Dr. C. G. Gruber's Guide for Reformed Hipsters, aka enough using a Serre-ish doc.

THE REGULAR REPRESENTATION

The regular representation R is defined by

$$R(g)h = [g\,h] \quad \forall\, h,g \in G$$

(left action; group product; g fixed; h dummy; h varies through all elements)

then $[g]$ turns g into a "row-hat" under $\circ$: $([g]\circ[h]) \Leftrightarrow (h^{-1}g)$

SCHUR'S LEMMA

R^1 and R^2 are irreps of G and

$$\left(\text{e.g. } \begin{array}{c} R^1(g) \\ 1\times 1 \end{array} \begin{array}{c} T \\ 1\times 2 \end{array} = \begin{array}{c} T \\ 1\times 2 \end{array} \begin{array}{c} R^2(g) \\ 2\times 2 \end{array} \quad \forall g \in G\right)$$

$\Downarrow$

T defines an isomorphism between classes of R^1 and R^2 (and has no 0s)

OR

T defines an isomorphism between classes of R^1 and R^2 (and has no 0s) in R^1 and R^2 equivalent and T is square

Assumptions For Schur's Lemma

Suppose $R^1(g)\,T = T\,R^2(g) \quad \forall g \in G$

i.e.: $\begin{array}{c} R^1(g) \\ n\times n \end{array} \begin{array}{c} T \\ n\times m \end{array} = \begin{array}{c} T \\ n\times m \end{array} \begin{array}{c} R^2(g) \\ m\times m \end{array}$

Suppose also that R^1 and R^2 are irreps.

Find: $\forall v \in \mathrm{Image}(T) \Rightarrow \exists\, x \in V$ s.t. $v = Tx$
$\Rightarrow R^1(g)\,v = R^1(g)\,Tx \quad \forall g$
$\Rightarrow (R^1(g)\,T)\,x = (T\,R^2(g))\,x \quad \forall g$
$\Rightarrow (T\,R^2(g))\,x = R^1(g)\,Tx \quad \forall g$
$\Rightarrow R^1(g)\,v \in \mathrm{Image}(T) \quad \forall g$

Then $v \in \mathrm{Image}(T) \Rightarrow R^1(g)\,v \in \mathrm{Image}(T) \Rightarrow \mathrm{Image}(T)$ is a subspace of R^1

But since R^1 is irreducible, it has (by definition) no non-trivial subspaces.
$\therefore \mathrm{Image}(T)$ is a trivial subspace of V_1

$\left(R^1(g) \text{ on } \boxed{n\times n} \text{ will act on } V_1 \text{ [illegible]}\right)$

EITHER

$\mathrm{Image}(T) = V_1$ ..... OR ..... $\mathrm{Image}(T) = \{\vec{0}\}_{V_1}$

$T = \begin{array}{c} \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \end{array}$ $\{\vec{0}\}_{V_1}$

Find: $\forall v \in \ker(T) \Rightarrow T\,v = 0$
$\Rightarrow R^1(g)\,v = 0 \quad \forall g$
$\Rightarrow (R^1(g)\,T)\,v = 0 \quad \forall g$
$\Rightarrow (T\,R^2(g))\,v = 0 \quad \forall g$
$\Rightarrow T\,(R^2(g)\,v) = 0 \quad \forall g$
$\Rightarrow R^2(g)\,v \in \ker(T)$

Then $v \in \ker(T) \Rightarrow R^2(g)\,v \in \ker(T) \Rightarrow \ker(T)$ is a subspace of R^2

But since R^2 is irreducible, it has (by definition) no non-trivial subspaces.
$\therefore \ker(T)$ is a trivial subspace of V_2

$\left(R^2(g) \text{ on } \boxed{n\times n} \text{ will act on } V_2 \text{ [illegible]}\right)$

EITHER

$\ker(T) = V_2$ ..... OR ..... $\ker(T) = \{\vec{0}\}_{V_2}$

$T = \begin{array}{c} \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \end{array}$ $\{\vec{0}\}_{V_2}$

