# SU(2)

2 quark representation $\begin{bmatrix} u \\ d \end{bmatrix}$ w/ basis

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\&\; d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

* Invariance of strong interaction under $u \leftrightarrow d$ expressed as invariance under 'rotations' in abstract isospin space.

* Hermitian generators $\hat{\underline{G}}$ are Pauli matrices. (not really actually)

$$\hat{U} = \mathbb{1}_2 + i\underline{\varepsilon}\cdot\hat{\underline{G}} = \mathbb{1}_2 + i\underline{\varepsilon}\cdot\frac{\underline{\sigma}}{2} \Rightarrow \underline{I} = \frac{1}{2}\underline{\sigma}$$

Isospin

* same transformation properties as spin.

$$[T_i, T_j] = i\epsilon_{ijk}T_k \qquad T_\pm = T_1 \pm iT_2$$

$$[T^2, T_3] = 0 \qquad T_\pm|I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3 \pm 1)}\,|I, I_3 \pm 1\rangle$$

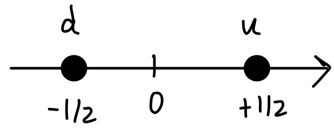
$$T^2 = T_1^2 + T_2^2 + T_3^2$$

eigenvalue of $T_3$!

* states labelled $|I, I_3\rangle$

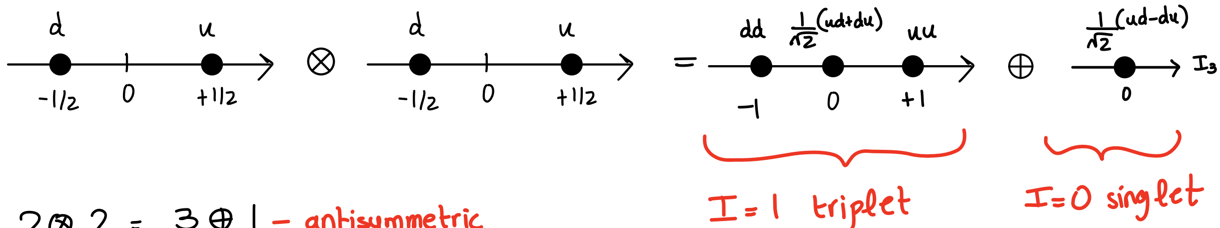
$$\rightarrow u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle$$

$$\rightarrow d = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

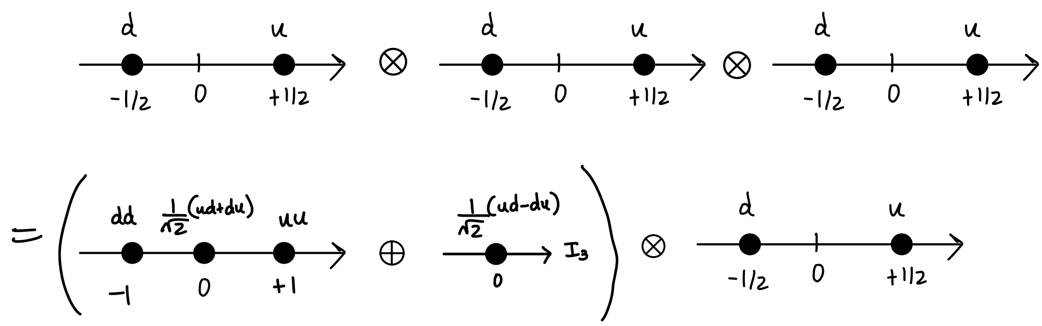



$\Rightarrow$ u, d form $I = +\frac{1}{2}$ isospin doublet.

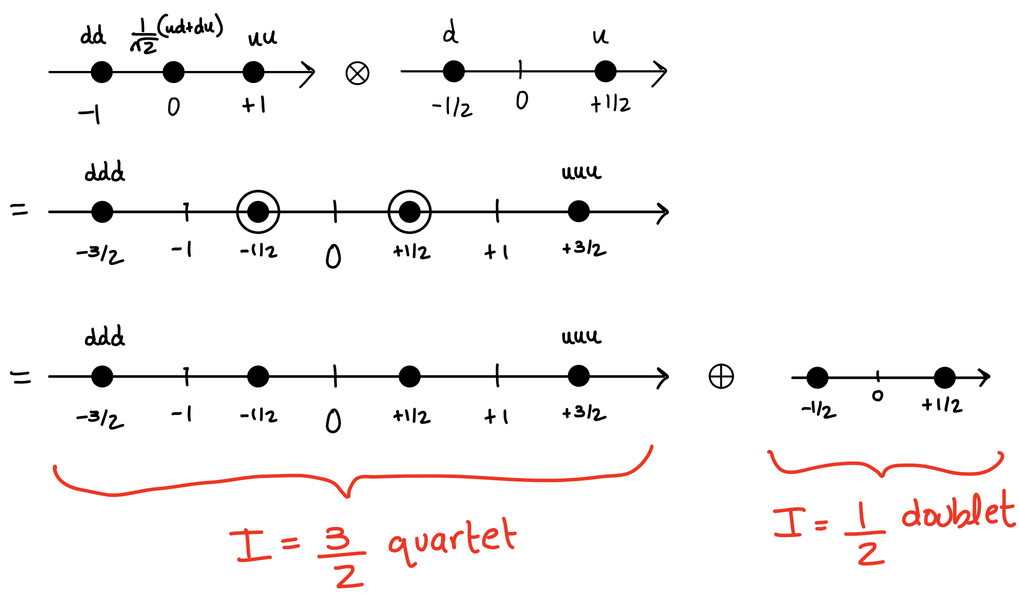
* total isospin $I$ labels multiplet & is not changed by ladder operators. States within a multiplet should be the same physically.




