Nuclear Physics Problem Sheet, Lent 2004

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Basic Nuclear Properties

1. The semi empirical mass formula for atomic masses may be written in the form

$$\mathrm{M(A,Z)c^2 = Zm_pc^2 + (A-Z)m_nc^2 + Zm_ec^2 - a_vA + a_sA^{\frac{2}{3}} + a_c\frac{Z^2}{A^{\frac{1}{3}}} + a_a\frac{(A-2Z)^2}{A} + \delta(A,Z)}$$

where $\mathrm{m_p, m_n}$ and $\mathrm{m_e}$ are the masses of the proton, neutron and electron respectively.

(a) Explain the physical significance of the various terms.

(b) Calculate a numerical value for the constant $\mathrm{a_c}$. State clearly any assumptions you make and explain how valid you expect these assumptions to be.

(c) With reference to the semi empirical mass formula explain why nuclear fission and fusion are possible.

(d) Use the semi-empirical mass formula to estimate the value of Z of the most stable isobar of a super-heavy nucleus of mass number 300.

(e) Estimate the energy released when a nucleus of $^{235}_{92}\mathrm{U}$ undergoes fission into the fragments $^{87}_{35}\mathrm{Br}$ and $^{145}_{57}\mathrm{La}$ with three prompt neutrons.

[$\mathrm{m_p = 938.3\ MeV/c^2, m_n = 939.6\ MeV/c^2, m_e = 0.511\ MeV/c^2}$; $\mathrm{a_v = 15.8\ MeV, a_s = 18.0\ MeV, a_a = 23.5\ MeV}$. Nuclear radius $\mathrm{R = R_0A^{1/3}}$; $\mathrm{R_0 = 1.2\ fm}$]

2. Define the terms *total cross-section* and *differential cross-section* for scattering processes.

A beam of neutrons with an intensity 10^5 particles per second traverses a foil of $^{235}\mathrm{U}$ with a thickness of 10^{-1} kg m^{-2}.[‡]

A neutron interacting with a $^{235}\mathrm{U}$ nucleus has three possible results.

(a) Elastic scattering of the neutron. The cross-section for this process is 10^{-2} b.

(b) The capture of the neutron followed by the emission of a γ-ray. The cross-section for this process is 70 b.

[‡] To determine the number of particles that interact one must know the density of target particles and the thickness of the target. In nuclear physics instead of giving two numbers, that just have to be multiplied, the target thickness multiplied by the target density is usually quoted. This gives a target "thickness" in units of mass/area. It is in fact also easier to directly calculate this number when making targets. A specific area can be defined by a frame and the weight is easily obtained.

(c) The capture of the neutron followed by the resulting nucleus undergoing fission i.e. the splitting of the nucleus into two nuclei of similar masses. The cross-section for this process is 200 b.

Determine:

(i) The attenuation of the neutron beam by the foil.

(ii) The number of fission reactions, caused by the incident beam, occurring per second in the foil.

(iii) The flux of elastically scattered neutrons at a point 10m from the foil and out of the beam, assuming that the neutrons are scattered isotropically.

3. A spherically symmetric nucleus has a radial charge density $\rho(r)$ which is normalised such that $\int \rho(r) d\tau = 1$. Show that the form factor can be expanded as:

$$F(q^2) = 1 - \frac{1}{6} q^2 \overline{R^2} + \ldots$$

where $\overline{R^2}$ is the mean square radius of the charge distribution. When elastic scattering of 200 MeV electrons from a gold foil is observed at 11°, it is found that the scattered intensity is 70% of that expected for a point nucleus. Calculate the r.m.s. radius of the gold nucleus.

For larger scattering angles ($> 50°$) it is found that the scattered intensity, instead of falling off monotonically with angle, exhibits definite structure. What does this imply about $\rho(r)$?

4. *In the lectures we mentioned that muonic atoms may be used to determine nuclear sizes. This question illustrates why this is possible.*

Consider the muonic transition 1s-2p in $^{48}_{22}$Ti

(a) What is the radius of the muonic 1st Bohr orbit in Ti? Compare the radius of a 1s electron orbit in Ti to the nuclear radius.

(b) What are the transition energies for the electronic and muonic 1s-2p transitions if the nucleus is considered to be a point charge?