Conclusion of Schur's Lemma

EITHER

$T = \begin{array}{c} \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \end{array}$ $\det(V_1)$ OR $\det(V_2)$

$\det(V_1)$ defines an isomorphism between classes of R^1 and R^2 (and has no 0s)
i.e. $R^1(g) = T\,R^2(g)\,T^{-1}$
in R^1 and R^2 equivalent and T is square

OR

$\mathrm{Image}(T) = V_2$ AND $\ker(T) = \{0 \in V_1\}$

T is square and invertible

$\det(V_1) = \det(V_2)$

EITHER

$\det(V_1) > \det(V_2)$ → Contradicts $\ker(T) = \{0 \in V_1\}$

$\det(V_1) < \det(V_2)$

Contradicts $\mathrm{Image}(T) = V_2$!

THIS IS A CORRESPONDENCE OF SCHUR'S LEMMA:

T commutes with every member of a single irrep $\Rightarrow$ $T = c\,\mathbb{1}$ for some c (Q)

$\Updownarrow$

ALTERNATIVE STATEMENT OF SAME CHARACTERISTIC

T commutes with every member of a single representation $\Rightarrow$ $T = c\,\mathbb{1}$ for some c (the representation is reducible)

PROOF:

Suppose that $\mathbb{1}$ is an irrep of G and that $R(g)\,T = T\,R(g) \quad \forall g \in G$.
Suppose $T^\dagger = T - \lambda\,\mathbb{1}$ where $\lambda \in \mathbb{C}$ is an eigenvalue of T. (T may contain lots of "bad" non-diagonals)

$\Rightarrow$ $R(g)\,T^\dagger = R(g)\,(T - \lambda\,\mathbb{1})$
$= R(g)\,T - \lambda\,R(g)$
$= T\,R(g) - \lambda\,R(g)$
$= T^\dagger\,R(g)$

$T^\dagger = 0$ OR $T^\dagger$ is invertible

But "$T^\dagger$ is invertible" is equivalent to: $T_X + T_X - 3\lambda\,X_X + 3\lambda_X - 3\lambda_X + 0$

$T = \lambda\,\mathbb{1} \Leftarrow T^\dagger = 0 \Leftarrow$

Can we extend this theorem to equivalent rather than identical reps? Would make Grad Orthog Thm stronger.

MINI THEOREM (S)

non-equivalent $g \in G$ (of necessarily unitary or indecomposable)

$$T(X, I^1, I^2) \stackrel{\text{def}}{=} \sum_g I^1(g)_X\, X\, I^2(g)$$

$\Downarrow$

$$T(X, I^1, I^2) \text{ satisfies the properties for Schur's lemma (ie. } I^1(g)\,T = T\,I^2(g) \quad \forall\, g \in G)$$

PROOF:

$$I^1(g)\,T = \sum_{g'} I^1(g)_X\, X\, I^2(g')$$
$$= \sum_{g'} I^1(g)_X\, X\, I^2(g)\, I^2(g^{-1})\, I^2(g)$$
$$= \sum_{g'} I^1(g)_X\, X\, I^2(g^{-1})\, I^2(g)$$
$$= \left(\sum_{g'} I^1(g)_X\, X\, I^2(g^{-1})\right) I^2(g)$$
$$= T\, I^2(g) \quad \text{(by rearrangement)}$$

MINI THEOREM (S)

If I^1 & I^2 are inequivalent irreps of G then by Schur's Lemma (S) & mini theorem (S):
$T(X, I^1, I^2) = 0 \quad \forall$ arbitrarily chosen X.
(for T as (S) shows)
in that manner

Choosing $(X)_{ij} \stackrel{\text{def}}{=} \delta_{ip}\,\delta_{iq}$:

$$0 = \left(T(X, I^1, I^2)\right)_{ij} = \sum_g I^1(g)_{ip}\, \delta_{pq}\, I^2(g)_{qj}$$
$$= \sum_g I^1(g)_{ip}\, \delta_{pq}\, I^2(g)_{qj}$$
$$= \sum_g I^1(g)_{ip}\, I^2(g)_{qj}$$

MINI THEOREM (S)

If I^1 & I^2 are inequivalent irreps of G
then $\sum_g I^1(g)_{ij}\, I^2(g^{-1})_{kl} = 0$ [sum for ANY type of rep]
and $\sum_g I^1(g)_{ij}\, I^2(g)_{kl} = 0$ [sum for UNITARY rep only]
$(\forall\, i,j,k,l)$.

