{\nu} N} \equiv \tfrac{1}{2}\left(\sigma^{\overline{\nu} \mathrm{p}} + \sigma^{\overline{\nu} \mathrm{n}}\right) = \frac{G_{\mathrm{F}}^2 s}{2\pi}\left[\tfrac{1}{3} f_{\mathrm{q}} + f_{\overline{\mathrm{q}}}\right]$$

where s is the neutrino-nucleon centre of mass energy squared, and $f_{\mathrm{q}} = f_{\mathrm{u}} + f_{\mathrm{d}}$ and $f_{\overline{\mathrm{q}}} = f_{\overline{\mathrm{u}}} + f_{\overline{\mathrm{d}}}$ are the average momentum fractions carried by u and d quarks and antiquarks.

c) Estimate the average fractions of the nucleon momentum carried by quarks, antiquarks and gluons.

[Take $G_{\mathrm{F}} = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$.]

20. The figure below shows measurements of the cross section $\mathrm{d}\sigma/\mathrm{d}Q^2$ from the H1 experiment at HERA for the neutral current (NC) processes $\mathrm{e^-p \to e^-X}$ and $\mathrm{e^+p \to e^+X}$, and the charged current (CC) processes $\mathrm{e^-p} \to \nu_\mathrm{e}\mathrm{X}$ and $\mathrm{e^+p} \to \overline{\nu}_\mathrm{e}\mathrm{X}$, with unpolarised incoming $\mathrm{e^+}$ or $\mathrm{e^-}$ and proton beams:



a) Draw Feynman diagrams for the quark-level processes which contribute to CC $\mathrm{e^-p} \to \nu_\mathrm{e}\mathrm{X}$ and $\mathrm{e^+p} \to \overline{\nu}_\mathrm{e}\mathrm{X}$ scattering. (Neglect the s, c, b and t quark flavours).

b) The HERA data extends to values of $Q^2 > m_\mathrm{W}^2$. Starting from the parton model cross sections $\mathrm{d}^2\sigma/\mathrm{d}x\mathrm{d}y$ for (anti)neutrino-nucleon scattering derived in the lectures for $Q^2 \ll m_\mathrm{W}^2$, explain why the CC cross sections can be written down directly as

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}(\mathrm{e^+p} \to \overline{\nu}_\mathrm{e}\mathrm{X}) = \frac{G_\mathrm{F}^2 m_\mathrm{W}^4}{2\pi x(Q^2+m_\mathrm{W}^2)^2} x\left[\overline{u}(x) + (1-y)^2 d(x)\right]$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}(\mathrm{e^-p} \to \nu_\mathrm{e}\mathrm{X}) = \frac{G_\mathrm{F}^2 m_\mathrm{W}^4}{2\pi x(Q^2+m_\mathrm{W}^2)^2} x\left[u(x) + (1-y)^2 \overline{d}(x)\right]$$

c) Explain why the $\mathrm{e^-p}$ CC cross section is always higher than the $\mathrm{e^+p}$ CC cross section.

d) Explain why the CC cross sections become approximately constant as Q^2 decreases, while the NC cross sections grow indefinitely large. Account approximately for the observed slope of the NC cross sections at low values of Q^2.

e) Explain why the NC cross sections become similar in magnitude to the CC cross sections at high values of $Q^2 \sim m_\mathrm{Z}^2$.

f) (optional) Explain why the two NC cross sections are equal at low Q^2, but differ at high Q^2.

NEUTRINO OSCILLATIONS

21. In the Daya Bay experiment (`arXiv:1203.1669` and `arXiv:1310.6732`) electron antineutrinos from six nuclear reactors were observed in six detectors in three experimental halls, some $\approx 0.5\,\mathrm{km}$ and some $\approx 1.5\,\mathrm{km}$ distant from the reactors. The nuclear reactors emit electron antineutrinos of mean energy $E \approx 3\,\mathrm{MeV}$, and the detectors can resolve their energy to within a few percent.

