Appendices

A – Momentum modes

From the discretisation of position space we can determine, through a Fourier transform, that of momentum space:

$$\Delta x = \frac{L}{N}$$
$$\Delta p = \frac{2\pi}{L}$$

As position is periodical, so is momentum (note, for $a > \frac{N}{2}$, momentum is in a negative direction):

$$x_a = a\Delta x = \frac{La}{N}$$
$$p_a = a\Delta p = \frac{2\pi a}{L}$$
$$x_{a+N} = x_a + L = x_a$$
$$p_{a+N} = p_a + \frac{2\pi}{\Delta x} = p_a$$

B – Verifying commutation relations

Substituting our field expansions into the commutation relation gives:

$$\left[\hat{\phi}_{x_a}, \hat{\pi}_{x_b}\right] = -\frac{i}{4L^2}\sum_{n,m}\frac{1}{\omega_{p_n}}\{\left(\hat{a}_{p_n}e^{-ip_n x_a} + \hat{a}^\dagger_{p_n}e^{ip_n x_a}\right)\left(\hat{a}_{p_m}e^{-ip_m x_b} - \hat{a}^\dagger_{p_m}e^{ip_m x_b}\right)$$
$$-\left(\hat{a}_{p_m}e^{-ip_m x_b} - \hat{a}^\dagger_{p_m}e^{ip_m x_b}\right)\left(\hat{a}_{p_n}e^{-ip_n x_a} + \hat{a}^\dagger_{p_n}e^{ip_n x_a}\right)\}$$

The term in braces is:

$$\left[\hat{a}_{p_n}, \hat{a}_{p_m}\right]e^{-i(p_n x_a + p_m x_b)} + \left[\hat{a}^\dagger_{p_n}, \hat{a}_{p_m}\right]e^{i(p_n x_a - p_m x_b)} + \left[\hat{a}^\dagger_{p_m}, \hat{a}_{p_n}\right]e^{i(p_m x_b - p_n x_a)}$$
$$+\left[\hat{a}^\dagger_{p_n}, \hat{a}^\dagger_{p_m}\right]e^{+i(p_n x_a + p_m x_b)}$$

Assuming the commutation relations for the creation and annihilation operators, this is just:

$$-2L\omega_{p_n}\delta_{p_n p_m}\left(e^{i(p_n x_a - p_m x_b)} + e^{i(p_m x_b - p_n x_a)}\right)$$

So we find:

$$\left[\hat{\phi}_{x_a}, \hat{\pi}_{x_b}\right] = \frac{i}{2L}\sum_n\left(e^{ip_n(x_a - x_b)} + e^{-ip_n(x_a - x_b)}\right) = \frac{i}{L}\sum_n \cos(p_n(x_a - x_b))$$

Using the identity:

$$\sum_{n=0}^{N-1}\cos\left(\frac{2\pi n}{N}I\right) = \begin{cases} N, & I = 0 \\ 0, & 0 < I \leq (N-1) \end{cases}$$

We find, as required:

$$\left[\hat{\phi}_{x_a}, \hat{\pi}_{x_b}\right] = \frac{i}{L}N\delta_{x_a x_b} = \frac{i}{\Delta x}\delta_{x_a x_b}$$

C – Energy-momentum relation

Applying the definition of the derivative twice gives:

$$\nabla^2\phi_{x_a} = \nabla\left(\frac{\phi_{x_{a+1}} - \phi_{x_a}}{\Delta x}\right) = \frac{1}{\Delta x^2}\left(\phi_{x_{a+1}} - 2\phi_{x_a} + \phi_{x_{a-1}}\right)$$

So the Euler-Lagrange equations become:

$$\ddot{\phi}_{x_a} - \frac{1}{\Delta x^2}\left(\phi_{x_{a+1}} - 2\phi_{x_a} + \phi_{x_{a-1}}\right) + m^2{\phi_{x_a}}^2 = 0$$

Now we substitute in the trial field solution:

$$\phi_{x_a} \propto e^{i\left(p_b x_a + \omega_{p_b}t\right)}$$
$$\ddot{\phi}_{x_a} \propto -{\omega_{p_b}}^2\phi_{x_a}$$
$$\phi_{x_{a+1}} \propto e^{ip_b\Delta x}\phi_{x_a}$$
$$\phi_{x_{a-1}} \propto e^{-ip_b\Delta x}\phi_{x_a}$$

$$-{\omega_{p_b}}^2 - \frac{1}{\Delta x^2}\left(e^{ip_b\Delta x} + e^{-ip_b\Delta x} - 2\right) + m^2 = 0$$

$${\omega_{p_b}}^2 = m^2 - \frac{2}{\Delta x^2}(\cos(p_b\Delta x) - 1) = m^2 + \frac{4}{\Delta x^2}\left(\sin\left(\frac{p_b\Delta x}{2}\right)\right)^2$$