MINI THEOREM (S)

If I^1 & I^2 are IDENTICAL irreps of G
then by Schur's Lemma (S) & mini theorem (S):
$T(X, I^1, I^2) = c\,\mathbb{1} \quad \forall$ arbitrarily chosen X, for some c.
(for T as (S) shows)
in that manner

Choosing $(X)_{ij} \stackrel{\text{def}}{=} \delta_{ip}\,\delta_{iq}$:

$$c\,\delta_{ij} = \left(T(X, I^1, I^2)\right)_{ij} = \sum_g I^1(g)_{ip}\, \delta_{pq}\, I^2(g)_{qj}$$
$$= \sum_g I^1(g)_{ip}\, \delta_{pq}\, I^2(g)_{qj}$$
$$= \sum_g I^1(g)_{ip}\, I^2(g)_{qj}$$

(where c is independent of i & j.)

Can determine c by taking trace over i & j:

$$c\,n = \sum_g I^1(g)_{ip}\, I^2(g)_{qj}$$
$$\Rightarrow \boxed{c\,n\,\mathrm{Dim}(I^1)} = \sum_g I^1(g^{-1})_{qp} \quad \text{(via } I^1, I^2\text{)}$$
$$\Rightarrow c\,n\,\mathrm{Dim}(I^1) = |G|\,\delta_{pq}$$
$$\Rightarrow c = \frac{|G|}{\mathrm{Dim}(I^1)}\,\delta_{pq}$$

MINI THEOREM (S)

If I^1 & I^2 are IDENTICAL irreps of G
then $\sum_g I^1(g)_{ij}\, I^2(g^{-1})_{kl} = \frac{|G|}{\mathrm{Dim}(I^1)}\,\delta_{ik}\,\delta_{jl}$ [sum for ANY type of rep]
and $\sum_g I^1(g)_{ij}\, I^2(g)_{kl} = \frac{|G|}{\mathrm{Dim}(I^1)}\,\delta_{ik}\,\delta_{jl}$ [sum for UNITARY rep only]
$(\forall\, i,j,k,l)$.

GRAND ORTHOGONALITY THEOREM (S)

If I^1 & I^2 are IDENTICAL irreps of G
then $\sum_g I^1(g)_{ij}\, I^1(g^{-1})_{kl} = \frac{|G|}{\mathrm{Dim}(I^1)}\,\delta_{ik}\,\delta_{jl}$ [sum for ANY type of rep]
and $\sum_g I^1(g)_{ij}\, I^1(g)_{kl} = \frac{|G|}{\mathrm{Dim}(I^1)}\,\delta_{ik}\,\delta_{jl}$ [sum for UNITARY rep only]
where (i.e. one of each) $\delta_{ik} = \begin{cases} 1 & \text{if } i = k \text{ (IDENTICAL)} \\ 0 & \text{if } i \neq k \text{ (INCOHERENT)} \end{cases}$;

$\Downarrow$

$$\sum_g I^1(g)_{ij}\, I^1(g^{-1})_{kl} = \frac{|G|}{\mathrm{Dim}(I^1)}\,\delta_{ik}\,\delta_{jl}, \quad \frac{|G|}{\mathrm{Dim}(I^1)}\,\mathrm{Dim}(I^1)\,\delta_{ik}$$
$$\therefore \sum_g \chi^1(g)\,\chi^1(g^{-1}) = |G|\,\delta_{ik} \quad \text{(Irrep IDENTICAL implies Unitary)}$$
$$\sum_g \chi^1(g)\,\chi^1(g) \quad \text{(Unitary INCOHERENT implies Irrep)}$$

