Part II Nuclear Physics

Handout 3

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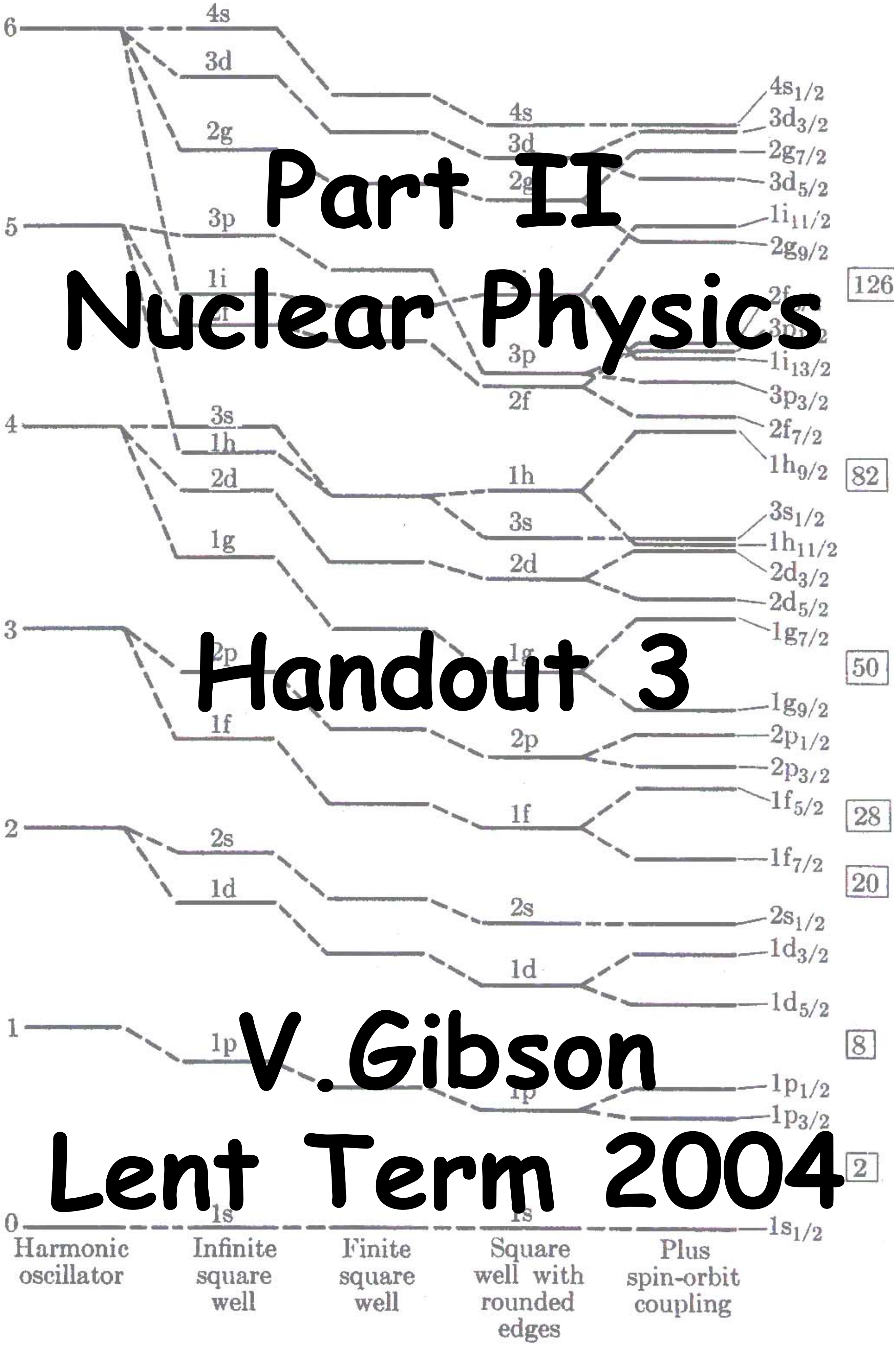
Lent Term 2004