Using our definition of momentum this becomes:

$${\omega_{p_b}}^2 = m^2 + \frac{4}{\Delta x^2}\left(\sin\left(\frac{b\pi}{N}\right)\right)^2$$

In the limit of $N \to \infty$ or $\Delta x \to 0$ this becomes the standard energy-momentum relation:

$$\lim_{\Delta x\to 0}\left({\omega_{p_b}}^2\right) = m^2 + \frac{4}{\Delta x^2}\left(\frac{p_b\Delta x}{2} + \cdots\right)^2 = m^2 + {p_b}^2$$

D – Simplifying the Hamiltonian

First find the derivative of the field:

$$\nabla\phi_{x_a} = \frac{1}{L\Delta x}\sum_n \frac{1}{2\omega_{p_n}}\left(\hat{a}_{p_n}e^{-ip_nx_{a+1}} + \hat{a}^\dagger_{p_n}e^{ip_nx_{a+1}} - \hat{a}_{p_n}e^{-ip_nx_a} - \hat{a}^\dagger_{p_n}e^{ip_nx_a}\right)$$

$$\nabla\phi_{x_a} = \frac{1}{L\Delta x}\sum_n \frac{1}{2\omega_{p_n}}\left(\left(e^{-ip_b\Delta x} - 1\right)\hat{a}_{p_n}e^{-ip_nx_a} + \left(e^{ip_b\Delta x} - 1\right)\hat{a}^\dagger_{p_n}e^{ip_nx_a}\right)$$

Use the identity:

$$e^{ix} - 1 = 2ie^{\frac{ix}{2}}\sin\left(\frac{x}{2}\right)$$

To obtain:

$$\nabla\phi_{x_a} = -\frac{2i}{L\Delta x}\sum_n \frac{1}{2\omega_{p_n}}\sin\left(\frac{p_n\Delta x}{2}\right)\left(\hat{a}_{p_n}e^{-ip_n\left(x_a+\frac{\Delta x}{2}\right)} - \hat{a}^\dagger_{p_n}e^{ip_n\left(x_a+\frac{\Delta x}{2}\right)}\right)$$

Substitute this into the Hamiltonian:

$$\mathrm{H} = \frac{1}{8L^2}\Delta x\sum_{a,n,m}\left(-\left(\hat{a}_{p_n}e^{-ip_nx_a} - \hat{a}^\dagger_{p_n}e^{ip_nx_a}\right)\left(\hat{a}_{p_m}e^{-ip_mx_a} - \hat{a}^\dagger_{p_m}e^{ip_mx_a}\right)\right.$$
$$- \frac{4\sin\left(\frac{p_n\Delta x}{2}\right)\sin\left(\frac{p_m\Delta x}{2}\right)}{\Delta x^2\omega_{p_n}\omega_{p_m}}\left(\hat{a}_{p_n}e^{-ip_n\left(x_a+\frac{\Delta x}{2}\right)} - \hat{a}^\dagger_{p_n}e^{ip_n\left(x_a+\frac{\Delta x}{2}\right)}\right)\left(\hat{a}_{p_m}e^{-ip_m\left(x_a+\frac{\Delta x}{2}\right)}\right.$$
$$\left.\left. - \hat{a}^\dagger_{p_m}e^{ip_m\left(x_a+\frac{\Delta x}{2}\right)}\right) + \frac{m^2}{\omega_{p_n}\omega_{p_m}}\left(\hat{a}_{p_n}e^{-ip_nx_a} + \hat{a}^\dagger_{p_n}e^{ip_nx_a}\right)\left(\hat{a}_{p_m}e^{-ip_mx_a} + \hat{a}^\dagger_{p_m}e^{ip_mx_a}\right)\right)$$