$\Downarrow$

Weak Row Orthogonality Theorem

$$\left\{\begin{array}{l} \frac{1}{|G|}\sum_g \chi^1(g)\,\chi^2(g) = \delta_{ik} = \begin{cases} 1 & \text{if } I^1 \text{ \& } I^2 \text{ are IDENTICAL} \\ 0 & \text{if } I^1 \text{ \& } I^2 \text{ are INCOHERENT} \end{cases} \\ \frac{1}{|G|}\sum_g \chi^1(g)\,\chi^1(g^{-1}) = \delta_{ik} = \begin{cases} 1 & \text{if } I^1 \text{ \& } I^2 \text{ are IDENTICAL} \\ 0 & \text{if } I^1 \text{ \& } I^2 \text{ are INCOHERENT} \end{cases} \end{array}\right\} \begin{array}{l} \text{UNITARY IRREPS} \\ \text{IRREPS} \end{array}$$

BUT

If I^1 and I^2 are EQUIVALENT then $\exists\, T$ s.t. $I^1(g) = T\,I^2(g)\,T^{-1} \quad \forall g$
$\therefore \chi^1(g) = \mathrm{Tr}(I^1(g)) = \mathrm{Tr}(T\,I^2(g)\,T^{-1}) = \mathrm{Tr}(I^2(g)) = \chi^2(g)$
Hence, final orthogonality statement can be made stronger:

Strong Row Orthogonality Theorem (D)

$$\left\{\begin{array}{l} \frac{1}{|G|}\sum_g \chi^1(g)\,\chi^2(g) = \delta_{ik} = \begin{cases} 1 & \text{if } I^1 \text{ \& } I^2 \text{ are EQUIVALENT} \\ 0 & \text{if } I^1 \text{ \& } I^2 \text{ are INCOHERENT} \end{cases} \\ \frac{1}{|G|}\sum_g \chi^1(g)\,\chi^1(g^{-1}) = \delta_{ik} = \begin{cases} 1 & \text{if } I^1 \text{ \& } I^2 \text{ are EQUIVALENT} \\ 0 & \text{if } I^1 \text{ \& } I^2 \text{ are INCOHERENT} \end{cases} \end{array}\right\} \begin{array}{l} \text{UNITARY IRREPS} \\ \text{IRREPS} \end{array}$$

The above result (D) has established the

ROW ORTHOGONALITY of CHARACTER TABLES for

unitary representations of finite groups.

What about

COLUMN ORTHOGONALITY ?

Now that we have shown that the rows of the character table are orthogonal to each other, the proof of column orthogonality brings you showing that the character table (when there is one column per conjugacy class) is square.

Why?

Schematically

$M\,M^\dagger = \mathbb{1}$ is a way of saying that the rows of M are orthogonal (totally unitary), and

$M^\dagger\,M = \mathbb{1}$ . . . . . COLUMNS . . . . . .

yet if SQUARE matrix $(M\,M^\dagger = \mathbb{1}) \Leftrightarrow (M^\dagger\,M = \mathbb{1})$!

This schematic description breaks down a few small details, however, which we now address:

① we want to impose row orthogonality not orthonormality

② our notion of row orthogonality (Gorard row orthogonality theorem (D)) has a row over group elements, whereas the character table has one column per equivalence class. Hence our notion of row orthogonality in the character table requires our "inner product" to include a weighted sum (weighted by the number in each equivalence class $n_1, n_2, \ldots, n_k$)

③ The sum in Schur-Row Orthogonality Theorem (S) requires us to use an $\langle\cdot,\cdot\rangle_G$ inner product (which it requires us to use a UNITARY representation if we are to apply it directly to a "normal" character table.

→ Finally, all these things together, if "M" is the matrix representing a character table built from UNITARY IRREPS of a finite group, then (D) (i.e. "GRAND ORTHOGONALITY") can be written in matrix form as

the k stand-in T matrix (D) — $|G|$ needs for (D)

$$M\,D\,M^\dagger = |G|\,\mathbb{1}$$

$\Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ accomplishes (D)

for some positive diagonal matrix D, and furthermore, our claim is that squareness of M (yet to be proved) is sufficient for us to deduce that $M^\dagger\,M = \text{Diagonal}$ (TT) which is the desired result that the character table has orthogonal columns.