(c) Show that the electromagnetic potential felt by the muon at radius r inside a uniformly charged sphere of radius R and charge Ze is

$$V(r) = \frac{-Ze^2}{4\pi\varepsilon_o R}\left(\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right) \qquad r \leq R \qquad (1)$$

(d) The radial part of the Coulomb wavefunction for the 1s state is $2(Z/a_0)^{3/2}e^{-Zr/a_0}$ where a_0 is the Bohr radius $(4\pi\varepsilon_0\hbar^2/me^2)$. Using this and the potential in (c) determine the shift in the energy level for an electron in a 1s state. Write down the expression that must be evaluated to obtain the shift in the energy level for a muon 1s state. Explain why the evaluation of this will differ from the electron case and make some sensible comments about what you expect to find, without doing the integral. (This expression is tedious to evaluate for a muon)

You may assume that the 1s wave function does not change due to finite nuclear size. Comment on whether you think this is a good assumption.

(e) How large do you expect the shift in the 2p level to be compared to the shift in the 1s level?

5. A sample of gold is exposed to a neutron beam of constant intensity such that 10^{10} neutrons per second are absorbed in the reaction

$$n + {}^{197}\text{Au} \rightarrow {}^{198}Au + \gamma$$

The nuclide ^{198}Au undergoes β decay to ^{198}Hg with a mean lifetime of 4 days.

(a) How many atoms of ^{198}Au will be present after 6 days of irradiation?

(b) How many atoms of ^{198}Hg will be present after 6 days assuming that the neutron beam has no effect on the Hg?

(c) What is the equilibrium number of ^{198}Au nuclei?

6. The decay chain ^{139}Cs$\rightarrow$^{139}Ba$\rightarrow$^{139}La is observed for an initially pure sample of 1mCi of ^{139}Cs. The half life of ^{139}Cs is 9.5 minutes and that of ^{139}Ba is 82.9 minutes; ^{139}La is stable. What is the maximium ^{139}Ba activity, and when does it occur?

(A common unit of activity is the Curie. 1 Curie = 1 = 3.7x10^{10} disintegrations per second.)

The Nucleon Potential

7. *This question is practice for type B questions on the exam papers*

Describe what may be deduced about the internucleon potential by considering the following facts.

(i) Deuteron spin = 1

(ii) Deuteron parity even

(iii) Deuteron magnetic dipole moment = +0.857 μ_N

(iv) Deuteron electric quadruple moment = +2.83x10^{-31} m^2

(v) No excited states are observed for the Deuteron.

(vi) No bound p-p or n-n states are observed

What further interaction cannot be deduced from the above information, but is apparent in nucleon nucleon scattering experiments when beam energies are of the order of 100MeV? Explain how this type of experiment demonstrates this interaction.

8. In lectures the n-p potential was modelled as a square well of width b and depth V_0. A more realistic model would include a repulsive core, e.g.:

$$V(r) = \infty(r < c);\ V(r) = -V_0(c < r < c + b);\ V(r) = 0(r > c + b).$$

(a) Show that the binding energy of the deuteron does not depend on c. Therefore, as in lectures, the values of $V_0 = 33.7$ MeV and $b = 2.1$ fm are consistent with the binding energy.

(b) Derive an expression for the s-wave phase shift δ_0 for scattering of neutrons of energy E.

(c) If $b = 0$ the problem reduces to "hard sphere" scattering. Show that in this case the scattering cross-section is given by $\sigma = 4\pi c^2$ at low energy.

(d) Experimentally, the triplet np phase shift δ_0 falls to zero for E$\sim$ 300 MeV. Use this to estimate the value of the hard core radius c.

(e) Show that the scattering length (the amplitude of the scattered wavefunction) is given by

$$a = b + c - \tan(k_0 b)/k_0$$

where $k_0 = \sqrt{2\mu V_0}/\hbar$ and μ is the np reduced mass, and evaluate a.

9. Show that the operator:

$$\hat{\pi}_s = \frac{1}{4} - \frac{\mathbf{\hat{s}_n} \cdot \mathbf{\hat{s}_p}}{\hbar^2}$$

projects out the singlet (S=0) part of the np wavefunction i.e. it gives zero when operating on an S=1 state and unity when operating on an S=0 state. Likewise

$$\hat{\pi}_t = \frac{3}{4} + \frac{\mathbf{\hat{s}_n} \cdot \mathbf{\hat{s}_p}}{\hbar^2}$$

projects out the triplet part of the wavefunction.