Section IV
The Nuclear Shell Model

Nuclei with values of "Magic Numbers"

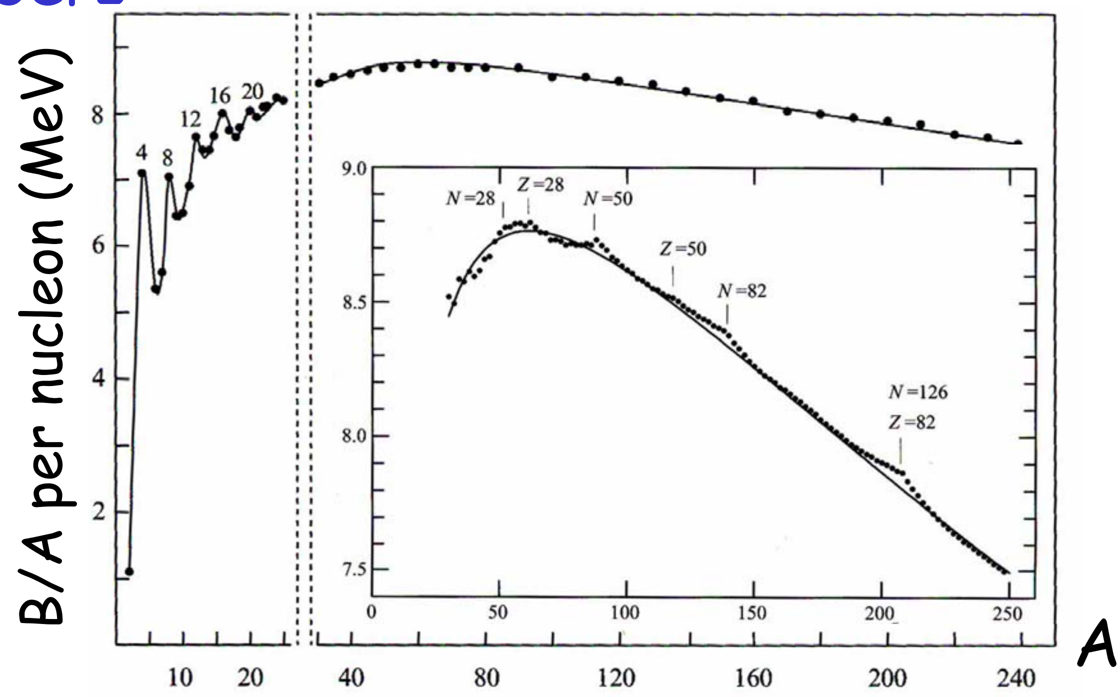
Z and/or N = 2, 8, 20, 28, 50, 82, 126

are very stable and show significant departures from the average nucleus behaviour.

- Abundance of isotopes and isotones for magic numbers (see Segrè chart)

  e.g. Z=20 6 stable isotopes (average =2)
  Z=50 10 " " (" =4)

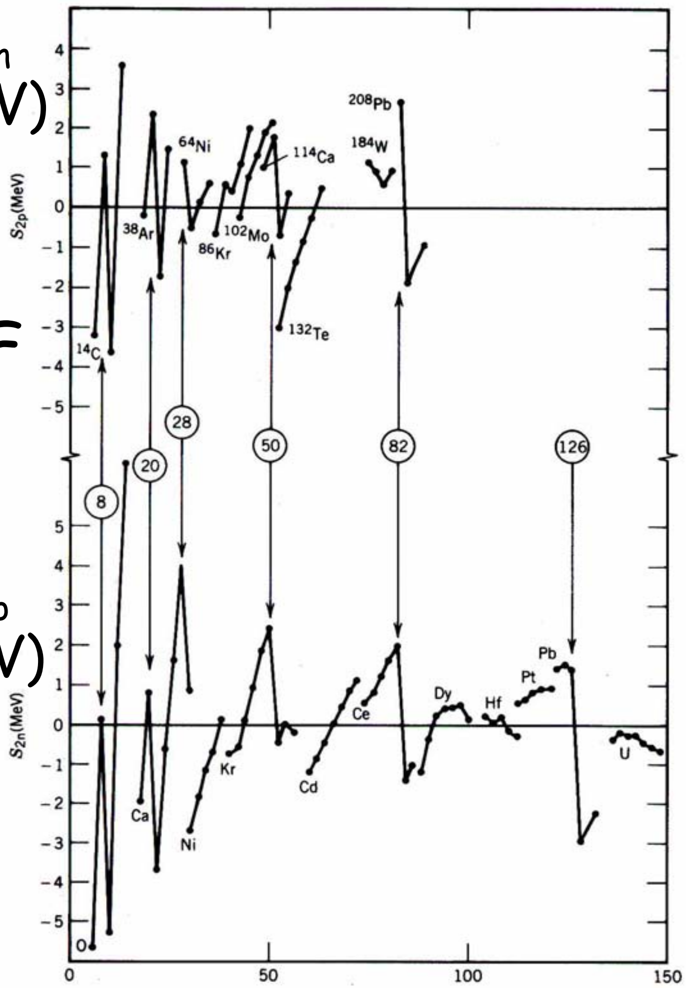
- B/A and separation energies large for magic numbers



