Theoretical Background for "Higgs Simulator"

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Abstract

Within the framework of classical field theory, the Lagrangian used in the "Higgs Simulator" (by Dr C. G. Lester, 2013) is systematically motivated and developed. The propagation of wave packets and the Manton instanton, which are visualised in "Higgs Simulator", are discussed as solutions to the Euler-Lagrange equations. Finally the numerical implementation, as well as remaining problems with the initialisation of the wave packets, are outlined.

1 Introduction

The Higgs boson is formally most adequately studied in the realm of quantum field theory. However, classical field theory has proven itself to show valuable insight in the physical phenomena involved. A classical field theory distinguishes itself from quantum theory by the fact that the wave function described does not lend itself to a probabilistic interpretation.

The "Higgs Simulator" was written with the intent to visualise the dynamics of multiple interacting fields in a classical field theory.

In the following, we will be considering the Lagrangian density $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ (abbreviated to "Lagrangian") for real scalar fields ϕ_i in (1+1)-dimensional space-time.[1] We will be using natural units such that $\hbar = c = 1$ and the Minkowski metric $g^{\mu\nu} = \mathrm{diag}(1, -1)$ so that the gradient takes the form $\partial^\mu \equiv (\frac{\partial}{\partial t}, -\frac{\partial}{\partial x})$, leading to identities such as $(\partial_\mu \phi)(\partial^\mu \phi) \equiv (\frac{\partial \phi}{\partial t})^2 - (\frac{\partial \phi}{\partial x})^2$ and $\partial_\mu \partial^\mu \phi \equiv \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2}$.

By varying the action $S = \int \mathcal{L} d^2x$, the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \partial_\mu \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i]}, \tag{1}$$

for each field ϕ_i are obtained, where the index μ runs over 0 and 1 since we are in (1+1)-dimensional space-time.

The next section systematically motivates and develops the Lagrangian. In Section (3), the algorithms used in the program are presented. Section (4) contains the conclusions. In Section (2.5), a summary of the main physical consequences of the Lagrangian is given.

2 Developing the Lagrangian

In Sections (2.1)–(2.4) special cases to which the full Lagrangian reduces are considered. These are then combined in Section (2.5). Section (2.6) treats the Manton instanton.

2.1 Massless particles

The Lagrangian of the field corresponding to a massless free particle is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi), \tag{2}$$

which has only a kinetic term.

Substituting into the Euler-Lagrange equation (1) yields

$$\partial_\mu \partial^\mu \phi = 0, \tag{3}$$

which, when written in non-covariant form, $\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2}$, is the wave equation with phase velocity $c = 1$. Therefore, any wave packet which is a solution to Equation (3) does not disperse and has group velocity c. This wave packet corresponds to a particle which is a photon.

In the program "Higgs Simulator", this field has the name `Photon`. In the `Default`, `Intermediate` and `Odd` settings, the wave packet in this field has the fastest group velocity, compared with the wave packets in the other fields.

[1] A real scalar field has one degree of freedom. We could also treat complex scalar fields, which have two degrees of freedom. Furthermore it can be shown that complex scalar fields can carry charge (to be precise, a non-vanishing Noether charge), so by restricting ourselves to real scalar fields, we only consider uncharged fields.

2.2 Massive particles

The previous Lagrangian can be extended to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2, \tag{4}$$

where m^2 is, for now, an arbitrary scalar.

Substituting into the Euler-Lagrange equation, we obtain the Klein-Gordon equation,

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0, \tag{5}$$

which can be solved by Fourier transforming to give

$$[k^2 - m^2]\tilde{\phi}(k) = 0, \tag{6}$$

so that, for a non-vanishing field $\tilde{\phi}(k)$,

$$k^2 \equiv (k^0)^2 - (\boldsymbol{k})^2 = m^2, \tag{7}$$

where we have defined $k^2 \equiv k_\mu k^\mu$, and k^0 and $\boldsymbol{k}$ are the time- and space-like components of k^μ, respectively.

Rearranging, this is the dispersion relation,

$$k^0 = \sqrt{\boldsymbol{k}^2 + m^2}, \tag{8}$$

governing the propagation of wave packets. Therefore wave packets which satisfy the Klein-Gordon equation disperse, in contrast to the photon field, and travel at a group velocity less than the speed of light.

Physically, ϕ is the (relativistic) wave function of a particle. Consequently, the energy E and momentum $\boldsymbol{p}$ of the particle are related to the frequency k^0 and wave vector $\boldsymbol{k}$ by

$$E = k^0 \quad \text{and} \quad \boldsymbol{p} = \boldsymbol{k}, \tag{9}$$

from which we see that Equation (8) is precisely the energy-momentum invariant, $E = \sqrt{\boldsymbol{p}^2 + m^2}$. Indeed, this justifies identifying the (at first arbitrary) parameter m with the mass.

In the program "Higgs Simulator", this field has the name `Massive`. In the `Default`, `Intermediate` and `Odd` settings, the wave packet in this field has a slower group velocity than the wave packet in the field `Photon`, as expected, since massive particles travel more slowly than photons, see Figure (1).