$2 \otimes 2 = \underset{\text{symmetric}}{3} \oplus 1$ – antisymmetric




① Building on Triplet:

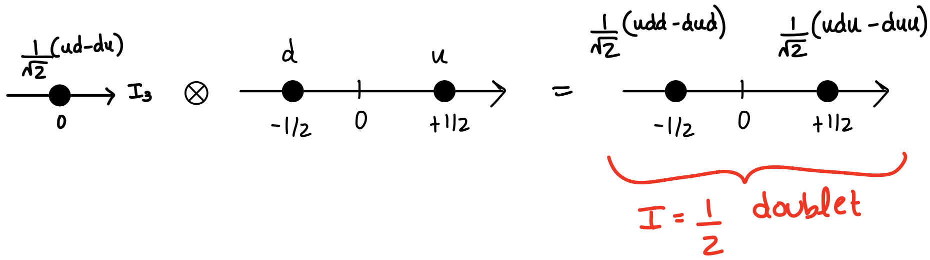



→ quartet states obtained by acting ladder operators on $|ddd\rangle$ or $|uuu\rangle$.

→ doublet states are orthogonal to corresponding $I_3$ states in quartet. & must be normalised.

→ doublet is $M_S$,
- symmetric under $1 \leftrightarrow 2$
- no defined symmetry under $1 \leftrightarrow 3$ & $2 \leftrightarrow 3$.

## ② Building on singlet



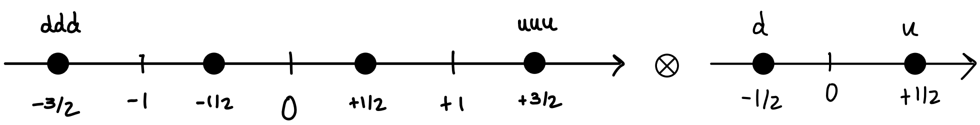

→ $M_A$ doublet : • antisymmetric under $1 \leftrightarrow 2$

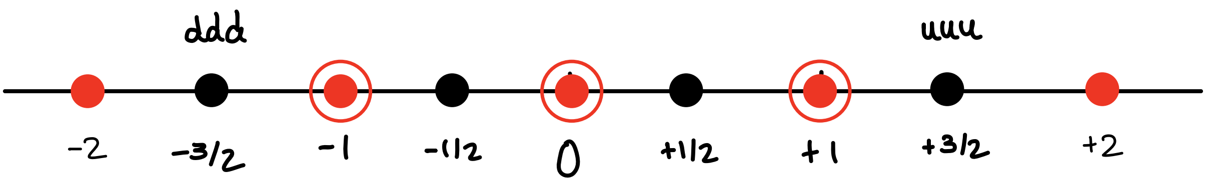
. no defined symmetry under $2 \leftrightarrow 3$ or $1 \leftrightarrow 3$

$$2 \otimes 2 \otimes 2 = (3 \oplus 1) \otimes 2 = 4 \oplus 2 \oplus 2$$

What about $2 \otimes 2 \otimes 2 \otimes 2$ ?

$$\begin{aligned}(4 \oplus 2 \oplus 2) \otimes 2 &= (4 \otimes 2) \oplus (2 \otimes 2) \oplus (2 \otimes 2) \\ &= (4 \otimes 2) \oplus (3 \oplus 1) \oplus (3 \oplus 1)\end{aligned}$$




→ ○ are old states in the quartet

→ ○ are the new states.

✯ We get a pentaplet and a triplet:

$$= 5 \oplus 3 \oplus 3 \oplus 3 \oplus 1 \oplus 1$$

→ you can also do $2 \otimes 2 \otimes 2 \otimes 2 = (3 \oplus 1) \otimes (3 \oplus 1)$

$$= (3 \otimes 3) \oplus 3 \oplus 3 \oplus 1$$

initial symmetric triplets we obtained.