a) Show that neutrino oscillations associated with the (solar) mass-squared difference $|\Delta m_{12}^2| \approx 7 \times 10^{-5}\,\mathrm{eV}^2$ can be neglected for the Daya Bay experiment, and that

$$P(\overline{\nu}_\mathrm{e} \to \overline{\nu}_\mathrm{e}) \approx 1 - \sin^2 2\theta_{13} \sin^2 \Delta_{23}$$

where

$$\Delta_{23} \equiv \frac{\Delta m_{23}^2 L}{4E} \, .$$

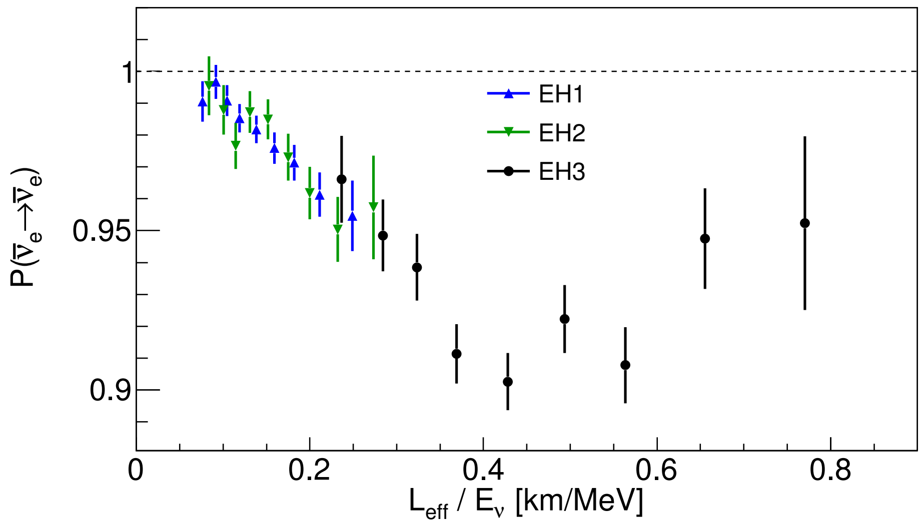
b) In the limit $|\Delta m_{23}^2| \gg (E/L)$, explain why a given measurement, P, of the survival probability $P(\overline{\nu}_\mathrm{e} \to \overline{\nu}_\mathrm{e})$ determines the neutrino mixing to be $\sin^2 2\theta_{13} = 2(1-P)$.

c) In the limit $|\Delta m_{23}^2| \ll (E/L)$, show that a given measurement, P, of the survival probability $P(\overline{\nu}_\mathrm{e} \to \overline{\nu}_\mathrm{e})$ determines the neutrino mixing to be $\sin^2 2\theta_{13} \propto 1/(\Delta m_{23}^2)^2$, with constant of proportionality $(1-P)(4E/L)^2$.

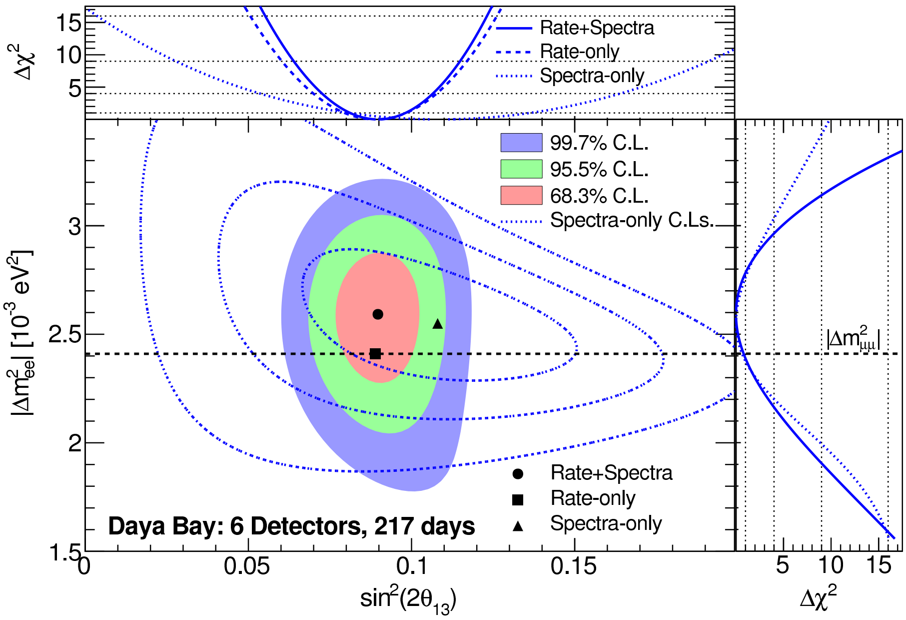
d) The third experimental hall is a (weighted) distance of $1.63\,\mathrm{km}$ from the reactor complex. A detector here sees a fractional deficit in the number of electron antineutrinos of 0.071 ± 0.010, compared to that expected from the neutrino fluxes of the reactors. Place a lower bound on the value of $\sin^2 2\theta_{13}$.