Now rewrite this using the following relation:

$$\sum_{n=0}^{N-1} e^{-i\frac{2\pi n}{N}I} = \begin{cases} N, & I = 0 \\ 0, & 0 < I \le (N-1) \end{cases}$$

$$\mathrm{H} = \frac{1}{8L^2}N\Delta x\sum_n\left\{\left(\hat{a}_{p_n}\hat{a}_{p_n} + \hat{a}^\dagger_{p_n}\hat{a}^\dagger_{p_n}\right)\left(\frac{m^2}{{\omega_{p_n}}^2} - 1 - \frac{4\left(\sin\left(\frac{p_n\Delta x}{2}\right)\right)^2}{(L\Delta x)^2{\omega_{p_n}}^2}\right)\right.$$
$$\left. + \left(\hat{a}_{p_n}\hat{a}^\dagger_{p_n} + \hat{a}^\dagger_{p_n}\hat{a}_{p_n}\right)\left(\frac{m^2}{{\omega_{p_n}}^2} + 1 + \frac{4\left(\sin\left(\frac{p_n\Delta x}{2}\right)\right)^2}{\Delta x^2{\omega_{p_n}}^2}\right)\right\}$$

Finally, use the energy-momentum relation to simplify this to give:

$$\mathrm{H} = \frac{1}{4L}\sum_n\left(\hat{a}_{p_n}\hat{a}^\dagger_{p_n} + \hat{a}^\dagger_{p_n}\hat{a}_{p_n}\right) = \sum_n\left(\frac{\hat{a}^\dagger_{p_n}\hat{a}_{p_n}}{2L} + \frac{\omega_{p_n}}{2}\right)$$

E – Fock state orthogonality

For a general commutation relation of the form:

$$\left[\hat{a}_p, \hat{a}^\dagger_q\right] = X\delta(p-q)$$

We find:

$${\hat{a}_p}^Y\hat{a}^{\dagger Y}_q = {\hat{a}_p}^{Y-1}\left(\hat{a}^\dagger_q\hat{a}_p + X\right)\hat{a}^{\dagger Y-1}_q = {\hat{a}_p}^{Y-1}\hat{a}^\dagger_q\hat{a}_p\hat{a}^{\dagger Y-1}_q + X{\hat{a}_p}^{Y-1}\hat{a}^{\dagger Y-1}_q$$

$$\hat{a}_p{}^Y \hat{a}_q^{\dagger Y} = \hat{a}_p{}^{Y-1} \hat{a}_q^{\dagger Y} \hat{a}_p + YX \hat{a}_p{}^{Y-1} \hat{a}_q^{\dagger Y-1}$$

So that the value of a normalisation constant can be determined:

$$\langle 0|\hat{a}_p{}^Y \hat{a}_q^{\dagger Y}|0\rangle = YX\langle 0|\hat{a}_p{}^{Y-1} \hat{a}_q^{\dagger Y-1}|0\rangle = Y!\, X^Y$$

$$|Y\rangle = \frac{\hat{a}_q^{\dagger Y}}{\sqrt{Y!\, X^Y}}|0\rangle$$

Given the indifference of the operators to those in different modes, multiple values of p leads to equivalent normalisation constants that are just multiplied together. For our commutation relations this gives:

$$|p\rangle = \frac{\left(\hat{a}_{p_0}^{\dagger}\right)^{n_0}\left(\hat{a}_{p_1}^{\dagger}\right)^{n_1} \dots \left(\hat{a}_{p_{N-1}}^{\dagger}\right)^{n_{N-1}}}{\sqrt{\left(2\omega_{p_0}\right)^{n_0}\left(2\omega_{p_1}\right)^{n_1} \dots \left(2\omega_{p_{N-1}}\right)^{n_{N-1}}}\sqrt{L^{n_0}L^{n_1} \dots L^{n_{N-1}}}\sqrt{(n_0!)(n_1!) \dots (n_{N-1}!)}}|0\rangle$$

F – Eigenvalues of the Free Hamiltonian

Repeated use of the commutators between the Free Hamiltonian and the creation operators yields:

$$\mathrm{H}\hat{a}_{p_m}{}^{\dagger} = \omega_{p_m}\hat{a}_{p_m}^{\dagger} + \hat{a}_{p_m}^{\dagger}\mathrm{H}$$

$$\mathrm{H}\left(\hat{a}_{p_m}^{\dagger}\right)^Y = Y\omega_{p_m}\left(\hat{a}_{p_m}^{\dagger}\right)^Y + \left(\hat{a}_{p_m}^{\dagger}\right)^Y\mathrm{H}$$