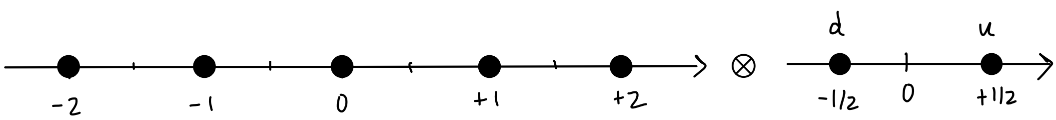
One can also do qqqqq states (would not be surprised if this comes up in an exam)

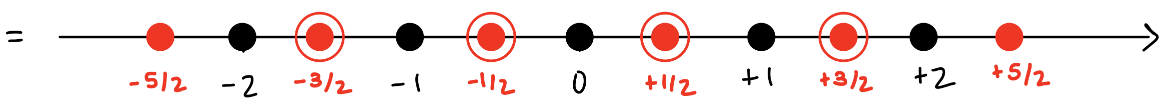

$$= (5 \oplus 3 \oplus 3 \oplus 3 \oplus 1 \oplus 1) \otimes 2$$

already know this from adding a quark to the triplet

$$= (5 \otimes 2) \oplus (3 \otimes 2) \oplus (3 \otimes 2) \oplus (3 \otimes 2) \oplus (1 \otimes 2) \oplus (1 \otimes 2)$$

$$= (5 \otimes 2) \oplus (4 \oplus 2) \oplus (4 \oplus 2) \oplus (4 \oplus 2) \oplus 2 \oplus 2$$




→ $I = 5/2$ sextet and $I = 3/2$ quartet.

$$= (6 \oplus 4) \oplus (4 \oplus 2) \oplus (4 \oplus 2) \oplus (4 \oplus 2) \oplus 2 \oplus 2$$

$$= 6 \oplus 4 \oplus 4 \oplus 4 \oplus 4 \oplus 2 \oplus 2 \oplus 2 \oplus 2 \oplus 2$$

→ if we have an SU(2) symmetry of colour, then qqqqq would not exist, since there would be no colour singlet as seen above.

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Returning to useful stuff, like actual baryons (qqq)

In an SU(2) (ud) flavour symmetry:

$$2 \otimes 2 \otimes 2 = (3 \oplus 1) \otimes 2 = \underbrace{4}_{S} \oplus \underbrace{2}_{M_S} \oplus \underbrace{2}_{M_A}$$

★ Total wavefunction of baryons must be antisymmetric under under interchange of any 2 quarks.

$$|\Psi\rangle = \phi_{flavour}\, \chi_{spin}\, \alpha_{colour}\, \eta_{space}$$

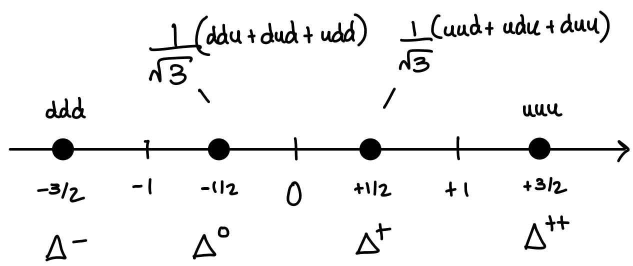
★ $\underline{\alpha_{colour}}$ in SU(3) colour symmetry for $qqq$ bound states is <u>antisymmetric</u>. (must be a colour singlet)

★ For lowest mass, ground state baryons, $\underline{L=0}$ and so <u>spatial</u> wavefunction is <u>symmetric</u> $(-1)^L$.

$\Rightarrow$ need $\phi_f \chi_s$ to be symmetric.

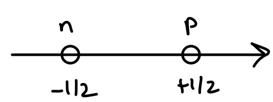
<u>2 ways</u>

1) $\phi(S)\chi(S)$ $\Rightarrow$ combine $S = \frac{3}{2}$ symmetric quartet and $I = \frac{3}{2}$ symmetric quartet:




$S = 3/2$
$I = 3/2$
$J = L + S = 3/2$

2) $\frac{1}{\sqrt{2}}\left(\phi(M_S)\,\chi(M_S) + \phi(M_A)\,\chi(M_A)\right)$



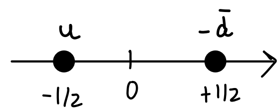

## Antiquarks & mesons

Antiquarks live in the conjugate representation of SU(2) flavour $\bar{2}$:

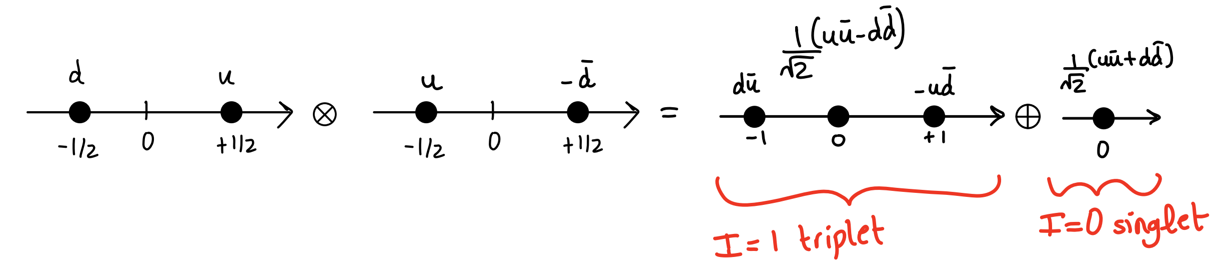
$$\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$

where the basis is

$$\bar{d} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$




## Light (ud) mesons




# SU(3)

SU(3) flavour (uds) symmetry is not exact.

SU(3) colour (rgb) symmetry is exact

Now the quark live in the fundamental 3-representation:

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

where the basis is:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore \begin{bmatrix} u' \\ d' \\ s' \end{bmatrix} = \hat{U} \begin{bmatrix} u \\ d \\ s \end{bmatrix}$, where $\hat{U}$ = 3x3 unitary matrix.

$\hat{U} = \mathbb{1}_3 + i\underline{\alpha}\cdot\hat{\underline{a}}$, where $\hat{\underline{a}}$ are the generators.

→ $U^\dagger U = \mathbb{1}_3$ imposes 9 constrainst, so we get 9 matrices forming U(1) group.

→ 1 matrix is just identity × phase.

→ $\underline{\underline{8}}$ remaining matrices form SU(3) group :

$$\hat{\underline{G}} = \underline{T} = \frac{1}{2}\underline{\lambda}$$ , where $\lambda$ are the 8 Gell-mann matrices.

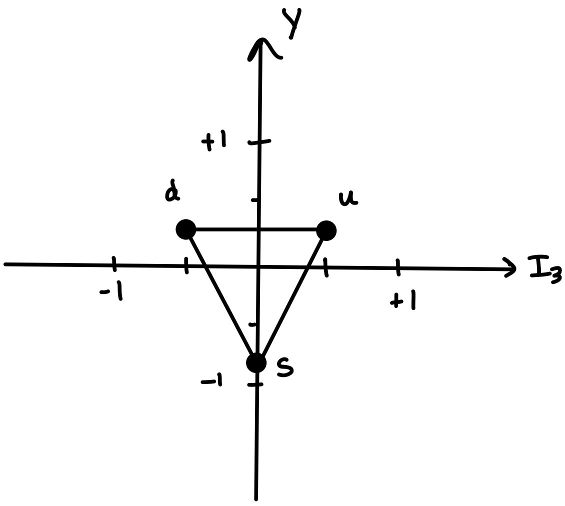
3rd component

$I_3 = \frac{1}{2}\lambda_3$ is Isospin operator

$Y = \frac{1}{\sqrt{3}}\lambda_8$ is hypercharge operator

$$\begin{cases} T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2) & u \leftrightarrow d \\ V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5) & u \leftrightarrow s \\ U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7) & d \leftrightarrow s \end{cases}$$

States labelled by $(I_3, Y)$ in 2D plane.

