Fermi's Golden Rule

$$\Gamma_{fi} = 2\pi \, |T_{fi}|^2 \rho(E_f)$$

$$T_{fi} = \langle f|\hat{H}_{int}|i\rangle + \sum_{|j\rangle \neq |i\rangle} \frac{\langle f|\hat{H}_{int}|j\rangle\langle j|\hat{H}_{int}|i\rangle}{E_i - E_j} + \dots$$

Flux + Phase-space connection to cross sections σ, $\frac{d\sigma}{dt}$ etc.

(+) → "Old Fashioned Perturbation Theory"

Issue

100% Valid ✓ But calculating T_{fi} is not convenient.
One source of big problems is normalisation: $\langle\psi|\psi\rangle = 1$ (or const).

(Volumes length contract by factor γ under BOOST)

Part of resolution is state renormalisation:

Single particle state:
$$|\psi'\rangle = \sqrt{2E}\,|\psi\rangle$$
so that $\int \langle\psi'|\psi'\rangle \, dV = 2E \int \langle\psi|\psi\rangle dV = 2E = \gamma \times \text{const}$

Multi-particle state:
$$|a',b',c'\rangle = \sqrt{2E_a}\sqrt{2E_b}\sqrt{2E_c}\,|a,b,c\rangle$$

Turns out to make $\langle\psi_1',\psi_2'\dots|\hat{H}_{int}|\dots\psi_{n-1}',\psi_n'\rangle$ Lorentz invariant .... see why later.

Motivates this definition

$$M_{abc\dots de} \overset{def}{\equiv} \langle a'b'c'\dots|\hat{H}_{int}|\dots d'e'\rangle$$
$$T_{abc\dots de} \overset{def}{\equiv} \langle abc\dots|\hat{H}_{int}|\dots de\rangle$$

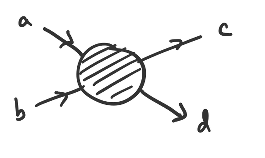
CONSEQUENCE

$$M_{abc\dots de} = \sqrt{2E_a}\sqrt{2E_b}\dots\sqrt{2E_e}\;T_{abc\dots de}$$

....only later see why this is useful.

Consider $T_{fi} = \langle f|\hat{H}_{int}|i\rangle + \sum_{|j\rangle\neq|i\rangle} \frac{\langle f|\hat{H}_{int}|j\rangle\langle j|\hat{H}_{int}|i\rangle}{E_i - E_j} + \dots$.

For special case of



($\underline{a}$ = shorthand for $\underline{p}_a$)

Mom Cons ⇒ $\underline{a} + \underline{b} = \underline{c} + \underline{d}$

"$|i\rangle$" = $|\underline{a},\underline{b}\rangle$
"$|f\rangle$" = $|\underline{c},\underline{d}\rangle$

What are states $|j\rangle$ being summed over?
Everything!

$$|j\rangle \in \{|0\rangle, |a\text{ piano}\rangle, |\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\rangle, |\text{Elon Musk}\rangle, |j_x\rangle, |j_y\rangle, \dots\}$$

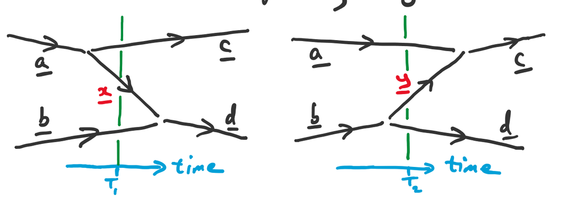
In this giant list, consider just **two special entries**:

"$|j_x\rangle$" = $|\underline{c}, \underline{b}, \underline{x} = \underline{a} - \underline{c}\rangle$, x = "particle with mass m & mom $\underline{x}$"
"$|j_y\rangle$" = $|\underline{a}, \underline{d}, \underline{y} = \underline{b} - \underline{d}\rangle$, y = "particle with mass m & mom $\underline{y}$"

