Magmas, Monoids, Quasigroups, etc

Thursday, 25 February 2021 09:46

TAKING MEAN

$x \circ y = \frac{1}{2}(x+y)$

$x, y \in \mathbb{R}$.

No identity.
Not associative.
Divisible, since:
$x \circ a = b \Rightarrow x = 2b - a$
$(b/a = a \backslash b = 2b - a)$

Note: The word "invertibility" does not appear until after "identity" appears, as an identity is needed to define inverses. So what is divisibility?

["Left division" exists if $a \circ x = b$ is solvable and unique. In that case $x = a \backslash b$.
"Right division" " " $x \circ a = b$ " " " " " " " " $x = b / a$.] ← [If both of these exists the magma is "divisible" and is said to be a quasigroup] ← [This is equivalent to Latin square property] ← [This means not only that $\begin{cases} a \circ (a \backslash b) \equiv b \\ (b/a) \circ a \equiv b \end{cases}$ but also that $\begin{cases} a \backslash (a \circ b) \equiv b \\ (b \circ a)/a \equiv b \end{cases}$ $\forall a, b$]

NON-EMPTY STRING CONCATENATION WITH REVERSAL:

"moo" $\circ$ "cow" = "moowoc"

"moo" $\circ\ x$ = "potato" has no solution (so not divisible)

OR

$x \circ y \to \bar{x}$ on BOOLEANS

A STRANGE OPERATION ON STRINGS LIKE THIS ONE:

$x \circ y = \begin{cases} y & \text{if } x = \text{"moo"} \\ x & \text{if } y = \text{"moo"} \\ x\text{-then-}y\text{-then-"X"} & \text{concatenation otherwise} \end{cases}$ } so "moo" is identity

eg "hello" $\circ$ "there" = "hellothereX"
"hello" $\circ$ "moo" = "hello" = "moo" $\circ$ "hello"
("a" $\circ$ "b") $\circ$ "c" = "abX" $\circ$ "c" = "abXcX"
"a" $\circ$ ("b" $\circ$ "c") = "a" $\circ$ "bcX" = "abcXX"
← unequal so not associative
There is no x s.t. $x \circ$ "cow" = "moo" (so no inverse)

weak inverse

$ABA = A$

↳ of this.

SUBTRACTION

or DIVISION of non-zero $\mathbb{R}$ or $\mathbb{Q}$.

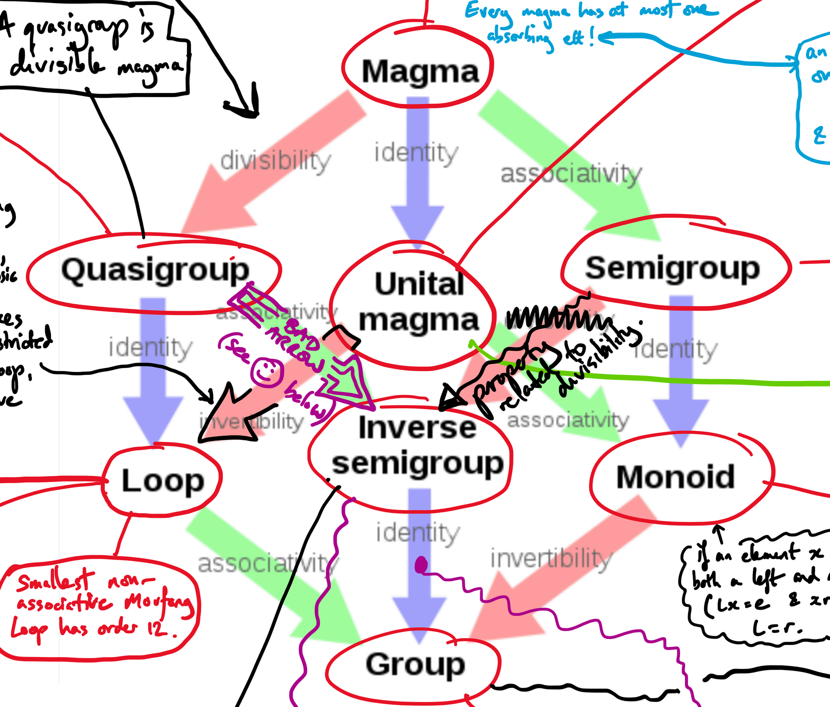
	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

sub mod 3

A quasigroup is a divisible magma

Every magma has at most one absorbing elt!

