2015 NEXT LECTURES

Effective Field Theory

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1 Avant propos

Let us suppose that we wish to describe some physical system on large distance and time scales. Suppose, furthermore, that the system exhibits some kind of random, local (or short-distance) fluctuations (for example, these fluctuations may be the ones inherent in quantum mechanics). The formalism for describing such a system is called 'effective field theory' and is the subject of these lectures.

Note that it is already something of a miracle that such a theory exists at all. Experience tells us that systems can be extremely complicated on short-distance scales. Even though we are not so arrogant as to try to describe that short-distance physics, we know that that physics is there and that it is what gives rise to the long-distance physics that we *do* wish to describe.

To give an example, consider QCD. Not quantum chromodynamics, but quantum *cow* dynamics. Scientists now know that a cow, viewed at short-distance scales, is a very complicated object indeed, with multiple stomachs made of cells made of proteins made of atoms made of electrons and nuclei made of quarks made of goodness-knows-what. These quarks and electrons interact with each other (and with the quarks and electrons in other cows) via the complicated quantum dynamics of QED and QCD (the other, chromo, version).

Viewed in this way, the problem of the computation of cow-cow scattering looks like a very hard problem indeed.

But viewed from far enough away (at large enough distance scales), a cow behaves, for all intents and purposes, like a point particle of mass M, with no internal dynamics at all. Moreover, when we scatter 2 cows off each other, we see a very simple, contact interaction (albeit with some rather complicated final states, corresponding to inelastic scattering).

This example makes it clear that the desired miracle sometimes does happen – one doesn't need to know about gauge theory in order to study long-distance cow-cow scattering. This is just as well, if you are a physicist. Indeed, I call the miracle the 'miracle of physics', because it is the basic reason why physicists have ever been able to make *any* progress and why physics enjoys the hegemony that it does today: without the miracle, we could never get started on tackling a physical system with a given length scale (e.g. on a desk in a lab), without first worrying about all the other physics taking place on all other distance scales throughout the Universe.

Enough philosophy. What are the ingredients of an effective field theory? Clearly, we need some degrees of freedom. These will be represented by space-time fields. The dynamics of the physical system may well be invariant under some group of symmetries (such as space-time translations, rotations, or Lorentz boosts), in which case we will need to specify how the group acts on the fields. We will then write the most general dynamics (in the form of an action) for the fields that is invariant under the group action. We do this not because of a desire to be as general as possible; rather, we will find that the short-distance fluctuations will, of its own accord, generate the most general dynamics consistent with the symmetries.[1]

You might be thinking that this sounds a lot like quantum field theory (QFT). It is. In QFT, the lore is that one decides on the fields and symmetries, and then writes down the most general *renormalizable* action for the fields that is consistent with the symmetries. The insistence on renormalizability guarantees that one has a theory which can be used to make predictions on *all* length scales, including arbitrarily short ones. This is not only rather arrogant, but also rather pointless, because no one has yet done an experiment on an arbitrarily short distance scale! So EFT is really just the correct way to do QFT. Unfortunately, it receives rather scant treatment in the QFT textbooks. Fortunately, there are lots of excellent lecture notes available [1–4] and I encourage you to read as many of them as possible. My goal here is not to repeat what others have said already, but rather to give you the basic outline and then illustrate the principles and pitfalls via several examples, namely the Standard Model of particle physics, the non-linear sigma model, and the quantum theory of perfect fluids. Other instructive examples that are discussed in lecture notes elsewhere are the Euler-Heisenberg lagrangian of low-energy QED, Landau's theory of Fermi liquids [1], and the effective theory of heavy quarks [2, 3].

2 Notation and conventions

As usual, $\hbar = c = 1$, and our metric is mostly[2] mostly-minus: $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. We will exclusively use 2-component left-handed Weyl fermions. In the Standard Model for example, the fermions are $\psi \in \{q, u^c, d^c, l, e^c\}$. Kinetic terms are written as $i\overline{\psi}\overline{\sigma}^\mu \partial_\mu \psi$ and a Dirac mass term for ψ and χ is written as $\psi \cdot \chi +$ h. c.. See [5] for more details.

3 Modus Operandi

3.1 QFT redux

I assume that you know all about bog-standard QFT.[3] There, the rules of the game are that we decide upon a set of fields and a group of symmetries acting upon them, and then write the most general renormalizable action involving them. You well know, I hope, that in terms of their canonical or engineering dimensions, this necessarily restricts us to terms in the action of dimension four or less. The number of such terms is finite (if the number of fields is). We assign each term an arbitrary coefficient (though the coefficients of kinetic

[1] This is sometimes called 'Gell-Mann's Totalitarian Principle': everything which is not forbidden is compulsory.

[2] When we study the EFT of a perfect fluid in the last lecture, we'll switch to mostly-plus. Sorry!

[3] Only joking: no one knows *all* about QFT. But I hope that you at least know the basics.

terms can be set, without loss of generality, to one, if the fields are complex, or one-half, if they are real). Given that there is a finite number of such parameters (n say), we have the possibility of constructing a physical theory, in the sense that once we have made n suitable measurements to fix the values of the parameters, we can start to make predictions for the results of other measurements.

Things are not quite so straightforward in practice, because when we try to fix the values of the *bare* parameters, we find that they have to be infinite. But in a renormalizable theory, these infinites can be absorbed into finite, scale-dependent, *renormalized* parameters, such that all relations between physical observables are finite, and we have a *bona fide* physical theory.

The appearance of infinities nevertheless caused great headaches for the founding fathers of QFT. They arise because of loop diagrams in QFT, whose short-distance contributions involve divergent integrals. We thus call them *UV divergences*. But to actually get a divergence requires us to assume that the theory is valid on arbitrarily short distance scales, way beyond those that we actually probe in experiments. This seems overly arrogant and liable to result in hubris. Indeed, it runs contrary to what we have observed in all previous instances in physics, namely that physical theories only ever have some limited region of validity.[4]

3.2 Effective field theory: naïve approach

The point of departure for EFT is to humbly accept that any given theory is likely to have some short-distance or UV cut-off, Λ beyond which it is invalid. We should not dare to extrapolate beyond this cut-off. If we don't, then we will never encounter any UV divergences, and so the problems that plagued the founding fathers of QFT seem to have completely disappeared!

In its place, a new problem appears. The good side of insisting on renormalizability (that is, a theory valid on all scales), was that it necessarily restricted the dimensions of operators that can appear in the action and hence implies that the theory has a finite number, n, of parameters and hence is predictive, once we have made n measurements. If we give up on renormalizability, but still write down all operators consistent with the symmetry (if we don't quantum fluctuations will generate them anyway . . .), then we will have to include infinitely many. (Proof: consider any operator that is invariant under the symmetry; the mth power of the operator is also invariant, for any $m \in \mathbb{Z}$.) If each of these operators has an arbitrary coefficient, then we need to do infinitely many measurements before we can start to make predictions. This is not a theory!

We find a way out of the impasse *à la* George Orwell, by declaring that 'all operators are equal, but some are more equal than others'. How? Since we are interested in the physics at large-distance scales, it may be that some operators are more important at large-distances than others. This is indeed the case, and in fact it turns out the usual QFT dimensional analysis gives us a measure of how important operators are, relative to the kinetic term of the free theory (which governs the size of typical fluctuations).

[4] One day, of course, some bright spark might write down a theory of everything, in which case they would be quite justified in extrapolating in this way. But this can only happen once!

Consider, as an example, relativistic scalar field theory in D spacetime dimensions.[5] In units where $\hbar = c = 1$, the action is dimensionless and so the kinetic term, $(\partial\phi)^2$, has energy dimension D. Since the derivatives have unit (energy) dimension, the field ϕ must have dimension $D/2 - 1$. An operator $\mathcal{O}_{p,q}$ made up of p fields and q derivatives then has dimension $p(\frac{D}{2} - 1) + q$ and appears in the action as

$$S \subset \int d^D x \, \frac{g_{p,q}}{\Lambda^{p(\frac{D}{2}-1)+q-D}} \mathcal{O}_{p,q}, \tag{3.1}$$

where we have written the coupling in terms of the cut-off scale Λ and a dimension*less* coefficient, $g_{p,q}$. Now, we see that if we consider a field configuration of energy E, the contribution of the operator $\mathcal{O}_{p,q}$ to the action is, on dimensional grounds, given by

$$S \subset \; g_{p,q} \left(\frac{E}{\Lambda}\right)^{p(\frac{D}{2}-1)+q-D}. \tag{3.2}$$

If the exponent $p(\frac{D}{2}-1)+q-D > 0$, then the operator becomes less and less important at energies below the cut-off and we call it *irrelevant*. If $p(\frac{D}{2}-1)+q-D < 0$ then the operator becomes more and more important at $E < \Lambda$ and we call it *relevant*. If $p(\frac{D}{2}-1)+q-D = 0$ (which includes the kinetic term itself), then the operator is equally important as the kinetic term at low energies and we call it *marginal*.

Before going further, let us make two remarks. The first remark is that in a non-relativistic theory, we will need to count dimensions of space and time separately. The second remark is that our counting of dimensions and our decision of which operators are (ir)relevant is contingent on our singling out a particular term as 'the' kinetic term. This is a natural thing to do, since a kinetic term is present in all dynamical theories and sets the scale for the typical size of fluctuations in the theory. But there is no obvious definition of what a 'kinetic term' actually is and a given theory might have multiple kinetic term candidates. In such a case, one should proceed by computing the dimensions of operators with respect to each of these terms individually; it may turn out that different kinetic terms dominate in different regimes of distance and time scales.

Now let us return to the main thrust. We have discovered that, of the infinitely many operators that we may write in the action, some are more important than others at the large distance and time scales in which we are interested. Can we use this to make a predictive theory? The answer is no, strictly speaking. But we can use it to make a theory which is almost as good, in that we can use it to make predictions to an arbitrarily high degree of precision, provided that we are willing to do enough donkey work.

What we do is to write out the most general action, but including operators only up to some finite dimension Δ.[6] This truncated theory has only a finite number of arbitrary coefficients so we can use it to make predictions, once we have made enough measurements. (Clearly we will need to do more and more measurements as we increase D and herein lies some of the aforementioned donkey work.) But we will not be able to make exact

[5] We will return to this example repeatedly in the sequel.

[6] A renormalizable theory, then, corresponds to the special case with $\Delta = D$.

predictions, because we have neglected operators in the theory whose dimensions exceed Δ. Comparing with 3.2, we see that in computing the action (or indeed any other observable), we have only included contributions of $O\left((\frac{E}{\Lambda})^{p(\frac{D}{2}-1)+q-D}\right)$ compared to the leading ones and so this is the accuracy of our prediction.