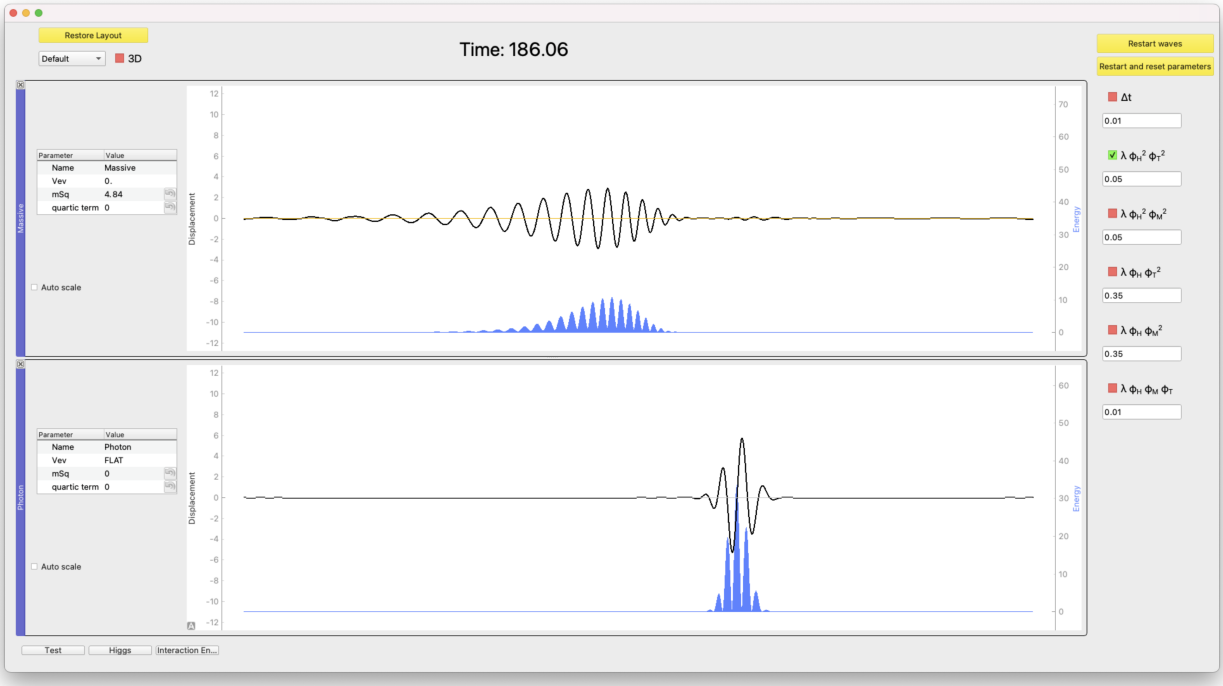


Figure 1: Screenshot from "Higgs Simulator". The two plots show the `Photon` and `Massive` fields. In each plot, the black curve is the displacement plotted against the x-axis, while the blue curve shows the energy density. The simulation time t is given in units of inverse energy (or equivalently in units of inverse mass) since natural units are used.
At $t = 0$, both fields are initialised to Gaussian wave packets, see Section (3.2). After some time has elapsed, the wave packet in the field `Massive` can be seen to lag behind the field `Photon`, as the wave packet in `Massive` has a smaller group velocity. It can also be seen that the wave packet in `Massive` disperses, while the one in `Photon` does not.

2.3 The Higgs field

We introduce the Lagrangian of the Higgs field by including higher-order self-interaction terms[2]:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + 2B\phi^2 - A\phi^4, \tag{10}$$

where A and B are real constants, chosen such that the potential $V(\phi) = -2B\phi^2 + A\phi^4$ remains lower-bounded[3], which is required so that the energy E (defined in Equation (11) below) does not diverge.

(The reader might wonder why we distinguish between the "mass term" $\frac{1}{2}m^2\phi^2$ of Equation (5) and the "self-interaction term" of Equation (10). Indeed, the mass term is technically also a self-interaction term as it involves powers of the field itself. However the physical interpretation differs: the mass term gives rise to a mass, and we require it to be non-negative for the energy to be bounded from below, but the self-interaction term allows the parameter B to take on arbitrary values, which causes spontaneous symmetry breaking discussed below.)

We can Legendre-transform the Lagrangian to obtain the Hamiltonian $\mathcal{H}$, which we identify with the energy density. The energy of the entire wave is then given by the integral thereof:

$$E = \int d^2x \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + V(\phi) \right]. \tag{11}$$

We now want to work out the ground-state, $\phi(x) = \tilde{\phi}(x)$, which minimises the energy E. Since both terms in the kinetic term are positive semi-definite, the energy is minimised when the field is both time-independent, $\frac{\partial \phi}{\partial t} = 0$, and homogeneous in space, $\frac{\partial \phi}{\partial x} = 0$. Therefore, ϕ can be taken to be a constant $\phi(x) = \tilde{\phi}$. Minimising E is then equivalent to minimising the potential $V(\tilde{\phi})$.

For reasons apparent later, we will choose $A > 0$ and $B > 0$, so that the potential takes the W-shape shown in Figure (2).[4]

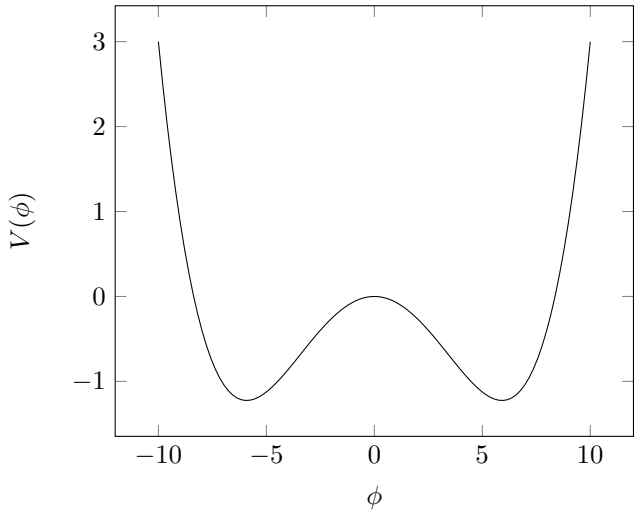


Figure 2: Potential $V(\phi) = -2B\phi^2 + A\phi^4$.

We see that the global minimum of $V(\tilde{\phi})$ is attained at

$$\tilde{\phi} = \pm\sqrt{\frac{B}{A}}. \tag{12}$$

Hence, at minimum energy, the Higgs field takes on one of these values everywhere. As the ground state does not display the symmetry of the governing Lagrangian anymore, $\phi(x) \to \phi(-x)$, the symmetry is said to be *spontaneously broken*. The value $\tilde{\phi}$ is also referred to as the *vacuum expectation value* (vev) of the field. Without loss of generality, we will choose the field to be in the state $\tilde{\phi} = +\sqrt{B/A}$ from this point onwards. This choice is arbitrary, but once it has been made, the Higgs field will remain in that state as there is an energy associated with switching from one minimum to the other.

[2] See [1], Chapter 2.7, for a discussion of why a polynomial in ϕ is sensible.

[3] $V(\phi)$ is lower-bounded if $A > 0$ (and B arbitrary) or, if $A = 0$ and $B < 0$.

[4] If ϕ were a complex scalar field, the potential $V_{\text{comp}}(\phi) \equiv -2B|\phi|^2 + A|\phi|^4$ would take the shape of the "mexican hat" potential in the complex plane. Qualitatively, this would give the same behaviour as in the case for real scalar fields, i.e. making a particle massive, however one might argue that having two isolated minima is different from the potential minimum being a circle in the complex plane. In the case of the complex scalar field there is continuous $U(1)$ symmetry (i.e. invariance under global rotations and reflections in the complex plane), while the real scalar field exhibits discrete symmetry under a parity transformation.

If energy is added to the Higgs field, there will be small perturbations around $\tilde{\phi}$, which are the Higgs bosons[5]. We can make this more precise by letting $\phi(x) = \tilde{\phi} + \epsilon(x)$ and expanding:

$$\mathcal{L}'(\epsilon) \equiv \mathcal{L}(\tilde{\phi} + \epsilon) \tag{13}$$

$$= \frac{1}{2}(\partial_\mu \epsilon)(\partial^\mu \epsilon) - 4B\epsilon^2 - 4A\tilde{\phi}\epsilon^3 - A\epsilon^4 + B^2/A, \tag{14}$$

where the additive constant can be neglected, in principle.

If ϵ is small, terms of higher order than ϵ^2 can be neglected, and we see that the Lagrangian has the form as that for a massive particle (compare with Equation (5)). This is the mass of a Higgs boson travelling in the potential valley[6] of Figure (2).

In the program "Higgs Simulator", this field has the name `Higgs`. In the `Default`, `Intermediate` and `Odd` settings, there is a wave packet around the vacuum expectation value. In each of these settings, the values of the constants A and B are chosen differently so that `Default` has the deepest and most narrow potential $V(\phi)$ and `Odd` has the least deep and narrow potential $V(\phi)$, while `Intermediate`'s potential function is in between. As a consequence, the wave packet of the Higgs field is smallest in the setting `Default`, and largest in `Odd`, while it has an intermediate size in `Intermediate`, see Figure (3).