an absorbing elt $\emptyset$ is one for which $x \circ \emptyset = \emptyset$ & $\emptyset \circ x = \emptyset\ \forall x$



	0	1	2	3	4
0	0	1	2	3	4
1	1	4	0	2	3
2	2	3	4	1	0
3	3	0	1	4	2
4	4	2	3	0	1

- Has Identity (0)
- Is Latin square (so is quasigroup)
- Is non-associative $[(1.1).1 = 4.1 = 2;\ 1.(1.1) = 1.4 = 3]$
- L & R inverses differ $[1.2 = 0 = 2.4]$

$L_{inv} \neq R_{inv}$

Another misleading arrow. All loops are unital magmas, but adding the classic "L=R inverse" to a unital magma makes something more restricted (i.e. "beyond") a Loop, since Loops can have $L_{inv} \neq R_{inv}$.

NON-ZERO OCTONIONS UNDER MULTIPLICATION

Smallest non-associative Moufang Loop has order 12.

(Bol Loop)

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	7	5	0	6	2	4	3
2	2	6	7	5	0	3	1	4
3	3	0	6	7	5	4	2	1
4	4	5	0	6	7	1	3	2
5	5	2	3	4	1	7	0	6
6	6	4	1	2	3	0	7	5
7	7	3	4	1	2	6	5	0

- $\begin{bmatrix} 3\circ(4\circ5) = 3\circ1 = 0 \\ (3\circ4)\circ5 = 5\circ5 = 7 \end{bmatrix}$ so non-associative
- 0 = identity
- Latin square, so divisible.

∴ is non-associative Loop.
(Inverse exists ($L_{inv} = R_{inv}$))

NON-EMPTY-STRING CONCATENATION

"moo" $\circ$ "cow" = "moocow"

OR

ADDITION ON POSITIVE INTEGERS

STRING CONCATENATION

For L & R!
identity = "" (empty string)
"moo" $\circ$ "cow" = "moocow"

If an element x has both a left and right inverse $(Lx = e\ \&\ xr = e)$ then $L = r$.

By definition, a magmal identity e works from both L & R: $ex = xe = x\ \ \forall x$

Such identities are Unique:
Proof: If e_1 & e_2 are magmal identities:
$x = e_1 x\ \forall x$ & $y e_2 = y\ \forall y$
let $x = e_2$ | let $y = e_1$
∴ $e_2 = e_1 e_2$ | ∴ $e_1 e_2 = e_1$
∴ $e_1 = e_2$

ADDITION

An Inverse Semigroup S is a semigroup for which, for every $x \in S\ \exists$ a unique $y \in S$ s.t. $xyx = x$ and $yxy = y$.

An inverse semigroup need not be a quasigroup. See ☺ below.

SEMIGROUP EXAMPLE

☺	x	y
x	x	y
y	y	y

Has two idempotent elements ($x^2 = x$ & $y^2 = y$) so is not a group

Is associative since $x(xx) = (xx)x = x$ and $(a(bc) = (ab)c = y$ for all a, b, c not all x)

This example is NOT a quasigroup, however, as this is not a latin square. ($yt = y$ has solns $t = x$ & $t = y$)

So not all inverse semigroups are quasigroups!

☺ is an example of an inverse semigroup which is not a quasigroup

☺ is an inverse semigroup since both of these are shown

Is there a unique t satisfying $\begin{Bmatrix} txt = t \\ xtx = x \end{Bmatrix}$? Yes! only $t = x$ works.

Is there a unique t satisfying $\begin{Bmatrix} tyt = t \\ yty = y \end{Bmatrix}$? Yes! only $t = y$ works.

WHAT IS THIS?

The operation $a \circ b = |a - b|$ on the elements in $\{0, 1, 2\}$

$(1.1).2 = 2$
$1.(1.2) = 0$
Not Assoc!

	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

It satisfies all the axioms of a group except associativity
YET IT IS NOT A LOOP

Every Inverse Semigroup with an Identity is a Group.