Which, using the results in Appendix E and the enforced zero eigenvalue of the vacuum state, gives:

$$E_p = \sum_a n_a \omega_{p_a}$$

G - Operator effects on general states

If we separate a general state into its normalisation factors and creation operators, then apply another creation operator:

$$|\alpha\rangle = f_\alpha \hat{O}_\alpha|0\rangle$$

$$\hat{a}_{p_a}^{\dagger}|\alpha\rangle = f_\alpha \hat{O}_{\alpha+p_\alpha}|0\rangle = \frac{f_\alpha}{f_{\alpha+p_\alpha}} f_{\alpha+p_\alpha}\hat{O}_{\alpha+p_\alpha}|0\rangle = \frac{f_\alpha}{f_{\alpha+p_\alpha}}|\alpha + p_a\rangle$$

Using the definition of our orthonormal states to calculate the prefactor we can rewrite this:

$$\frac{f_\alpha}{f_{\alpha+p_\alpha}} = \sqrt{2(n_a+1)L\omega_{p_a}}$$

$$\hat{a}_{p_a}^{\dagger}|\alpha\rangle = \sqrt{2(n_a+1)L\omega_{p_a}}|\alpha + p_a\rangle$$

Repeating this process for an annihilation operator, using the commutation relation multiple times to move the annihilation operator to act on the vacuum state, we find:

$$\hat{a}_{p_a}|\alpha\rangle = \hat{a}_{p_a} f_\alpha \hat{O}_\alpha|0\rangle = n_a\left(2L\omega_{p_a}\right) f_\alpha \hat{O}_{\alpha-p_\alpha}|0\rangle$$

$$\hat{a}_{p_a}|\alpha\rangle = \frac{n_a\left(2L\omega_{p_a}\right) f_\alpha}{f_{\alpha-p_\alpha}} f_{\alpha-p_\alpha}\hat{O}_{\alpha-p_\alpha}|0\rangle = \frac{n_a\left(2L\omega_{p_a}\right) f_\alpha}{f_{\alpha-p_\alpha}}|\alpha - p_a\rangle$$

We find the prefactor in the same way as before:

$$\frac{n_a\left(2L\omega_{p_a}\right) f_\alpha}{f_{\alpha-p_\alpha}} = \frac{2n_aL\omega_{p_a}}{\sqrt{2n_aL\omega_{p_a}}} = \sqrt{2n_aL\omega_{p_a}}$$

Which gives an equation very similar to that for creation operators:

$$\hat{a}_{p_a}|\alpha\rangle = \sqrt{2n_aL\omega_{p_a}}|\alpha - p_a\rangle$$

We can confirm these equations by noting that they reproduce the commutation relation:

$$\left[\hat{a}_{p_a}, \hat{a}_{p_a}^{\dagger}\right]|\alpha\rangle = \left(2(n_a+1)L\omega_{p_a} - 2n_aL\omega_{p_a}\right)|\alpha\rangle = 2L\omega_{p_a}|\alpha\rangle$$

H – Number of basis states

For a particle-biased labelling method, the total number of states for up to Y particles is:

$$1 + N + \frac{N(N+1)}{2} + \frac{N(N+1)(N+2)}{6} + \cdots + \frac{N(N+1) \dots (N+Y-1)}{Y!}$$

Which is of the order N^Y.

I – Relativistic Effects

The maximum momentum of a particle in our system is given by:

$$p_{max} = N\Delta p = \frac{2\pi}{\Delta x}$$

This can also be written as:

$$p_{max} = \gamma_{max} m v_{max}$$

We can solve for v_{max} (in natural units):

$$\gamma_{max}{}^2 = \frac{1}{1 - v_{max}{}^2} = \frac{4\pi^2}{\Delta x^2 m^2 v_{max}{}^2}$$

$$v_{max}{}^2(\Delta x^2 m^2 + 4\pi^2) = 4\pi^2$$

$$v_{max} = \frac{1}{\sqrt{1 + \left(\frac{m\Delta x}{2\pi}\right)^2}}$$

This is only much less than 1 (the speed of light in natural units) for:

$$m\Delta x \gg 2\pi$$

J - Power Iteration method convergence

Expanding a state in terms of the Hamiltonian eigenbasis:

$$\mathrm{H}|\alpha\rangle = E_\alpha|\alpha\rangle$$

$$|s_0\rangle = \sum_\alpha c_\alpha |\alpha\rangle$$

Applying the algorithm once (ignoring the renormalisation):

$$|s_1\rangle = (\tau I - \mathrm{H}) \sum_\alpha c_\alpha |\alpha\rangle = \sum_\alpha c_\alpha (\tau - E_\alpha) |\alpha\rangle$$