Notes:

① (mom cons $\underline{p}_{initial} = \underline{p}_{final}$) ⇒ ($\underline{a} + \underline{b} = \underline{c} + \underline{d}$) ⇒ ($\underline{x} = -\underline{y}$)
∴ x & y are heading in "opposite directions,"
and $E_x = \sqrt{m^2 + |\underline{x}|^2} = \sqrt{m^2 + |\underline{y}|^2} = E_y$,

② x & y have correct 3-momenta to conserve momentum in the following ways:



③ 4-momentum is not conserved, though!
(i.e. if $E_{T_1} \overset{def}{\equiv} E_c + E_x + E_b$
and $E_{T_2} \overset{def}{\equiv} E_a + E_y + E_d$
then $(a^\mu + b^\mu = c^\mu + d^\mu) \Rightarrow (E_a + E_b = E_c + E_d)$
but $(a^\mu + b^\mu = c^\mu + d^\mu) \not\Rightarrow (E_a + E_b = E_{T_1})$
and $(a^\mu + b^\mu = c^\mu + d^\mu) \not\Rightarrow (E_a + E_b = E_{T_2})$).

A feel for terms in the interaction Hamiltonian

$$\hat{H}_{int} \overset{?}{=} e\hat{\phi}\hat{\psi}\hat{\psi} \;\overset{?}{+}\; e\hat{\phi}\hat{\phi}\hat{\phi} \;\overset{?}{+}\; \overset{?}{\dots}$$

$$\hat{\phi} \sim \int \hat{a}_p + \hat{a}_p^\dagger \, dp$$
$$\hat{\psi} \sim \int \hat{b}_p + \hat{b}_p^\dagger \, dp$$

∴
$|0\rangle \sim |0\rangle$
$\hat{\phi}|0\rangle \sim \int |p\rangle \, dp$
$\hat{\phi}\hat{\phi}|0\rangle \sim |0\rangle + \int |p,q\rangle \, dp\,dq$
$\hat{\phi}\hat{\phi}\hat{\phi}|0\rangle \sim \int |p\rangle\, dp + \int |p,q,r\rangle \, dp\,dq\,dr$
$\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}|0\rangle \sim |0\rangle + \int |p,q\rangle\, dp\,dq + \int |p,q,r,s\rangle \, dp\,dq\,dr\,ds$

Since "$|i\rangle$" = $|\underline{a},\underline{b}\rangle$ and "$|f\rangle$" = $|\underline{c},\underline{d}\rangle$ consider the following M_{fi}:

$$M_{fi} = \sqrt{2E_a}\sqrt{2E_b}\sqrt{2E_c}\sqrt{2E_d}\; T_{fi}$$

$$= \sqrt{2E_a}\sqrt{2E_b}\sqrt{2E_c}\sqrt{2E_d}\left(\langle \underline{c},\underline{d}|\hat{H}_{int}|\underline{a},\underline{b}\rangle + \sum_{|j\rangle\neq|i\rangle}\frac{\langle \underline{c},\underline{d}|\hat{H}_{int}|j\rangle\langle j|\hat{H}_{int}|\underline{a},\underline{b}\rangle}{(E_a+E_b) - E_j} + \dots\right)$$

$$= \langle \underline{c}',\underline{d}'|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle + \sum_{|j\rangle\neq|i\rangle}\frac{\langle \underline{c}',\underline{d}'|\hat{H}_{int}|j\rangle\langle j|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle}{(E_a+E_b) - E_j} + \dots$$

$$= \langle \underline{c}',\underline{d}'|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle + \sum_{|j\rangle\neq|i\rangle}\frac{\langle \underline{c}',\underline{d}'|\hat{H}_{int}|j\rangle\langle j|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle}{(E_a+E_b) - E_j} + \dots$$