Separation energies-SEMF

S_{2n} (MeV)

S_{2p} (MeV)



Nucleon number

- Doubly magic nuclei extremely stable
- Energies in α, β decay high when daughter nucleus is magic
- Nuclear radius, small change with Z,N at magic numbers.
- 1st excited states for magic numbers higher than neighbours
- Spontaneous neutron emitters have magic number+1
- Odd A nuclei have small quadrupole moment when magic

Parallels with atomic behaviour as electron shells fill

$\Rightarrow$ Fermi Gas Model

Atom

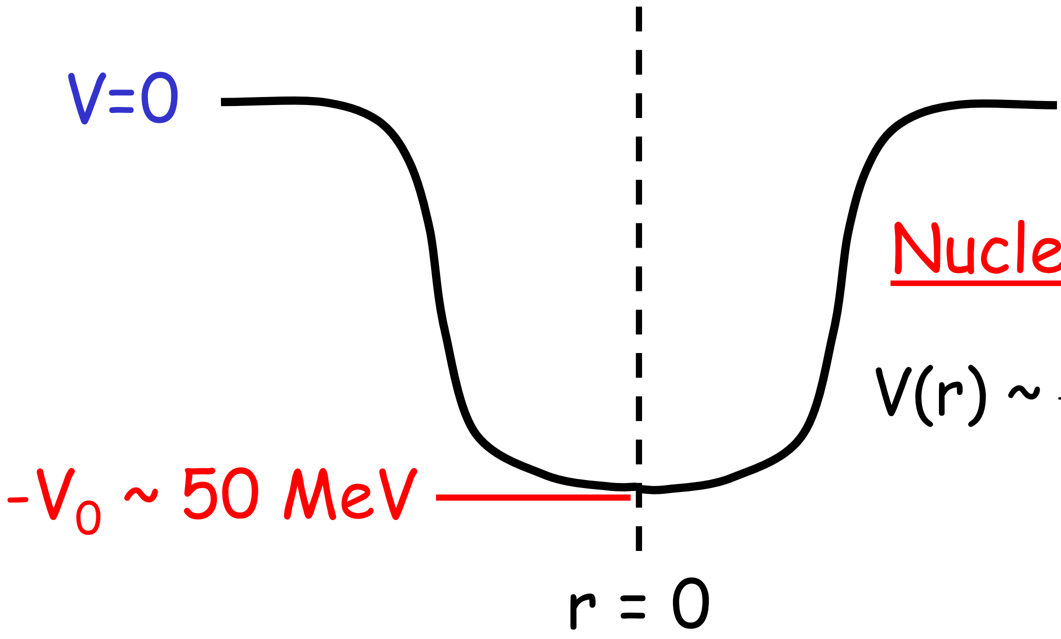
- Electrons move independently in central potential V(r)~1/r (coulomb field of nucleus).
- Shells filled according to Pauli exclusion principle.
- Properties of atom defined by valence electrons
- Energy levels obtained by solving Schrödinger equation for central potential.

$$E_n \sim \frac{1}{n^2}$$ n=principal quantum number

- Magic number Z $\rightarrow$ noble gas atoms

Nucleus

Nucleons move in a net nuclear potential that represents the average effect of interactions with the other nucleons in the nucleus.