The proof that (D) ⇒ (TT) for square M is trivial:

(D) $\Rightarrow M\,D\,M^\dagger = |G|\,\mathbb{1}$
$\Rightarrow \frac{1}{|G|}\,M\,D\,M^\dagger = \mathbb{1}$ (now $|G| > 0$)
$\Rightarrow \frac{1}{|G|}\,D\,M^\dagger\,M = \mathbb{1}$ (since M is square and so $\frac{1}{|G|}\,D$ is left-inv. left and its right inverse)
$\Rightarrow M^\dagger\,M = |G|\,D^{-1}$ ("D^{-1}" exists and is diagonal since "D" is diagonal and positive definite)
$\Rightarrow (M^\dagger\,M \text{ is diagonal})$
$\Rightarrow$ (TT).

We therefore set about proving that:

$\#(\text{irreducible IRREPS}) = \#(\text{CONJUGACY CLASSES})$

i.e.: $N_{\text{irreps}} = N_{\text{class}}$

as this will complete the proof that the columns of a character table are orthogonal (provided the group is finite, thus we are dealing with a square M)

N_{class}

CHARACTER TABLE

	C_1	C_2	C_3	C_4
χ^1	χ_1	χ_2	χ_3	χ_4
χ^2	χ_1	χ_2	χ_3	χ_4
χ^3	χ_1	χ_2	χ_3	χ_4

M_{irreps}

M MATRIX (will be shown to be square!)

$N_{\text{irreps}} = N_{\text{row vectors}}$ (since these are orthogonal by (D))
$= \dim(\text{RANK}(M))$
$= \text{CALC RANK}(M)$ (by row matrix rank)
$\leq N_{\text{row vectors}}$
$\leq N_{\text{class}}$

Find we show that $N_{\text{irreps}} \leq N_{\text{class}}$ PROOF

If only remains to show $N_{\text{irreps}} \geq N_{\text{class}}$.

However, the proof of this is much longer↓

STEP 1: CREATE SOME HYPOTHESES.

CLASS FUNCTION

A function $\phi : G \to \mathbb{C}$ st. $\phi(g) = \phi(hgh^{-1}) \quad \forall\, g,h \in G$.

DEFINE SOME PRODUCTS

$$\langle\langle \phi, \psi \rangle\rangle \stackrel{\text{def}}{=} \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}\,\psi(g)$$

ϕ, ψ: ϕ dependent, ψ independent; ϕ independent, ψ dependent; $\phi, \psi \in \mathbb{C}$

$$\langle\langle \phi, \psi \rangle\rangle \stackrel{\text{def}}{=} \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}\,\overline{\psi(g)}$$

$\phi, \psi \in G$; $\phi, \psi \in \mathbb{C}$

$\langle\langle \phi, \psi \rangle\rangle$ is a unitary "dot-prod" of the $\phi(g)$ matrices — the gps, weighted by conjugacy class.

The centralizer of an element z of a group G is the set of elements of G which commute with z.

$$C_G(z) = \{x \in G,\ xz = zx\}.$$

Likewise, the centralizer of a subgroup H of a group G is the set of elements of G which commute with every element of H.

$$C_G(H) = \{x \in G,\ \forall\, h \in H,\ xh = hx\}.$$

STEP 2: PROVE SOME INTERMEDIATE RESULTS

MINI THEOREM (A)

If ϕ is a class function and ψ is a class function of ψ then $\langle\langle \phi, \psi \rangle\rangle = \frac{1}{\mathrm{Dim}(G)}\langle\langle \phi, \psi \rangle\rangle$ (dim(G))

PROOF

$$\phi(g)\,\langle\langle \phi, \psi \rangle\rangle\, \phi(g) = \phi(g)\,\frac{1}{|G|}\sum_{h\in G} \overline{\phi(h)}\,\phi(h)\,\phi(h)$$
$$= \frac{1}{|G|}\sum_{h\in G} \overline{\phi(h)}\,\phi(g^{-1}h)$$
$$= \frac{1}{|G|}\sum_{h\in G} \overline{\phi(h)}\,\phi(h^{-1}g)$$
$$= \frac{1}{|G|}\sum_{h\in G} \overline{\phi(h)}\,\phi(h^{-1}g)$$