Four remarks are now in order. Firstly, we note that our predictions automatically become arbitrarily accurate as we go to arbitrarily large distance scales, *viz.* $E \to 0$. It is in this sense that we have a theory for physics on large distance scales. Secondly, we note that we can improve the accuracy of our theory at fixed energy E by truncating at higher order in the operator expansion. To do this, one needs to find all the invariant operators up to a given dimension (in general, this number grows exponentially with the dimension), to calculate the theory predictions including all these operators, and to perform more measurements (at the higher accuracy) to fix the extra parameters. This is a lot of donkey work. Thirdly, we remark that once we hit energies E of the order of the cut-off Λ, no amount of donkey work is going to help us, because all neglected terms become equally important. The expansion breaks down completely and so the cut-off Λ really does deserve its name. Finally, we remark that we are not free to choose the value of Λ arbitrarily. The predictions of the theory for experimental observables depend, via (3.2), on Λ. And so we can use measurements to determine the value of Λ in a given theory.

3.3 Effective field theory, *comme il le faut*

So far, we implied that the way to do EFT is to impose a hard UV cut-off on the theory, such that the UV divergences coming from loop integrals in the theory do not appear. If we do this (as most lecture notes, *&c* do), then whilst we end up with a theory that is manifestly finite, we also end up with a theory that is completely useless for making predictions. The problem is that higher-dimension operators give contributions that are suppressed when they appear in tree-level Feynman diagrams, but not when they are inserted into loops. It is easy to see schematically why this happens. At tree-level, the only powers of Λ that appear in amplitudes are those coming from the denominators in (3.1). So the presence of higher-dimension operators always leads to suppression of amplitudes by factors of $E/|Lambda < 1$. But when we inset higher-dimension operators into loops, we get additional powers of Λ in the numerators of amplitudes, coming from the fact that we cut-off the loop momenta at Λ. With enough loops, we can always arrange for more powers of Λ in the numerator than in the denominator, meaning that the contributions of higher-dimension operators will be unsuppressed. But then we are not at liberty to simply truncate the operator expansion and ignore operators above a certain dimension!

We can see the phenomenon explicitly using our favourite example of scalar field theory. Consider 1-loop corrections to the dimension-4 operator $\lambda\phi^4$. The EFT Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\phi(\partial^2 + m^2)\phi - \frac{1}{4!}\lambda\phi^4 - \frac{1}{6!}\frac{c_6}{\Lambda^2}\phi^6 - \frac{1}{2\cdot 4!}\frac{c_8}{\Lambda^4}\phi^4(\partial\phi)^2 - \dots,$$

where the dimensionless coefficients $c_6, c_8, \dots$ are $O(1)$. With momentum cut-off Λ, we get loop diagrams of similar size from *all* operators.[7] Indeed,

$$\delta\lambda_{\text{1-loop}} \supset \frac{c_6}{\Lambda^2}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2} \sim \frac{c_6}{\Lambda^2}\frac{\Lambda^2}{16\pi^2} \sim \mathrm{O}(1), \tag{3.3}$$

$$\delta\lambda_{\text{1-loop}} \supset \frac{c_8}{\Lambda^4}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{k^2}{k^2-m^2} \sim \frac{c_8}{\Lambda^4}\frac{\Lambda^4}{16\pi^2} \sim \mathrm{O}(1),\ \&c. \tag{3.4}$$

Thus we find that predictivity is lost using such a cut-off, since we need to consider loops containing all operators to calculate at any given order in the momentum expansion of the Lagrangian.

The solution to this problem is, in fact, very simple: we need to replace the UV cut-off Λ with a *mass independent* regulator, such as dimensional regularization. Then, the only mass scales that can appear in the numerators of diagrams correspond to light masses or momenta, with the renormalization scale appearing only in logarithms. For the EFT of a scalar, for example,

$$\frac{c_6\mu^{2\epsilon}}{\Lambda^2}\int\frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}}\frac{1}{k^2-m^2} \sim \frac{c_6}{\Lambda^2}\frac{m^2}{16\pi^2}\frac{1}{\epsilon} - \frac{c_6}{\Lambda^2}\frac{m^2}{16\pi^2}\ln(\frac{m^2}{\mu^2}), \tag{3.5}$$

$$\frac{c_8\mu^{2\epsilon}}{\Lambda^4}\int\frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}}\frac{k^2}{k^2-m^2} \sim \frac{c_8}{\Lambda^4}\frac{m^4}{16\pi^2}\frac{1}{\epsilon} - \frac{c_8}{\Lambda^4}\frac{m^4}{16\pi^2}\ln(\frac{m^2}{\mu^2}),\ \&c, \tag{3.6}$$

where μ is the renormalization scale.

A mass independent scheme thus preserves the original momentum expansion: contributions from higher dimension operators are suppressed, even in loops. If we consider all operators up to dimension Δ, we are guaranteed a result accurate to $O\left((E/\Lambda)^{\Delta-4}\right)$, where E is the energy scale of the process, at any loop order.

3.4 Topsy-turvy EFT

We can motivate the EFT idea in a completely different way, by showing that starting from a renormalizable QFT at high energies, the low energy theory is equivalent to an EFT.

Suppose, for example, that we start with the renormalizable SM, and consider only energies and momenta well below the weak scale, $\sim 10^2$ GeV. We can never produce W, Z, or h bosons on-shell and so we can simply do the path integral with respect to these fields (we 'integrate them out', to use the vernacular). We will be left with a path integral for the light fields, but with a complicated lagrangian that is non-local in space and time. But since we are only interested in low energies and momenta, we can expand in powers of the spacetime derivatives (and the fields) to obtain an infinite series of *local* lagrangian operators, which become less and less important as we go down in (energy-)momentum.

At tree-level, this procedure just corresponds to replacing the fields using their classical equations of motion, and expanding $\frac{-1}{q^2-m_W^2} = \frac{1}{m_W^2} + \frac{q^2}{m_W^4} + \dots$. It is already clear that our expansion breaks down for momenta comparable to m_W, so that the theory is naturally equipped with a cut-off scale m_W.

[7] An apparently simple solution to this problem would be to use a lower cut-off $\Lambda' < \Lambda$ for the loop integral. But doing so generates operators with derivatives of size $\frac{\partial}{\Lambda'}$ under the renormalization group flow, thereby reducing the regime of validity of the EFT as a whole to $p \lesssim \Lambda'$.

In particular, the leading operator we get by the above process will be the 4-fermion operator in Fermi's theory of beta decay, with coefficient $\sim \frac{1}{m_W^2}$. By measuring the decay constant, G_F, we are able to estimate the cut-off m_W. All of this is in accord with what we discussed above.

3.5 The scourge of relevant operators

Now is the time for us to acknowledge the presence of an elephant in the room. In renormalizable QFT, the problems come from irrelevant operators. They are non-renormalizable and lead to uncontrollable divergences in loop diagrams. But in EFT, irrelevant operators are completely benign. There are no divergences, and instead the irrelevant operators are, well, irrelevant. Or at least, *largely* irrelevant, in that they give small, corrections to physics at energy scales well below the cut-off.

In EFT, the problems come rather from relevant operators. These become increasingly important at low energies, and indeed (**??**) shows that they dominate the physics. But this invalidates our assumption that the physics is dominated by the kinetic term, and so invalidates our operator expansion. All we can say is that the physics of the system at low energies is likely to be completely different from that 'predicted' by the original EFT.

To examine this in more detail, let us start with a relatively trivial case. Consider scalar field theory in 4-d. The symmetries allow a mass term $\propto \phi^2$ in the lagangian. This has dimension 2 (meaning that we can write its coefficient as $g_{2,0}\Lambda^2$, with $g_{2,0}$ being dimensionless) and it gives contributions of size $g_{2,0}\Lambda^2/E^2$ relative to the kinetic term. There are then two possibilities. Either $g_{2,0} \gtrsim 1$, in which case this term always dominates the kinetic term. We should redo our scaling arguments above, taking ϕ^2 to be the dominant term at low energies. If we do, we will find that all other operators are irrelevant. At low energies therefore, the dynamics is dominated by the term ϕ^2. Classically, we find that $\phi = 0$ and there is no dynamics at all. We obtain a consistent theory of nothing! The alternative is that $g_{2,0} \ll 1$, in which case there is a regime of energies in which the kinetic term dominates and our EFT is valid. But then the question arises of how we can end up with a theory in which $g_{2,0} \ll 1$. Indeed, starting from a generic short-distance theory of dynamics at the scale Λ we will invariably end up with $g_{2,0} \sim 1$ in the low energy EFT. Again, a simple example suffices to illustrate the general point: consider a theory with two scalar fields ϕ and Φ, where ϕ is assumed light compared to Φ, which has mass M. If we integrate out the field Φ to obtain the low-energy EFT for the light scalar ϕ, we will find that loops of Φ give corrections to the mass of ϕ of order M.

Thus, to end up with a small mass for ϕ, we need to delicately arrange the tree-level and loop contributions (which correspond to physics on differing length scales) in order to obtain a cancellation in the resulting value. This is called an *unnatural fine-tuning*.

Unfortunately, this issue is not just an academic one: the Standard Model of Particle Physics features just such a scalar field (the Higgs field) and it is a mystery to us why its mass is so light compared to the short-distance theory that gives rise to the SM.

Finally, note that it is quite possible to have relevant operators than are not mass terms, but rather correspond to interactions. Are these bad too? They certainly are, because they represent interactions that become arbitrarily strong at low energy. Perturbation theory thus breaks down completely. All it is safe to say is that the degrees of freedom and symmetries that we assumed in formulating our EFT are completely unsuitable for describing the physical system at low energies.

You already know a good example of this, namely QCD, where the coupling is marginal at tree-level, but acquires an anomalous dimension and becomes relevant at one-loop. The coupling thus becomes strong at low energies, and the low energy degrees of freedom (mesons and baryons) are completely unlike the quarks and gluons of QCD.

4 First example: The Standard Model and beyond

We have already described how Fermi's theory of beta decay is just a low-energy EFT description of a more complete, short-distance theory, *viz.* the SM. Now that we know about EFT and how it works, it seems reasonable to suppose that every QFT we have to hand is really just a low-energy EFT description of some more fundamental theory. Let us suppose that the SM itself is just an effective, low-energy description of some more complete BSM theory, and see what the consequences may be.

Following the rules above, the fields and the (gauge) symmetries of the EFT should be exactly the same as in the SM, but we should no longer insist on renormalizability. For operators up to dimension 4, we simply recover the SM. But at dimensions higher than 4, we obtain new operators, with new physical effects. As a striking example of these effects, we expect that the accidental baryon and lepton number symmetries of the SM will be violated at some order in the expansion, and that protons will decay.[8]

We don't know what the BSM theory actually is yet, and so when we write down the EFT, we should allow the coefficients of the operators in the expansion to be arbitrary. While we don't know the actual values of the coefficients, we can estimate their size using dimensional analysis, since we expect the expansion to break down at energies of order the cut-off, Λ. So the natural size of coefficients is typically just an $O(1)$ number in units of Λ, which is precisely how we wrote them above.

4.1 Mathematical interlude on vector spaces

Now we wish to write down the most general set of operators up to a given dimension. Before doing so, it is useful to notice that the operators of a given dimension form a vector space, V, and so we can simplify things by choosing a basis for this space. This is not so straightforward as it sounds (and indeed, disputes about it still erupt in the literature from time to time), because of equivalences between operators. In particular, any two operators that are equal up to a total divergence may be considered equal (since they give the same contribution at any order in perturbation theory), as may operators that differ by terms that vanish when the equations of motion hold, because such pieces give zero contribution to S-matrix elements (see, *e. g.*, [6]).[9]

For a simple example [7], consider a scalar field theory, in which we allow only operators that are even in ϕ and set the mass term to zero, for simplicity. The lagrangian at dimension

[8] Let us hope that we can finish the lecture before they do so!

