Appendix D: Interaction via Particle Exchange

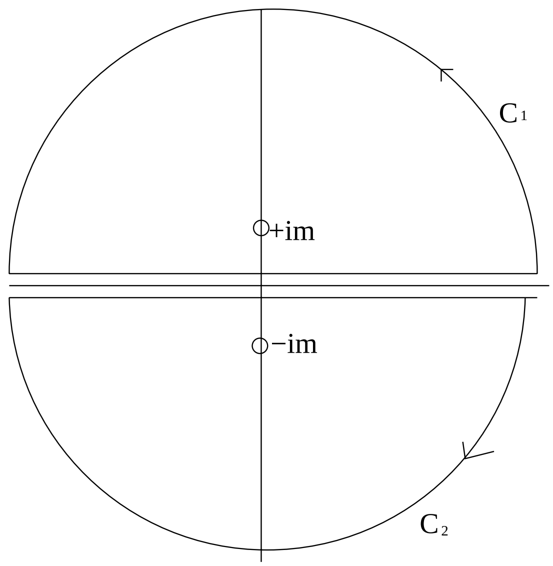
We need to evaluate the following integral in order to determine the energy shift when in state i when a particle of mass m is exchanged between particle 1 and particle 2,

$$\Delta E_i^{1\to 2} = -\frac{g^2}{2(2\pi)^2}\int_0^\infty \frac{p^2}{p^2+m^2}\frac{\exp(\imath pr)-\exp(-\imath pr)}{\imath pr}dp$$

Start by rewriting

$$\Delta E_i^{1\to 2} = -\frac{1}{2}\frac{g^2}{2(2\pi)^2}\int_{-\infty}^\infty \frac{p}{p^2+m^2}\frac{\exp(\imath pr)-\exp(-\imath pr)}{\imath r}dp$$

using the fact that the integrand is even in p. The integrand has poles at $p = \pm\imath m$ (see the figure). The integrals with the $\exp(\imath pr)$ and $\exp(-\imath pr)$ terms are performed separately. This is because one chooses an infinite semi-circular contour to do the integration over, in such a way that on the circular piece the contribution from infinity vanishes. This happens if the integrand contains a decaying exponential in $|p|$. For $\exp(\imath pr)$, this happens for $p = +\imath|p|$ and so one closes the contour in the upper half plane (C_1 in the figure). For $\exp(-\imath pr)$, we want $p = -\imath|p|$, and so close the contour in the lower half plane (C_2 in the figure).



The whole integral is thus:

$$-\frac{g^2}{2(2\pi)^2}\left[\oint_{C_1}\frac{p}{p^2+m^2}\frac{\exp(\imath pr)}{\imath r}dp - \oint_{C_2}\frac{p}{p^2+m^2}\frac{\exp(-\imath pr)}{\imath r}dp\right].$$

The residue of the pole at $p = \imath m$ in the first integrand is:

$$\lim_{p\to\imath m}\frac{(p-\imath m)}{(p-\imath m)(p+\imath m)}\frac{p}{\imath r}\exp(\imath pr) = \frac{1}{2\imath r}\exp(-mr)$$

and the residue of the pole at $p = -\imath m$ in the second integrand is:

$$\lim_{p\to -\imath m}\frac{(p+\imath m)}{(p-\imath m)(p+\imath m)}\frac{-p\exp(-\imath pr)}{\imath r} = -\frac{1}{2\imath r}\exp(-mr).$$

Cauchy's residue theorem tells us that the contour integral over an anti-clockwise contour is $2\pi\imath$ multiplied by the sum of the residues of the poles enclosed by the contour. For a clockwise contour, there is an additional minus sign. Noting that C_1 is anti-clockwise, and C_2 is clockwise, one has:

$$\begin{aligned} \Delta E_i^{1\to 2} &= -\frac{g^2}{2(2\pi)^2} 2\pi\imath \left[\frac{\exp(-mr)}{2\imath r} + \frac{\exp(-mr)}{2\imath r}\right] \\ &= \underline{-\frac{g^2}{8\pi}\frac{\exp(-mr)}{r}} \end{aligned}$$

as given in the notes.