The deficit is observed to monotonically decrease for neutrinos of energy greater than $4\,\mathrm{MeV}$ average. What bound does this place on Δm_{23}^2?

e) The plot below shows the ratio of the number of observed to number of expected electron antineutrinos, as a function of the effective detector-reactor distance L_eff over the observed neutrino energies E_ν. It comprises data from all the detectors in the three experimental halls. Estimate values for $\sin^2 2\theta_{13}$ and Δm_{23}^2.



f) Sketch your results of parts (d) and (e) on a plot of the values of $\sin^2 2\theta_{13}$ and Δm^2_{23}, as fitted to the data by the Daya Bay collaboration.



22. a) It was shown in the lectures (see Equation (14) of Handout 12) that a general expression for the probability that an initial $\nu_{\rm e}$ oscillates into a ν_μ is

$$P(\nu_{\rm e} \to \nu_\mu) = 2\sum_{i<j} \mathrm{Re}\left(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j}\left[e^{-i(E_i-E_j)t} - 1\right]\right) \, .$$

Show that

$$P(\nu_{\rm e} \to \nu_\mu) = -4\sum_{i<j} \mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j}) \sin^2\Delta_{ij} + 2\sum_{i<j} \mathrm{Im}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j}) \sin 2\Delta_{ij}$$

where

$$\Delta_{ij} \equiv \frac{(m_i^2 - m_j^2)L}{4E} \equiv \frac{\Delta m^2_{ij}L}{4E} \, .$$

b) Use the unitarity of the PMNS matrix to show that

$$\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e3}U_{\mu 3}) = -\mathrm{Im}(U_{e2}U^*_{\mu 2}U^*_{e3}U_{\mu 3}) = -\mathrm{Im}(U_{e1}U^*_{\mu 1}U^*_{e2}U_{\mu 2}) \equiv -J, \text{ say} \, .$$

c) Hence show that

$$P(\nu_{\rm e} \to \nu_\mu) = -4\sum_{i<j} \mathrm{Re}(U_{ei}U^*_{\mu i}U^*_{ej}U_{\mu j}) \sin^2\Delta_{ij} + 8J \sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23}$$

[You may wish to use the trigonometric identity

$$\sin A + \sin B - \sin(A+B) = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{A+B}{2} \, .]$$

d) The standard parameterisation of the PMNS matrix is

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. Show that, in this parameterisation,

$$J = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\sin\delta$$

and find the maximum possible value of $|J|$ given the present experimental knowledge of the mixing angles θ_{12}, θ_{23} and θ_{13}.

e) The conversion probabilities for antineutrinos are obtained by replacing U by U^*. Show that

$$P(\nu_{\rm e} \to \nu_\mu) - P(\overline{\nu}_{\rm e} \to \overline{\nu}_\mu) = 16J \sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23} \, .$$

f) It is proposed to build a "neutrino factory" to search for evidence of CP violation in neutrino oscillations; $P(\nu_{\rm e} \to \nu_\mu) \neq P(\overline{\nu}_{\rm e} \to \overline{\nu}_\mu)$. A neutrino factory would produce an intense beam of neutrinos with typical energy $10\,\mathrm{GeV}$. Roughly how far away should a neutrino detector be positioned to optimise the chances of observing CP violation, and how large an effect might be expected ?

CP VIOLATION AND THE CKM MATRIX

23. a) Draw Feynman diagrams for the decays $\mathrm{K}^0 \to \pi^+\pi^-$ and $\overline{\mathrm{K}}^0 \to \pi^+\pi^-$, and for the decays $\mathrm{K}^0 \to \pi^0\pi^0$ and $\overline{\mathrm{K}}^0 \to \pi^0\pi^0$.

b) Draw Feynman diagrams for the decays $\mathrm{K}^0 \to \pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\overline{\mathrm{K}}^0 \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$, and explain why the decays $\overline{\mathrm{K}}^0 \to \pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\mathrm{K}^0 \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$ cannot occur.

c) How does the decay rate for each of the above decays depend on the Cabibbo angle θ_C ?