[9] Alternatively, they can be removed by a field redefinition in the path integral.

4 is then

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4. \tag{4.1}$$

At dimension 6, three operators present themselves, namely $\phi^6, (\partial^2\phi)^2$, and $\phi^2(\partial\phi)^2$. Only one of these is independent. Indeed, we have that

$$(\partial^2\phi)^2 - \lambda^2\phi^6 = [(\partial^2\phi) - \lambda\phi^3][(\partial^2\phi) + \lambda\phi^3] \tag{4.2}$$

and the second term on the right vanishes when the equations of motion hold.

Similarly, integrating by parts we have that

$$\phi^2(\partial\phi)^2 = -\phi\partial_\mu(\phi^2\partial^\mu\phi) = -\phi^3\partial^2\phi - 2\phi^2(\partial\phi)^2 \tag{4.3}$$

which implies that

$$3\phi^2(\partial\phi)^2 = -\phi^3\partial^2\phi \tag{4.4}$$

and thus that

$$3\phi^2(\partial\phi)^2 - \lambda\phi^6 = -\phi^3(\partial^2\phi + \lambda\phi^3). \tag{4.5}$$

Thus, we see that $\phi^2(\partial\phi)^2 \sim \lambda^2\phi^6$ and $3\phi^2(\partial\phi)^2 \sim \lambda\phi^6$.

The best way to deal with these equivalences is as follows. Write each equivalence in the form $A = 0$, where A is an operator in the vector space V and let $U \subset V$ be the linear span (i.e. all linear combinations) of the As. Now we form equivalence classes in V by identifying any two operators that differ by an operator in U. So, for example, if $B = C + A$, then we regard B and C as equivalent operators and write $[B] = [C]$, where $[B]$ denotes the class containing B. In doing so, we form the quotient space, V/U, of equivalence classes. This is itself a vector space, with zero vector $[0] = U$, where 0 is the zero vector in V.

As you can see, identifying a true basis of operators is not easy, even for the simple example of scalar field theory. Fortunately, there will soon be a computer programme that will do it for you [8] (at least for a theories like the SM whose symmetries only include factors of $SU(N)$ and $U(1)$).

It is common in the literature to see a further subdivision of operators (or, rather, equivalence classes of operators) into those that can be generated at n-loop level in a renormalizable UV completion, where $n \in \{0, 1, 2, 3, \dots\}$. The rationale for doing this is that, if the new physics couplings are of $O(1)$, then each additional loop leads to suppression factor of $\sim 4\pi$, lowering the scale of new physics (i.e. the cut-off) that is required to generate a contribution of a given size.

What the literature does not tell you, sadly, is that the classes of operators generated at a given loop level do *not* form a vector subspace, in general.[10] Thus, it is meaningless to set experimental limits on the scale of new physics by taking an arbitrary linear combination of, say, classes of operators that can be generated at tree level: the resulting class of operators is not necessarily tree-level generated. This doesn't stop people doing it though!

[10] For a counterexample, consider scalar field theory in $d = 6$. The scalar field ϕ has dimension 2 and so the ϕ^3 interaction is marginal. At dimension 8, the only class of operators is $[\phi^4]$. The operator $+[\phi^4]$ can be generated at tree level, but the operator $-[\phi^4]$ can only be generated at one-loop level. So the tree level operators cannot form a vector subspace. For more details, see [8].

4.2 Back to the SM

Getting back to the SM, let's start by reminding ourselves of the form of the lagrangian. Recall that the SM is a gauge theory with gauge symmetry $SU(3)\times SU(2)\times U(1)$, together with matter fields comprising 15 Weyl fermions and one complex scalar, carrying irreps of $SU(3)\times SU(2)\times U(1)$. The fermions consist of 3 copies (the different families or flavours or generations) of 5 fields, $\psi \in \{q, u^c, d^c, l, e^c\}$, carrying reps of $SU(3)\times SU(2)\times U(1)$ as listed in Table 1. The scalar field, H, carries the $(1, 2, -\frac{1}{2})$ rep of $SU(3)\times SU(2)\times U(1)$.

Field	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
q	3	2	$+\frac{1}{6}$
u^c	$\overline{3}$	1	$-\frac{2}{3}$
d^c	$\overline{3}$	1	$+\frac{1}{3}$
l	1	2	$-\frac{1}{2}$
e^c	1	1	$+1$

Table 1. Fermion fields of the SM and their $SU(3)\times SU(2)\times U(1)$ representations.

The lagrangian can be written on a single line (just!). It is, schematically,

$$\mathcal{L} = i\overline{\psi}_i \overline{\sigma}^\mu D_\mu \psi_i - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \lambda^{ij}\psi_i\psi_j H^{(c)} + \text{h. c.} + |D_\mu H|^2 - V(H), \tag{4.6}$$

where i, j label the different families and a labels the different gauge fields. There are 5 fermion irreps $\psi \in \{q, u^c, d^c, l, e^c\}$, with 3 copies of each, corresponding to the 3 families. There are really 12 gauge fields: 8 in an adjoint of $SU(3)$, 3 in an adjoint of $SU(2)$, and 1 for $U(1)$. The covariant derivative D_μ contains the gauge couplings g_s, g, and g', with the gauge group generators in the appropriate reps. The fermion kinetic terms (but not the Yukawa couplings) are invariant under a $U(3)^5$ global symmetry. The Yukawa interactions can be written more explicitly as

$$\mathcal{L} = \lambda^u q H^c u^c + \lambda^d q H d^c + \lambda^e l H e^c + \text{h. c.} \tag{4.7}$$

The λ^i are 3 3×3 complex matrices (in family space).

The Higgs potential is given by

$$V(H) = \mu^2 H^\dagger H + \lambda (H^\dagger H)^2. \tag{4.8}$$

Ugly or not, the renormalizable SM does an implausibly good job of describing the data, reaching the per mille level in individual measurements and with an overall fit (to hundreds of measurements) that cannot be denied: the SM is undoubtedly correct, at least in the regime in which we are currently probing it (see [5] for more details, at a similar level to these lectures). What does this imply, if the SM is really just an EFT, with a cut-off Λ? Since the operators with dimension up to 4 already do an excellent job of describing the measurements (which are themselves very precise), we must conclude that the effects of higher-dimension operators are very small. In other words, Λ must be very large. How large? Well, each experimental measurement that agrees with the SM predictions can be translated into a rough lower bound on Λ, once we make the reasonable assumption that the dimensionless coefficients are of order 1. In this way, we obtain some very stringent bounds on Λ, reaching up to 10^{15} GeV or so! This is way beyond the reach of the LHC.

4.3 Accidental symmetries and proton decay

One miracle of the SM is that it has accidental symmetries. These are symmetries of the lagrangian that are not put in by *fiat*, but arise accidentally from the field content and *other* symmetry restrictions, and the insistence on renormalizability. Once we allow operators with higher dimensions in the EFT, we will find that these accidental symmetries get broken, with sometimes spectacular consequences for physics.

A simple example of an accidental symmetry is parity in QED. The most general, Lorentz-invariant, renormalizable lagrangian for electromagnetism coupled to a Dirac fermion Ψ may be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + iaF_{\mu\nu}\tilde{F}^{\mu\nu} + i\overline{\Psi}\not{D}\Psi + \overline{\Psi}(m + i\gamma^5 m_5)\Psi, \tag{4.9}$$

where both the term involving $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}$ and the term involving γ^5 naïvely violate parity. However, the former term is a total derivative and so does not contribute to physics at any order in perturbation theory. The latter term can be removed by a chiral rotation $\psi \to e^{i\alpha\gamma^5}\psi$ to leave a parity-invariant theory with fermion mass $\sqrt{m^2 + m_5^2}$. So we find that the lagrangian is invariant under parity, even though we did not require this in the first place. The same is true of charge conjugation symmetry. Note that if we had not insisted on renormalizability, we could write dimension-six terms like $\overline{\Psi}\gamma^\mu\gamma^5\Psi\overline{\Psi}\gamma_\mu\Psi$, which do violate parity.

As we already alluded to above, the SM lagrangian is accidentally invariant under a $U(1)_B$ baryon number symmetry (an overall rephasing of all quarks) and three $U(1)$ lepton number symmetries, corresponding to individual rephasings of the three different lepton families (which contains an overall lepton number symmetry $U(1)_L$ as the diagonal subgroup). Either $U(1)_B$ or $U(1)_L$ symmetry, together with Lorentz invariance, prevents the proton from decaying. Indeed, a putative final state must (by Lorentz invariance, which implies the fermion number is conserved mod 2) contain an odd number of fermions lighter than the proton. The only such states carry lepton number but not baryon number, whereas the proton carries baryon number but not lepton number.

Again, once we allow higher dimension operators, we will find that lepton and baryon number are violated (by operators of dimension five or six, respectively), meaning that the proton can decay. Similarly, generic theories of physics BSM will violate them and hence will be subject to strong constraints.

There is another interesting accidental symmetry of the SM, which is only approximate. This is called *custodial symmetry*. Consider the Higgs sector. The Higgs is a complex $SU(2)$ doublet, and so there are four real fields. The kinetic terms therefore have an $O(4)$ symmetry. Let us now consider how this symmetry gets broken when we switch on the various couplings.

One of the miracles of group theory is that the Lie algebra of the group $O(4)$ is the same as that of the group $SU(2) \times SU(2)$. So the Higgs fields can be thought of as carrying 2 $SU(2)$ symmetries, rather than the single $SU(2)_L$ of the standard model. It is usual to call the other symmetry $SU(2)_R$, so the Higgs carries a $(2,2)$ rep of $SU(2)_L \times SU(2)_R$. Now, when we switch on the $SU(2)_L$ gauge coupling g, we still have global symmetry $SU(2)_L$ (because the gauge symmetry includes constant gauge transformations, which are the same as the global ones) and we still have global symmetry $SU(2)_R$, because this factor is independent of $SU(2)_L$. So the full $SU(2)_L \times SU(2)_R$ remains unbroken.

What is more, this $SU(2)_L \times SU(2)_R$ is also unbroken when we switch on the Higgs potential, because $V(H)$ is only a function of $|H|^2 = h_1^2 + h_2^2 + h_3^2 + h_4^2$, which is manifestly invariant under $O(4)$.

The Yukawa couplings do break $SU(2)_L \times SU(2)_R$,[11] as does the coupling to the Z (which couples to the combination $T_L^3 + T_R^3$). So the correct statement is that the SM is invariant under $SU(2)_L \times SU(2)_R$ in the limit that $\lambda^u = \lambda^d, g' = 0$.

When the Higgs gets a VEV, the $SU(2)_L \times SU(2)_R$ is broken to the diagonal $SU(2)_V$ combination of the 2 original $SU(2)$s. This approximate symmetry implies a relation between m_W and m_Z that holds automatically in the SM, but does not hold in generic theories BSM. Again, see [5] for more details.