The singlet and triplet n-p scattering lengths can be measured by comparing neutron scattering from ortho-hydrogen (in which the two protons have S=1) and para-hydrogen (S=0). Thus, show that when a neutron scatters coherently from the two protons in a hydrogen molecule, the total scattering length is given by:

$$\hat{\mathbf{a}} = a_s(\hat{\pi}_{s_1} + \hat{\pi}_{s_2}) + a_t(\hat{\pi}_{t_1} + \hat{\pi}_{t_2}) = \frac{1}{2}a_s + \frac{3}{2}a_t + (a_t - a_s)\frac{\hat{\mathbf{s}}_{\mathbf{n}} \cdot \hat{\mathbf{s}}_{\mathbf{H}}}{\hbar^2}$$

where a_s and a_t are the singlet and triplet scattering lengths, the suffixes 1 and 2 refer to the two protons, and $\hat{\mathbf{s}}_{\mathbf{H}} = \hat{\mathbf{s}}_{\mathbf{p_1}} + \hat{\mathbf{s}}_{\mathbf{p_2}}$ is the spin of the two protons.

Hence show that $a = \frac{1}{2}a_s + \frac{3}{2}a_t$ for scattering from para-hydrogen, $a = 2a_t$ for scattering from ortho-hydrogen when when the total spin $n - H_2$ is 3/2 and $a = \frac{3}{2}a_s + \frac{1}{2}a_t$ for scattering from ortho-hydrogen when the total $n - H_2$ spin is 1/2. From counting spin states the probabilities of total spin 1/2 and 3/2 are in the ratio 1:2. Thus calculate the ratio $\sigma_{\text{ortho}}/\sigma_{\text{para}}$ given $a_s = -23.7$ fm and $a_t = 5.41$ fm and compare with the experimental value $\sigma_{\text{ortho}}/\sigma_{\text{para}}= 34\pm2$.

The Nuclear Shell Model

10. What are *magic numbers* ? Outline the basis of the nuclear shell model and show how it accounts for magic numbers. How can the shell model be used to predict the spins, parities and magnetic dipole moments of nuclear ground states ?

Using the nuclear shell model determine the spins and parities of the ground states of the nuclides listed below and compare them with experimental values. In some cases the magnetic moments in nuclear magnetons are given. In these cases also determine the magnetic moments predicted by the shell model. Comment on any discrepancies you find.

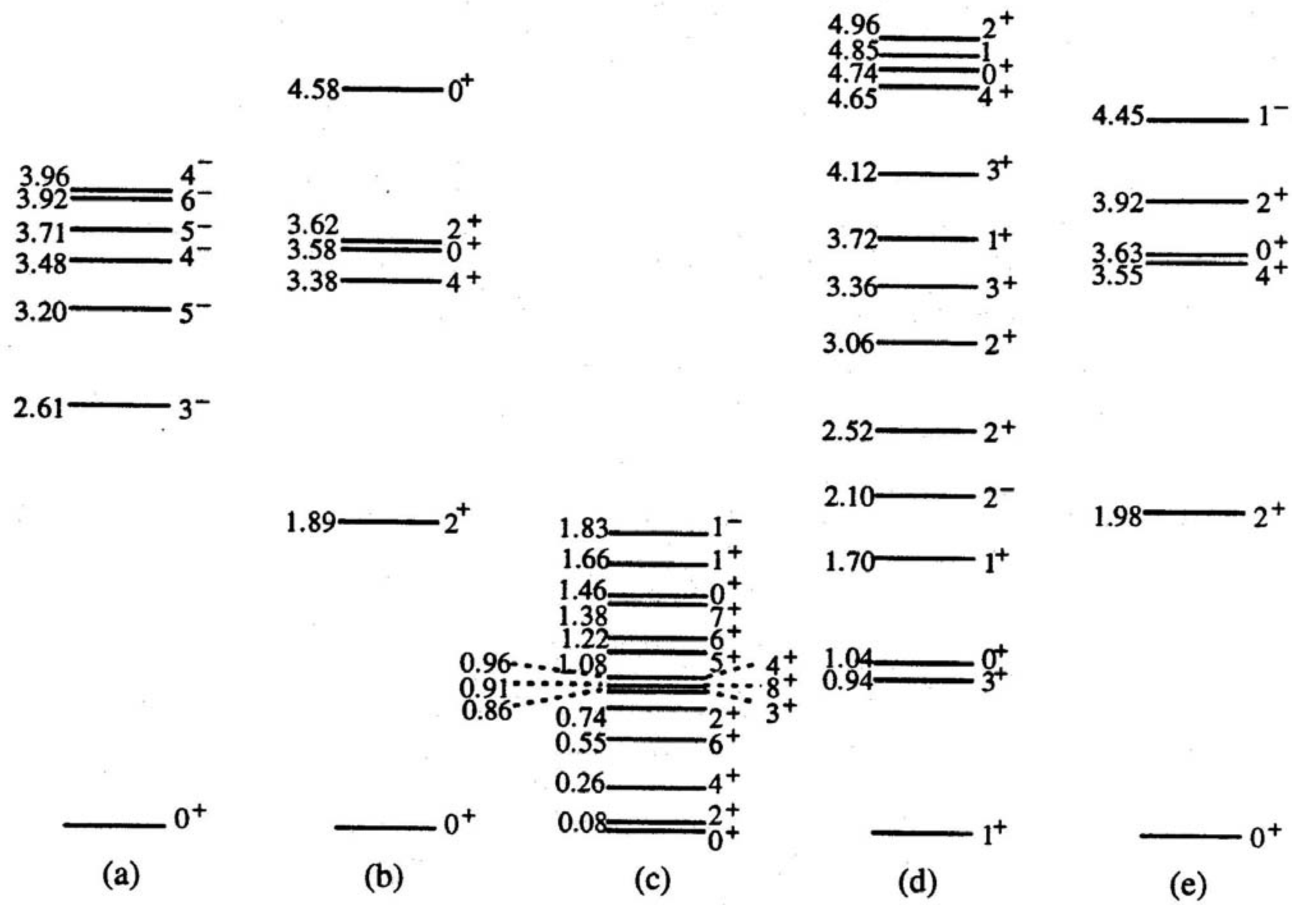
${}^{3}_{2}$He	${}^{9}_{4}$Be	${}^{7}_{3}$Li	${}^{12}_{6}$C	${}^{13}_{6}$C	${}^{15}_{7}$N	${}^{17}_{8}$O	${}^{23}_{11}$Na	${}^{131}_{54}$Xe	${}^{207}_{82}$Pb
-2.13	-1.17	3.26		0.70	-0.28	-1.89			