24. In the CPLEAR experiment at CERN, neutral kaons are produced in low energy proton-antiproton collisions via the channels $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\overline{\mathrm{K}}^0$ and $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\mathrm{K}^0$. The strangeness of the initial $\overline{\mathrm{K}}^0$ or K^0 is tagged by the charge of the accompanying K^+ or K^-, and the K^0 or $\overline{\mathrm{K}}^0$ is subsequently detected via decays into the semileptonic final states $\pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$.

a) Draw Feynman diagrams for the reactions $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\overline{\mathrm{K}}^0$ and $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\mathrm{K}^0$, and explain why the reactions $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\mathrm{K}^0$ and $\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\overline{\mathrm{K}}^0$ cannot occur.

b) Show that, for a system which is initially in a pure K^0 state, the decay rates R_+ and R_- to the semileptonic final states $\pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$ depend on the proper decay time t as

$$R_+ \equiv \Gamma(\mathrm{K}^0_{t=0} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) = N_{\pi e\nu}\tfrac{1}{4}\left[e^{-\Gamma_\mathrm{S}t} + e^{-\Gamma_\mathrm{L}t} + 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}\cos\Delta mt\right]$$
$$R_- \equiv \Gamma(\mathrm{K}^0_{t=0} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}) \approx N_{\pi e\nu}\tfrac{1}{4}\left[1 - 4\mathrm{Re}\epsilon\right]\left[e^{-\Gamma_\mathrm{S}t} + e^{-\Gamma_\mathrm{L}t} - 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}\cos\Delta mt\right]$$

where $\Gamma_S = 1/\tau_S$, $\Gamma_L = 1/\tau_L$, $\Delta m = m_\mathrm{L} - m_\mathrm{S}$, ϵ is the CP violation parameter, and $N_{\pi e\nu}$ is an overall normalisation constant. Show that the corresponding expressions for a system which is initially in a pure $\overline{\mathrm{K}}^0$ state are

$$\bar{R}_+ \equiv \Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) \approx N_{\pi e\nu}\tfrac{1}{4}\left[1 + 4\mathrm{Re}\epsilon\right]\left[e^{-\Gamma_\mathrm{S}t} + e^{-\Gamma_\mathrm{L}t} - 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}\cos\Delta mt\right]$$
$$\bar{R}_- \equiv \Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}) = N_{\pi e\nu}\tfrac{1}{4}\left[e^{-\Gamma_\mathrm{S}t} + e^{-\Gamma_\mathrm{L}t} + 2e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}\cos\Delta mt\right] .$$

c) The figure overleaf shows a measurement from the CPLEAR experiment of the asymmetry

$$A_{\Delta m} \equiv \frac{(R_+ + \overline{R}_-) - (\overline{R}_+ + R_-)}{(R_+ + \overline{R}_-) + (\overline{R}_+ + R_-)}$$

as a function of the proper decay time $\tau = t$ (plotted in units of the $\mathrm{K_S}$ lifetime $\tau_S = 0.9 \times 10^{-10}$ s). Show that $A_{\Delta m}$ is given by

$$A_{\Delta m} = \frac{2\cos(\Delta mt)\, e^{-(\Gamma_\mathrm{S}+\Gamma_\mathrm{L})t/2}}{e^{-\Gamma_\mathrm{S}t} + e^{-\Gamma_\mathrm{L}t}}$$

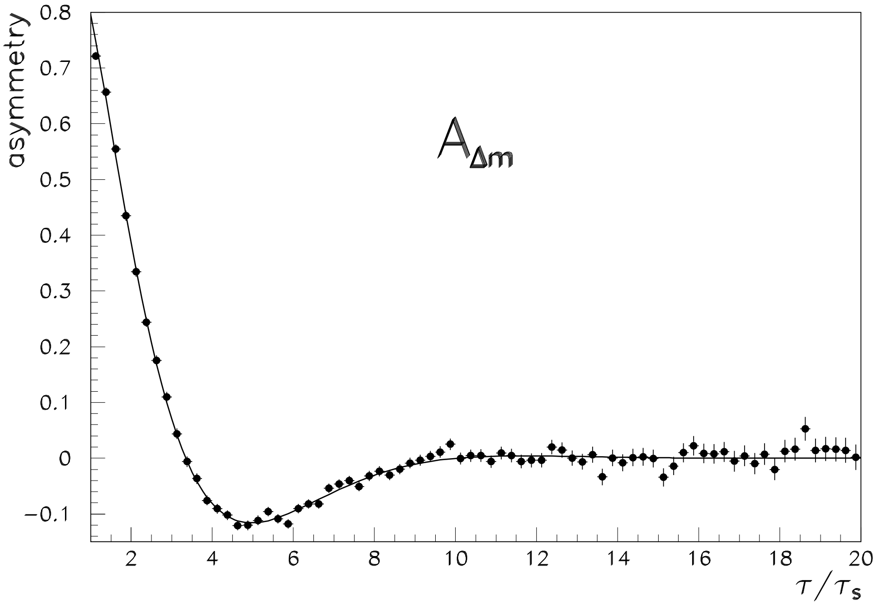
and obtain an estimate of the mass difference Δm.