2.4 Interaction with the Higgs field

We now consider a massless field, ϕ_0, which interacts with a Higgs field, ϕ_2, through an interaction term[7] $\lambda {\phi_0}^2 {\phi_2}^2$:

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2)}_{\text{kinetic term}} + \underbrace{2B{\phi_2}^2 - A{\phi_2}^4}_{\text{self-interaction}} - \underbrace{\lambda {\phi_0}^2 {\phi_2}^2}_{\text{interaction}}. \tag{15}$$

Note that this is simply the sum of the massless and the Higgs Lagrangians, Equations (3) and (10), plus the interaction term. If the interaction term were not there, substituting into the Euler-Lagrange equations (1) for the two fields would give two decoupled equations, which amounts to no interaction. Including the interaction term couples the two Euler-Lagrange equations, therefore giving rise to an interaction. This justifies calling it the interaction term.

The effective potential, which appears in the Euler-Lagrange equations for both ϕ_0 and ϕ_2, is

$$V(\phi_0, \phi_2) = -2B{\phi_2}^2 + A{\phi_2}^4 + \lambda {\phi_0}^2 {\phi_2}^2. \tag{16}$$

Substituting the Lagrangian into the Euler-Lagrange equations, we get

$$\partial_\mu \partial^\mu \phi_2 = -\frac{\partial}{\partial \phi_2}\left(-2B{\phi_2}^2 + A{\phi_2}^4 + \lambda {\phi_0}^2 {\phi_2}^2\right) \quad \text{and} \tag{17}$$

$$\partial_\mu \partial^\mu \phi_0 = -\frac{\partial}{\partial \phi_0}\left(\lambda {\phi_0}^2 {\phi_2}^2\right). \tag{18}$$

Looking first at Equation (17), we have previously seen that the Higgs field undergoes small perturbations around a non-zero vacuum expectation value, $\tilde{\phi_2}$. If the field ϕ_0 only performs small perturbations around its minimum-energy state, $\tilde{\phi_0} \approx 0$, the potential simplifies to $V \approx -2B{\phi_2}^2 + A{\phi_2}^4$. Hence, the self-interaction term of the Higgs field is only perturbed slightly by the interaction term and the Higgs field will still be close to its vacuum expectation value.

Looking at Equation (18), i.e. from the perspective of the massless field ϕ_0, the interaction term now plays the role of a mass term, since ϕ_2 is effectively a constant:

$$\lambda {\phi_0}^2 {\phi_2}^2 \approx \underbrace{\lambda \tilde{\phi_2}^2}_{\text{const.}} {\phi_0}^2. \tag{19}$$

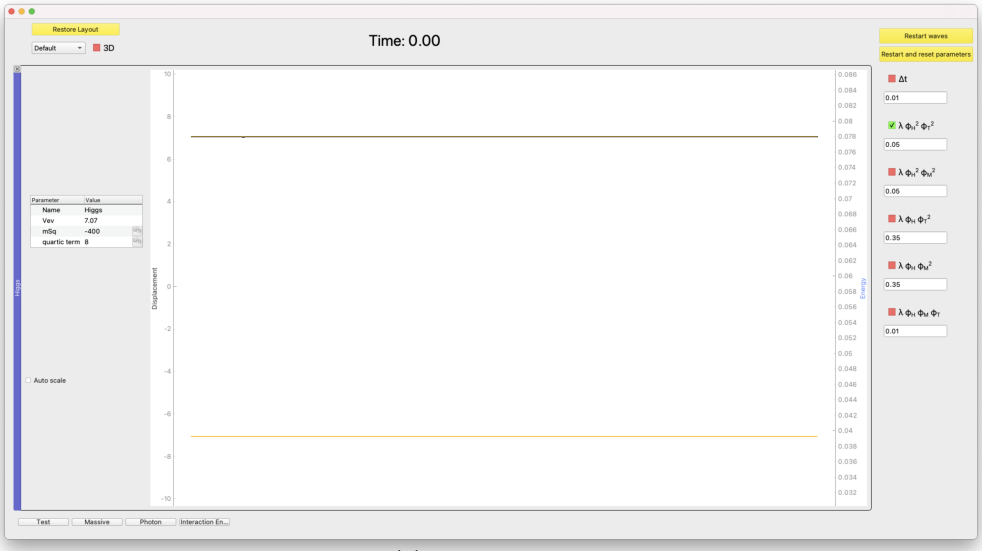
Thus, for small perturbations around the minimum-energy state, the initially massless field ϕ_0 has acquired a mass purely from interaction with the Higgs field.

In the setting `Silly 1` the Higgs field is initialised by setting its value everywhere to a constant much smaller than 1. This causes the Higgs field to "roll down" the W-shaped potential and subsequently perform oscillatory motion, which is however disturbed by the presence of the wave packet `Test`. If the Higgs field would not interact with any other fields, the entire Higgs field would perform an oscillation about the vacuum expectation value.

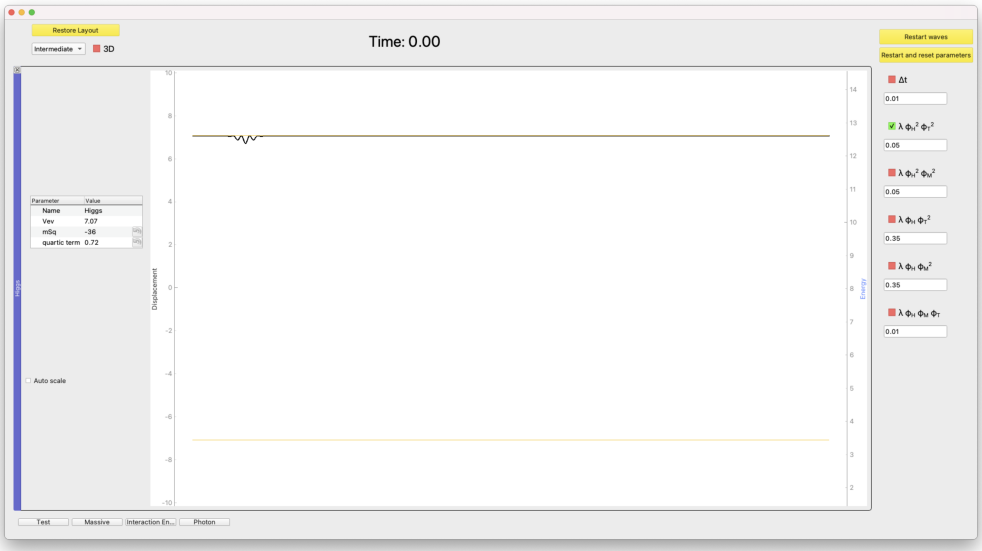
[5] After second quantisation, there will be zero-point fluctuations anyway, but in the realm of classical field theory, we for now imagine adding in energy "manually".

[6] In the case of a complex scalar field with a mexican hat potential, there will be two bosons, a massless one and a massive one, corresponding to the two principal curvatures at the minimum of the potential. Here, for real scalar fields, we only have one principal curvature at the bottom of the potential valley, therefore giving one massive boson.

[7] This interaction term is added in "manually". However, such an interaction term appears naturally, e.g., in the case of the electromagnetic field interacting with a complex scalar Higgs field. In that case, the covariant derivative, which is a necessary modification of the derivative to make the Lagrangian invariant under local gauge transformations, gives rise to such a cross term. For details, see [1], Chapter 6.