Use ϕ is a class function

THE REARRANGEMENT

$$= \frac{1}{|G|}\sum_{h\in G} \overline{\phi(h)}\,\phi(h)$$
$$= \langle\langle \phi, \phi \rangle\rangle \quad \forall g$$

$\therefore \langle\langle \phi, \phi \rangle\rangle\, \phi(g) = \phi(g)\,\langle\langle \phi, \phi \rangle\rangle \quad \forall g$

$\therefore$ Schur's Lemma (Gorard (D)) applies to $\langle\langle \phi, \phi \rangle\rangle$

$\therefore \langle\langle \phi, \phi \rangle\rangle = \lambda\,\mathbb{1}$ for some λ.
(since ϕ is only fixed by arbitrary group)

MINI THEOREM (B)

If $\langle\langle \phi, \psi \rangle\rangle = 0 \quad \forall$ irreps ψ, then $\phi \equiv 0$ (ie. $\phi(g) = 0 \quad \forall g$)

PROOF:

Suppose $\phi : G \to \mathbb{C}$ st. $\langle\langle \phi, \psi \rangle\rangle = 0 \quad \forall$ irreps ψ.
For such a ϕ, consider $\langle\langle \phi, R \rangle\rangle$.
$\langle\langle \phi, R \rangle\rangle$ must also be zero, since any representation is a direct sum of irreducible representations.

i.e. Regular representation of G.

Should write out proof of this without the reliance on irreps.

So $0 = \langle\langle \phi, R \rangle\rangle = \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}\,R(g)$

gen. matrix — gen. identity

$\therefore 0\,[\mathbb{1}] = \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}\,R(g)\,[\mathbb{1}]$

Pre-fact gender $G \to \mathbb{R}^n$ — using the adjoint property of REGULAR REPRESENTATION

Pre-mult (for $i \in \mathbb{R}^n$)

$$\therefore 0 = \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}\,[\mathbb{1}\,\mathbb{1}]$$
$$= \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}\,[\mathbb{1}]$$
$$= \frac{1}{|G|}\left(\overline{\phi(e_1)}, \overline{\phi(e_2)}, \ldots \overline{\phi(e_n)}\right)^\dagger$$
$$\therefore \phi(g) = 0 \quad \forall\, g \in G$$
$$\therefore \phi \equiv 0. \quad \text{QED.}$$

MINI THEOREM (C)

$\langle\langle \psi_i^1, \psi_j^2 \rangle\rangle = \frac{1}{\mathrm{Dim}(\psi^1)}\,\delta_{ij}$ if ψ^1 is a UNITARY IRREP

(Gorard (D) (TT) for Unitary IRREPS)

PROOF:

$$\langle\langle \psi_i^1, \psi_j^2 \rangle\rangle = \frac{1}{|G|}\sum_g \psi^1(g)_{ii}\,\overline{\psi^2(g)_{jj}}$$
$$= \frac{1}{\mathrm{Dim}(\psi^1)}\,\delta_{ij}\,\delta_{ii}\,\delta_{jj} = \frac{1}{\mathrm{Dim}(\psi^1)}\,\delta_{ij}$$

Gorard GRAND ORTHOGONALITY THEOREM (D) Simple

$$\sum_g \psi^1(g)_{ij}\,\overline{\psi^2(g)_{kl}} = \frac{|G|}{\mathrm{Dim}(\psi^1)}\,\delta_{ik}\,\delta_{jl} \quad \text{[sum for UNITARY rep only]}$$

STEP 3: STATE AND PROVE THE MAIN RESULT USING THE INTERMEDIATE RESULTS.

MAJOR THEOREM (E)

Any class function of some irreducible ψ can be written as a linear combination of irreducible characters χ^ψ of inequivalent irreps ψ of the group.

$$\phi = \sum_\psi \langle\langle \phi, \chi^\psi \rangle\rangle\, \chi^\psi$$

Can be shown from (B) (Gorard (D))

That is to say, the characters of inequivalent irreps span the space of class functions.