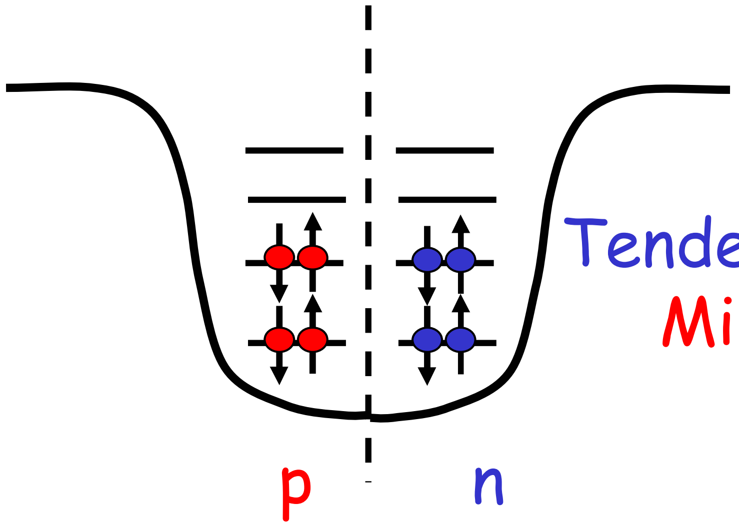
Nuclear Potential

$$V(r) \sim \frac{V_0}{\left(1+e^{(r-R)/s}\right)}$$

- Nuclear force short range → near centre constant potential
- Near surface density decreases → V(r) decreases
- V(r) for protons modified by the coulomb interaction (see α decay)

In the ground state, nucleons occupy energy levels of the nuclear potential to minimize the total energy without violating the Pauli principle.

The exclusion principle operates independently for protons and neutrons.



Postulate: nucleons in well-defined orbits with discrete energies.

Objection: nucleons are similar size to nucleus ∴ expect many collisions. How can there be well-defined orbits (c.f. electrons in atoms) ?

Pauli principle: if energy is transferred in a collision then nucleons must move to new states. However, all nearby states are occupied ∴ no collision. i.e. almost all nucleons in a nucleus move freely within nucleus if it is in it's ground state.

The Nuclear Shell Model

Treat each nucleon independently and solve Schrödingers equation for nuclear potential to obtain nucleon energy levels.

Consider spherically symmetric central potential e.g. $V(r) \sim -V_0\left(1+e^{(r-R)/s}\right)^{-1}$ (Q.M II course)

Solution of the form $\psi(\vec{r}) = R_{n\ell} Y_\ell^m(\vartheta,\varphi)$

Obtain 2 equations separately for radial and angular coordinates.

Radial equation:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2} + \frac{2M}{\hbar^2}\left(E - V(r)\right)\right] R_{n\ell}(r) = 0$$