$$= M_{cd;ab} + \frac{\langle \underline{c}',\underline{d}'|\hat{H}_{int}|j_x\rangle\langle j_x|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle}{(E_a+E_b) - E_{j_x}} \quad \leftarrow j_x \text{ term in sum}$$
$$+ \frac{\langle \underline{c}',\underline{d}'|\hat{H}_{int}|j_y\rangle\langle j_y|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle}{(E_a+E_b) - E_{j_x}} \quad \leftarrow j_y \text{ term in sum}$$
$$+ \sum_{|j\rangle\notin\{|i\rangle,|j_x\rangle,|j_y\rangle\}}\frac{\langle \underline{c}',\underline{d}'|\hat{H}_{int}|j\rangle\langle j|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle}{(E_a+E_b) - E_j} \quad \leftarrow \text{all other terms in sum}$$
$$+ \dots \quad \leftarrow \text{all other terms in other sums.}$$

call all of this "+ ..." in next line

$$= M_{cd;ab} + \frac{\langle \underline{c}',\underline{d}'|\hat{H}_{int}|\underline{c},\underline{b},\underline{x}\rangle\langle \underline{c},\underline{b},\underline{x}|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle}{(E_a+E_b)-(E_c+E_b+E_x)}$$
$$+ \frac{\langle \underline{c}',\underline{d}'|\hat{H}_{int}|\underline{a},\underline{d},\underline{y}\rangle\langle \underline{a},\underline{d},\underline{y}|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle}{(E_a+E_b)-(E_a+E_d+E_y)} + \dots$$

[If momenta were not discretised (not finite box) we would need some kind of $\int d^3c\, d^3b\, d^3x$ here, but we gloss over such details.]

$$= M_{cd;ab} + \frac{\langle \underline{c}',\underline{d}'|\hat{H}_{int}|\underline{c}',\underline{b}',\underline{x}'\rangle\langle \underline{c}',\underline{b}',\underline{x}'|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle}{E_a - E_c - E_x}\cdot\frac{1}{2E_c\,2E_b\,2E_x}$$
$$+ \frac{\langle \underline{c}',\underline{d}'|\hat{H}_{int}|\underline{a}',\underline{d}',\underline{y}'\rangle\langle \underline{a}',\underline{d}',\underline{y}'|\hat{H}_{int}|\underline{a}',\underline{b}'\rangle}{E_b - E_d - E_y}\cdot\frac{1}{2E_a\,2E_d\,2E_y} + \dots$$

[Cancellations shown rely on $\hat{H}_{int}$ having certain undiscussed properties. E.g. $\hat{H}_{int} = \hat{\phi}\hat{\phi}\hat{\psi}$ ← (x,y)-type field; $\hat{\phi}\hat{\phi}$ ← (a b c d)-type fields]

$$= M_{cd;ab} + \frac{\langle \underline{d}'|\hat{H}_{int}|\underline{b}',\underline{x}'\rangle\langle \underline{c}',\underline{x}'|\hat{H}_{int}|\underline{a}'\rangle}{E_a - E_c - E_x}\cdot\frac{1}{2E_x}$$
$$+ \frac{\langle \underline{c}'|\hat{H}_{int}|\underline{a}',\underline{y}'\rangle\langle \underline{d}',\underline{y}'|\hat{H}_{int}|\underline{b}'\rangle}{E_b - E_d - E_y}\cdot\frac{1}{2E_y} + \dots$$

$$= M_{cd;ab} + \frac{M_{d;bx}\,M_{cx;a}}{E_a - E_c - E_x}\cdot\frac{1}{2E_x}$$
$$+ \frac{M_{c;ay}\,M_{dy;b}}{E_c - E_a - E_y}\cdot\frac{1}{2E_y} + \dots$$

[Since $(E_a+E_b = E_c+E_d) \Rightarrow (E_b - E_d = E_c - E_a)$]

$$\sim h\,\delta_{\underline{c}+\underline{d},\underline{a}+\underline{b}} + \frac{g_{bd}\delta_{\underline{d},\underline{b}+\underline{x}}\, g_{ac}\delta_{\underline{c}+\underline{x},\underline{a}}}{(E_a - E_c - E_x)\, 2E_x}$$
$$+ \frac{g_{ac}\delta_{\underline{c},\underline{a}+\underline{y}}\, g_{bd}\delta_{\underline{d}+\underline{y},\underline{b}}}{(E_c - E_a - E_x)\, 2E_x} + \dots$$

(Since $E_x = E_y$.)