ASIDE:

If a class function of χ is writable in the form $\phi = \sum_\psi \chi^\psi$ then it is about trivial to show that

$$\phi = \sum_\psi \langle\langle \phi, \chi^\psi \rangle\rangle\, \chi^\psi$$

All the difficulty proving the MAJOR THEOREM (Gorard from) showing that the stated form holds for any ϕ.

SECOND ASIDE

It is also possible to show a stronger result, namely that the χ^ψ are a basis for the space of class functions. We will not need this stronger result, though, so we do not bother to prove it.

PROOF

To prove the above, it is sufficient to show that

$$\Phi_\phi = 0 \quad \forall \text{ class functions } \phi$$

where

$$\Phi_\phi \stackrel{\text{def}}{=} \phi - \sum_\psi \langle\langle \phi, \chi^\psi \rangle\rangle\, \chi^\psi. \quad (1)$$

We prove that $\Phi_\phi = 0$ by showing below that

$$\langle\langle \Phi_\phi, \lambda \rangle\rangle = 0 \quad \forall \text{ irreps } \lambda$$

so that MINI THEOREM (B) applied to Φ_ϕ.

$$\langle\langle \Phi_\phi, \lambda^\psi \rangle\rangle = \frac{1}{|G|}\sum_g \overline{\Phi_\phi(g)}\,\lambda^\psi(g) \quad \text{(from (1))}$$
$$= \frac{1}{|G|}\sum_g \left(\overline{\phi(g)} - \sum_\psi \langle\langle \phi, \chi^\psi \rangle\rangle\, \overline{\chi^\psi(g)}\right)\lambda^\psi(g)$$
$$= \frac{1}{|G|}\sum_g \overline{\phi(g)}\,\lambda^\psi(g) - \sum_\psi \langle\langle \phi, \chi^\psi \rangle\rangle\, \frac{1}{|G|}\sum_g \overline{\chi^\psi(g)}\,\lambda^\psi(g)$$
$$= \langle\langle \phi, \lambda^\psi \rangle\rangle - \sum_\psi \langle\langle \phi, \chi^\psi \rangle\rangle\, \langle\langle \chi^\psi, \lambda^\psi \rangle\rangle \quad \text{(for (B))}$$
$$= \langle\langle \phi, \lambda^\psi \rangle\rangle - \sum_\psi \langle\langle \phi, \chi^\psi \rangle\rangle\, \frac{\delta_{\psi\psi}}{\mathrm{Dim}(\lambda^\psi)} \quad \text{(by mini Theorem (C))}$$
$$= \langle\langle \phi, \lambda^\psi \rangle\rangle - \langle\langle \phi, \chi^\psi \rangle\rangle\, \frac{1}{\mathrm{Dim}(\lambda^\psi)} \quad \text{(Replace UNITARY IRREP)}$$
$$= \langle\langle \phi, \lambda^\psi \rangle\rangle - \langle\langle \phi, \lambda^\psi \rangle\rangle \quad \text{(by mini Theorem (C))}$$
$$= 0 \quad \text{QED.}$$

STEP 4: FINAL ARGUMENT

There are at least N_{class} linearly independent class functions (Eg. the "δ-like" such L.I. class function could be one which maps elts of the "i"-th conjugacy class to 1 and maps all other elements of G to 0.)

Major theorem (E) tells us that the characters χ^ψ of the inequivalent unitary irreps λ^ψ span the above L.I. class function, hence there must be at least as many irreps (Nirreps) there are L.I. class functions i.e.

i.e.: $N_{\text{irreps}} \geq N_{\text{class}}$ QED.

THIS COMPLETES THE PROOF THAT $N_{\text{irreps}} = N_{\text{class}}$

IGNORE THIS BOX

Def: $\langle s, t \rangle \stackrel{\text{def}}{=} s^{-1}\,t^{-1}$ ← the "inner" Another rep

and: $[s, t] \stackrel{\text{def}}{=} \sum_i \langle A(s)_i, A(t)_i \rangle$.