d) Show that the time-reversal asymmetry

$$A_T \equiv \frac{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \mathrm{K}^0) - \Gamma(\mathrm{K}^0_{t=0} \to \overline{\mathrm{K}}^0)}{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \mathrm{K}^0) + \Gamma(\mathrm{K}^0_{t=0} \to \overline{\mathrm{K}}^0)}$$

is independent of the decay time t and that

$$A_T \approx 4\mathrm{Re}(\epsilon) = 4|\epsilon|\cos\phi .$$



25. a) Draw the Feynman (box) diagrams responsible for $\mathrm{K}^0-\overline{\mathrm{K}}^0$, $\mathrm{D}^0-\overline{\mathrm{D}}^0$, $\mathrm{B}^0_\mathrm{d}-\overline{\mathrm{B}}^0_\mathrm{d}$ and $\mathrm{B}^0_\mathrm{s}-\overline{\mathrm{B}}^0_\mathrm{s}$ mixing.

[The K^0, D^0, B^0_d and B^0_s mesons have quark content $\mathrm{d\overline{s}}$, $\mathrm{c\overline{u}}$, $\mathrm{d\overline{b}}$ and $\mathrm{s\overline{b}}$, respectively.]

b) The mass difference Δm between the mass eigenstates resulting from mixing in neutral meson systems is proportional to the magnitude of the matrix element derived from the box diagrams: $\Delta m \propto |M_{\mathrm{fi}}|$. For $\mathrm{K}^0-\overline{\mathrm{K}}^0$ mixing, for example, the box diagrams involving virtual quarks of flavour q and q′, with masses m_q and $m_{q'}$, lead to the prediction

$$\Delta\,\mathrm{mK} \approx \frac{G_\mathrm{F}^2}{3\pi^2} f_\mathrm{K}^2\,\mathrm{mK}\left|V_\mathrm{qd}V^*_\mathrm{qs}V_\mathrm{q'd}V^*_\mathrm{q's}\right| m_q m_{q'}$$

where f_K is a constant and the V_{ij} are CKM matrix elements. Show that the dominant contribution to $\Delta\,\mathrm{mK}$ comes from the box diagram containing two virtual charm quarks. Estimate $\Delta\,\mathrm{mK}$ and compare with experiment. [Take $f_\mathrm{K} = 100\,\mathrm{MeV}$.]

THE Z BOSON

26. Consider the decay of the Z^0 to a fermion-antifermion pair, $\mathrm{Z}^0 \to \mathrm{f\overline{f}}$, where the fermion couples to the Z^0 with vector and axial vector coupling constants c_V and c_A:

a) Use the Feynman rules to show that the matrix element for the decay $\mathrm{Z}^0 \to \mathrm{f\overline{f}}$ can be written in the form

$$\begin{aligned} M_\mathrm{fi} &= c_\mathrm{L}\cdot g_\mathrm{Z}\epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(1-\gamma^5)v(p_4) + c_\mathrm{R}\cdot g_\mathrm{Z}\epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(1+\gamma^5)v(p_4) \\ &\equiv c_\mathrm{L}\cdot M_\mathrm{L} + c_\mathrm{R}\cdot M_\mathrm{R} \end{aligned}$$

where p_1 is the Z^0 4-momentum, p_3 and p_4 are the 4-momenta of the fermion and antifermion, and $c_\mathrm{L} = \frac{1}{2}(c_\mathrm{V}+c_\mathrm{A})$, $c_\mathrm{R} = \frac{1}{2}(c_\mathrm{V}-c_\mathrm{A})$.

b) Assuming the fermion mass can be neglected, draw diagrams illustrating the spin configurations which result in non-zero values of $M_{\rm L}$ and $M_{\rm R}$.

c) Use the results of the calculation of the $\rm W^- \to e^-\overline{\nu}_e$ decay rate in the lectures to show that, for unpolarised $\rm Z^0$'s,

$$\langle |M_{\rm fi}|^2 \rangle = \tfrac{2}{3} g_{\rm Z}^2 m_{\rm Z}^2 (c_{\rm L}^2 + c_{\rm R}^2)$$

and hence that the decay rate is

$$\Gamma(\rm Z^0 \to f\bar{f}) = \frac{g_{\rm Z}^2 m_{\rm Z}}{48\pi}(c_{\rm V}^2 + c_{\rm A}^2)\ .$$

27. a) Use the result of question 26 to compute the total width of the $\rm Z^0$, and compare to experiment. [Take $\sin^2\theta_{\rm W} = 0.23$, and remember that quarks have three colour states].

b) What will be the value of

$$R = \frac{\sigma(\rm e^+e^- \to hadrons)}{\sigma(\rm e^+e^- \to \mu^+\mu^-)}$$

at the peak of the $\rm Z^0$ resonance ?

c) Calculate the cross section for $\rm e^+e^- \to Z^0$ at the resonance peak, and show that the cross-section for $\rm e^+e^- \to \mu^+\mu^-$ is increased by a factor of ≈ 200 relative to the QED cross section.

d) The width $\Gamma(\rm Z^0 \to b\overline{b})$ has been measured at LEP to be 0.378 GeV. Show that the weak isospin of the $\rm b$ quark is compatible with a value of -0.5. Explain why this result effectively guaranteed the existence of the top quark, even before it was directly discovered.