11. The quantum of vibrational excitation of a $J^P = 0^+$ nucleus (a "phonon") carries $J^P = 2^+$, and behaves as a boson. By enumerating the possible m_J values which can be formed from two such phonons, show that two-phonon excitations of a nucleus should have $J^P = 0^+, 2^+$ and 4^+. Similarly show that the three-phonon excitations have $J^P = 0^+, 2^+, 3^+, 4^+$ and 6^+.

12. The diagram below shows the low-lying energy levels for the nuclides:

${}^{18}_{10}$Ne ${}^{166}_{68}$Er ${}^{18}_{9}$F ${}^{208}_{82}$Pb ${}^{18}_{8}$O

The schemes are drawn to the same scale, with energies (in MeV) with respect to the ground state and the spin and parity (J^P) values given for each level. Identify which schemes belong to which nuclei and explain as fully as you can which features of the levels support your choices.



The Decay of Nuclei

13. A nucleus with A=200 can decay by the emission of α particles. The α particles are observed with two different energies, 4.687 MeV and 4.650 MeV. Neither of these decays populates the ground state of the daughter nucleus, but each is followed by a γ ray of energy 266 and 305 keV respectively. No other γ rays are seen.

    (a) From this information construct a decay scheme.

    (b) The decaying parent state has spin 1 and negative parity and the daughter has ground state $J^P = 0^-$. Explain why there is no direct α decay to the ground state.

14. Neglecting coulomb effects, show that the average kinetic energy carried off by the electron in β decay is $Q/2$ when the electron is highly relativistic, and $Q/3$ when the electron is non-relativistic. *Hint: Start from the electron momentum spectrum in β decay derived in lectures.*

15. Outline the Fermi theory of β decay and explain the principal assumptions made.

Explain the difference between Fermi and Gamow-Teller transitions and between super-allowed, allowed and forbidden decays.

Classify each of the following examples of β decay according to whether the decay is super-allowed, allowed, 1st forbidden etc., and whether Fermi or Gamow-Teller matrix elements are involved.

$$n \to p$$

$${}^{6}_{2}\mathrm{He}(0^+) \to {}^{6}_{3}\mathrm{Li}(1^+) \qquad (\mathrm{ft}=830\mathrm{s})$$

$${}^{14}_{6}\mathrm{O}(0^+) \to {}^{14}_{7}\mathrm{N}^*(0^+) \qquad (\mathrm{ft}=3300\mathrm{s})$$