We see how the algorithm works and that for convergence we require:

$$\tau > E_{max}$$

K – Cantor's Pairing Function

For any pair of numbers $[a, b]$ we can use Cantor's Pairing Function to compute an unique associated index, z, and can also calculate the pair from that index. By applying the algorithm recursively, using b as the next z, we can associate any set of number lists of the same length with unique indices and vice versa. For example, with a list of 4 numbers we have:

$$z_0 \leftrightarrow [a_o, z_1] \leftrightarrow [a_o, a_1, z_2] \leftrightarrow [a_o, a_1, a_2, a_3]$$

By repeatedly applying this recursion to the grid produced by the pairing functions we go from:

$$\begin{array}{ccccccc} {}^0[0,0] & {}^1[1,0] & {}^3[2,0] & {}^6[3,0] & {}^{10}[4,0] & \dots \\ {}^2[0,1] & {}^4[1,1] & {}^7[2,1] & {}^{11}[3,1] & \dots & & \\ {}^5[0,2] & {}^8[1,2] & {}^{12}[2,2] & \dots & & & \\ {}^9[0,3] & {}^{13}[1,3] & \dots & & & & \\ {}^{14}[0,4] & \dots & & & & & \\ \dots & & & & & & \end{array}$$

To the following, where the triangular truncation mentioned is evident:

$$\begin{array}{ccccccc} {}^0[0,0,0,\dots] & {}^1[1,0,0,\dots] & {}^3[2,0,0,\dots] & {}^6[3,0,0,\dots] & {}^{10}[4,0,0,\dots] & \dots \\ {}^2[0,1,0,\dots] & {}^4[1,1,0,\dots] & {}^7[2,1,0,\dots] & {}^{11}[3,1,0,\dots] & \dots & & \\ {}^5[0,0,1,\dots] & {}^8[1,0,1,\dots] & {}^{12}[2,0,1,\dots] & \dots & & & \\ {}^9[0,2,0,\dots] & {}^{13}[1,2,0,\dots] & \dots & & & & \\ {}^{14}[0,1,1,\dots] & \dots & & & & & \\ \dots & & & & & & \end{array}$$

L – Conservation of probability

We start with a general state and apply the Schrodinger equation:

$$|\psi\rangle = \sum_p c_p |p\rangle$$

$$i\frac{\partial}{\partial t}|\psi\rangle = \mathrm{H}|\psi\rangle$$

Pre-multiplying by the original state gives:

$$\frac{\partial}{\partial t}\langle\psi|\psi\rangle = \left(\sum_q c_q^\dagger \langle q|\right)\left(\sum_p c_p \mathrm{H}|p\rangle\right) = \sum_{p,q} c_q^\dagger c_p \langle q|\mathrm{H}|p\rangle$$

$$-i\frac{\partial}{\partial t}\langle\psi|\psi\rangle = \sum_{p,q} c_p^\dagger c_q \langle p|\mathrm{H}|q\rangle = \sum_{p,q} c_q^\dagger c_p \langle q|\mathrm{H}|p\rangle = i\frac{\partial}{\partial t}\langle\psi|\psi\rangle$$

From which we can deduce:

$$\frac{\partial}{\partial t}\langle\psi|\psi\rangle = 0$$

$$\langle\psi|\psi\rangle = \sum_p |c_p|^2 = A$$

M – Conservation of energy

Working in the eigenbasis of the Hamiltonian, we can write a general state as:

$$\mathrm{H}|\alpha\rangle = E_\alpha|\alpha\rangle$$

$$|s(0)\rangle = \sum_\alpha c_\alpha |\alpha\rangle$$

Using the Heisenberg picture we can apply the Hamiltonian to this general state after some time has passed:

$$|s(t)\rangle = e^{-i\mathrm{H}t}|s(0)\rangle = \sum_\alpha c_\alpha\, e^{-iE_\alpha t}|\alpha\rangle$$

$$\mathrm{H}|s\rangle = \sum_\alpha E_\alpha c_\alpha\, e^{-iE_\alpha t}|\alpha\rangle$$

Pre-multiplying by the state at this time gives the expectation value of the state energy, assuming the state is normalised:

$$\langle s|\mathrm{H}|s\rangle = \sum_{\alpha,\beta} E_\alpha c_\alpha c_\beta^\dagger e^{it(E_\beta - E_\alpha)}\langle\alpha|\beta\rangle = \sum_\alpha E_\alpha |c_\alpha|^2 = \mathrm{E}_s$$