[$G_{\rm F} = 1.166 \times 10^{-5}\,\rm GeV^{-2}$.]

28. a) It was shown in the lectures that the centre of mass frame differential cross section $\mathrm{d}\sigma_{\mathrm{LR}}/\mathrm{d}\cos\theta$ for the process $\mathrm{e^+e^-} \to \mathrm{f\bar{f}}$ on the peak of the Z^0 resonance, for the case that the incoming electron is left-handed and the outgoing fermion is right-handed, is given by

$$\frac{\mathrm{d}\sigma_{\mathrm{LR}}}{\mathrm{d}\cos\theta} \propto (c_{\mathrm{L}}^{\mathrm{e}})^2 (c_{\mathrm{R}}^{\mathrm{f}})^2 (1-\cos\theta)^2 \; .$$

Show that the corresponding forward and backward cross sections $\sigma_{\mathrm{LR}}^{\mathrm{F}}$ and $\sigma_{\mathrm{LR}}^{\mathrm{B}}$ are given by

$$\sigma_{\mathrm{LR}}^{\mathrm{F}} \propto (c_{\mathrm{L}}^{\mathrm{e}})^2 (c_{\mathrm{R}}^{\mathrm{f}})^2, \qquad \sigma_{\mathrm{LR}}^{\mathrm{B}} \propto 7 (c_{\mathrm{L}}^{\mathrm{e}})^2 (c_{\mathrm{R}}^{\mathrm{f}})^2 \; ,$$

and write down similar expressions for the cross sections $\sigma_{\mathrm{RL}}^{\mathrm{F}}$, $\sigma_{\mathrm{RL}}^{\mathrm{B}}$, $\sigma_{\mathrm{LL}}^{\mathrm{F}}$, $\sigma_{\mathrm{LL}}^{\mathrm{B}}$, $\sigma_{\mathrm{RR}}^{\mathrm{F}}$, $\sigma_{\mathrm{RR}}^{\mathrm{B}}$.

b) The asymmetry $A_{\mathrm{LR}}^{\mathrm{FB}}$ is defined as

$$A_{\mathrm{LR}}^{\mathrm{FB}} \equiv \frac{(\sigma_{\mathrm{L}}^{\mathrm{F}} - \sigma_{\mathrm{L}}^{\mathrm{B}}) - (\sigma_{\mathrm{R}}^{\mathrm{F}} - \sigma_{\mathrm{R}}^{\mathrm{B}})}{(\sigma_{\mathrm{L}}^{\mathrm{F}} + \sigma_{\mathrm{L}}^{\mathrm{B}}) + (\sigma_{\mathrm{R}}^{\mathrm{F}} + \sigma_{\mathrm{R}}^{\mathrm{B}})}$$

where $\sigma_{\mathrm{L}} \equiv \sigma_{\mathrm{LL}} + \sigma_{\mathrm{LR}}$ and $\sigma_{\mathrm{R}} \equiv \sigma_{\mathrm{RL}} + \sigma_{\mathrm{RR}}$ are the total cross sections for left-handed and right-handed incoming electrons, respectively. Show that

$$A_{\mathrm{LR}}^{\mathrm{FB}} = \frac{3}{4} \frac{(c_{\mathrm{L}}^{\mathrm{f}})^2 - (c_{\mathrm{R}}^{\mathrm{f}})^2}{(c_{\mathrm{L}}^{\mathrm{f}})^2 + (c_{\mathrm{R}}^{\mathrm{f}})^2} \equiv \frac{3}{4} A_{\mathrm{f}} \; ,$$

and compare with the similar predictions for the asymmetries A_{LR} and A_{FB}.

c) Using a polarised electron beam, the SLD experiment has recently measured $A_{\mathrm{LR}}^{\mathrm{FB}}$ for the process $\mathrm{e^+e^-} \to \mathrm{c\bar{c}}$, and obtained the result $A_{\mathrm{c}} = 0.6712 \pm 0.0274$. Determine the corresponding value of $\sin^2\theta_{\mathrm{W}}$ and (optionally) its error.