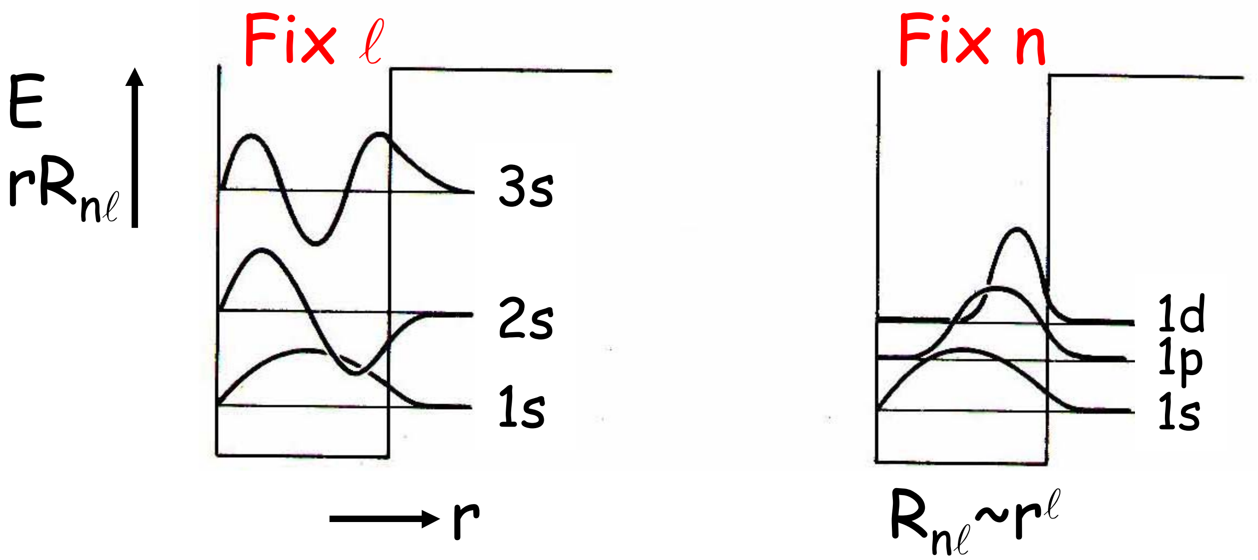
Allowed states specified by n, ℓ, m:

- **n** radial quantum number
  (N.B. different to atomic notation)
- **ℓ** orbital angular momentum q.n.
  Any ℓ for given n.
  (Atomic ℓ=n-1)
- **m** magnetic quantum number
  m = $-\ell$......$+\ell$

Energy levels increase with n and ℓ

Example

Notation $n\ell$



As n increases,
$rR_{n\ell}$ has more nodes, greater curvature and E increases.

As ℓ increases,
$rR_{n\ell}$ has more nodes, greater curvature and E increases.

Fill shells for both p and n

$$\text{Degeneracy} = (2s+1)(2\ell+1)$$
$$= 2(2\ell+1) \quad s=1/2$$

No central potential can reproduce magic numbers. Need to include non-central term

$\Rightarrow$ Spin-Orbit Interaction

Spin Orbit Interaction

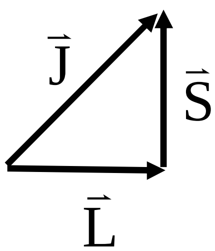
Evidence for a spin-orbit interaction comes from polarized nucleon scattering (see Handout 2).

1949 Mayer and Jensen included spin-orbit potential to explain magic numbers.

$$V(r) = V_C(r) + V_{so}(r)\vec{L}\cdot\vec{S}$$

V_{so} is -ve

Spin-orbit interaction splits ℓ levels



$$\vec{J} = \vec{L} + \vec{S}$$

$$J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$$

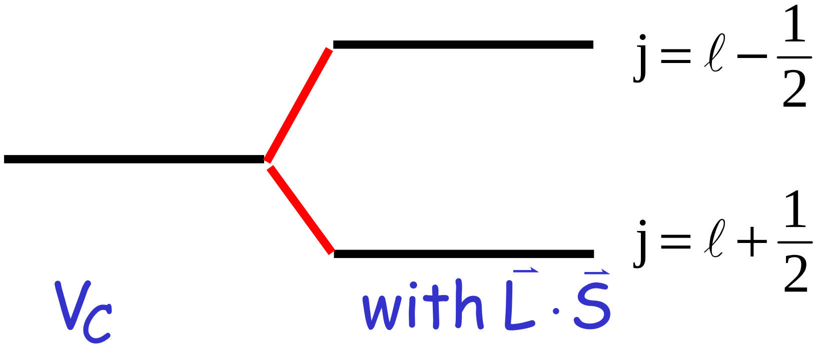
$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left[J^2 - L^2 - S^2\right]$$

$$\hat{L}\cdot\hat{S}\left|\psi\right\rangle = \frac{1}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1)\right]\left|\psi\right\rangle$$