Is an inverse semigroup with an identity a group?
Let S be an inverse semigroup with an identity e.
S is already associative, so it only remains for us to show that each element has a unique inverse.
The inverse semigroupness of S means that given $x \in S\ \exists$ unique $y \in S$ st. $\underbrace{xyx = x}_{①}$ & $\underbrace{yxy = y}_{②}$.
Note that ① & ② independently show that
$(xy)(xy) = (xy)$ & $(yx)(yx) = (yx)$.
∴ (xy) & (yx) are idempotent.
We know $e^2 = e$, so if we can show that there is only one idempotent element in S, then we will have shown that $xy = yx = e$ which will mean that $y = x^{-1}$ is a unique inverse of y.
Hence, it only remains to show that $(ii = i) \Rightarrow (i = e)$.
Suppose $ii = i$ then $iei = i$ (by insertion of identity).
But semigroupness implies that given $e \in S$ $\exists$ unique y st. $yey = y$. ∴ i in $iei = i$ is unique.
$iei = i$ is solved by $i \to e$. ∴ $i = e$ is unique soln.
This was what we needed to solve.

A group has exactly one idempotent element.
If $xx = x$ (ie x is idempotent)
then $(x^{-1}xx = x^{-1}x) \Rightarrow (ex = e)$
$\Rightarrow (x = e)$
∴ Every idempotent is the identity
(and we know that magmal identities are unique).

Minimal Fixes:

3: Consider putting something between Loop & Group to hold the mystery thing that is an extension of a unital magma by the addition of an inverse.

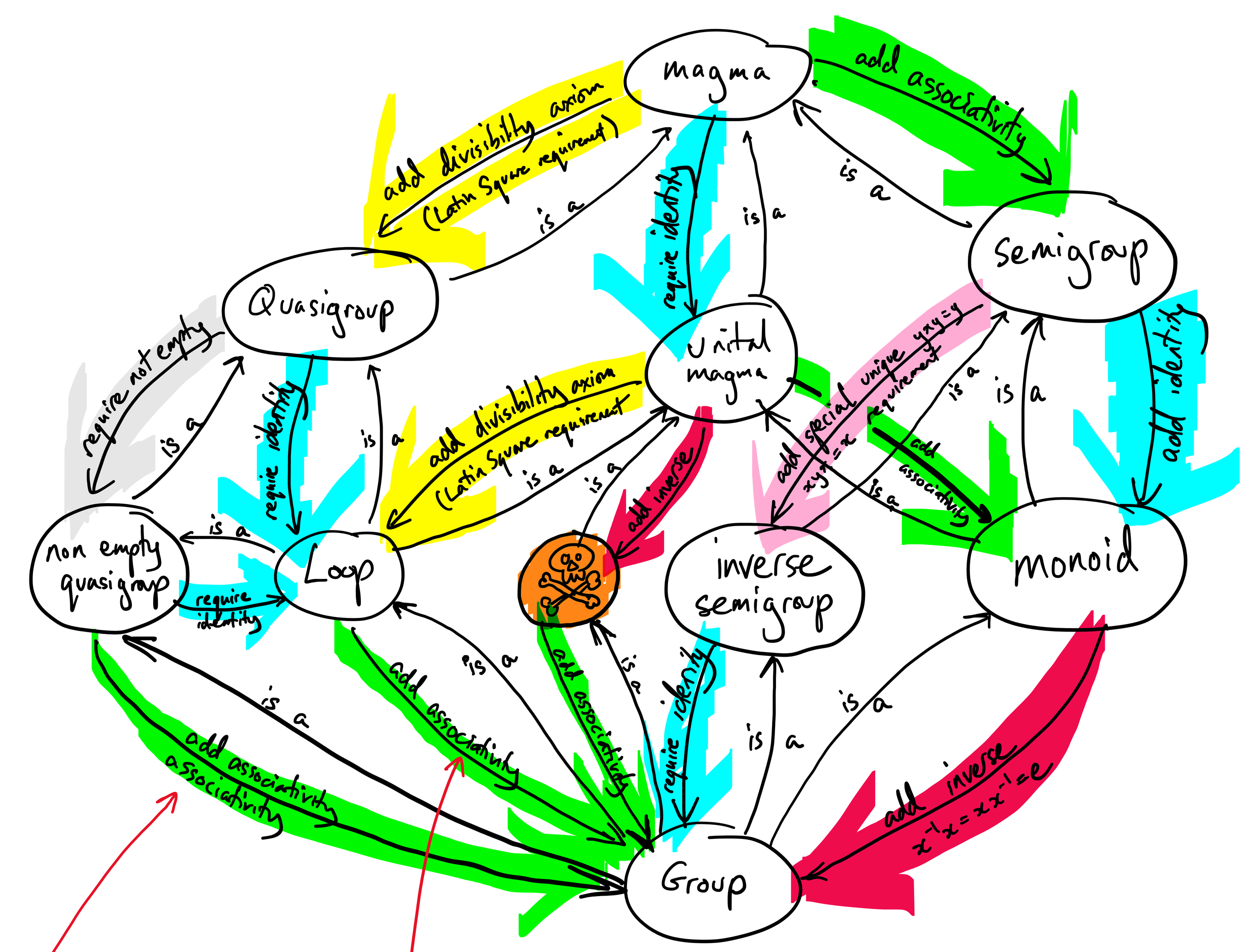
4: Identify what the mystery yellow thing is.