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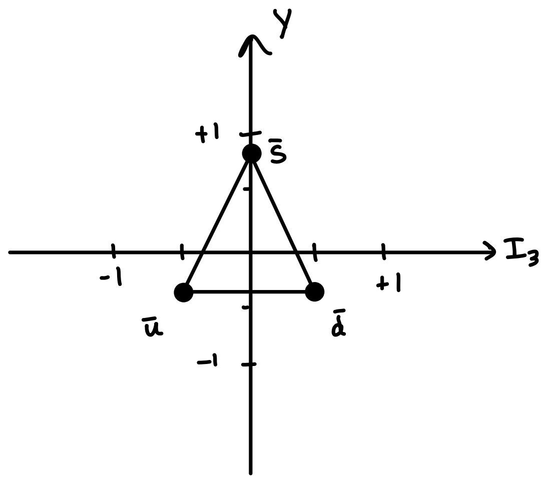
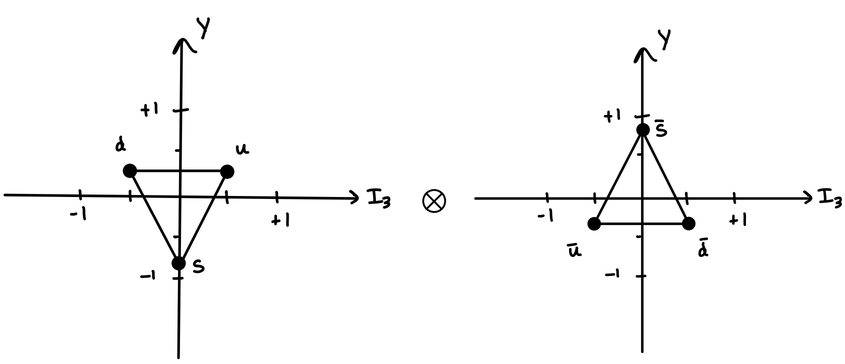
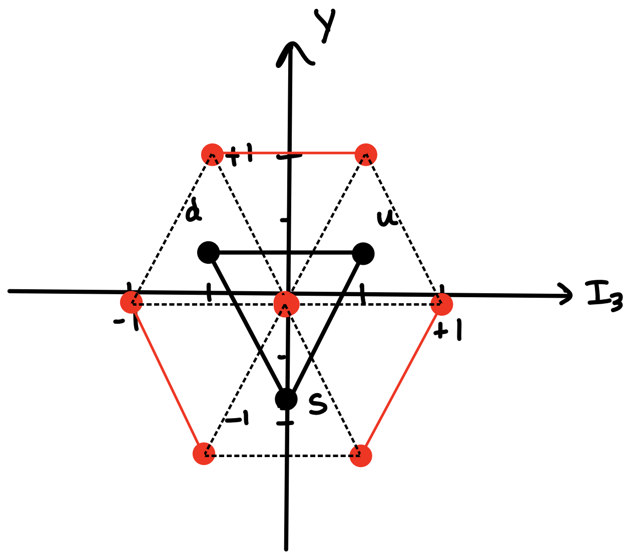



$\bar{3}$

## Light (uds) Mesons






→ 3 states in the middle.

→ 6 outer states

→ according to multiplet rules:

- once you have triangle, states do not increase going inside
- if you don't have triangle, states increase by one going inside.

→ can reach centre in 6 ways via ladder operators

→ only 2 linearly independent, so one state in middle not part of octet

→ $|u\bar{u}\rangle - |d\bar{d}\rangle$, $|u\bar{u}\rangle - |s\bar{s}\rangle$, $|d\bar{d}\rangle - |s\bar{s}\rangle$

→ $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \equiv \pi^0 = \psi_1$

✯ can get other state as $\psi_2 = \alpha(u\bar{u} - s\bar{s}) + \beta(d\bar{d} - s\bar{s})$

→ use $\langle\psi_2|\psi_2\rangle = 1$ & $\langle\psi_1|\psi_2\rangle = 0$

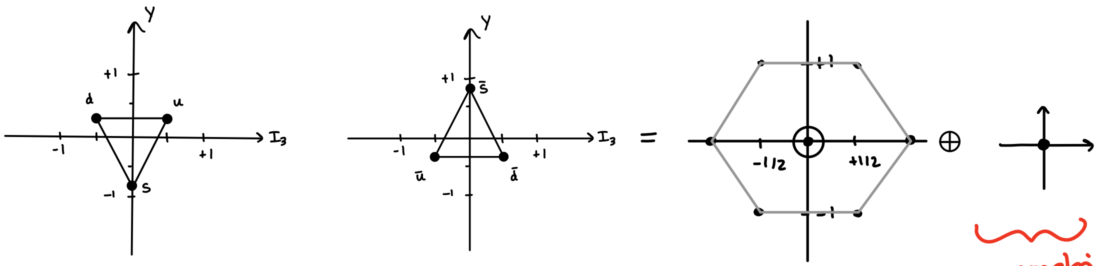
✯ can get singlet as $\langle\psi_2|\psi_3\rangle = 0$ & $\langle\psi_3|\psi_3\rangle = 1$

$\langle\psi_3|\psi_1\rangle = 0$

☆ remember singlet $\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$

or

$$\boxed{\psi_{singlet} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})}$$ → we obtain colour singlet wavefunction.