[If momenta are discrete (ie finite box) then
$M_{ab;cd} \sim h\,\delta_{\underline{a}+\underline{b},\,\underline{c}+\underline{d}}$ ($h = 0$ if no four-point interaction)
$M_{d;bx} \sim g_{bd}\,\delta_{\underline{d},\,\underline{b}+\underline{x}}$ ($g = 0$ if no three-point interaction). ← Kronecker deltas.
Continuous case would contain Dirac delta-functions ....
$\int M_{d;bx}\, d^3x \sim g\int \delta^3(\underline{d} - (\underline{b}+\underline{x}))\, d^3x$ etc.
(Such δ-funcs would actually also ensure $\underline{x} = -\underline{y}$, so no need to separate $|j_x\rangle$ and $|j_y\rangle$ either!
However – as this is only sketch such details are ignored.
See QFT & AQFT courses for full details!]

$$= h.1 + \frac{g_{ac}g_{bd}\,1}{(E_a - E_c - E_x)\,2E_x}$$
$$- \frac{g_{ac}g_{bd}\,.1}{(-E_c + E_a + E_x)\,2E_x} + \dots$$

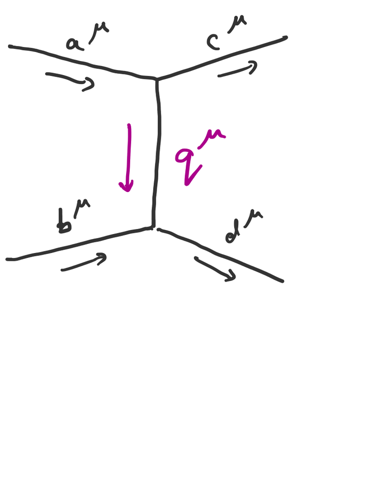
$$= h + \frac{g_{ac}g_{bd}}{2E_x}\,\frac{[(E_a - E_c) + E_x] - [(E_a - E_c) - E_x]}{((E_a - E_c) - E_x)((E_a - E_c) + E_x)} + \dots$$

$$= h + \frac{g_{ac}g_{bd}}{(E_a - E_c)^2 - E_x^2} + \dots$$

So now, and only now Define a quantity q^μ by:

$$\boxed{q^\mu \overset{def}{\equiv} a^\mu - c^\mu}$$ (4-mom conserving). ($a^\mu + b^\mu = c^\mu + d^\mu \Rightarrow q^\mu = d^\mu - b^\mu$ too)

Clearly $q^\mu = \begin{pmatrix} E_a - E_c \\ \underline{a} - \underline{c}\end{pmatrix}$ ∴ $q^2 = q^\mu q_\mu = (E_a - E_c)^2 - |\underline{a} - \underline{c}|^2$



$$\therefore M_{fi} \sim h + \frac{g_{ac}g_{bd}}{q^2 + |\underline{a} - \underline{c}|^2 - E_x^2} + \dots$$

$$= h + \frac{g_{ac}g_{bd}}{q^2 + |\underline{a} - \underline{c}|^2 - (m^2 + |\underline{x}|^2)} + \dots$$

$$= h + \frac{g_{ac}g_{bd}}{q^2 - m^2}. \quad \square$$

Could add structure of $\hat{H}_{int}$ .... to explain 3 & 4-point interactions