1 Supplement on Electroweak Unification

1.1 Notation 1: (common to most text-books and the lecture slides)

The respective electromagnetic, W_3 and weak-hypercharge currents are often written down in (in both text books and in the Part III course) in the following way:

$$j_\mu^{\text{em}} = e \cdot Q \cdot \bar{\psi}_L \gamma_\mu \psi_L + e \cdot Q \cdot \bar{\psi}_R \gamma_\mu \psi_R \tag{1}$$

$$j_\mu^{W_3} = g_W \cdot I_3^W \cdot \bar{\psi}_L \gamma_\mu \psi_L \tag{2}$$

$$j_\mu^{Y} = \frac{g'}{2} \cdot Y_L \cdot \bar{\psi}_L \gamma_\mu \psi_L + \frac{g'}{2} \cdot Y_R \cdot \bar{\psi}_R \gamma_\mu \psi_R \tag{3}$$

in which the absence of a $\bar{\psi}_R \gamma_\mu \psi_R$ term in the W_3 current emphasises that the $SU(2)_L$ weak-interaction does not couple to right-chiral states as they are in weak singlets.

1.1.1 What confuses (some) students about the above notation?

In general I_3^W can take both the value 0 (for right-chiral states) or one of $\pm\frac{1}{2}$ (for left-chiral states).

Both left- and right-chiral states appear in eqs (1) and (3). However: only left-chrial states appear in (2). This causes the I_3^W quantity in eq (2) to be only able take values in $\pm\frac{1}{2}$ **despite** the fact that the symbol I_3^W it will later be used in equations containing both left- and right-chiral particles and left- and right-chiral fields.

The lack of any flags or labels on I_3^W to remind us that it applies only to the left-chiral states causes confusion for some students. This confusion is made worse by the apparent lack of symmetry in the three currents above caused by the absence of a $\bar{\psi}_R \gamma_\mu \psi_R$ term in the W_3 current.

These issues motivate the following non-standard (but equivalent) notation for pedagogical purposes.

1.2 Notation 2: (alternative preferred by some students?)

Past experience in the Part III course has shown that some students find it easier to be presented first with a more symmetric looking set of equations, like those below:

$$j_\mu^{\text{em}} = e \cdot Q \cdot \bar{\psi}_L \gamma_\mu \psi_L + e \cdot Q \cdot \bar{\psi}_R \gamma_\mu \psi_R \tag{4}$$

$$j_\mu^{W_3} = g_W \cdot I_{3L}^W \cdot \bar{\psi}_L \gamma_\mu \psi_L + g_W \cdot I_{3R}^W \cdot \bar{\psi}_R \gamma_\mu \psi_R \qquad (\text{with } I_{3R}^W = 0) \tag{5}$$

$$j_\mu^{Y} = \frac{g'}{2} \cdot Y_L \cdot \bar{\psi}_L \gamma_\mu \psi_L + \frac{g'}{2} \cdot Y_R \cdot \bar{\psi}_R \gamma_\mu \psi_R. \tag{6}$$

Here two changes have been made. Firstly: the symbol I_3^W has been renamed I_{3L}^W so that its 'implicit' link to left-chiral particles is made more 'explicit' in the notation. Secondly: a new symbol I_{3R}^W has been introduced, *even though* it is defined to take the value zero at all times.

1.2.1 Comments on 'notation 2' and comparison to 'notation 1'

It should be remembered that despite 'notation 2' having introduced a notational separation between I_{3L}^W and I_{3R}^W, it still remains the case that both symbols represent 'the same quantity I_3^W', the third-component of weak isospin. The L/R suffices are not changing what is measured. The suffices are only serving to remind us what SORT of particle is having is I_3^W recorded.

Putting the above another way: use of 'notation 1' in an unambiguous way would require explicit labelling of particles' chiralities, e.g.:

$$I_3^W(\nu_L) = +\frac{1}{2}, \quad I_3^W(e_L) = -\frac{1}{2}, \quad I_3^W(e_R) = 0$$

whereas in 'notation 2' the labels can be dropped from the particles without loss of ambiguity:

$$I_{3L}^W(\nu) = +\frac{1}{2}, \quad I_{3L}^W(e) = -\frac{1}{2}, \quad I_{3R}^W(e) = 0.$$

In both notations we would have

$$Q(e) = -1, \quad Q(\nu) = 0, \quad Q(u) = \frac{2}{3}, \quad \text{etc.}$$

We do not (yet) claim to know what values will be taken by the quantities Y_L and Y_R, but over the next few pages their values will be deduced.