Note: $[A(s), A(t)] \equiv [s, t]$. Irrep repeat
ie $[\ ,\ ]$ is invariant in the above sense.

Note that for left-regular:

$$[s, A(t)_i] = \sum_i \langle A(s)_i, A(t)\,A(t)_i \rangle$$
$$= \sum_i \langle A(s)\,A(t)\,A(t^{-1})_i, A(t)\,A(t)_i \rangle$$
$$= \sum_i \langle A(s)\,A(t^{-1})_i, A(t)\,\mathbb{1} \rangle$$
$$= [A(t^{-1})\,s, t] \quad \text{(or right)} \quad \text{(ie. this is not the "inner" Another rep)}$$

$\therefore$ For unitary regular: $[s, A(t)] = [A(t)^\dagger\,s, t]$

Lecture notes claim that if you started without copying that your rep was unitary, but then demanded that it satisfy (factoring function rep)
$[s, A(t)] = [A(t)^\dagger\,s, t]$
then this would be equivalent to demanding that the representation be unitary.

seems unfairly plausible

Aside Theorem: unitary

The characters χ^ψ of the inequivalent/irreps λ^ψ of a group G are an orthonormal basis for the space of Class Functions on G.
$\mathrm{Cl}(G) = \{\phi : G \to \mathbb{C} \mid \phi(g) = \phi(hgh^{-1}),\ \forall\, h,g \in G\}$

i.e.: $\langle\langle \chi^i, \chi^j \rangle\rangle = \delta_{ij}$ (orthonormality)

& $\left(\sum_i c_i\,\chi^i(g) = 0\right) \Rightarrow \left(c_i = 0 \quad \forall\, i\right)$ (LI independence)

& $\left(\{\phi, \chi^i\} = 0 \quad \forall\, i\right) \Rightarrow \left(\phi = 0\right)$ (spans the space)

(SUPPRESSED)

Orthogonality relations [edit]

Main article: Schur orthogonality relations

The space of complex-valued class functions of a finite group G has a natural inner-product:

$$\langle \alpha, \beta \rangle := \frac{1}{|G|}\sum_{g\in G} \alpha(g)\overline{\beta(g)}$$

where $\overline{\beta(g)}$ means the complex conjugate of the value of β on g. With respect to this inner product, the irreducible characters form an orthonormal basis for the space of class-functions, and this yields the orthogonality relation for the rows of the character table:

$$\langle \chi_i, \chi_j \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

For $g, h \in G$ the orthogonality relation for columns is as follows:

$$\sum_{\chi_i} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} |C_G(g)|, & \text{if } g, h \text{ are conjugate} \\ 0 & \text{otherwise.} \end{cases}$$

where the sum is over all of the irreducible characters χ_i of G and the symbol $|C_G(g)|$ denotes the order of the centralizer of g.

For an arbitrary character χ_i, it is irreducible if and only if $\langle \chi_i, \chi_i \rangle = 1$.

The orthogonality relations can aid many computations including:

- Decomposing an unknown character as a linear combination of irreducible characters, i.e. # of copies of irreducible representation V_i in $V = \langle \chi_V, \chi_i \rangle$.
- Constructing the complete character table when only some of the irreducible characters are known.
- Finding the orders of the centralizers of representatives of the conjugacy classes of a group.
- Finding the order of the group, $|G| = |C_G(g)| * \sum_{\chi_i} \chi_i(g)\overline{\chi_i(g)}$, for any g in G.

If the irreducible representation V is non-trivial, then $\sum_{\chi} \chi(g) = 0$.

Here is the character table of C_3 = <σ>, the cyclic group with three elements and generator σ:

	(1)	(σ)	(σ^2)
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

where ω is a primitive third root of unity. The character table for general cyclic groups is (a scalar multiple of) the DFT matrix. Another example is the character table of S_3:

	(1)	(12)	(123)
χ_{triv}	1	1	1
χ_{sgn}	1	−1	1
χ_{stand}	2	0	−1

where (12) represents conjugacy class consisting of (12),(13),(23), and (123) represents conjugacy class consisting of (123),(132). To learn more about character table of symmetric groups, see 2Ed.