4.4 Beyond the SM - Effective field theory

Now let's reconsider the SM from the EFT viewpoint, cataloguing the operators of increasing dimension and describing their effects in turn.

4.5 $D = 0$: the cosmological constant

We have avoided mentioning it up to now, but clearly a constant term (which has dimension 0) is consistent with the symmetries of the SM. It has no effect until the SM is coupled to gravity, whereupon it causes the Universe to accelerate. On the one hand, this looks like good news, because the Universe is observed to accelerate. On the other hand, this is bad news because our estimate of the size of this operator coefficient (the operator is 1) is Λ^4, while the observed energy density is around $(10^{-3}\text{ eV})^4$. But the cut-off of the SM had better not be 10^{-3} eV, because if it were then we could certainly not use it to make predictions at LHC energies of several TeV. So either dynamics or a tuning makes the constant small. If we consider the Planck scale to be to be a real physical cut-off, then we need to tune at the level of 1 part in 10^{120}. It is fair to say, that despite $O(10^{120})$ papers having been written on the subject, no satisfactory dynamical solution has been suggested hitherto. An alternative is to argue that we live in a multiverse in which the constant takes many different values in different corners, and we happen to live in one which is conducive to life. Indeed, it has been argued [9] that if the constant were much larger and positive, structure could never form, while if it were too large and negative, the Universe would recollapse before life could appear. The flavour-of-the-month as regards how the multiverse itself arises is by a process of eternal inflation in string theory.

[11] A technical point: if $\lambda^u = \lambda^d$, then we can group u^c and d^c into an $SU(2)_R$ doublet, and $SU(2)_L \times SU(2)_R$ is restored.

4.6 $D = 2$: the Higgs mass parameter

The only other relevant operator in the SM is the Higgs mass parameter, which sets the weak scale. As above, the natural size for this is Λ. But we measure $v \sim 10^2$ GeV, leaving us with 2 options: either the natural cut-off of the SM is not far above the weak scale (in which case we can hope to see evidence for this, in the form of new physics, at the LHC) or the cut-off is much larger, and the weak scale is tuned, perhaps once again by anthropics.

4.7 $D = 4$: marginal operators

We have discussed these already in the context of the renormalizable SM, and there is nothing to add here.

4.8 $D = 5$: neutrino masses and mixings

Now things get more interesting. There is precisely one operator at $D = 5$, namely $\frac{\lambda^{ll}}{\Lambda}(lH)^2$, where λ^{ll} is a dimensionless 3×3 matrix in flavour space. Note that this operator violates the individual and total lepton numbers; moreover, it gives masses to neutrinos after EWSB, just as we observe. So, one might argue that it is no surprise that neutrino masses have been observed, since they represent the *leading* deviation from the SM, in terms of the operator expansion. Given the observed 10^{-3} eV^2 mass-squared differences of the neutrinos, we estimate $\Lambda \sim 10^{14}$ GeV. Thus, one could argue that while neutrino masses are undeniably, as one so often hears, evidence for physics BSM, they are also evidence that the SM is valid up to energy scales that are way, way beyond the reach of conceivable future colliders.

Even so, it is worthwhile to consider what theory might replace the EFT at Λ to give a UV completion, extending the regime of validity. One extremely simple possibility is to add to the SM a new fermion, ν^c, that is a singlet under $SU(3) \times SU(2) \times U(1)$. In fact we need at least 2 of these to generate the two observed neutrino mass-squared differences, and it seems plausible that there are 3 – one for each SM family.

We may then replace the $D = 5$ operator with the renormalizable Yukawa term $\lambda^\nu l H^c \nu^c$ (which is a Dirac mass term for neutrinos after EWSB), along with the Majorana mass term $m^\nu \nu^c \nu^c$. This leads to the so-called 'see-saw' mechanism, about which you may have heard.

4.9 $D = 6$: trouble at t'mill

Once we get to $D = 6$, a whole slew of operators appear. These include operators that violate baryon and lepton number, such as $\frac{qqql}{\Lambda^2}$ and $\frac{u^c u^c d^c e^c}{\Lambda^2}$ and which cause the proton to decay via $p \to e^+ \pi^0$. We can estimate a lower bound on Λ from the experimental bounds on the proton lifetime, $\tau_p > 10^{33}$ yr, as follows. The decay rate (which comes from the amplitude squared) is proportional to $\frac{1}{\Lambda^4}$ and the remaining dimensions must be supplied by phase space, giving a factor of m_p^5. Plugging in the numbers, we get $\Lambda > 10^{15}$ GeV. Again, the implication is that new physics either respects baryon or lepton number, or is a long way away.

There are also operators that give corrections to flavour-changing processes that are highly suppressed in the SM, because of the GIM mechanism. As an example, the operator $(s^c d)(d^c s)/\Lambda^2$ contributes to Kaon mixing and measurements of Δm_K and ϵ_K yield a bound of $\Lambda > 10^5$ TeV.

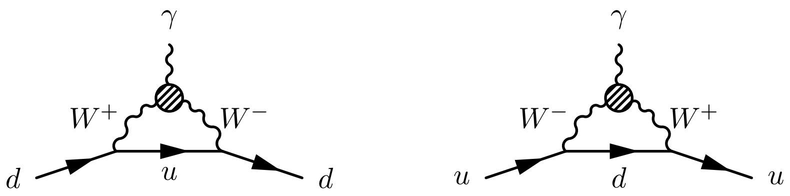

Figure 1. One-loop contributions of $\mathcal{O}_W$ (shaded blob) to the neutron EDM.

4.10 Two pitfalls

The SM affords a wonderful example of what goes wrong if one doesn't regularize using a mass-independent scheme. Consider the dimension 6 operator $\mathcal{O}_W \propto \frac{i\epsilon^{abc}}{3!} W^{a\mu}_{\ \ \ \nu} W^{b\nu}_{\ \ \ \lambda} \tilde{W}^{c\lambda}_{\ \ \ \mu}$. This operator violates CP and thus may be relevant for baryogenesis, so it is of interest to ask what the bound on its coefficient is. Now, the operator $\mathcal{O}_W$ contributes to the electric dipole moment of the neutron at one-loop, via the diagrams shown in Figure 1. Five sets of authors attempted this calculation in the literature, obtaining five different results, mostly because the authors were using a variety of regularization schemes. One set of authors even showed that essentially any answer could be obtained by a suitable choice of regularization! We know, of course, that only results obtained using a mass-independent regulator are reliable. In fact, this historical example affords us yet another illustration of a classic pitfall. Since the SM $SU(2) \times U(1)$ gauge invariance is broken in the vacuum, some authors have tried to argue that the correct way to write the EFT expansion is in terms of operators that respect only the unbroken subgroup of electromagnetism. In this example, once can write not only $\mathcal{O}_W$, but also an arbitrary superposition of the two electromagnetic invariants $\mathcal{O}_Z \equiv\equiv W^{+\mu}_{\ \ \ \nu} W^{-\nu}_{\ \ \ \lambda} \tilde{Z}^{\lambda}_{\mu}$ and $\mathcal{O}_\gamma \equiv W^{+\mu}_{\ \ \ \nu} W^{-\nu}_{\ \ \ \lambda} \tilde{F}^{\lambda}_{\mu}$. But it is easy to show that if the coefficients are proportional to $\frac{1}{\Lambda'^2}$, then the real cut-off of the EFT is not Λ', but rather is $\sqrt{v\Lambda'}$. This is completely obvious if we work in a manifestly $SU(2) \times U(1)$-invariant formalism, where the same physics can be described by including the dimension *eight* operator $H^\dagger W^\mu_\nu W^\nu_\lambda H \tilde{B}^\lambda_\mu$. See [10] for details.

5 Second Example: Non-linear sigma models and the composite Higgs

We have already argued that there is a basic problem with our canonical example of scalar field theory as an EFT: it contains a relevant operator, ϕ^2, requiring either an unnatural fine tuning of the parameters, or a breakdown of the EFT at low energies.

It turns out that it is possible to forbid this operator, and make a consistent EFT of scalar fields by means of additional symmetries, albeit in a non-trivial way. The trick is to make the scalar field a Goldstone boson.

You have probably encountered Goldstone bosons before in QFT in the context of 'spontaneous symmetry breaking'. This is a bad misnomer, because if the symmetry really were broken, we could not use it to forbid operators (like the mass term) in the lagrangian.

It is better to say that the symmetry is *non-linearly realized* in the vacuum. Let's do it properly (see [11] for more details).

Along the way, I'll illustrate the general results in the context of a specific example, called the minimal composite Higgs model (MCHM) [12].[12] This is one of the leading candidates for solving the electroweak hierarchy problem. For more details, see [15].

A general EFT theory of Goldstone bosons is called a *non-linear sigma model*. We suppose that there is a physical system with dynamics invariant under a continous (Lie) symmetry group G, but such that the ground state is invariant only under a proper subgroup $H \subset G$. Thus, if we act with an element $h \in H$ on the ground state, we get it back again. But if we act with a $U \in G$ but $U \notin H$, then we must obtain a different state. But this state must also be a ground state, because the dynamics is invariant under G. Thus the theory has a space of degenerate, inequivalent ground states.

In the MCHM, $G = SO(5)$ and $H = SO(4)$. $SO(n)$ is the group of $n \times n$ orthogonal matrices with unit determinant. It's Lie algebra is the vector space of $n \times n$ traceless, imaginary, Hermitian matrices. There are $\frac{n(n-1)}{2}$ such matrices and so $SO(5)$ is 10 dimensional and $SO(4)$ is 6 dimensional.

5.1 The coset space G/H of inequivalent ground states

How can we parameterise the space of ground states? Start with some ground state, Φ_0, pick two elements U and U' of G and consider the states $U\Phi_0$ and $U'\Phi_0$. Clearly these will be the same state if we can write $U' = Uh$, with $h \in H$, since we know that $h\Phi_0 = \Phi_0$. At this point it is useful to define an equivalence relation[13] by $U \sim U'$ if $\exists h \in H$ s. t. $U' = Uh$. The equivalence classes are called the *left cosets* of H in G, and there is one of them for every inequivalent ground state.

Now we can try to parameterize the space of cosets. A nice way to do so is to choose an orthonormal basis $\{T^{\tilde{a}}, X^a\}$ for the Lie algebra of G, such that $\{T^{\tilde{a}}\}$ are a basis for the Lie algebra of H. We may then parameterise the cosets (and hence the vacua) by $U = e^{i\phi^a X^a}$.