2 Making use of 'notation 2' to improve upon slide 491 in the Lecture Notes

If the B-boson (associated with weak hypercharge) and the W^3 (of the weak interaction) are to mix to make a photon-like A-boson and a Z-boson then the mixing may be expressed as:

$$A_\mu = +B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W$$
$$Z_\mu = -B_\mu \sin\theta_W + W_\mu^3 \cos\theta_W$$

which in terms of currents reads:

$$j_\mu^{\text{em}} = +j_\mu^Y \cos\theta_W + j_\mu^{W_3} \sin\theta_W \quad (7)$$
$$j_\mu^Z = -j_\mu^Y \sin\theta_W + j_\mu^{W_3} \cos\theta_W. \quad (8)$$

Substituting our 'notation 2' expressions for each current (eqs (4), (5) and (6)) into our equation (7) which relates those three currents yields:

$$
\begin{aligned}
e \cdot Q \cdot \bar{\psi}_L \gamma_\mu \psi_L + e \cdot Q \cdot \bar{\psi}_R \gamma_\mu \psi_R = \\
&= \cos\theta_W \left(\frac{g'}{2} \cdot Y_L \cdot \bar{\psi}_L \gamma_\mu \psi_L + \frac{g'}{2} \cdot Y_R \cdot \bar{\psi}_R \gamma_\mu \psi_R \right) \\
&+ \sin\theta_W \left(g_W \cdot I^W_{3L} \cdot \bar{\psi}_L \gamma_\mu \psi_L + g_W \cdot I^W_{3R} \cdot \bar{\psi}_R \gamma_\mu \psi_R \right) \qquad (9)
\end{aligned}
$$

which, if it is to be true for any fermion field ψ, will require the following to be true at all times:

$$
eQ = \cos\theta_W \frac{g'}{2} Y_L + \sin\theta_W g_W I^W_{3L} \qquad \text{and} \qquad (10)
$$

$$
eQ = \cos\theta_W \frac{g'}{2} Y_R + \sin\theta_W g_W I^W_{3R} \qquad (11)
$$

which we can write as the single[1] equation:

$$
eQ = \cos\theta_W \frac{g'}{2} Y_\bullet + \sin\theta_W g_W I^W_{3\bullet} \qquad (12)
$$

if we allow '$\bullet$' to stand for either R or L. No '$\bullet$' is needed on the Q since the electric charge of a particle does not depend on its chirality.

What we do not (yet) know is whether there are values for hypercharges $Y_\bullet$ and couplings g' which will allow us to satisfy eq (12). It is our job now to see if such associations exist. To determine whether it is possible we need to be clear about what is fixed and what is still free:

3 What is fixed and what is still free?

If we take the view point that the electric charges of every particle has already been fixed by prior experiments, then we are not free to fiddle with e (the modulus of the charge of the electron) or Q (the fractions like $2/3$ or -1). If we also take the view that the W-boson has been discovered then we are no longer free to fiddle with g_W or the $I^W_{3\bullet}$ quantities. We therefore regard the following quantities in the following set as 'fixed' or 'known':

$$
\text{knowns} = \{e, Q, g_W, I^W_{3\bullet}\}.
$$

However: if we take the view that we have just introduced the $U(1)_Y$-weak-hypercharge and are keen to see if it can be made to reproduce something like a photon (in addition to a further Z-boson, eventually) then we are free to fiddle with the values in

$$
\text{unknowns} = \{\theta_W, g', Y_\bullet\}
$$

[1] Note that we are only able to compress both identities into a single equation as we are using 'notation 2'. This is the benefit of 'notation 2'.

as none affect the properties of the W-boson which we have already studied.

We choose first to set the unknown θ_W in terms of the knowns e and g_W by using the following relationship:

$$e = g_W \sin\theta_W. \tag{13}$$

Having fixed θ_W with the above formula, we are then choose to fix the unkwnown g' by requiring that

$$e = g' \cos\theta_W. \tag{14}$$

The above choices leave only the quantities $Y_\bullet$ (yet) undetermined. We will shortly see what the $Y_\bullet$ values are ... but it should be emphasised that *a priori* we would not know that the choices just made are going to be good choices. It is only when we do succeed in finding good values for $Y_\bullet$ that we will know the choices just made were good choices.