(a) Setting `Default`



(b) Setting `Intermediate`

(c) Setting `Odd`

Figure 3: Screenshots from "Higgs Simulator". In each Figure, the plot shows the `Higgs` field. The black curve is the displacement of the field plotted against the x-axis, while the orange lines show the vacuum expectation values (`vev`) of the field, given by Equation (12). The simulation time t is given in units of inverse energy (or equivalently in units of inverse mass) since natural units are used. At the top left corner of the graphical user interface, one of the settings `Default`, `Intermediate` and `Odd` can be selected.
The wave packet of the field `Higgs` in the settings `Default`, `Intermediate` and `Odd` (with the vertical axis scaled the same in each setting) at time $t = 0$ is shown. The size of the wave packet in `Default` is the smallest (almost not visible) while it is largest in the setting `Odd` due to the different depths of the potentials. The potential can be changed by changing the parameters `mSq` and `quarticTerm`.

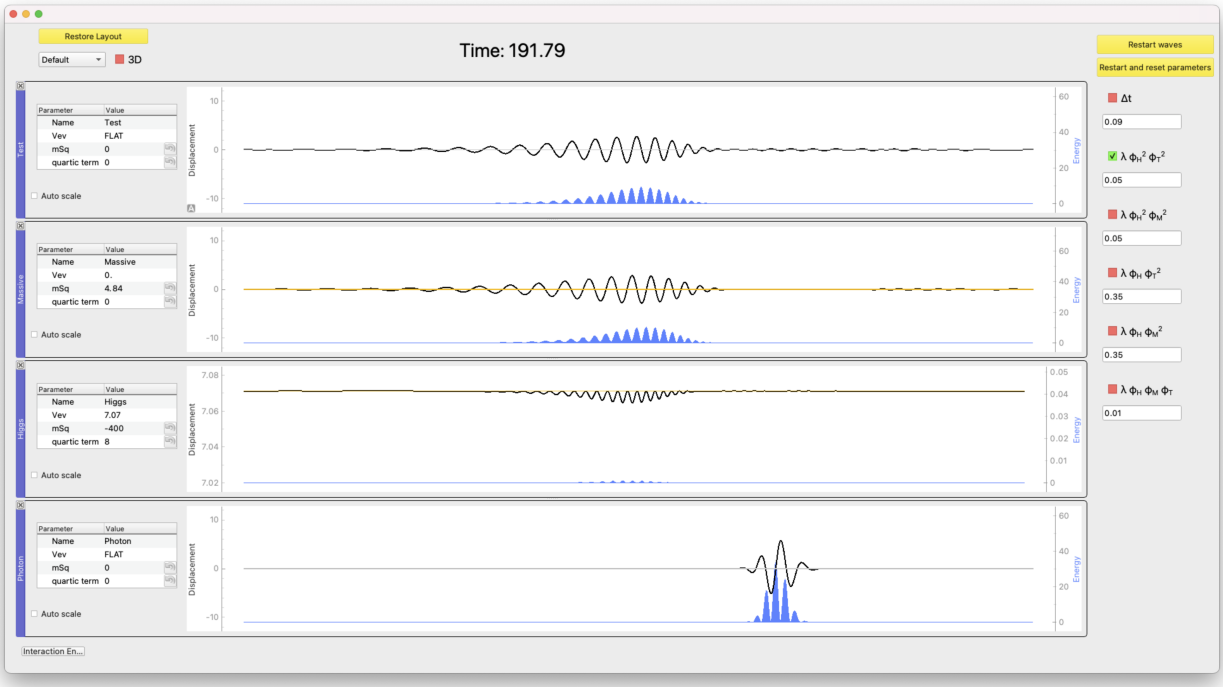


Figure 4: Screenshot from "Higgs Simulator". From top to bottom, the plots show the `Test`, `Massive`, `Higgs` and `Photon` fields. The black curve is the displacement of the field plotted against the x-axis and the blue curve is the energy density, while the orange lines show the vacuum expectation values (`vev`) of the fields. The simulation time t is given in units of inverse energy (or equivalently in units of inverse mass) since natural units are used.
At time $t = 0$, the fields `Test`, `Massive` and `Photon` are initialised as a Gaussian wave packet, as described in Section (3.2), and the field `Higgs` is initialised as described in Section (3.3). After some time has elapsed, the wave packets of both fields `Test` (first panel from the top) and `Massive` (second panel) have the same position and width, since they approximately have the same dispersion relation. However, `Test` has mass purely from interaction with the field `Higgs` (third panel) while `Massive` explicitly has a mass term in the Lagrangian. Both wave packets are slower than the wave packet of the field `Photon` (fourth panel), as expected.

2.5 The full Lagrangian

We are now in a position to piece together the full Lagrangian:

$$\mathcal{L} = T - V, \tag{20}$$

where the kinetic and potential terms are, respectively,

$$T = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) + \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) + \frac{1}{2}(\partial_\mu \phi_3)(\partial^\mu \phi_3) \quad \text{and} \tag{21}$$

$$V = \underbrace{\frac{1}{2}m^2 {\phi_1}^2}_{\text{masses}} + \underbrace{\lambda_{0022}\,\phi_0\phi_0\phi_2\phi_2}_{\text{interaction(s)}} + \underbrace{A{\phi_2}^4 - 2B{\phi_2}^2}_{\text{self-interactions}}, \tag{22}$$

where the subscripts on λ are used to uniquely identify the powers appearing in the interaction term.

We now consider each field in turn and summarise key results:

- ϕ_2 is the Higgs field, discussed in Section (2.3). The self-interaction term encourages this field to be at the minimum-energy state given by the vacuum expectation value $\tilde{\phi_2} = \pm\sqrt{\frac{B}{A}}$.
- ϕ_1 is a massive field as in Section (2.2). In this Lagrangian, it is not interacting with any other fields. It's dispersion relation is given by the energy-momentum invariant, $E = \sqrt{\boldsymbol{p}^2 + m^2}$.
- ϕ_0 is the massless ("test") field interacting with the Higgs field, ϕ_2, as in Section (2.4). By choosing the mass, m, to be equal to the vacuum expectation value of the Higgs field, we can make a wave packet travel at the same group velocity as the field ϕ_1. This is depicted in Figure (4).
- ϕ_3 is a massless (photon) field as discussed in Section (2.1) which is not interacting with any other fields.

The test field, ϕ_0, thus acquires mass by interaction with the Higgs field. We can think of ϕ_0 as a fermionic (e.g. electron) field. Why can it not be given a mass by including a mass term for it, like for ϕ_1? This would imply that Fermions are governed by the Klein-Gordon equation, which can only describe spin-0 particles[8]. For a description of spin-1/2 particles within classical field theory, we would have to turn to the Dirac equation.

The Hamiltonian corresponding to the Lagrangian, Equation (21), is

$$\mathcal{H} = \sum_{i=0}^{3} \frac{1}{2} \left[\left(\frac{\partial \phi_i}{\partial t} \right)^2 + \left(\frac{\partial \phi_i}{\partial x} \right)^2 \right] + V. \tag{23}$$

In the program "Higgs Simulator" we assign each wave an energy which is the kinetic term plus the self-interactions (which includes mass terms) in the Hamiltonian $\mathcal{H}$. Any other remaining cross-terms are referred to as the "interaction energy".