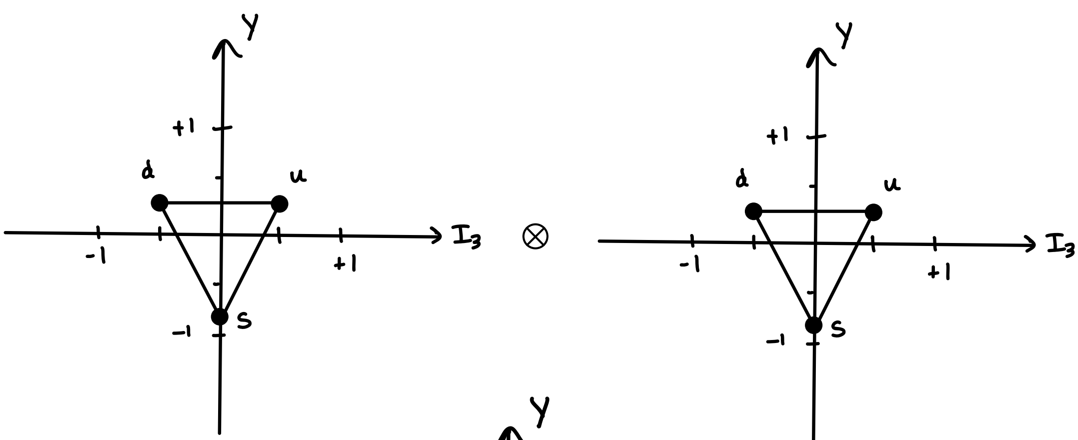


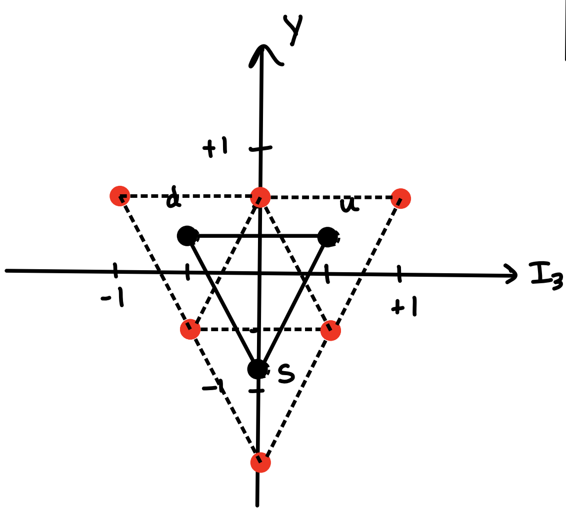

symmetric singlet

$$\boxed{3 \otimes \bar{3} = 8 \oplus 1}$$

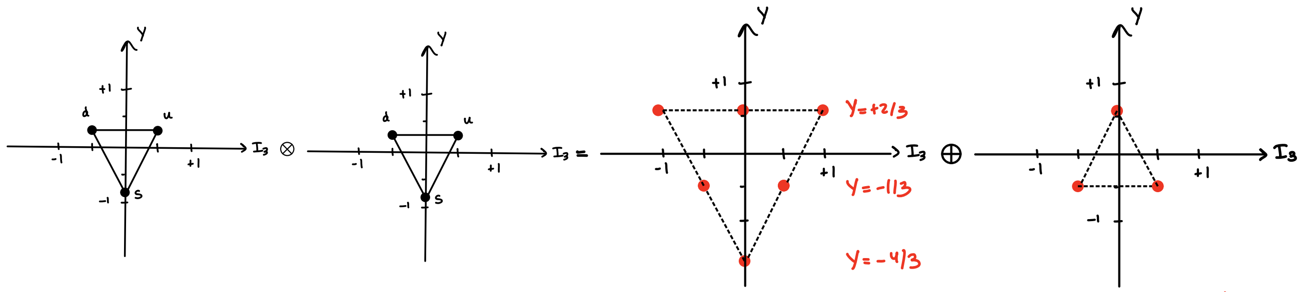
## Light (uds) Baryons

First add 2 quarks together:



→ shapes of dotted lines give us resulting multiplet shapes:

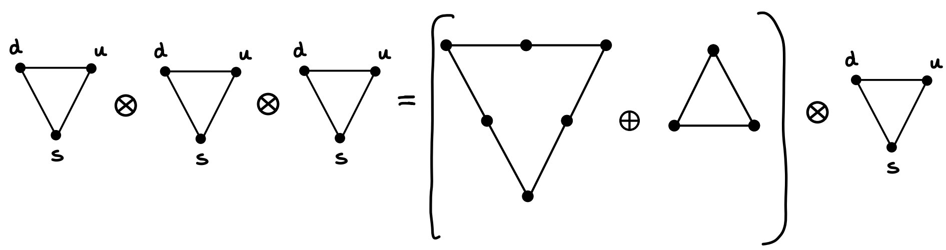



$$\boxed{3 \otimes 3 = 6 \oplus \bar{3}}$$

SYMMETRIC ANTISYMMETRIC

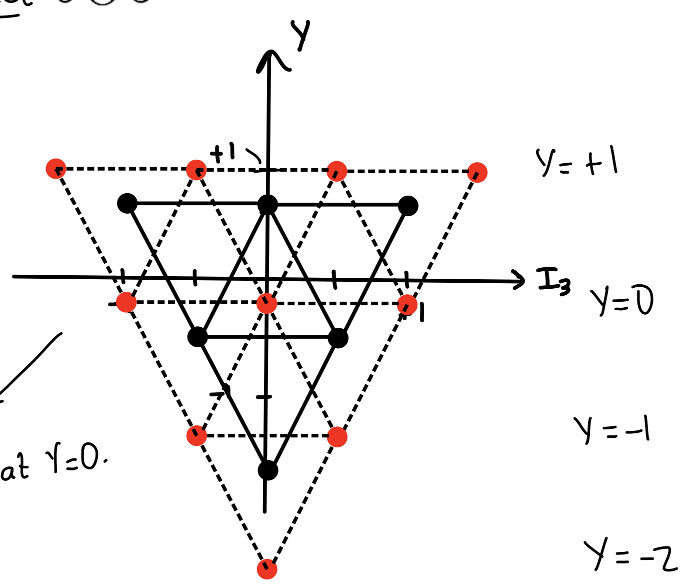
→ no singlet, so in SU(3) colour symmetry, no qq bound states.

Now add the 3rd quark...




$$\boxed{3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3}$$

1) <u>Building on sextet</u> $6 \otimes 3$

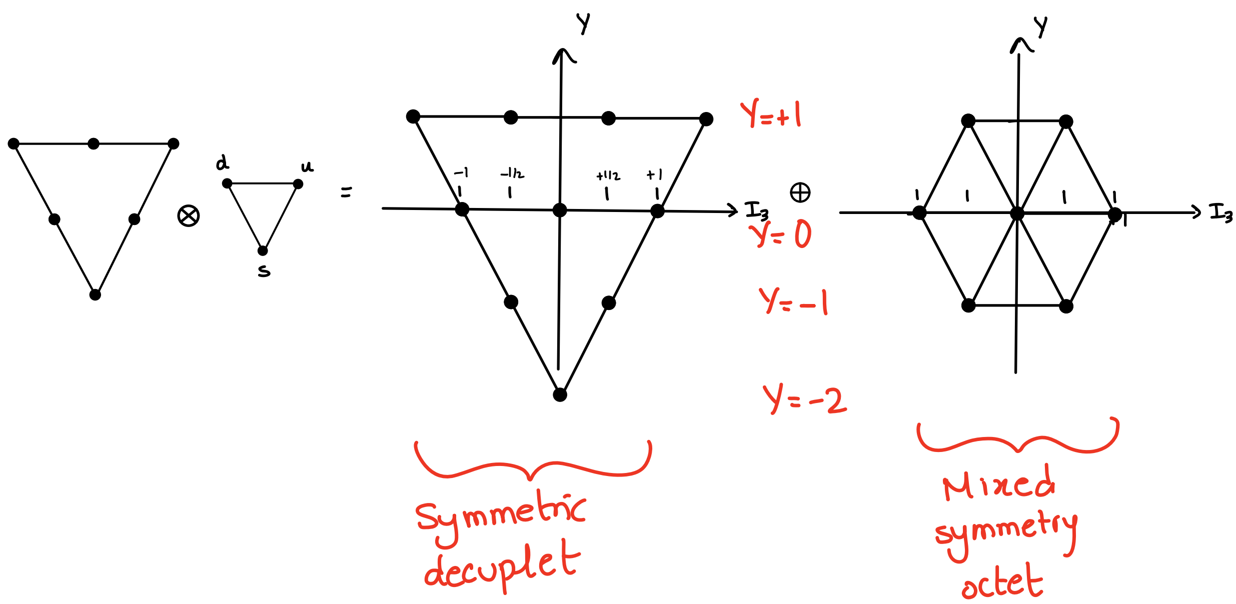



should be on $I_3$ axis at Y=0.

→ black dots & lines are our original sextet
→ red dots are positions of new states.
→ number of red dots obtained by counting nearest number black dots.