The coefficients have no time dependence, so we find the basis independent result:

$$\frac{\partial}{\partial t}(\langle s|\mathrm{H}|s\rangle) = \frac{\partial \mathrm{E}_s}{\partial t} = 0$$

O – Eigenstate decay

We can write our calculated state as the actual state plus some error decomposed into the eigenstate basis:

$$|c_{calc}\rangle = |c\rangle + \sum_d \epsilon_d |d\rangle$$

where the ϵ_d are small. Now we apply the Hamiltonian:

$$\mathrm{H}|c_{calc}\rangle = \mathrm{H}|c\rangle + \sum_d \epsilon_d \mathrm{H}|d\rangle = \lambda_c|c\rangle + \sum_d \epsilon_d \lambda_d |d\rangle$$

We can use this to write the error in the next calculated state for the Leapfrog method. First we must apply an Euler step:

$$|c_{calc}(\Delta t)\rangle = -i\Delta t\mathrm{H}|c_{calc}(0)\rangle + |c_{calc}(0)\rangle = (1 - i\Delta t\lambda_c)|c\rangle + \sum_d \epsilon_d(1 - i\Delta t\lambda_d)|d\rangle$$

Now we can apply a Leapfrog step:

$$|c_{calc}(2\Delta t)\rangle = -i2\Delta t\mathrm{H}|c_{calc}(\Delta t)\rangle + |c_{calc}(0)\rangle$$

$$|c_{calc}(2\Delta t)\rangle = \left(1 - 2i\Delta t\lambda_c - 2\Delta t^2{\lambda_c}^2\right)|c\rangle + \sum_d \epsilon_d\left(1 - 2i\Delta t\lambda_d - 2\Delta t^2{\lambda_d}^2\right)|d\rangle$$

Repeated application gives the state after a number of timesteps (to $O(\Delta t)$):

$$|c_{calc}(n\Delta t)\rangle \simeq (1 - ni\Delta t\lambda_c)|c\rangle + \sum_d \epsilon_d(1 - ni\Delta t\lambda_d)|d\rangle$$

The first term matches the expected phase rotation of the eigenstate to $O(\Delta t)$. We expect the Leapfrog method to be accurate to $O(\Delta t^2)$ and a full calculation of the above state would show that, for large n, the method here is accurate to this order. The moduli of the error components do not rise collinearly in time, as may have been expected. As a result the error components associated with larger eigenvalues dominate. As a rough approximation, for large n these moduli increase of the order:

$$\sim (n\lambda_d)^2$$

P – Negative energies

In general, we cannot say that the Free Hamiltonian ground state is no longer the Full Hamiltonian ground state without explicitly checking it. However, for the matrix considered here it can be proven that the ground state does indeed change. We can write our Full Hamiltonian in a certain way, using the labelling system described in section 3.5, because a ϕ^3 interaction cannot change particle number by an even number. Calculating the energy of a normalised state of the particular type shown gives (for small δ):

$$E_s = \left(\left(1 - \frac{\delta^2}{2}\right), -\delta, 0, 0, \dots\right)\begin{pmatrix} 0 & \mathrm{H}_{10} & \mathrm{H}_{20} & 0 & \cdots \\ \mathrm{H}_{10} & E_1 & 0 & \mathrm{H}_{31} & \cdots \\ \mathrm{H}_{20} & 0 & E_2 & \mathrm{H}_{32} & \cdots \\ 0 & \mathrm{H}_{31} & \mathrm{H}_{32} & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} \left(1 - \frac{\delta^2}{2}\right) \\ -\delta \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$E_s = \left(\left(1 - \frac{\delta^2}{2}\right), -\delta, 0, 0, \dots\right) \cdot \left(-\delta\mathrm{H}_{10}, \left(\left(1 - \frac{\delta^2}{2}\right)\mathrm{H}_{10} - \delta E_1\right), \left(1 - \frac{\delta^2}{2}\right)\mathrm{H}_{20}, 0, \dots\right)$$

$$E_s = -2\delta\mathrm{H}_{10} + \delta^2 E_1 + \delta^3\mathrm{H}_{10} \simeq -2\delta\mathrm{H}_{10}$$

We see that we can always find a negative energy state, as we have the freedom to choose δ. So the ground state of the Free Hamiltonian, which still has energy zero, is not the ground state of the Full Hamiltonian.

Q – Code structure