5. Consider
- inverse semigroup: semigroup for which $\forall x\ \exists$ unique y s.t. $x = xyx$ & $y = yxy$
- regular semigroup: " S " " $\forall a \in S\ \exists\ x$ s.t. $axa = a$ ← def 1
- or equivalently $\forall a \in S\ \exists$ at least one b s.t. $aba = a$ & $bab = b$ ← def 2

} supposedly equivalent.
is this supposed to be unique?

6. Understand purple box below my new diagram (box about cancellative inverse semigroups)

7. Add proof that loop + assoc = group



proof

proof

Proof that [Latin Squ] + [Non empty] + [Assoc] = [LOOP + ASSOC] (which is elsewhere proved to be group)

Non empty so contains elt "a".
① $ae = a$ has unique soln, call it "e".
(suspect / hope e = identity!)

$bE = b$ (suspect / hope $E = e$)
Note: $(xb) = a$ has unique soln.
∴ with that soln $x(bE) = xb$
$\Rightarrow (xb)E = xb$ (assoc)
$\Rightarrow aE = a \Rightarrow e = E$ by ①.
∴ e is a right identity which works for all elements.

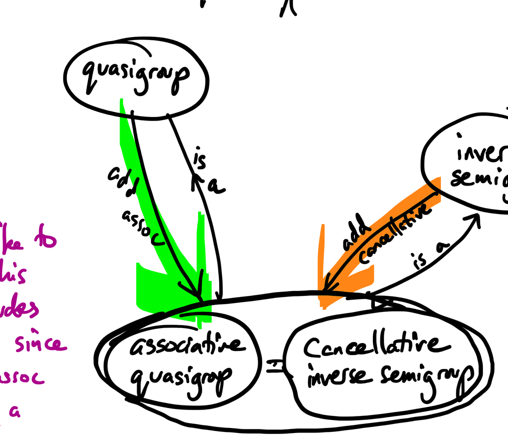
Consider LIKEWISE: ② $Fa = a$, defines a unique left identity which works for all elements.
In particular it works for e. I.e. $Fe = e$.
But e is a right identity ∴ $F = e$.
∴ There is an IDENTITY.

So now we have:
- LATIN SQUARE
- IDENTITY
- NON-EMPTY
- ASSOCIATIVE

But this is [LOOP + ASSOCIATIVE] which we already know is a group.

identity + latin square (uniquely solvable) + assoc.
Just need inverses.
$a.x = e$ is uniquely solvable ∴ R inverse exists
$a = e.a = (a.x).a = a.(x.a)$ (Assoc)
But $a = a.(x.a)$ is uniquely solvable for $(x.a)$ and is solved by $x.a = e$.
∴ $L_{inv} = R_{inv}$.

Furthermore: (3770047)
math stack exchange question claims:



I would like to check if this really excludes non-empty, since non empty assoc quasigroup is a group!