To see if we can find consistent values for $Y_\bullet$ with the above choices we substitute the definitions for θ_W in eq (13) and for g' in (14) back into eq (12) to yield:

$$eQ = e\frac{1}{2}Y_\bullet + eI^W_{3\bullet}$$

which is equivalent to

$$Q = \frac{1}{2}Y_\bullet + I^W_{3\bullet} \tag{15}$$

or

$$Y_\bullet = 2\left(Q - I^W_{3\bullet}\right). \tag{16}$$

Any of the above (and particularly the last, eq (16)) show that there evidently is a way of associating hypercharges $Y_\bullet$ to fermions that allows the electromagnetic current j^{em}_μ to take the desired form, and hence we have shown that there is not yet[2] a bar to our having the new (weak hypercharge) gauge group $U(1)_Y$ co-exist with the $SU(2)_L$ weak interaction.

[2]Another properties which we have not yet checked is that the photon stays massless while the Z-boson is not. While that also proves to be fine, discussion of it is beyond the scope of our Part III course.

4 What we can now deduce about the Z-boson's current

We may now return to our definition of j_μ^Z seen earlier in eq (8), but using results found since then. Specifically, (8) becomes:

$$\begin{aligned}
j_\mu^Z &= -\sin\theta_W \left(\frac{g'}{2} \cdot Y_L \cdot \bar{\psi}_L \gamma_\mu \psi_L + \frac{g'}{2} \cdot Y_R \cdot \bar{\psi}_R \gamma_\mu \psi_R \right) && \text{(using (6))} \\
&\quad + \cos\theta_W \left(g_W \cdot I_{3L}^W \cdot \bar{\psi}_L \gamma_\mu \psi_L + g_W \cdot I_{3R}^W \cdot \bar{\psi}_R \gamma_\mu \psi_R \right) && \text{(using (5))} \\
&= \sum_{\bullet \in \{L,R\}} \left(-\sin\theta_W \frac{g'}{2} Y_\bullet + \cos\theta_W g_W I_{3\bullet}^W \right) \bar{\psi}_\bullet \gamma_\mu \psi_\bullet && \text{(simplifying)} \\
&= \sum_{\bullet \in \{L,R\}} \frac{g_W}{\cos\theta_W} \left(-\frac{1}{2} \sin^2\theta_W \frac{g' \cos\theta_W}{g_W \sin\theta_W} Y_\bullet + \cos^2\theta_W I_{3\bullet}^W \right) \bar{\psi}_\bullet \gamma_\mu \psi_\bullet && \text{(rearranging)} \\
&= \sum_{\bullet \in \{L,R\}} \frac{g_W}{\cos\theta_W} \left(-\frac{1}{2} \sin^2\theta_W Y_\bullet + \cos^2\theta_W I_{3\bullet}^W \right) \bar{\psi}_\bullet \gamma_\mu \psi_\bullet && \text{(using eqs (13) and (14))} \\
&= \sum_{\bullet \in \{L,R\}} \frac{g_W}{\cos\theta_W} \left(\sin^2\theta_W (I_{3\bullet}^W - Q) + \cos^2\theta_W I_{3\bullet}^W \right) \bar{\psi}_\bullet \gamma_\mu \psi_\bullet && \text{(using eqs (16))} \\
&= \sum_{\bullet \in \{L,R\}} \frac{g_W}{\cos\theta_W} \left(I_{3\bullet}^W - Q \sin^2\theta_W \right) \bar{\psi}_\bullet \gamma_\mu \psi_\bullet && \text{(simplifying)}
\end{aligned}$$

and so defining

$$g_Z = \frac{g_W}{\cos\theta_W} \tag{17}$$

and

$$c_L = I_{3L}^W - Q \sin^2\theta_W \tag{18}$$
$$c_R = I_{3R}^W - Q \sin^2\theta_W \tag{19}$$

(which could also be written $c_\bullet = I_{3\bullet}^W - Q \sin^2\theta_W$) one sees that

$$j_\mu^Z = g_Z \sum_{\bullet \in \{L,R\}} c_\bullet \cdot \bar{\psi}_\bullet \gamma_\mu \psi_\bullet = g_Z (c_L \bar{\psi}_L \gamma_\mu \psi_L + c_R \bar{\psi}_R \gamma_\mu \psi_R). \tag{20}$$

4.1 Improvement on the lecture notes

It is hoped that 'notation 2' has made the definitions of c_L and c_R in eqs (18) and (19) easier to understand than the equivalent but less clear forms shown on slide 475 of the lecture notes.