APPENDIX E: LOCAL GAUGE INVARIANCE IN QED

Consider a non-relativistic charged particle in an electromagnetic field:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

where $\vec{E}$ and $\vec{B}$ can be written in terms of the vector and scalar potentials, $\vec{A}$ and ϕ:

$$\vec{B} = \vec{\nabla} \times \vec{A} \text{ and } \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}.$$

The classical Hamiltonian,

$$\vec{H} = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi,$$

can be used along with Schrödinger's equation to obtain

$$H\psi = \left[\frac{1}{2m}\left(-\imath\vec{\nabla} - q\vec{A}\right)^2 + q\phi\right]\psi(\vec{x},t) = \imath\frac{\partial\psi}{\partial t}(\vec{x},t). \quad (1)$$

where we have substituted $\vec{p} \to -\imath\vec{\nabla}$. We now need to show that Schrödinger's equation is invariant under the local guage transformation

$$\begin{aligned}\psi \to \psi' &= e^{\imath q\alpha(\vec{x},t)}\psi \\ \vec{A} \to \vec{A}' &= \vec{A} + \vec{\nabla}\alpha \\ \phi \to \phi' &= \phi - \frac{\partial\alpha}{\partial t}\end{aligned}$$

Substituting for ψ', $\vec{A}'$ and ϕ' in equation (1):

$$\left[\frac{1}{2m}\left(-\imath\vec{\nabla} - q(\vec{A} + \vec{\nabla}\alpha)\right)^2 + q(\phi - \frac{\partial\alpha}{\partial t})\right]e^{\imath q\alpha}\psi = \imath\frac{\partial}{\partial t}(e^{\imath q\alpha}\psi)$$

$$\left[\frac{1}{2m}\left(-\imath\vec{\nabla} - q\vec{A} - q\vec{\nabla}\alpha\right)^2 + q\phi - q\frac{\partial\alpha}{\partial t}\right]e^{\imath q\alpha}\psi = \imath\left(e^{\imath q\alpha}\frac{\partial\psi}{\partial t} + \imath q\psi\frac{\partial\alpha}{\partial t}e^{\imath q\alpha}\right).$$

The last terms on either side of the above equation cancel.
Now consider the $\left(-\imath\vec{\nabla} - q\vec{A} - q\vec{\nabla}\alpha\right)^2$ term. In order to show local gauge invariance, we need to show that

$$\left(-\imath\vec{\nabla} - q\vec{A} - q\vec{\nabla}\alpha\right)^2 e^{\imath q\alpha}\psi = \left(-\imath\vec{\nabla} - q\vec{A}\right)^2 e^{\imath q\alpha}\psi$$

or, equivalently,

$$\left(-\imath\vec{\nabla} - q\vec{A}'\right)^2\psi' = \left(-\imath\vec{\nabla} - q\vec{A}\right)^2 e^{\imath q\alpha}\psi.$$

Now,

$$\left(-\imath\vec{\nabla} - q\vec{A} - q\vec{\nabla}\alpha\right)^2 e^{\imath q\alpha}\psi = \left(-\imath\vec{\nabla} - q\vec{A} - q\vec{\nabla}\alpha\right)\cdot\left(-\imath\vec{\nabla} - q\vec{A} - q\vec{\nabla}\alpha\right)e^{\imath q\alpha}\psi$$

and

$$\vec{\nabla}\left(e^{\imath q\alpha}\psi\right) = e^{\imath q\alpha}\left(\vec{\nabla} + \imath q\vec{\nabla}\alpha\right)\psi.$$

Therefore,

$$\begin{aligned}\left(-\imath\vec{\nabla} - q\vec{A} - q\vec{\nabla}\alpha\right)e^{\imath q\alpha}\psi &= e^{\imath q\alpha}\left(-\imath\vec{\nabla} + q\vec{\nabla}\alpha - q\vec{A} - q\vec{\nabla}\alpha\right)\psi \\ &= e^{\imath q\alpha}\left(-\imath\vec{\nabla} - q\vec{A}\right)\psi\end{aligned}$$

and

$$\begin{aligned}\left(-\imath\vec{\nabla} - q\vec{A} - q\vec{\nabla}\alpha\right)^2\psi' &= \left(-\imath\vec{\nabla} - q\vec{A} - q\vec{\nabla}\alpha\right)e^{\imath q\alpha}\left(-\imath\vec{\nabla} - q\vec{A}\right)\psi \\ &= e^{\imath q\alpha}\left(-\imath\vec{\nabla} - q\vec{A}\right)^2\psi.\end{aligned}$$

Hence,

$$\underline{\left(-\imath\vec{\nabla} - q\vec{A}'\right)^2\psi' = e^{\imath q\alpha}\left(-\imath\vec{\nabla} - q\vec{A}\right)^2\psi}$$

and Schrödinger's equation is invariant under a local gauge transformation.