Single nucleon s = ±1/2, j = ℓ+1/2 or j = ℓ-1/2

$$j = \ell + \frac{1}{2} \qquad \hat{L}\cdot\hat{S}\left|\psi\right\rangle = \frac{1}{2}\ell\left|\psi\right\rangle \qquad V = V_C + \frac{1}{2}\ell V_{so}$$

$$j = \ell - \frac{1}{2} \qquad \hat{L}\cdot\hat{S}\left|\psi\right\rangle = -\frac{1}{2}(\ell+1)\left|\psi\right\rangle \qquad V = V_C - \frac{1}{2}(\ell+1) V_{so}$$



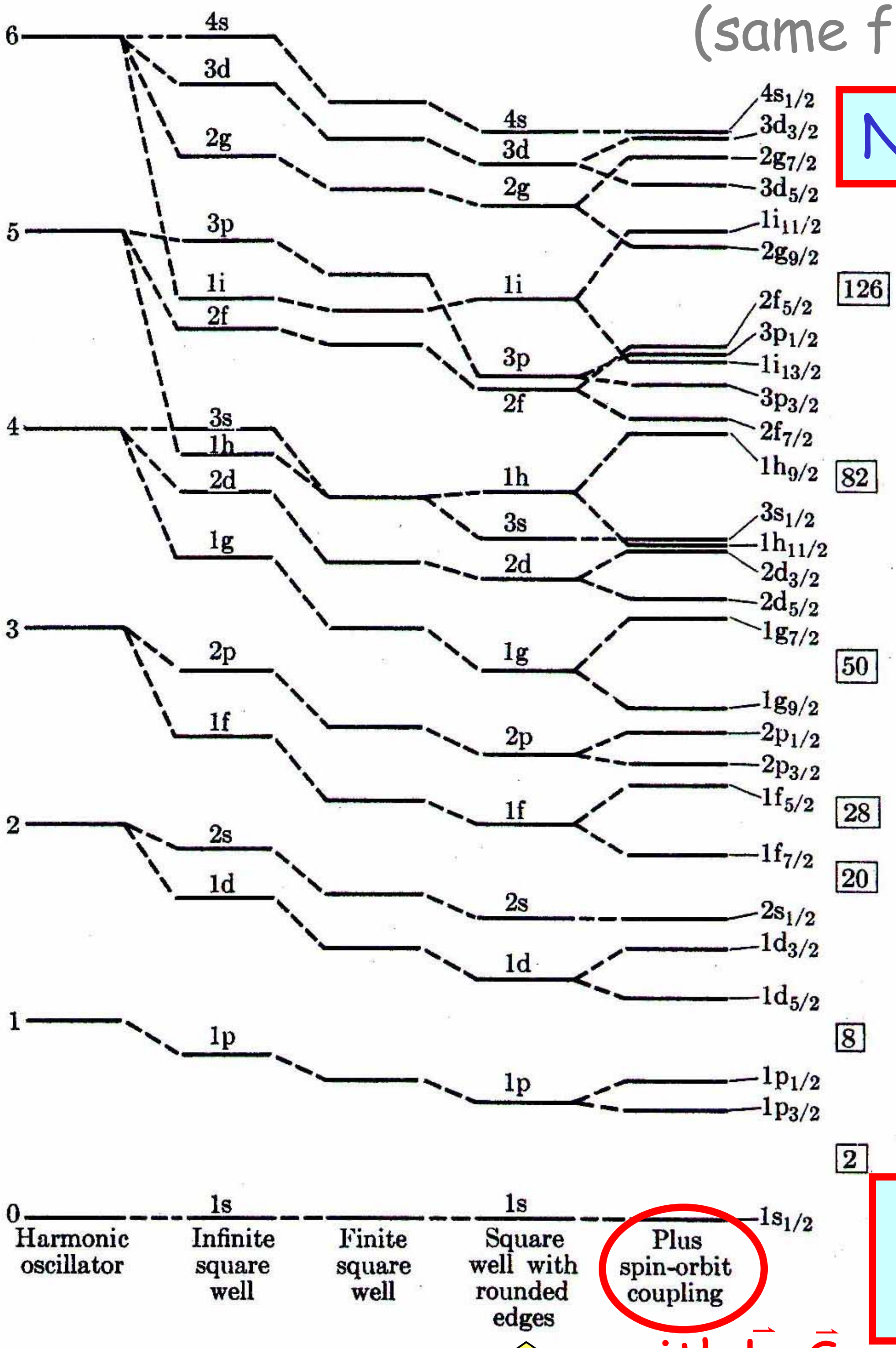
$$\Delta E = \frac{1}{2}(2\ell+1)\hbar^2$$

Shell Model Energy Levels

Splitting increases with increasing ℓ.