2.6 The Manton instanton

For the Higgs field Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi), \tag{24}$$

where $V(\phi) = -2B\phi^2 + A\phi^4$ with $A > 0$ and $B > 0$ has the W-shape in Figure (2), we look for a static solution, $\frac{\partial \phi}{\partial t} = 0$, which switches from one vacuum expectation value, $\phi(x) = -\tilde{\phi} \equiv -\sqrt{\frac{B}{A}}$, to the other, $\phi(x) = +\tilde{\phi}$.

The Euler-Lagrange equation gives the Klein-Gordon equation in a potential,

$$\partial_\mu \partial^\mu \phi = -\frac{\partial V}{\partial \phi}, \tag{25}$$

so that we have to solve the boundary value problem

$$\begin{cases} \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial V}{\partial \phi}, \\ \phi(x) \to +\tilde{\phi} \quad \text{as} \quad x \to +\infty, \\ \phi(x) \to -\tilde{\phi} \quad \text{as} \quad x \to -\infty, \\ \frac{\partial \phi}{\partial x} \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{26}$$

The solution is found to be

$$\phi(x) = \tilde{\phi} \tanh\left(\sqrt{2A}\tilde{\phi}(x - x_0) \right), \tag{27}$$

where x_0 is a constant of integration corresponding to the centre of the instanton. The solution is depicted in Figure (5), which can be imagined as a rope lying on a ridge.

The initial configuration of `Silly 3` of the program shows two such instantons (Figure (6)). They are initially far enough apart so that the boundary condition at $x \to \infty$ is satisfied.

Changing the reference frame under a Lorentz transformation must give another solution, as the Euler-Lagrange equations are Lorentz-invariant. Intuitively, if we move at a constant low velocity along the x-axis, we will see the stationary instanton moving in the opposite x-direction. Hence, an instanton moving at a constant velocity must be a general solution. At relativistic velocities, we will observe the instanton to be "squished" along the x-axis due to length contraction. Hence, fast-moving instantons will have their centre squished. A moving instanton is in fact constructed in the setting `Silly 3` of the program by perturbing one stationary instanton with a wave packet, setting the instanton into motion.

What does the instanton correspond to physically? To answer that, we inspect the energy density of a stationary instanton:

$$\mathcal{H} = \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + V(\phi) \tag{28}$$

$$= \frac{B^2}{A} \left\{ \operatorname{sech}^4 \left[\sqrt{2B}(x - x_0) \right] - 1 \right\}, \tag{29}$$

where the constant offset can be neglected. We see that the energy density is highly concentrated around the centre x_0 and falls off rapidly to zero. Therefore, the instanton corresponds to a point-like particle.

In fact, the collision of two instantons shown in the Setting `Silly 3` of the program can be interpreted as two particles colliding, annihilating each other, and producing another pair of instantons travelling in different directions. The initial configuration can be seen in Figure (6).

In the setting `Silly 2`, the Higgs field is initialised to a sinusoidal curve and subsequently evolves to show a shape similar to that of a number of equally spaced Manton instantons next to each other while oscillating about the vacuum expectation values and being disturbed by the interactions with the field `Test`. The initial configuration is shown in Figure (7).

For a full discussion of this solution, including the stability under small perturbations, we refer to [1], Chapter 7.1.

[8] The Klein-Gordon equation describes spin-0 particles because it implies that ϕ is a Lorentz-invariant (scalar) function.

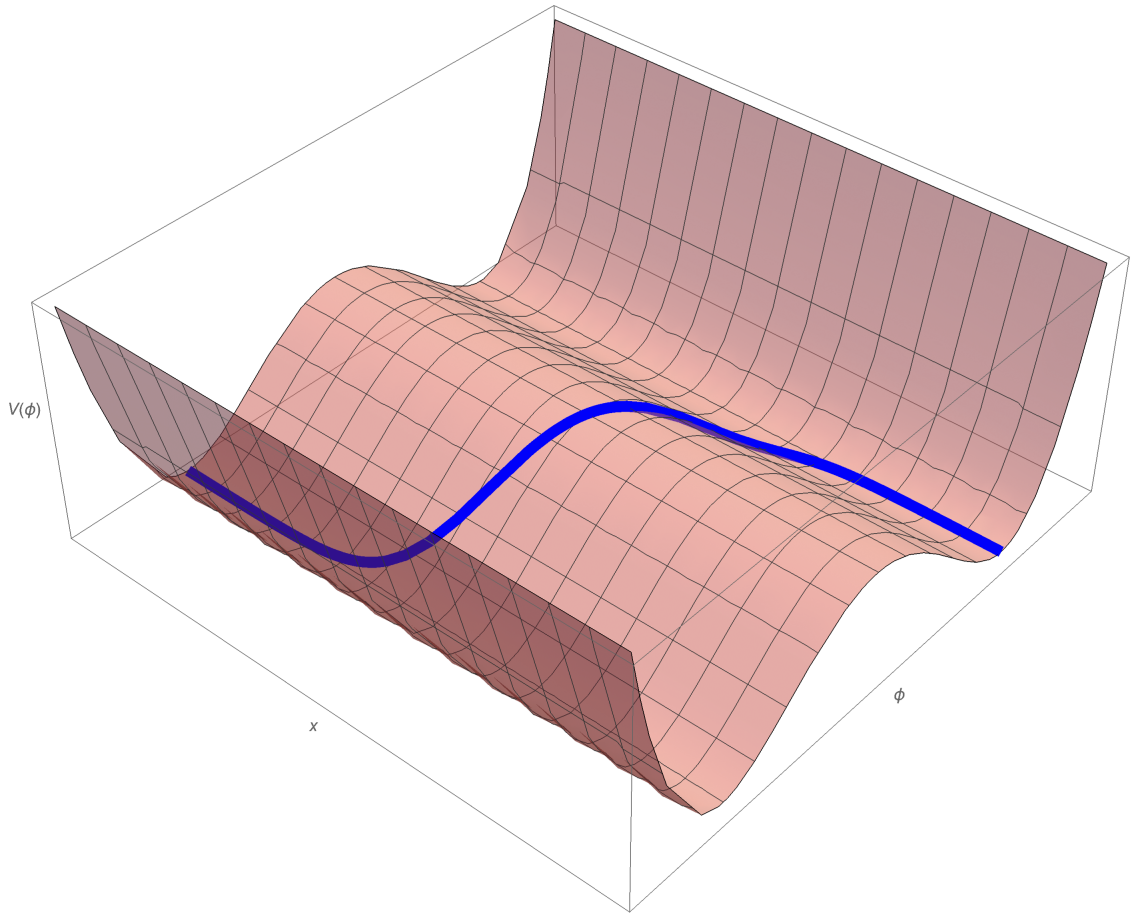


Figure 5: Instanton solution (blue curve) in the potential valley (red surface). Intuitively, this can be imagined as a rope (blue curve) lying on a ridge (red surface).