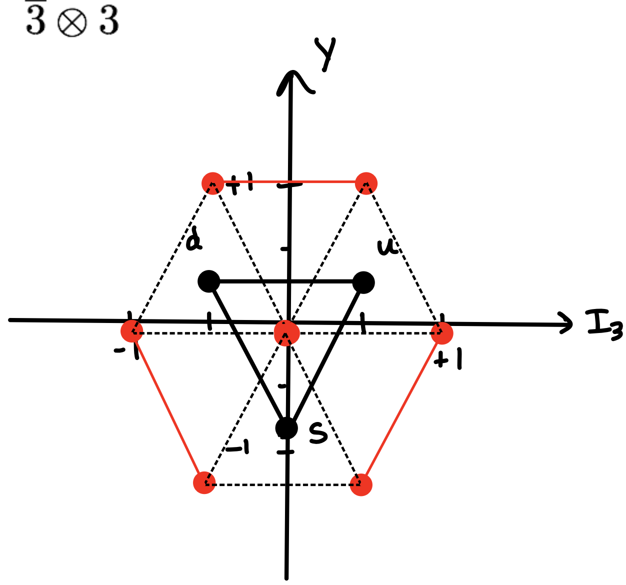
OR

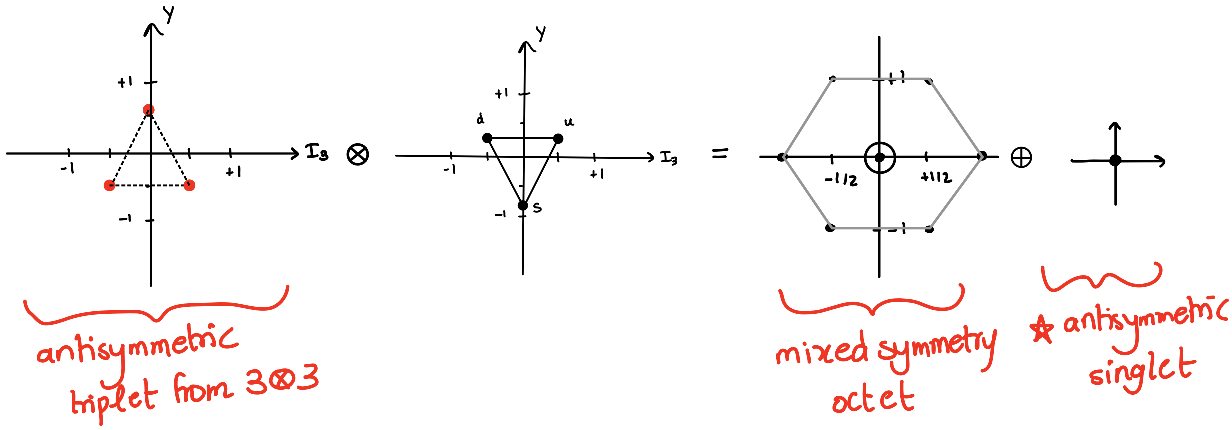
→ we know triangle will have 1 state each on periphery and one in the middle (don't increase no. of states when on triangles).
→ Can see another octet which will one state each on periphery and 2 in the middle (when shape is not triangle, increase one state going in).




$$\boxed{6 \otimes 3 = 10 \oplus 8}$$

Building on triplet $\bar{3} \otimes 3$






$$\boxed{\bar{3} \otimes 3 = 8 \oplus 1}$$

$$\begin{aligned}
\therefore 3 \otimes 3 \otimes 3 &= (6 \oplus \bar{3}) \otimes 3 \\
&= (6 \otimes 3) \oplus (\bar{3} \otimes 3) \\
&= (10 \oplus 8) \oplus (8 \oplus 1) \\
&= 10 \oplus 8 \oplus 8 \oplus 1
\end{aligned}$$

symmetric — Mixed — antisymmetric singlet

☆ $$\boxed{\psi_{singlet} = \frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)}$$

↳ colour singlet from SU(3) colour symmetry
⇒ qqq bound states possible.

→ cyclic permutations of uds have +ve signs.
cyclic permutations of usd have -ve signs.

# SUMMARY

## $\mathrm{SU}(2)$

BARYONS

$\psi_{singlet} = \frac{1}{\sqrt{2}}(rr - gg)$

$$
\begin{aligned}
2 \otimes 2 &= 3_S \oplus 1_A \\
2 \otimes 2 \otimes 2 &= (3_S \oplus 1_A) \otimes 2 \\
&= (3_S \otimes 2) \oplus (1_A \otimes 2) \\
&= (4_S \oplus 2_{MS}) \oplus (2_{MA}) \\
&= 4_S \oplus 2_{MS} \oplus 2_{MA}
\end{aligned}
$$

MESONS

$\psi_{singlet} = \frac{1}{\sqrt{2}}(r\bar{r} + g\bar{g})$

$$2 \otimes \overline{2} = 3 \oplus 1_S$$

## $\mathrm{SU}(3)$

Mesons

$\psi_{singlet} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$

$$3 \otimes \overline{3} = 8 \oplus 1_S$$

Baryons

$$
\begin{aligned}
3 \otimes 3 &= 6_S \oplus \overline{3}_A \\
3 \otimes 3 \otimes 3 &= \left(6_S \oplus \overline{3}_A\right) \otimes 3 \\
&= (6_S \otimes 3) \oplus \left(\overline{3}_A \otimes 3\right) \\
&= (10_S \oplus 8_M) \oplus (8_M \oplus 1_A) \\
&= 10_S \oplus 8_M \oplus 8_M \oplus 1_A
\end{aligned}
$$

$\psi_{singlet} = \frac{1}{\sqrt{6}}(rgb - rbg + brg - grb + gbr - bgr)$

* mesons are bosons
* baryons are fermions...

* Overall parity of 2 particle system in a state w/ orbital ang mom $L$:

$$P = P_1 \cdot P_2 \cdot (-1)^L$$