(same for all nuclei)

Notation $n\ell_j$



$\Sigma(2j+1)$

Magic Numbers

Degeneracy $(2j+1)$

with $\vec{L}\cdot\vec{S}$

Saxon-Woods potential

$$V(r) \sim -V_0\left(1+e^{(r-R)/s}\right)^{-1}$$

Predictions of the Shell Model

(1) Magic Numbers : The Shell Model successfully predicts the origin of the magic numbers.

(2) Spin and Parity : (ground state)

- Near closed shells

Even-Even Nuclei : $J^P = 0^+$

Even-Odd Nuclei : J^P given by unpaired nucleon or hole

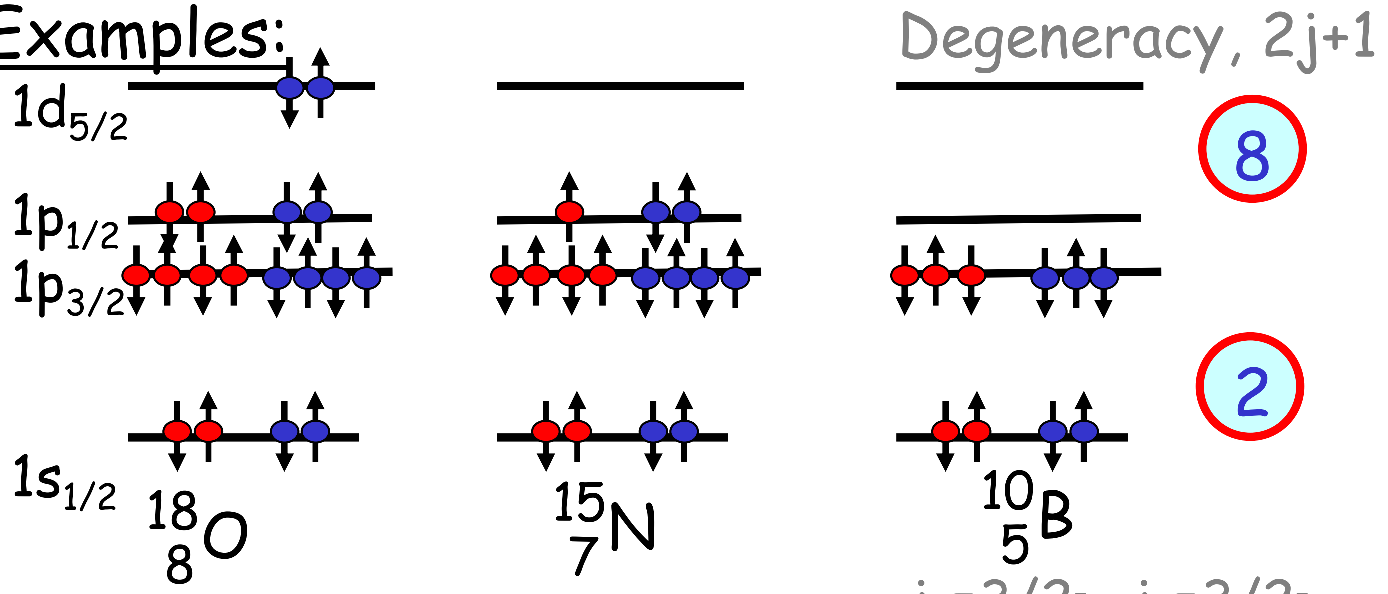
Odd-Odd Nuclei : Unpaired p and n

jj coupling $\vec{J} = \vec{j}_p \oplus \vec{j}_n$

$|j_p - j_n| \leq J \leq j_p + j_n$

Parity = $(-1)^{\ell_p}(-1)^{\ell_n}$

Examples:



$J^P=0^+$ (obs)

$J^P=1/2^-$ (obs)

$j_p=3/2^-$, $j_n=3/2^-$

$J^P=0^+,1^+,2^+,3^+$

observed $J^P=3^+$

➢ The pairing interaction between identical nucleons is not described by a spherically symmetric potential or by the spin-orbit interaction.