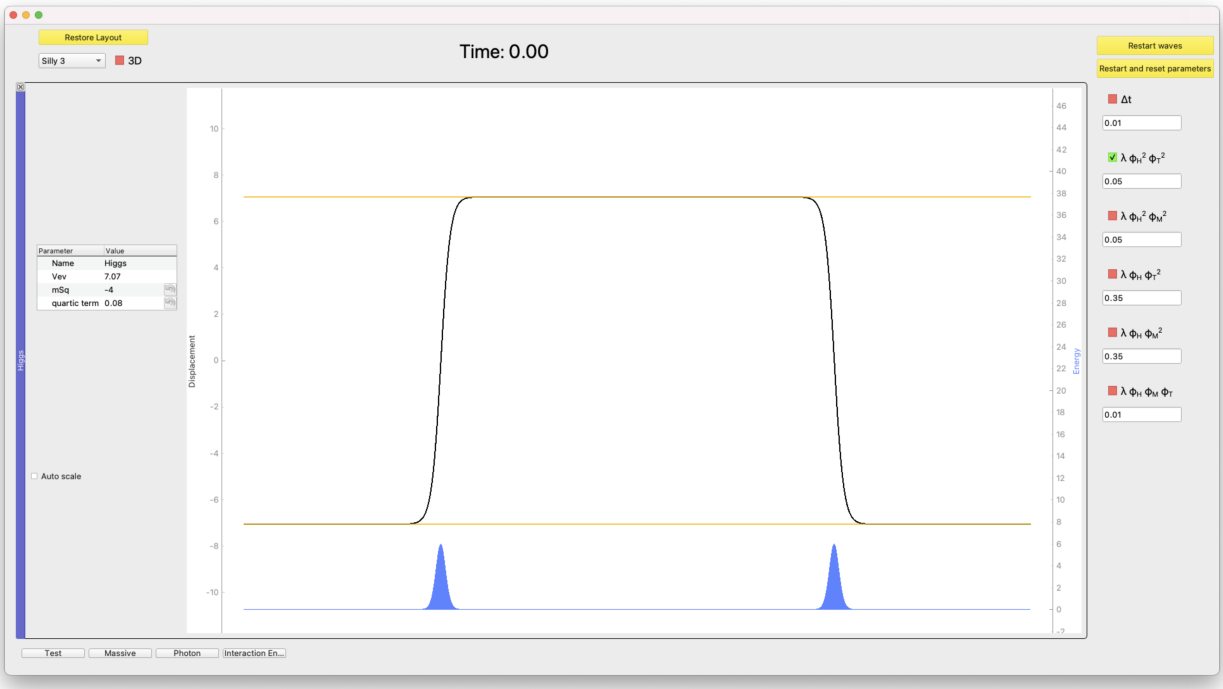


Figure 6: Initial configuration of the `Higgs` field in the `Silly 3` setting, as shown in "Higgs Simulator". (The setting `Silly 3` can be selected in the top left corner of the graphical user interface.) Two stationary instantons (black line) are next to each other. The energy density (blue filled plot) is sharply peaked at the centre of the each instanton, giving them particle character. The vacuum expectation values (`vev`) of the Higgs field are the orange lines.

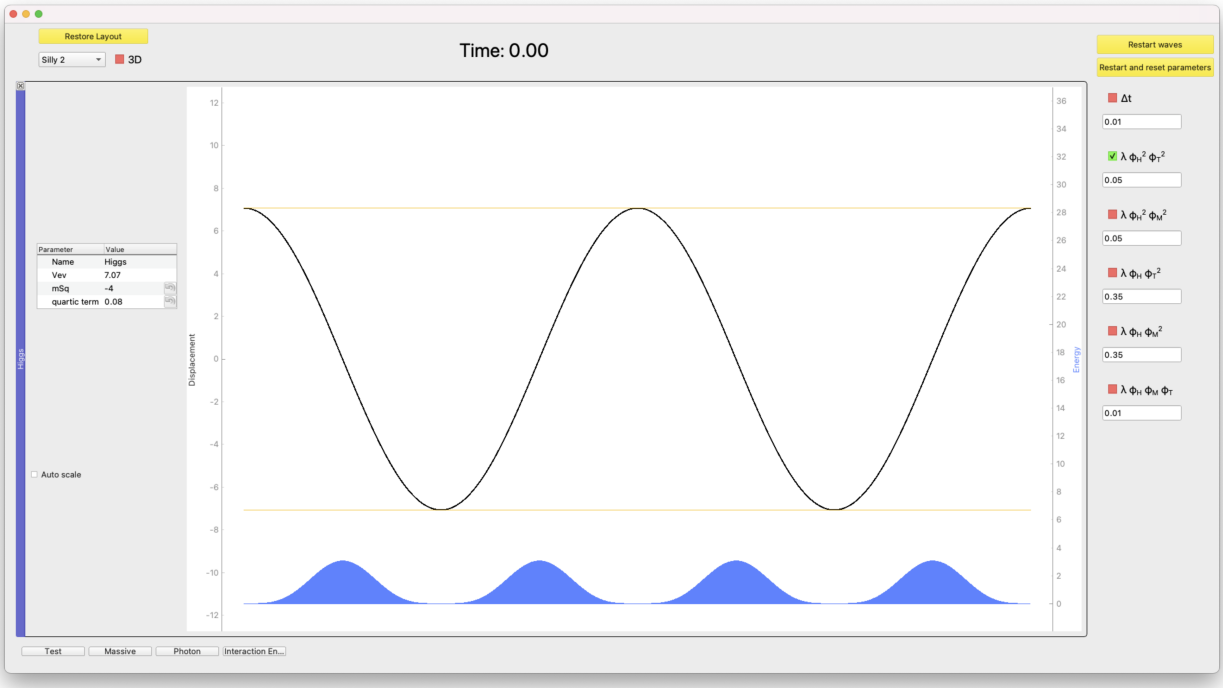


Figure 7: Initial sinusoidal configuration of the `Higgs` field in the `Silly 2` setting, as shown in "Higgs Simulator". (The setting `Silly 2` can be selected in the top left corner of the graphical user interface.) The black curve is the displacement of the Higgs field, the blue curve is the energy density and the orange lines are the vacuum expectation values (`vev`) of the Higgs field. When the simulation is started, parts of the sine wave "roll down" the potential minimum towards the vacuum expectation values so that the peaks of the sine wave flatten out.

3 Numerical simulation

For the numerical simulation, we will consider the most general Lagrangian and derive the differential equation to be integrated numerically. We switch notation to explicitly denote space, time, momentum and the Fourier transform of a wave Φ by x, t, p and $\tilde{\Phi}$, respectively.

The "Higgs Simulator" can simulate any Lagrangian of the form described in Section (3.1). In the graphical user interface, the Lagrangian simulated can be modified by modifying the values of the parameters λ and checking/unchecking the checkboxes in the right-hand column of the graphical user interface and/or by modifying the values of `mSq` and `quarticTerm` next to each plot.

3.1 The general Lagrangian

We now move on to the general case of n interacting waves with m arbitrary interactions of polynomial type. The Lagrangian is given by

$$\mathcal{L} = \underbrace{\sum_{i=0}^{n-1}\left[\frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i)\right]}_{n \text{ kinetic terms}} - \left\{\underbrace{\sum_{i=0}^{n-1}\left[\frac{1}{2}m_i^2\phi_i^2+\frac{1}{4}q_i\phi_i^4\right]}_{n \text{ self-interaction terms}} + \underbrace{\sum_{i=0}^{m-1}\left[\lambda_{i_0 i_1 i_2 \ldots i_{n-1}}\,\phi_0^{i_0}\phi_1^{i_1}\ldots\phi_{n-1}^{i_{n-1}}\right]}_{m \text{ interaction terms}}\right\}, \tag{30}$$

which is the Lagrangian implemented in "Higgs Simulator".