THE TOP QUARK

29. a) The top quark decays into final states containing 1) two quarks and an antiquark, or 2) a quark, a lepton and an antilepton. List the possible final states of each type and draw the generic leading order Feynman diagram for these decays. Explain why the total top quark decay rate is dominated by the rate for the decay $\mathrm{t} \to \mathrm{W}^+\mathrm{b}$ into a real W^+ boson and b quark.

b) Use the Feynman rules to show that the matrix element for the decay $\mathrm{t} \to \mathrm{W}^+\mathrm{b}$ is given by

$$M_{\mathrm{fi}} = \frac{g_{\mathrm{W}}}{\sqrt{2}} \epsilon^*_\mu(p_4)\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_1)$$

where p_1 is the 4-momentum of the top quark and p_3 and p_4 are the 4-momenta of the b quark and W^+, respectively.

c) Consider the decay $\mathrm{t} \to \mathrm{W}^+\mathrm{b}$ in the top quark rest frame, with the b quark travelling in the $+z$ direction. Neglect the b quark mass. Draw diagrams illustrating the two spin configurations which are allowed in this case. Show that, when the top quark spin points in the $+z$ direction, the matrix element M_{fi} is given by

$$M_\uparrow = -g_{\mathrm{W}}\sqrt{2m_{\mathrm{t}}p^*}$$

where $p^* = (m_{\mathrm{t}}^2 - m_{\mathrm{W}}^2)/2m_{\mathrm{t}}$ is the magnitude of the three-momenta of the W^+ and the b quark. Show that when the top quark spin points in the $-z$ direction, the matrix element becomes

$$M_\downarrow = -g_{\mathrm{W}}\frac{m_{\mathrm{t}}}{m_{\mathrm{W}}}\sqrt{m_{\mathrm{t}}p^*}\ .$$

d) Explain why the decay of an unpolarised sample of top quarks must be isotropic, and show that the total decay rate in this case is

$$\Gamma = \frac{G_{\mathrm{F}}m_{\mathrm{t}}^3}{8\pi\sqrt{2}}\left(1-\frac{m_{\mathrm{W}}^2}{m_{\mathrm{t}}^2}\right)^2\left(1+\frac{2m_{\mathrm{W}}^2}{m_{\mathrm{t}}^2}\right)\ .$$

e) Calculate the top quark lifetime. Use the uncertainty principle to estimate a typical hadronisation timescale and comment on the result.

THE HIGGS BOSON

30. a) Use the Feynman rules to show that the matrix element for the decay $\mathrm{H} \to \mathrm{W^+W^-}$ is

$$M_{\mathrm{fi}} = -g_{\mathrm{W}} m_{\mathrm{W}} g_{\mu\nu} \epsilon^{\mu}(p_2)^* \epsilon^{\nu}(p_3)^*$$

where p_2 and p_3 are the 4-momenta of the $\mathrm{W^-}$ and $\mathrm{W^+}$, respectively.

b) Show that $M_{\mathrm{fi}} = -g_{\mathrm{W}} m_{\mathrm{W}}$ when both W bosons are left-handed or both are right-handed, that $M_{\mathrm{fi}} = (g_{\mathrm{W}}/m_{\mathrm{W}})(\frac{1}{2}m_{\mathrm{H}}^2 - m_{\mathrm{W}}^2)$ when both W bosons are longitudinally polarised, and that $M_{\mathrm{fi}} = 0$ for the six remaining combinations of W boson spin states.

c) Show that the $\mathrm{H} \to \mathrm{W^+W^-}$ decay rate is

$$\Gamma(\mathrm{H} \to \mathrm{W^+W^-}) = \frac{G_{\mathrm{F}} m_{\mathrm{H}}^3}{8\pi\sqrt{2}} \sqrt{1 - 4\lambda^2}\left(1 - 4\lambda^2 + 12\lambda^4\right)$$

where $\lambda = m_{\mathrm{W}}/m_{\mathrm{H}}$.