In the MCHM, a suitable basis for $\{T^{\tilde{a}},\}$ is any set of linearly independent traceless, imaginary, Hermitian matrices with zeros in the fifth row and column. A particularly convenient choice is

$$T^{\tilde{a}_L} = \frac{i}{2}\left[\frac{1}{2}\epsilon^{\tilde{a}bc}(\delta_i^b\delta_j^c - \delta_j^b\delta_i^c) + (\delta_i^{\tilde{a}}\delta_j^4 - \delta_j^{\tilde{a}}\delta_i^4)\right] \tag{5.1}$$

$$T^{\tilde{a}_R} = \frac{i}{2}\left[\frac{1}{2}\epsilon^{\tilde{a}bc}(\delta_i^b\delta_j^c - \delta_j^b\delta_i^c) - (\delta_i^{\tilde{a}}\delta_j^4 - \delta_j^{\tilde{a}}\delta_i^4)\right] \tag{5.2}$$

This choice is convenient, because when we work out the Lie brackets, we find that the $T^{\tilde{a}_L}$ and $T^{\tilde{a}_R}$ form two independent copies of the $SU(2)$ algebra. We thus learn that, at least at the level of the Lie algebra, the group $SO(4)$ is equivalent to $SU(2) \times SU(2)$. We say that they are *locally isomorphic*. This is important, because you will recall from (**??**) than the

[12] For less minimal models, see [13, 14].

[13] A relation $\sim$ between pairs of elements of a set $\{a, b, c, \dots\}$ is called an equivalence relation if (i) $a \sim a$, (ii) $a \sim b \implies b \sim a$, and (iii) $a \sim b$ and $b \sim c \implies a \sim c$. It then follows that the sets of elements that are equivalent to each other, called the equivalence classes, partition the original set.

SM has an approximate accidental custodial symmetry $SU(2) \times SU(2)$, which we would like to build in to any theory beyond the SM.

A suitable basis for $\{X^a\}$ are the matrices

$$X^a = -\frac{i}{\sqrt{2}} \left[(\delta_i^a \delta_j^5 - \delta_j^a \delta_i^5) \right] \tag{5.3}$$

Note that there are 10-6=4 linearly independent matrices and that this is also the number of scalar fields ϕ^a in the theory.

Why are we making such a big effort to parameterize the inequivalent ground states of the theory? Suppose we now promote the parameters ϕ^a to spacetime fields $\phi^a(x)$. $\phi^a(x)$ = constant corresponds to a ground state, but by making $\phi^a(x)$ vary arbitrarily slowly in spacetime, we obtain an excitation of the theory that is arbitrarily close to the ground state, and hence has abritrarily small energy. We can now try to build an EFT for these low-energy excitations.

How do we build the EFT? Clearly the appropriate degrees of freedom are the fields $\phi^a(x)$ and the appropriate symmetry is G, but how does it act on the fields $\phi^a(x)$? Under a G transformation with $\Omega \in G$, we know that a ground state Φ transforms to $\Omega\Phi$. But every ground state Φ can be written as $U\Phi_0$ with $U = e^{i\phi^a X^a}$. Thus we have that $e^{i\phi^a X^a}\Phi_0 \mapsto e^{i\phi'^a X^a}\Phi_0 \equiv \Omega e^{i\phi^a X^a}\Phi_0$. Now, here we must be careful. It is tempting to conclude that the appropriate transformation law is $e^{i\phi'^a X^a} = \Omega e^{i\phi^a X^a}$, but this is not so. Whilst we know that $\Omega e^{i\phi^a X^a}$ is an element of G, we do not know that we can write it in the form $e^{i\phi'^a X^a}$! In general it will take the form $e^{i(\phi'^a X^a + \psi^{\tilde{a}} T^{\tilde{a}})}$. But there is an easy fix. Since $h\Phi_0 = \Phi_0$ for any $h \in H$, we also have that $e^{i\phi^a X^a}\Phi_0 \mapsto e^{i\phi'^a X^a}\Phi_0 \equiv \Omega e^{i\phi^a X^a} h\Phi_0$ and by choosing a suitable h, we can remove the piece $e^{i\psi^{\tilde{a}} T^{\tilde{a}}}$. Note that the required h will depend on both U and Ω.

To summarise, the action of an element Ω of the symmetry group G on the fields $U(x)$ is given by

$$U(x) \mapsto \Omega U(x) h(\Omega, U(x)). \tag{5.4}$$

Note that this is a non-linear transformation (because of the dependence of h on U), which is why we say that the symmetry G is non-linearly realized on the fields $U(x)$.

5.2 Building the EFT lagrangian

We now want to build the most general action for the EFT, consistent with the G symmetry. This looks like a formidable task, because of the complicated, non-linear way (5.4) in which the fields $U(x)$ transform. But again there is a trick, which is to first build objects that transform only under the subgroup H. To do so, consider the object $U^{-1}\partial_\mu U$. Evidently, since Ω is constant, this transforms as

$$U^{-1}\partial_\mu U \mapsto h^{-1}(U^{-1}\partial_\mu U)h + h^{-1}\partial_\mu h. \tag{5.5}$$

Notice that the dependence on Ω has disappeared. Now, $U^{-1}\partial_\mu U$ and the thing into which it transforms take values in the Lie algebra of G. Thus we can decompose them in our basis

$\{T^{\tilde{a}}, X^a\}$ for the Lie algebra. We then have that[14]

$$U^{-1}\partial_\mu U = (U^{-1}\partial_\mu U)_H + (U^{-1}\partial_\mu U)_X \tag{5.6}$$

and we can decompose the transformation law as

$$(U^{-1}\partial_\mu U)_X \mapsto h^{-1}(U^{-1}\partial_\mu U)_X h, \tag{5.7}$$

$$(U^{-1}\partial_\mu U)_H \mapsto h^{-1}(U^{-1}\partial_\mu U)_H h + h^{-1}\partial_\mu h. \tag{5.8}$$

These two pieces are more transparent: $(U^{-1}\partial_\mu U)_X$ is an object that transforms homegeneously under H, while $(U^{-1}\partial_\mu U)_H$ transforms like a covariant derivative under H.

We can now start to build invariants out of the coset fields using $(U^{-1}\partial_\mu U)_X$ in the following way. We first note that the fields ϕ^a actually transform as a representation under H. What representation? Well, the elements of the Lie algebra of G carry the adjoint representation of G in general. But the fields ϕ^a only transform under the subgroup $H \subset G$, so we should first decompose the adjoint representation of G into its irreducible representations (irreps) under H. Finally, the fields ϕ^a are projections on the subspace of the Lie algebra that is orthogonal to the Lie algebra of H and so we should remove the irrep that corresponds to the adjoint irrep of H. We are left with the rep, R, under which $(U^{-1}\partial_\mu U)_X$ transforms. Whenever the tensor product of n copies of R contains a singlet, we can write an invariant in the action involving n copies of $(U^{-1}\partial_\mu U)_X$.

In particular, it is a theorem that by taking just two copies of R, we can form a singlet for every real irrep that is contained in R.

For the MCHM, the adjoint rep of $SO(5)$ is 10 dimensional (the same as the dimension of the Lie algebra). Under the $SO(4) \simeq SU(2) \times SU(2)$ subgroup it decomposes as $\mathbf{10} \to (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{2})$. But $(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$ is just the adjoint rep of $SO(4) \simeq SU(2) \times SU(2)$, so we see that the 4 fields $\phi^a(x)$ must transform as a $(\mathbf{2},\mathbf{2})$ of $SU(2) \times SU(2)$. This is precisely how the Higgs field of the SM transforms under the custodial $SU(2))L \times SU(2)_R$ symmetry (**??**), and so the Goldstone bosons of the $SO(5)/SO(4)$ non-linear sigma model have just the representation to play the role of the SM Higgs!

So there is just one irrep in this case, and we can form just one singlet that is quadratic in derivatives. It takes the form

$$-f^2 \mathrm{tr}(U^{-1}\partial_\mu U)_X^2 = \frac{1}{2}\partial_\mu h^a \partial^\mu h^a + \dots \tag{5.9}$$

where we have now written $U = e^{ih^a X^a/f}$, including a dimensionful scale f so that the Higgs field has the canonical unit dimension of a scalar field in 4-d. At leading order, we get precisely the kinetic terms of the Higgs field in the SM. But at higher order we get terms with two derivatives and higher powers of Higgs fields. These are, of course, non-renormalizable, but we don't care any more, because we are doing EFT.

Note that we can also put in terms with more derivatives, by taking more copies of $(U^{-1}\partial_\mu U)_X$. Each factor adds one more derivative and in the EFT spirit that we expect

[14]Note that, even though $U = e^{i\phi^a X^a}$, it does not follow that $(U^{-1}\partial_\mu U)_H = 0$, because the generators $\{X^a\}$ do not close into themselves under the Lie bracket operation.

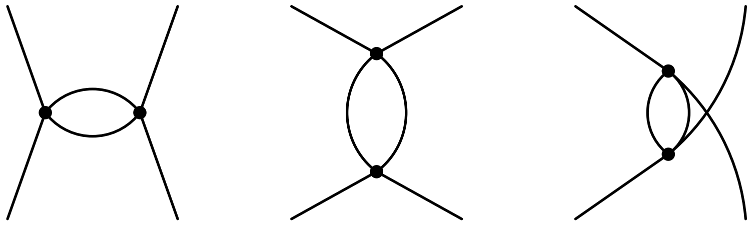

Figure 2. One-loop contributions to $hh \to hh$.

all operators to become equally important at the cut-off, they should be accompanied by a factor of the cut-off Λ, we thus get that

$$\mathcal{L} \sim f^2 (U^{-1}\partial_\mu U)^2_X + \frac{f^2}{\Lambda^2}(U^{-1}\partial_\mu U)^4_X + \dots \tag{5.10}$$

5.3 Estimate of the cut-off scale

Now, f and Λ are both dimensionful scales in the theory. We have already seen that non-renormalizable terms involving extra powers of the scalar fields are suppressed by powers of f, and so it must be that f is related to the cut-off Λ, somehow. We shall now argue that it is unreasonable to suppose that Λ is much greater than $4\pi f$. The argument goes as follows (see [**?**] for more details). The leading order term in (5.10) contains a quartic interaction that goes like (Fourier transforming to momentum space)

$$\frac{p^2 h^4}{f^2}. \tag{5.11}$$

Consider the 3 one-loop diagrams in Figure 2, contributing to $hh \to hh$, with two insertions of this vertex. By dimensional analysis, the loop integral naïvely goes like

$$\int d^4 k \frac{k^2 k^2}{f^4 k^2 k^2}, \tag{5.12}$$

which is quartically divergent. However, the group theory factors must be such that this contribution gives zero when summed over the 3 diagrams, because such a divergence would have to be cancelled by a counterterm of the form h^4 with no derivatives, but this is not allowed by the symmetry. When one works it out carefully, one finds that the contribution is indeed zero by the Jacobi identity. There is also a sub-leading piece which contains two powers of the external momenta p and goes like

$$p^2 \int d^4 k \frac{k^2}{f^4 k^2 k^2}. \tag{5.13}$$

This is quadratically divergent, but the divergence can be absorbed by the term (5.15) itself.

Finally, there is a logarithmically divergent piece of size

$$\frac{p^4}{f^4 (4\pi)^2} \log \mu, \tag{5.14}$$

where the 4π comes from the integration over a hypersphere. We get similar contributions at tree-level from a piece

$$\frac{p^4 h^4}{\Lambda^2 f^2}. \tag{5.15}$$

coming from the second term in (5.10). Now, it cannot be the case that $\Lambda \gg 4\pi f$, because if this were true for one choice of renormalization scale, if would not be true for another that differed by $O(1)$. Thus we conclude that $\Lambda \lesssim 4\pi f$.