Substituting into the Euler-Lagrange equation for the field $\phi_k(x,t)$ and switching to non-covariant notation, we obtain

$$\frac{\partial^2\phi_k}{\partial t^2} = \frac{\partial^2\phi_k}{\partial x^2} - \left\{\sum_{i=0}^{n-1}\left[m_i^2\phi_i + q_i\phi_i^3\right] + \sum_{i=0}^{m-1}\left[\lambda_{i_0 i_1 i_2 \ldots i_{n-1}}\, i_k \left(\prod_{\substack{j=0\\ j\neq k}}^{n-1}\phi_j^{i_j}\right)\phi_k^{i_k-1}\right]\right\}. \tag{31}$$

This is a second-order differential equation which is numerically integrated in the program using a Leapfrog integrator. The boundary conditions are chosen to be periodic, so that we are simulating waves on a circle. In the program, the self-interaction parameters m^2 and q are labelled as `mSq` and `quarticTerm`, respectively.

What remains is picking initial values $\phi_i(x, t=0)$ and $\left.\frac{\partial\phi_i}{\partial t}\right|_{t=0}$ for all values of x.

3.2 Initialisation of wave packets

Any non-Higgs field $\Phi(x,t)$ is initialised as a Gaussian wave packet, which is constructed as a Fourier transform,

$$\Phi(x,t=0) = \int_{-\infty}^{\infty} \tilde{\Phi}(p) e^{-ipx} \mathrm{d}p, \tag{32}$$

of the Gaussian envelope

$$\tilde{\Phi}(p) = \frac{1}{\sqrt{2\pi p_{\text{spread}}^2}} \exp\left[-\frac{1}{2}\left(\frac{p-p_{\text{central}}}{p_{\text{spread}}}\right)^2\right]. \tag{33}$$

For any time $t > 0$, each harmonic wave evolves individually, i.e.

$$\Phi(x,t) = \int_{-\infty}^{\infty} \tilde{\Phi}(p) e^{i(\omega(p)t - px)} \mathrm{d}p, \tag{34}$$

where $\omega(p)$ is the dispersion relation, given by

$$E_n = \omega_n = \sqrt{\left(\sin\left(\frac{\pi k_n}{N}\right)\frac{2}{\Delta x}\right)^2 + m^2}. \tag{35}$$

Therefore, at $t = 0$, the velocity of the wave packet is initialised as

$$\frac{\partial \Phi}{\partial t}(x, t=0) = \int_{-\infty}^{\infty} i\omega(p)\, \tilde{\Phi}(p) e^{-ipx} \mathrm{d}p. \tag{36}$$

We now discuss the practical implementation of this as a computer program. Because the code can only work with a discrete x-axis of N points with lattice spacing Δx and a discrete p-axis with spacing $\Delta p = \frac{2\pi}{N\Delta x}$, the Fourier transform has to be approximated by a (Riemann) sum.

Substituting $\mathrm{d}p \approx \Delta p$ and $p \approx n\Delta p$ with $n \in \mathbb{Z}$, we obtain

$$\Phi(x,t=0) \approx \sum_{n=-\infty}^{\infty} \tilde{\Phi}(n\Delta p) e^{-in\Delta p x} \Delta p. \tag{37}$$

Furthermore, the function $\tilde{\Phi}(p)$ is localised around $p = p_{\text{central}}$, so to a good approximation, the sum can run over a window of width N such that $n \in [n_{\text{lower}}, n_{\text{upper}}]$, where $n_{\text{lower}} = \left\lfloor \frac{p_{\text{central}}}{\Delta p} \right\rfloor - \lfloor \frac{N}{2} \rfloor$ and $n_{\text{upper}} = \left\lfloor \frac{p_{\text{central}}}{\Delta p} \right\rfloor + \lfloor \frac{N}{2} \rfloor$:

$$\Phi(x,t=0) \approx \sum_{n=n_{\text{lower}}}^{n_{\text{upper}}} \tilde{\Phi}(n\Delta p) e^{-in\Delta p x} \Delta p, \tag{38}$$

$$= \sum_{n=0}^{N-1} \tilde{\Phi}\left((n+n_{\text{lower}})\Delta p\right) e^{-i(n+n_{\text{lower}})\Delta p x} \Delta p. \tag{39}$$

(The window was chosen to be of length N as this then becomes a discrete Fourier transform between two complex vectors of length N, for which the FFT algorithm can be used.)

This produces a wave packet centred at $x = 0$. For a wave packet at $x = x_{\text{central}}$, the function has to be shifted so that

$$\Phi(x,t=0) = \sum_{n=0}^{N-1} \tilde{\Phi}\left((n+n_{\text{lower}})\Delta p\right) e^{-i(n+n_{\text{lower}})\Delta p (x - x_{\text{central}})} \Delta p. \tag{40}$$

Finally, this can be cast into a form which can be used by a FFT by relabelling $\Delta p = \frac{2\pi}{N\Delta x}$ and $x = m\Delta x$, where $m \in [0,\, N-1]$, and expanding the exponential:

$$\Phi(m\Delta x, t=0) = e^{-i\frac{2\pi}{N} n_{\text{lower}} (m - \frac{x_{\text{central}}}{\Delta x})} \sum_{n=0}^{N-1} \underbrace{\tilde{\Phi}\left((n+n_{\text{lower}})\Delta p\right) \Delta p e^{i\frac{2\pi}{N} n \frac{x_{\text{central}}}{\Delta x}}}_{\equiv \tilde{\Phi}_{\text{modified}}[n]} e^{-i\frac{2pi}{N} nm}. \tag{41}$$

Defining the FFT as a mapping between the N-dimensional complex vectors $a[n]$ and its transform $A[m]$ by

$$A[m] = \text{FFT(a[n])} = \sum_{n=0}^{N-1} a[n] \exp\left(-2\pi i \frac{mn}{N}\right) \quad \text{with} \quad m \in \{0, ..., N-1\}, \tag{42}$$

it is seen that one can take the FFT of $\tilde{\Phi}_{\text{modified}}[n]$ and evaluate it at $\frac{x}{\Delta x}$:

$$\Phi(x,t=0) = e^{-i\frac{2\pi}{N} n_{\text{lower}} (\frac{x}{\Delta x} - \frac{x_{\text{central}}}{\Delta x})} (\text{FFT}(\tilde{\Phi}_{\text{modified}}[n]))[\frac{x}{\Delta x}]. \tag{43}$$

(Note: This has been implemented in Python using numpy's FFT and gives the correct wave packet within an absolute error of 10^{-4} of the Java version, which implements a summation of plane waves.)

3.3 Initialisation of the Higgs field

The problem of initialising the Higgs wave, which interacts with other waves, is that the additional term in the potential changes the vacuum expectation value. The vacuum expectation value now varies as a function of x and may in fact not even exist if there is no global minimum of the effective potential.