d) For $\mathrm{H} \to \mathrm{Z^0Z^0}$ decays, an extra factor of $\frac{1}{2}$ is required to account for the fact that the final state contains two identical particles. Show that

$$\Gamma(\mathrm{H} \to \mathrm{Z^0Z^0}) = \tfrac{1}{2}\Gamma(\mathrm{H} \to \mathrm{W^+W^-})|_{(m_{\mathrm{W}} \to m_{\mathrm{Z}})} \,.$$

e) For $\mathrm{H} \to \mathrm{f\bar{f}}$ decays into a fermion-antifermion pair, the decay rate is

$$\Gamma(\mathrm{H} \to \mathrm{f\bar{f}}) = N_c \frac{G_{\mathrm{F}}}{\sqrt{2}} \frac{m_{\mathrm{f}}^2 m_{\mathrm{H}}}{4\pi} \left(1 - \frac{4m_{\mathrm{f}}^2}{m_{\mathrm{H}}^2}\right)^{3/2}$$

where N_c is the number of colour degrees of freedom of the fermion f of mass m_{f} [See Tripos paper, Jan 2002, for a derivation of this result]. Compute the $\mathrm{H} \to \mathrm{W^+W^-}$, $\mathrm{H} \to \mathrm{Z^0Z^0}$ and $\mathrm{H} \to \mathrm{t\bar{t}}$ branching ratios and the total Higgs width Γ for a Higgs mass of $500\,\mathrm{GeV}$. [Note that the decay rates into $\mathrm{f\bar{f}}$ final states other than $\mathrm{H} \to \mathrm{t\bar{t}}$ are negligibly small since $m_{\mathrm{f}} \ll m_{\mathrm{t}}$.]

NUMERICAL ANSWERS

0. a) $L = 1.98\ m$, $M = 79$ kg. BMI $= 20.1$

1. d) $\sqrt{s} = 300\,$GeV; $E = 48000\,$GeV

4. $\Gamma(\rho \to \pi\pi)/\Gamma(\mathrm{K}^* \to \mathrm{K}\pi) = 3.46$; expt $= 2.98$

5. a) $\tau_\pi = 3.0 \times 10^{16}\,$GeV^{-1} = $1.97 \times 10^{-8}\,$s; expt $= 2.6 \times 10^{-8}\,$s

   b) from phase space alone: $\Gamma(\pi^+ \to \mathrm{e}^+\nu_\mathrm{e})/\Gamma(\pi^+ \to \mu^+\nu_\mu) = 2.34$

11. a) $\lambda = 0.84\,$GeV; b) 0.81 fm; c) $\approx 0.68\,$fm

12. b) $x \approx 0.09$, $Q^2 \approx 610\,\mathrm{GeV}^2$, $y \approx 0.075$; c) $M_\mathrm{X} \approx 78\,\mathrm{GeV}$

    d) relative probabilities that scattering is from u, d, $\overline{\mathrm{u}}$, $\overline{\mathrm{d}}$ are

$$\mathrm{u} : \mathrm{d} : \overline{\mathrm{u}} : \overline{\mathrm{d}} \approx 0.73 : 0.12 : 0.12 : 0.04\ .$$

    e) the F_1 term contributes only $\approx 0.3\%$ of events.

13. d) $4.7^\circ < \theta < 21.3^\circ$

15.

18. $(Q^2)_\mathrm{max} \approx 750\,\mathrm{GeV}^2$

19. $f_\mathrm{q} \approx 0.41$, $f_{\overline{\mathrm{q}}} \approx 0.08$, $f_\mathrm{g} \approx 0.51$

21. d) $\sin^2\theta_{13} > 0.051$ at 97.5% C.L., $|\Delta m^2_{23}| < 3.0 \times 10^{-3}\,\mathrm{eV}^2$; e) $\sin^2\theta_{13} = 0.09$, $|\Delta m^2_{23}| = 2.6 \times 10^{-3}\,\mathrm{eV}^2$

22. d) $|J|_\mathrm{max} = 0.053$; f) about $5000\,\mathrm{km}$, $|\Delta P|_\mathrm{max} \approx 0.04$

25. b) $\Delta\,\mathrm{mK} \sim 2 \times 10^{-12}$ MeV;

27. a) $\Gamma_Z = 2.3\,$GeV; b) $R = 20.1$; c) 61 nb

28. c) $\sin^2\theta_\mathrm{W} \approx 0.230 \pm 0.008$

29. e) $\tau \approx 4.0 \times 10^{-25}\,\mathrm{s}$, $\tau_\mathrm{had} \sim \times 10^{-23}\,\mathrm{s}$

30. e) $\mathrm{BR}(\mathrm{H} \to \mathrm{W}^+\mathrm{W}^-) = 55.8\%$, $\mathrm{BR}(\mathrm{H} \to \mathrm{Z}^0\mathrm{Z}^0) = 26.7\%$, $\mathrm{BR}(\mathrm{H} \to \mathrm{t}\overline{\mathrm{t}}) = 17.5\%$;
    $\Gamma = 62.9\,\mathrm{GeV}$