5.4 Pseudo-Goldstone bosons

So far, we have built a consistent EFT of Goldstone bosons, in which the usual problematic mass term operators are forbidden by non-linearly realized symmetries. We have also found a specific model in which the Goldstone bosons transform as a $(\mathbf{2},\mathbf{2})$ of an $SU(2)\times SU(2)$ symmetry, just like the Higgs field of the SM.

But we are still rather a long way from a model that can describe Nature. Indeed, although we currently know rather little about the Higgs boson, we do know that it is rather a long way from being a Goldstone boson! It has a mass of 125 GeV, and it couples to gauge fields and to SM fermions. Our Goldstone bosons have none of these features, being massless, and coupled only to themselves, via derivative interactions.

To see how to solve these problems, we start by noting that there is no way that $SO(4) \simeq SU(2)\times SU(2)$, let alone $SO(5)$, can be an exact symmetry of Nature. We already know, for example, that the custodial $SU(2)\times SU(2)$ is only approximate, being broken both by Yukawa interactions and by the gauging of the hypercharge. But if $SU(2)\times SU(2)$ (and $SO(5)$) are only approximate, then the Goldstone bosons of the $SO(5)/SO(4)$ model will only approximately be Goldstone bosons and will only approximately be massless, etc. They will, to use the lingo, become *pseudo-Goldstone bosons*.

So the question is: can we somehow break $SO(4)$ and $SO(5)$ in a small way, by introducing gauge interactions and couplings of the Goldstone bosons to fermions, and thus end up with something much closer to the SM?

The answer is: Yes, we can! I am going to show you how to do properly for the gauge interactions only, and sketch how it goes for the couplings to fermions. This sounds like a bit of a cop out, but I really am going to do it properly for the gauge couplings, and indeed we will obtain a result which cannot be found elsewhere in the literature.

So, let us attempt the following. Starting with the $SO(5)/SO(4)$ non-linear sigma model, we will try to gauge the $SU(2)_L\times U(1)_Y$ subgroup of $SO(4)\simeq SU(2)_L\times SU(2)_R$, where $Y = T^3_R$. We expect that, as a result, the Goldstone bosons will acquire a potential (like the SM Higgs) and we shall derive its general form.

We will do this by using the trick of spurions. Specifically, suppose we wish to gauge a subgroup K of the group G. (In the MCHM, $K = SU(2)_L\times U(1)_Y$.) We will start by pretending that K is not a subgroup of G, but rather is separate, so the full theory has $K\times G$ invariance, where K is a local symmetry (meaning that we have a gauge field for it) and G is the global symmetry of the sigma model. We will then introduce a spurionic field $g^{A\alpha}$ (which we call the *gauge coupling spurion*), which transforms as an adjoint under

K (with index α) and as an adjoint under G (with index A). We will declare that in the vacuum, $g^{A\alpha}$ has expectation value given by[15]

$$\langle g^{A\alpha} \rangle = g\delta^{A\alpha}. \tag{5.16}$$

Now we can see how to write down a potential for the Goldstone bosons. Under G, the field $g^{A\alpha}$ transforms as an adjoint. This is conveniently expressed by defining $g^{\alpha} = g^{A\alpha}T^{A}$, s. t. the transformation law is $g^{\alpha} \mapsto \Omega g^{\alpha}\Omega^{-1}$. We now observe that the object $\tilde{g}^{\alpha} \equiv U^{-1}g^{\alpha}U \mapsto h^{-1}\tilde{g}^{\alpha}h$ and transforms not under G, but under H, so we can easily build invariants from it!

Now, $\tilde{g}^{\alpha}$ is an adjoint of G, so to see how it decomposes under H, we just need to do the decomposition of the adjoint of G under H. For the MHCM, we get $\mathbf{10} \to (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{2})$.

The object $\tilde{g}^{\alpha}$ still transforms under K, but we can get a K invariant by forming the quadratic object $\tilde{g}^{\alpha}\tilde{g}^{\alpha}$. This transforms as the product of two adjoints of G and we already know that we can get one H-invariant for each real irrep of H that appears in the decomposition of the adjoint of G. However, the sum of all these terms is just the trace of $\tilde{g}^{\alpha}\tilde{g}^{\alpha}$, which is a constant, independent of the PGBs. Thus, we obtain our final result, which is that *the number of independent potential terms is one fewer than the number of real irreps of H in the adjoint rep of G.* For the MCHM, there are 3 real irreps, *viz.* $(\mathbf{3}, \mathbf{1}), (\mathbf{1}, \mathbf{3})$, and $(\mathbf{2}, \mathbf{2})$ and hence 2 independent terms in the potential. I compute them in the Appendix. They are

$$V(h) = 2A(3g^2 \cos^4 \frac{h}{2f} + g'^2 \sin^4 \frac{h}{2f}) + 2B(3g^2 \sin^4 \frac{h}{2f} + g'^2 \cos^4 \frac{h}{2f}), \tag{5.17}$$

where, as always, A and B are arbitrary parameters in the EFT, to be fixed by measurements.

A few remarks now follow. Firstly, note that these potential terms depend quadratically on the gauge couplings. They are thus the dominant contributions for small couplings, corresponding to a weak breaking of the G symmetry.

Secondly, there is a variant of the MCHM in which the $SO(4)$ symmetry is enlarged to $O(4)$, so as to protect the theory from overly large contributions to the decay rate for $Z \to b\bar{b}$ [16]. The reducible rep $(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$ of $SO(4)$ is actually an irrep of $O(4)$ [17] and so in this case there is just a single potential term, given by

$$V(h) = A(3g^2 + g'^2) \sin^2 \frac{h}{f}. \tag{5.18}$$

This is the expression that you will find everywhere in the literature, even for the $SO(4)$ case.

[15] In fact, we can choose a different constant of proportion, g, for each simple factor in K, but we ignore this subtlety for now. We shall need it later, however, because $SU(2)_L \times U(1)_Y$ has two simple factors.

5.5 Composite Higgs

We are still quite a long way from a realistic composite Higgs model. For example, with $A, B > 0$ in (5.17), we have a minimum at the origin, and so we can't break the electroweak symmetry as needed.

This can be fixed though, once we add another source of breaking by coupling the pseudo-Goldstone bosons to fermions. These coupings must be present, because we know that the Higgs (which is here part of the strongly coupled sector) couples to fermions (and gives them mass after EWSB). There are two ways in which we can imagine the couplings arising. The first is much like the SM Yukawa couplings, in that the strong sector couples to fermion bi-linears. Schematically,

$$\mathcal{L} \supset \frac{q\mathcal{O}_h u^c}{\Lambda^{d-1}} + \dots, \tag{5.19}$$

where $\mathcal{O}_h$ is some operator in the strong sector of arbitrary dimension d with the right quantum numbers to couple to SM fermions.

However, to this EFT lagrangian we should also add other operators that are compatible with the symmetries of the theory. Amongst these are

$$\mathcal{L} \supset \frac{qqqq}{\Lambda^2} + \Lambda^{4-d'}\mathcal{O}_h^\dagger \mathcal{O}_h. \tag{5.20}$$

The first of these is responsible for flavour changing neutral currents; for these to be small enough, $\Lambda > 10^{3-5}$ TeV. But then, in order to get a mass as large as that of the top from the operator in (5.19), we need to choose d to be rather small: $d \lesssim 1.2 - 1.3$ [18]. Next, we need to worry about the second operator in (5.20). In order not to de-stabilize the hierarchy, its dimension, d', had better be greater than four, rendering it irrelevant.[16] So what is the problem? The limit in which $d \to 1$ corresponds to a free theory (for which the operator $\mathcal{O}_h$ is just the Higgs field h), and in that limit $d' \to 2d \to 2$. So in order to have an acceptable theory, we need a theory containing a scalar operator $\mathcal{O}_h$ (with the right charges) with a dimension that is close to the free limit, but such that the theory is nevertheless genuinely strongly-coupled, with the dimension of $\mathcal{O}_h^\dagger \mathcal{O}_h$ greater than four. We have very good evidence that such a theory cannot exist [19].

In the other approach, we imagine that the elementary fermions couple linearly to fermionic operators of the strong sector [20]. Schematically, the lagrangian is

$$\mathcal{L} \sim q\mathcal{O}_{q^c} + u^c\mathcal{O}_u + \mathcal{O}_{q^c}\mathcal{O}_q + \mathcal{O}_{u^c}\mathcal{O}_u + \mathcal{O}_{q^c}\mathcal{O}_H\mathcal{O}_u \tag{5.21}$$

(where I have left out the Λs) and the light fermion masses arise by mixing with heavy fermionic resonances of the strong sector, which feel the electroweak symmetry breaking. The beauty of this mechanism is that fermion masses can now be generated by relevant operators (*cf.* the operator that generates masses in (5.19), which is at best marginal, since $d > 1$); this means that one can, in principle, send Λ to infinity and the problems with

[16] It is, perhaps, instructive to see how the hierarchy problem of the SM is cast in this language. There, $\mathcal{O}_h$ corresponds to the Higgs field h, with dimension close to unity, whilst $\mathcal{O}_h^\dagger \mathcal{O}_h$ is the Higgs mass operator, with dimension close to 2.

flavour physics can be completely decoupled. There is even a further bonus, in that the light fermions of the first and second generations, which are the ones that flavour physics experiments have most stringently probed, are the ones that are least mixed with the strong sector and the flavour-changing physics that lies therein. In this model, the observed SM fermions are mixtures of elementary and composite fermions, with the lightest fermions being mostly elementary, and the top quark mostly composite. The scenario therefore goes by the name of *partial compositeness.*

It turns out (see, *e. g.*, [15]) that the fermions can give negative corrections to the mass-squared in the Higgs potential, and thus result in EWSB. Since the top quark Yukawa is somewhat bigger than the gauge couplings, this is (at least naïvely) the most likely outcome.

We now have something approaching a realistic model of EWSB via strong dynamics. Having built it up, we should now do our best to knock it down.

A first problem is that no one actually knows how to get a pattern of $SO(5) \to SO(4)$ global symmetry breaking out of an explicit strongly-coupled gauge theory coupled to fermions.[17]

A second problem is the S-parameter. We have argued that the necessary suppression can be obtained if v turns out to be somewhat smaller than f, the scale of strong dynamics. Well, v is obtained by minimizing the Higgs potential $V(h)$, which contains contributions of very roughly equal size, but opposite in sign, from the top quark and gauge bosons. Thus it is possible to imagine that there is a slight cancellation due to an accident of the particular strong dynamics, such that the v that emerges is small enough. A measure of the required tuning is $\frac{v^2}{f^2}$, and the observed S-parameter requires tuning at the level of ten *per cent* or so.

The third problem concerns flavour physics. To argue, as we have done above, that the flavour problem can be decoupled, is not the same as arguing that it *is* solved. To do that, one needs to find an explicit model which possesses all the required operators, with the right dimensions. Needless to say, our ignorance of strongly-coupled dynamics means we have no idea whether such a model exists. Certainly, in all cases that have been studied (either models with large rank of the gauge group, or lattice studies), there *is* a problem with flavour constraints.