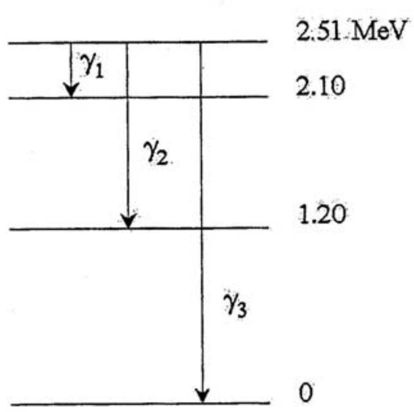
$${}^{35}_{16}\mathrm{S}(\tfrac{3}{2}^+) \to {}^{35}_{17}\mathrm{Cl}(\tfrac{3}{2}^+) \qquad (\mathrm{ft}=1\mathrm{x}10^5\mathrm{s})$$

$${}^{36}_{17}\mathrm{Cl}(2^-) \to {}^{36}_{18}\mathrm{Ar}(0^+)$$

$${}^{76}_{35}\mathrm{Br}(1^-) \to {}^{76}_{34}\mathrm{Se}(0^+)$$

$${}^{137}_{55}\mathrm{Cs}(\tfrac{7}{2}^+) \to {}^{137}_{56}\mathrm{Ba}(\tfrac{3}{2}^+)$$

16. A nucleus in an excited state at 2.51MeV can decay by emission of three γ rays as shown:



The transitions labelled γ_1, γ_2 and γ_3 are known to proceed via magnetic dipole, electric dipole and electric quadrupole transitions respectively. No other transitions to the ground state, of comparable intensities, are observed. Given that the ground state has a spin of $\frac{3}{2}^+$ what are the most probable spins and parities of these three excited states?

Nuclear Reactions

17. Deuterons with energy 600 keV impinge on a deuterium target. Protons emitted at an angle of 50° to the incident beam are found to have an energy of 3.866 MeV. Calculate the Q value of the reaction $^{2}H(d,p)^{3}H$ and hence determine for ^{3}H the value of the difference between the atomic mass, M of an isotope and its mass number A, i.e. (M-A). (M-A is usually what is quoted in mass tables and is known as the mass excess.)

[(M-A)=7.289 MeV for ^{1}H and (M-A)=13.136 MeV for ^{2}H.]

18. If $\sigma_{\rm abs}$ is the cross-section for the absorption of a thermal neutron of velocity v by a proton and $\sigma_{\rm dis}$ is the cross-section for the inverse process in which a γ ray causes the disintegration of a deuteron into a neutron and proton of relative velocity v, show, starting from Fermi's Golden rule, that

$$\sigma_{abs}/\sigma_{dis} = 6(E/Mvc)^2$$

where M is the mass of the proton or neutron (assumed to be equal) and E is the energy of the γ ray in the γ-deuteron centre of momentum frame.

19. The maximum value of the radiative capture cross-section for neutrons in ^{123}Te (J=1/2 in its ground state) is 75 kb and is reached at a neutron energy of 2.2 eV, where the elastic width Γ_n is 0.0104 eV and the radiative width Γ_γ is 0.105 eV. What is the elastic cross-section at resonance and what is the spin of the compound nucleus formed?

20. A mixture of ^{235}U and graphite is to be used for some experiments. The graphite is contaminated with 1 p.p.m. by weight of ^{10}B. What is the maximum fraction by weight of ^{235}U in the mixture if the multiplication factor at infinite volume is not to exceed unity ? The absorption cross-sections for ^{12}C, ^{10}B, and ^{235}U at thermal energies are 0.04 b, 3800 b and 700 b respectively, where fission accounts for 580 b of the cross-section in ^{235}U. Assume that 2.5 neutrons are produced per fission, and that all reactions take place at thermal energies.

Numerical answers

1. (b) 0.718 MeV; (c) 113; (e) 160 MeV
2. (i) 99,313 particles/s; (ii) 510 /s; (iii) 2.04x10^{-5} particles/m^2/s
3. 5 fm.
4. (a) μ 11.7 fm, e 2400 fm, Nucleus 4.4 fm

   (b) e 4.9 KeV, μ 1.02 MeV

   (d) e ΔE 1.8x10^{-5} keV, (μ ΔE 0.36 MeV estimate)
5. (a) 2.7x10^{15}; (b) 2.5x10^{15}; (c) 3.46x10^{15}
6. 0.087mCu, 33.5 minutes

8. (b) $\delta_0 = \cot^{-1}(\frac{k}{k'} \cot kb) - k'(b + c)$ where $k = \sqrt{2\mu(V_0 + E)}/\hbar$, $k' = \sqrt{2\mu E}/\hbar$ and μ is the reduced mass.

   (d) 0.11 fm

   (e) 5.57 fm

9. 31

17. 4.03 MeV, 14.95 MeV

19. 7400 b

20. 1.2×10^{-3}