If we naively initialise the Higgs wave at the vacuum expectation value, discarding any interactions from the potential, we would see backward and forward radiation in form of "recoil waves" travelling in the backward and forward direction. The recoil waves are undesirable. They can be eliminated, to first order, by minimising the effective potential which includes the interaction terms. However this does not eliminate the recoil waves fully because the new minimum influences the wave which the Higgs field is interacting with, therefore changing the vacuum expectation value again. The problem is that we have to find wave packets for all interacting waves simultaneously.

An iterative solution might be the following (by Dr C. G. Lester, private conversation):

1: Set the Higgs field to the vacuum expectation value (calculated from the self-interaction only).
2: Set the fields (which the Higgs field is interacting with) to the initial conditions of Section (3.2).
3: Let the simulation run for a time t_i.
4: Set all fields to zero outside the range of the desired wave packets, therefore setting the recoil waves to zero.
5: Let the simulation run for a time $-t_i$, i.e. backwards to return to $t = 0$. The wave packets in all fields will now produce no radiation at time t_i and are therefore better candidates for the initial conditions.
6: Go back to Step (2) and repeat, i.e. only reset the non-Higgs fields, but using a larger time interval, $t_{i+1} > t_i$.

A problem with the implementation of this algorithm is that the times t_i are not known but must be cleverly chosen. Furthermore, the windows outside of which the fields should be set to zero are not known a priori either.

Due to these problems, the initialisation procedure chosen in the program is to find the global minimum of the effective Higgs field potential.

3.3.1 Interaction terms of type $\lambda w^2 h^2$

Suppose the Lagrangian $\mathcal{L} = T - V$ has the potential term $V(h) = \alpha h^4 - \beta h^2 + \lambda h^2 w^2$, where $w(x)$ is the non-Higgs field which has been initialised according to Section (3.2) and is thus a fixed function of x, and $h(x)$ is the Higgs field.

Note that we are including the interaction term in the potential.

We seek to minimise $V(h)$ with respect to h, i.e. $\left.\frac{\partial V}{\partial h}\right|_{h=\tilde{h}} = 0$, giving

$$\tilde{h} = \sqrt{\frac{\beta - \lambda w^2}{2\alpha}}, \tag{44}$$

where we have chosen the positive solution. If the expression inside the square root turns negative at a particular value of x, we set the wave $h(x)$ to zero at that value of x.

Differentiating with respect to time, we obtain the initial velocity of the Higgs wave,

$$\left.\frac{\partial \tilde{h}}{\partial t}\right|_{t=0} = -\frac{\lambda w}{2\alpha \tilde{h}} \frac{\partial w}{\partial t}. \tag{45}$$

3.3.2 General interaction terms

Suppose we have a general Lagrangian in which the Higgs field, $h(x)$, is interacting with n other waves, $\phi_k(x)$, through m interaction terms, so that the potential term takes the form

$$V(h) = \alpha h^4 - \beta h^2 + \sum_{i=0}^{m-1} \left[\lambda_i \left(\prod_{k=0}^{n-1} \phi_k^{k_i} \right) h^{n_i} \right] \tag{46}$$

where all $n_i \geq 1$ and $k_i \geq 1$ and α, β and λ_i are constants. Note that the n_i do not have to be distinct, which makes it necessary to regroup the terms to represent $V(h)$ as a polynomial with, say, ascending powers in h.

We seek solutions to

$$\left.\frac{\partial V}{\partial h}\right|_{h=\tilde{h}} = 0. \tag{47}$$

As it is not possible to solve the resulting polynomial in h in general analytically, we seek the solutions numerically using the following algorithm, which returns the position of the global minimum of a polynomial, $p(x) = \sum_i a_i x^i$, in the domain $D_x = [x_{\min}, x_{\max}]$[9]:

[9]The expression "$A \leftarrow B$" denotes assignment of B to the variable A.

Algorithm 1 Find global minimum of polynomial

function ARGMIN-POLY($p(x)$, $[x_{\min}, x_{\max}]$)
    Take the derivative $d(x) \leftarrow \frac{dp}{dx}$
    Calculate the list of roots $r[i] \leftarrow \text{roots}(d(x))$
    Append the domain boundaries $[x_{\min}, x_{\max}]$ to $r[i]$
    Apply $p(x)$ to each value of $r[i]$ and store the results in the list $p_r[i]$
    Find the minimum value $m \leftarrow \min_i(p_r[i])$
    Return the values (multiple if degenerate) in $r[i]$ at which $p(x)$ attains this minimum: $r[p_r[i] == m]$
end function

To stay consistent, we take the largest returned value if there are multiple return values. This value will be the value $\tilde{h}$ which minimises $V(h)$ at a particular value of x. The initialisation procedure has to minimise $V(h)$ for every value of x.

By taking the time derivative of Equation (47) and rearranging, we obtain the initial velocity of the Higgs field,

$$\frac{\partial \tilde{h}}{\partial t}\Big|_{t=0} = -\frac{\sum_{i=0}^{m-1} n_i \lambda_i \left[\frac{\partial}{\partial t}\prod_{k=0}^{n-1}\phi_k^{k_i}\right]\tilde{h}^{n_i-1}}{12\alpha\tilde{h}^2 - 2\beta + \sum_{i=0}^{m-1}\begin{cases}0, & \text{if } n_i = 1\\ n_i(n_i-1)\lambda_i\left[\prod_{k=0}^{n-1}\phi_k^{k_i}\right]\tilde{h}^{n_i-2}, & \text{if } n_i > 1\end{cases}}, \tag{48}$$

where $\tilde{h}(x)$ is the solution to Equation (47). Note that the time derivative of the product, $\frac{\partial}{\partial t}\prod_{k=0}^{n-1}\phi_k^{k_i}$, has to be written out explicitly by using the product rule before being able to do calculations.

3.4 Summary

The following algorithm is used for the simulation:

Algorithm 2 Simulation

1: Declare the waves ϕ_i and, for each wave, set the self-interaction parameters m^2 and q.
2: Declare the interaction terms λ_i.
3: Label the waves which are to be initialised as Higgs waves as h_i.

4: Initialise the non-Higgs waves with initial displacement $\phi_i(x, t = 0)$ and velocity $\partial_t\phi_i(x, t = 0)$ given by Equation (32) and (36), respectively.
5: Initialise the position of the Higgs waves by solving Equation (47) for each x.
6: Initialise the velocity of the Higgs waves, given by Equation (48).

7: Run the simulation by numerically integrating Equation (31).

4 Conclusions

The governing Lagrangian used in the program "Higgs Simulator" was motivated and the Euler-Lagrange equations were derived. Two types of solutions, wave packets and Manton instantons, were discussed analytically.

For wave packets in a massive field, the dispersion relation was derived. The effect of the Higgs mechanism on a massless field was discussed and compared to the case of the massive field. By comparing the group velocities of wave packets in both fields, the numerical simulation confirmed that a massless field can acquire a mass purely by interaction with the Higgs field.

Stationary and moving Manton instantons were constructed in the numerical simulation, showing the expected behaviour.

First-order corrections to the initialisation of the Higgs field were successfully implemented, although an initialisation algorithm which completely eliminates backward and forward radiation remains to be found.

References

[1] V. Rubakov. *Classical Theory of Gauge Fields.* Princeton University Press, 1999.