Despite these problems, composite Higgs models seem just as good (or just as bad) as solutions to the hierarchy problem as supersymmetric models, and so they deserve thorough investigation at the LHC. This itself is not so easy to do. Naïvely, the obvious place to look for deviations is in the Higgs sector itself, for example in the couplings of the Higgs boson to other particles. However, we know that (since such models reproduce the SM in the limit $v^2/f^2 \to 0$) the deviations must be proportional to v^2/f^2 and hence at most 10 % or so. Such deviations are hard to see at the LHC, and even at a future e^+e^- collider. Perhaps a better way is to look for the composite partners of the top quark, which must be not too heavy in order to reproduce the observed Higgs mass. Many suggestions for how to do so have been put forward and the experiments are beginning to implement them. See,

[17] The breaking $SO(6) \to SO(5)$ [13] is easier to achieve, since $SO(6) \simeq SU(4)$, and unitary groups are easier to obtain.

e.g. [17, 21] and refs. therein for more details.

6 Third example: The quantum theory of fluids

In this lecture, I describe a rather different EFT, namely that of a perfect fluid[18] [22]. This is of interest in its own right, since classical fluid phenomena are among the most rich and fascinating in Nature. We will show that the quantum EFT based on the same degrees of freedom and symmetries is a sensible theory. Presumably, the quantum phenomena of this theory are even more fascinating than those of a classical fluid, and so it is of interest to explore the predicitions of the theory and search for evidence of systems that behave in this way in Nature.

The fluid EFT is also of interest because it is a rather non-trivial example of an EFT. In one sense, it is just a theory of Goldstone bosons like the non-linear sigma models we discussed in the last lecture. But it is more complicated, because the symmetry group is infinite-dimensional and because the non-linearly realized symmetries include spacetime symmetries. In particular, Lorentz invariance is non-linearly realized in the ground state and so we must take care in formulating the EFT.

6.1 Parameterization of a perfect fluid

We begin by discussing how to parameterize a fluid and its dynamics. Let the fluid occupy some spatial manifold M (e.g. $\mathbb{R}^2$) and choose some co-ordinates x^i thereon. At $t = 0$, we can label each fluid particle by the co-ordinates of the point in M that it occupies. Call these Eulerian co-ordinates, ϕ^i. As time evolves, the fluid particle will move around in M and we can denote its position at time t by $x^i(\phi^j, t)$. Alternatively (assuming the map is invertible, which requires that the fluid does not cavitate or interpenetrate), then we can also describe the fluid's configuration by the map $\phi^i(x^j, t)$. We choose to think about things this way, since we can then think of the ϕ^i as 2 scalar fields living in spacetime (x^j, t).

Note that the classical ground state corresponds to each fluid particle sitting at rest. So the classical ground state is given by $\phi^i = x^i$. Later, it will be useful to consider small fluctuations about the classical ground state, which we write as $\phi^i = x^i + \pi^i(x^j, t)$. Again, the π^i can be thought of as 2 scalar fields on spacetime.

6.2 Action principle and classical fluid dynamics

We have now identified the degrees of freedom for the EFT. We next wish to identify the symmetries. We do this essentially by guessing and showing that the resulting action reproduces the behaviour of a perfect fluid in the classical limit.

The action of a fluid has been known for a long time [23], but it is hard to find in the fluid mechanics textbooks,[19] where it is usual to derive the fluid equations of motion from conservation of energy and momentum.

[18] We take the fluid to be perfect because otherwise we expect to see dissipative or viscous behaviour. But the quantum theory would then presumably be non-unitary.

[19] One place you can find it is in [24].

The action is the most general one consistent with the following symmetries. Firstly, we require that the action be invariant under Poincaré transformations of x. This is because we expect the underlying dynamics of a fluid to be Poincaré invariant (though its ground state, with the fluid sat still in some frame, is not!). Secondly, we require that the action be invariant under area-preserving diffeomorphisms of the co-ordinates ϕ. This is because such transformations simply correspond to different labellings of the fluid particles.

At leading order, the lagrangian (in 2+1-d spacetime) can then be written as[20]

$$\mathcal{L} = -w_0 f(\sqrt{B}), \tag{6.1}$$

where $B = \det \partial_\mu \phi^i \partial^\mu \phi^j$, f is any function s. t. $f'(1) = 1$, and w_0 sets the overall dimension. It is easy to check that B is indeed invariant under the desired symmetries, and therefore so is $f(\sqrt{B})$.

Since the theory is invariant under spacetime translations, Noether's theorem tells us that there is a conserved energy-momentum tensor. It may be written as

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p\eta_{\mu\nu}, \tag{6.2}$$

where

$$\rho = w_0 f, \tag{6.3}$$

$$p = w_0(\sqrt{B} f' - f), \tag{6.4}$$

$$u^\mu = \frac{1}{2\sqrt{B}} \epsilon^{\mu\alpha\beta} \epsilon_{ij} \partial_\alpha \phi^i \partial_\beta \phi^j. \tag{6.5}$$

Eq. (6.2) is, of course, the standard form for the energy-momentum tensor of a fluid with density ρ, pressure p and 3-velocity u^μ of the fluid particles. Thus our action does indeed reproduce a classical fluid. Note that different choices for the function f lead to different relations between the pressure and density of the fluid, meaning that we are able to describe a fluid with an arbitrary equation of state. Note also that these physical quantities are invariant under the area-preserving diffeomorphisms of ϕ, which correspond to physically-equivalent relabellings of the fluid particles.

6.3 Effective field theory: IR divergences

Let's now try to study small fluctuations about the classical ground state and see if we can make a consistent EFT. So, expanding $\phi^i = x^i + \pi^i$, we get

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\dot{\pi}^2 - c^2[\partial\pi]^2) - \frac{(3c^2 + f_3)}{6}[\partial\pi]^3 + \frac{c^2}{2}[\partial\pi][\partial\pi^2] + \frac{(c^2+1)}{2}[\partial\pi]\dot{\pi}^2 - \dot{\pi}\cdot\partial\pi\cdot\dot{\pi} \\ &- \frac{(f_4 + 3c^2 + 6f_3)}{24}[\partial\pi]^4 + \frac{(c^2+f_3)}{4}[\partial\pi]^2[\partial\pi^2] - \frac{c^2}{8}[\partial\pi^2]^2 + \frac{(1-c^2)}{8}\dot{\pi}^4 - c^2[\partial\pi]\dot{\pi}\cdot\partial\pi\cdot\dot{\pi} \\ &- \frac{(1-3c^2-f_3)}{4}[\partial\pi]^2\dot{\pi}^2 + \frac{(1-c^2)}{4}[\partial\pi^2]\dot{\pi}^2 + \frac{1}{2}\dot{\pi}\cdot\partial\pi\cdot\partial\pi^T\cdot\dot{\pi} + \dots,\end{aligned} \tag{6.6}$$

[20] Our metric is now mostly-plus. If you don't like it, sue me!

where $f_n \equiv d^n f/d\sqrt{B}^n|_{B=1}$, $c \equiv \sqrt{f_2}$, and $[\partial\pi]$ is the trace of the matrix $\partial^i \pi^j$, *&c.*

This is the sort of expression that is liable to give one a heart attack, so let's break it down and approach it bit by bit.

The first thing to notice is that all the terms have derivatives in them, either with respect to space or time or both. So this is a theory of Goldstone bosons, albeit a funny one.

Next, let's look at the quadratic piece in π. It is just

$$\frac{1}{2}(\dot{\pi}^2 - c^2[\partial\pi]^2). \tag{6.7}$$

What do we learn from this? Recall that there are two scalar degrees of freedom, π^i, with $i \in \{1, 2\}$. Suppose we choose a fluctuation mode with energy ω and wavevector $(k^1, k^2) = (k, 0)$. For the mode π^1, which is longitudinally polarised (it's in the same direction as $\vec{k}$), the lagrangian is just

$$\frac{1}{2}(\omega^2 - c^2k^2)(\pi^1)^2, \tag{6.8}$$

meaning that the dispersion relation for longitudinally polarised modes in the fluid is just $\omega^2(k) = c^2k^2$. Longitudinally polarised excitations of a fluid are called sound waves, and we learn that they have speed c. This is why we defined $c \equiv \sqrt{f_2}$ above.

For the mode π^2, which is transversely polarised (it's orthogonal to $\vec{k}$), we instead get

$$\frac{1}{2}\omega^2(\pi^2)^2, \tag{6.9}$$

meaning that the dispersion relation is just $\omega^2(k) = 0$. This looks a bit odd at first, but (at least classically) there is no problem. A transversely polarized small fluctuation is just an infinitesimal version of a fluid vortex. The dispersion relation says firstly that the energy of such vortices is independent of k that is independent of the size of the vortex. It also says that the energy of such a vortex is zero. Both of these statements make sense. Indeed (as you can check for yourself whilst sitting in the bath) it is possible to firstly make a vortex of arbitrary size with arbitrarily low energy, simple by placing ones hands the required distance apart and stirring the bathwater arbitrarily slowly.

So, classically, there is no problem. But there is a problem when we start trying to do EFT. In particular, the spacetime propagator for the transverse modes is given by $\int d\omega d^2k \; e^{i(\omega t + k\cdot x)}/\omega^2$ and this is undefined, because of the pole at $\omega = 0$. Note how this differs from normal scalar field theory, where the propagator is given by $\int d\omega d^2k \; e^{i(\omega t+k\cdot x)}/(\omega^2 + k^2)$, which is perfectly well-defined.

This obstruction to quantization was noted some time ago by Endlich *et al.* [25]. They tried to fix it up by adding a small sound speed c_T for the transverse modes, computing S-matrix elements, and then sending $c_T \to 0$. Unfortunately they found that everything they computed diverged as $c_T \to 0$.

In fact, it is not hard to see why this is the case. The fields π^i, just like the fields ϕ^i, are not physical. They correspond to arbitrary labellings of the fluid elements. We cannot, therefore, reasonably insist that correlation functions of them make sense. The correlation functions of them are, in fact, infra-red divergent (the propagator, for example, diverges because of the pole at $\omega = 0$) and this is a common feature in theories that are formulated in terms of unphysical degrees of freedom. The most obvious example occurs in gauge theories, where the gauge fields themselves are unphysical, and indeed we find IR divergences whenever we attempt to calculate correlation functions of the gauge fields. Another example arises in non-linear sigma models, exactly like those we studied in the last lecture, but in 2-d, where the propagator is $\int d\omega dk\ e^{i(\omega t + k\cdot x)}/(\omega^2 + k^2)$ and is also IR divergent. In these other theories, the solution to the problem of IR divergences is well known: they cancel when we compute correlation functions of physical quantities, such as gauge invariants in the case of gauge theories.

Does this work for the fluid as well? Indeed it does. There, the physical quantities are invariants under area-preserving diffeomorphisms, like the fluid's density, pressure, and 4-velocity. The latter is given in terms of the π fields, at leading order, by $u^i \propto \dot{\pi}^i$ and $u^0 \propto [\partial\pi]$ and it is almost trivial to check that the 2-point functions of these physical quantities are well behaved [22].

Indeed, we find that

$$\begin{aligned} \langle [\partial\pi][\partial\pi]\rangle &= \frac{ik^2}{\omega^2 - c^2k^2}, \\ \langle \dot{\pi}^i[\partial\pi]\rangle &= \frac{i\omega k^i}{\omega^2 - c^2k^2}, \\ \langle \dot{\pi}^i\dot{\pi}^j\rangle &= i\delta^{ij} + \frac{ic^2k^ik^j}{\omega^2 - c^2k^2}. \end{aligned} \tag{6.10}$$

The only poles are at $\omega = ck$ and the disappearance of poles at $\omega = 0$ implies that the spacetime Fourier transforms are well-defined.

The calculations for higher-point, tree-level correlation functions are much more involved, but the cancellations have been checked in a number of cases. See [22] for more details.

6.4 Effective field theory: UV divergences

Now we have got the IR divergences under control, we can look at the UV behaviour of the EFT. We would like to show that the EFT expansion makes sense, in that there is a regime of large distance and time scales (not necessarily the same, since the ground state is not Lorentz-invariant) in which the effects of higher dimension operators and loops are suppressed.

To check this, we compute the one-loop contribution to the 2-point function of the observable $\sqrt{B}u^0 - 1$. The diagrams, shown in Fig. 3, feature both IR and UV divergences, which we regularize by computing the integrals in $D = 1 + 2\epsilon$ time- and $d = 2 + 2\epsilon$ space-dimensions. We wish to show that the UV divergences can be absorbed in higher order counterterms and that the expansion in energy and momenta is valid in some non-vanishing region.

Fortunately, in the case at hand, we can be sure that the answer must be finite as $\epsilon \to 0$ (if the theory is consistent). This is because we can show by dimensional analysis

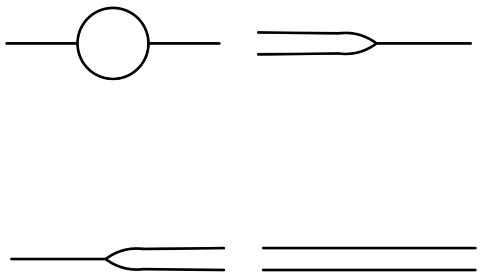

Figure 3. Diagrams for the correlator $\langle(\sqrt{B}u^0 - 1)(\sqrt{B}u^0 - 1)\rangle$.

that there can be no counterterms! Indeed the Feynman rules that follow from (6.6) imply that the 1-loop diagrams must contain 3 more powers of energy or momentum than the tree-level diagrams (because every π is always accompanied by a derivative). Now, since the correlator can only be a function of K^2 (where $icK \equiv \omega$) and k^2 (by time-reversal and rotation invariance, respectively), the 1-loop contribution necessarily contains radicals of K^2 and k^2. But higher order counterterms can only yield tree-level contributions that are rational functions of K^2 and k^2 and so cannot absorb divergences in the 1-loop contribution.

The actual computation is a pig to do, but the answer is pretty simple. One gets [22]

$$\begin{aligned}\frac{9Kk^6(1+c^4)}{64(K^2+k^2)^2} - \frac{k^4}{1024c^4(K^2+k^2)^{\frac{5}{2}}} \\ \times \Big[c^4(1-c^2)^2(19k^4 - 4K^2k^2 + K^4) \\ - 2f_3c^2(1+c^2)k^2(5k^2+14K^2) + f_3^2(3k^4+8K^2k^2+8K^4)\Big],\end{aligned}$$

which is indeed finite, as consistency demands. Moreover, there are no poles at $K = 0$ and the Fourier transform is well defined. Note that the only pole is at $K^2 + k^2 = 0 \implies \omega^2 = c^2k^2$, implying that the sound speed is not renormalized at one-loop.

We can use this formula to estimate the region of validity of the EFT expansion in energy-momentum, by comparing the absolute values of the tree-level and 1-loop results. The estimate depends on the values of the $O(1)$ coefficients c^2 and f_3; results for typical values (in units of the overall scale w_0) in shown in Fig. 4. As required, the loop contribution is always smaller than the tree-level one for large enough distances and times, suggesting that the EFT expansion does indeed make sense. Notice also that there is no suggestion that the locus of convergence of the EFT expansion is some simple curve, such as $\Lambda_\omega = c\Lambda_k$!

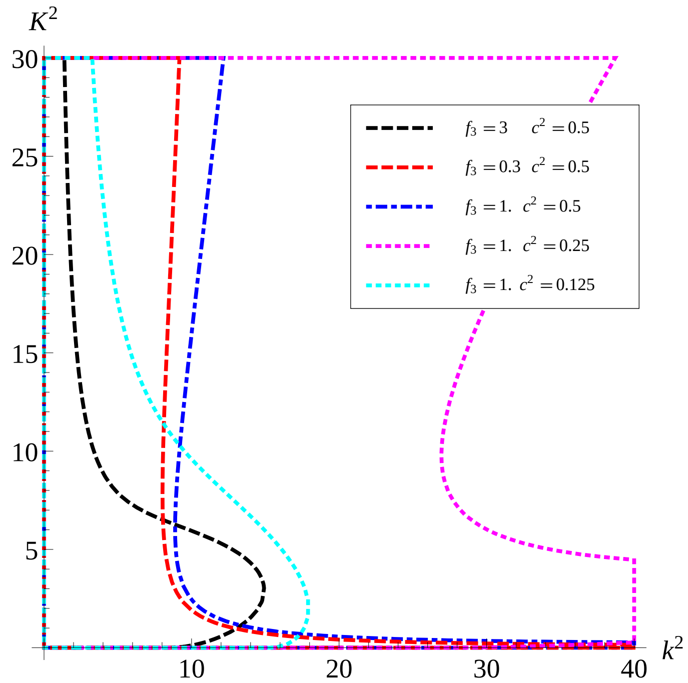

Figure 4. Contours of equal 1-loop and tree-level contributions to $\langle(\sqrt{B}u^0 - 1)(\sqrt{B}u^0 - 1)\rangle$, for various values of c and f_3. Figure reproduced from [22].

A Appendix: Potential terms in the minimal composite Higgs model

Here, we explicitly construct the potential invariants that arise from gauge the electroweak subgroup in the $SO(5)/SO(4)$ MCHM. We use the basis for $SO(5)$ generators given in [12]. We denote the generators in $H = SU(2)_L \times SU(2)_R$ as L^a, R^b and the remaining generators as X^c.

The coset representative is, in an obvious notation,

$$U = e^{i\sqrt{2}h\cdot X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin h}{h} \begin{pmatrix} 0 & h \\ -h^T & 0 \end{pmatrix} + \frac{\cos h - 1}{h^2} \begin{pmatrix} hh^T & 0 \\ 0 & h^2 \end{pmatrix}. \tag{A.1}$$

Now, the gauge coupling spurion is an adjoint of G (and K), so we may represent the G-action by $\Omega : g \to \Omega g \Omega^{-1}$. The combination $\tilde{g} \equiv U^{-1} g U$, then transforms (reducibly) under H alone.

As described in the main text, for each irreducible representation under $SO(4)$, we can build an $SO(5)$-invariant. To do so, we need to reduce $\tilde{g}$ into its components carrying irreps. of $SO(4)$. This is easy: $\tilde{g}$ is an element of the Lie algebra of $SO(5)$, so we may expand it as

$$\tilde{g}^\alpha = \lambda^{A\alpha} L^A + \rho^{B\alpha} R^B + \mu^{C\alpha} X^C \tag{A.2}$$

and the three irreps. of $SO(4)$ are carried precisely by the projections of $\tilde{g}$ onto the subalgebras corresponding to $SU(2)_L$ and $SU(2)_R$, together with their complement in $SO(5)$.

The projection itself is trivial, since our basis of generators for $SO(5)$ was chosen to be orthogonal with respect to the trace operation. Thus,

$$\lambda^{A\alpha} = \text{tr} L^A \tilde{g}^\alpha \tag{A.3}$$
$$\rho^{A\alpha} = \text{tr} R^A \tilde{g}^\alpha \tag{A.4}$$

each transform amongst themselves under $SO(4)$. (So, of course, does the projection onto the X subalgebra, but the sum of all three invariants is coset-independent.)

Let us now consider, as a first example, gauging the whole of $SO(4)$, but with different couplings, g_L and g_R, for the two simple subgroups. We label the generators of $K = SO(4)$ by $\{L'^\alpha, R'^\beta\}$, such that the VEV of the gauge coupling spurion may be written as

$$\langle g \rangle \equiv \langle g^\gamma T'^\gamma \rangle = g_L L'^\alpha \otimes L^\alpha + g_R R'^\beta \otimes R^\beta, \tag{A.5}$$

with

$$\langle g^\gamma \rangle = \begin{cases} g_L L^\alpha, \ \text{if} \gamma \in \{\alpha\} \\ g_R R^\beta, \ \text{if} \gamma \in \{\beta\} \\ 0, \ \text{else.} \end{cases} \tag{A.6}$$

Thus, we find that

$$\lambda^{A\alpha} = g_L \text{tr} L^A U^{-1} L'^\alpha U + g_R \text{tr} L^A U^{-1} R'^\alpha U \equiv \lambda^{A\alpha}_{L'} + \lambda^{A\alpha}_{R'} \tag{A.7}$$
$$\rho^{A\alpha} = g_L \text{tr} R^A U^{-1} L'^\alpha U + g_R \text{tr} R^A U^{-1} R'^\alpha U \equiv \rho^{A\alpha}_{L'} + \rho^{A\alpha}_{R'} \tag{A.8}$$

This is starting to look exceedingly unpleasant, but salvation comes in the form of a *deus ex Mathematica*:

$$\lambda^{A\alpha}_L \lambda^{A\beta}_L = g_L^2 \cos^4 \frac{h}{2} \delta^{\alpha\beta} \tag{A.9}$$
$$\lambda^{A\alpha}_R \lambda^{A\beta}_R = g_R^2 \sin^4 \frac{h}{2} \delta^{\alpha\beta} \tag{A.10}$$
$$\rho^{A\alpha}_L \rho^{A\beta}_L = g_L^2 \sin^4 \frac{h}{2} \delta^{\alpha\beta} \tag{A.11}$$
$$\rho^{A\alpha}_R \rho^{A\beta}_R = g_R^2 \cos^4 \frac{h}{2} \delta^{\alpha\beta} \tag{A.12}$$

The two invariants that can appear in the Higgs potential are then given by

$$\langle \lambda^{A\alpha} \lambda^{A\alpha} \rangle = \text{tr} \lambda \lambda^T \propto 3 g_L^2 \cos^4 \frac{h}{2} + 3 g_R^2 \sin^4 \frac{h}{2} \tag{A.13}$$
$$\langle \rho^{A\alpha} \rho^{A\alpha} \rangle = \text{tr} \rho \rho^T \propto 3 g_R^2 \cos^4 \frac{h}{2} + 3 g_L^2 \sin^4 \frac{h}{2} \tag{A.14}$$

A similar computation with

$$\langle g \rangle \equiv \langle g^\gamma T'^\gamma \rangle = g L'^\alpha \otimes L^\alpha + g' R'^3 \otimes R^3, \tag{A.15}$$

yields (5.17).

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