The lowest energy state for two identical nucleons is when they have opposite spins (↑↓) and anti-parallel orbital angular momenta (j,m and j,-m ⇒ Total J=0). ⇒ Overall antisymmetric wavefunction

The pairing energy increases with increasing ℓ of nucleons.

Example: $^{207}_{82}Pb$

Expect odd neutron in $2f_{5/2}$ subshell.
Pairing interaction: energetically favourable for the $2f_{5/2}$ neutron and a neutron from $3p_{1/2}$ to pair and leave hole in $3p_{1/2}$. ⇒ $J^P=1/2^-$

➢ Away from closed shells

More than one nucleon can contribute and Q large ⇒ V(r) no longer spherically symmetric.

Example: $^{23}_{11}Na$ Q large

3 protons in $1d_{5/2}$, expect $J^P=5/2^+$
All 3 protons contribute ⇒ $J^P=1/2^+$ (obs)

(3) Magnetic Dipole Moments :

- Even-even nuclei : J=0 $\Rightarrow \mu = 0$

- Odd A : μ due to unpaired nucleon or hole

Single nucleon

$$\vec{\mu} = \frac{\mu_N}{\hbar}\left(g_\ell \vec{L} + g_s \vec{S}\right) \quad \text{and} \quad \vec{J} = \vec{L} + \vec{S}$$

$$\mu_N = \frac{e\hbar}{2m_p}$$ = nuclear magneton

Thus, $\vec{\mu}$ not parallel to $\vec{J}$. However, angle between $\vec{\mu}$ and $\vec{J}$ is constant, because

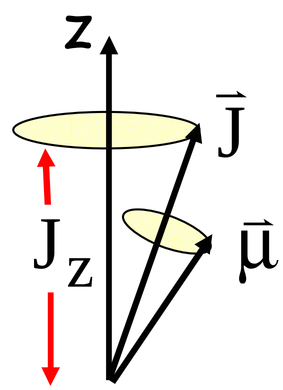
$$\cos\vartheta \sim \vec{\mu}\cdot\vec{J} \sim g_\ell \vec{L}\cdot\vec{J} + g_s \vec{S}\cdot\vec{J}$$
$$= \frac{1}{2}\left[g_\ell (L^2 + J^2 - S^2) + g_s (S^2 + J^2 - L^2)\right]$$

and J^2, L^2 and S^2 all constants of motion.

Hence, the nuclear magnetic moment (projection of $\vec{\mu}$ along z-axis)

$$\mu = \frac{\vec{\mu}\cdot\vec{J}}{|\vec{J}|}\frac{J_z}{|\vec{J}|}$$

project $\vec{\mu}$ onto $\vec{J}$ then $\vec{J}$ onto $\vec{z}$



$$\therefore \mu = \mu_N \frac{m_J}{2j(j+1)} \{ g_\ell [\ell(\ell+1) + j(j+1) - s(s+1)] + g_s [s(s+1) + j(j+1) - \ell(\ell+1)] \}$$

Hence,

$$\mu = g_J \, \mu_N \, J \quad \text{for } m_J = J \text{ and}$$

$$g_J = \frac{1}{2j(j+1)} \{ g_\ell [\ell(\ell+1) + j(j+1) - s(s+1)] + g_s [s(s+1) + j(j+1) - \ell(\ell+1)] \}$$

Two possibilities for a single nucleon (s=1/2):

$j = \ell + 1/2$ $\qquad g_J = g_\ell + \frac{g_s - g_\ell}{2\ell + 1}$

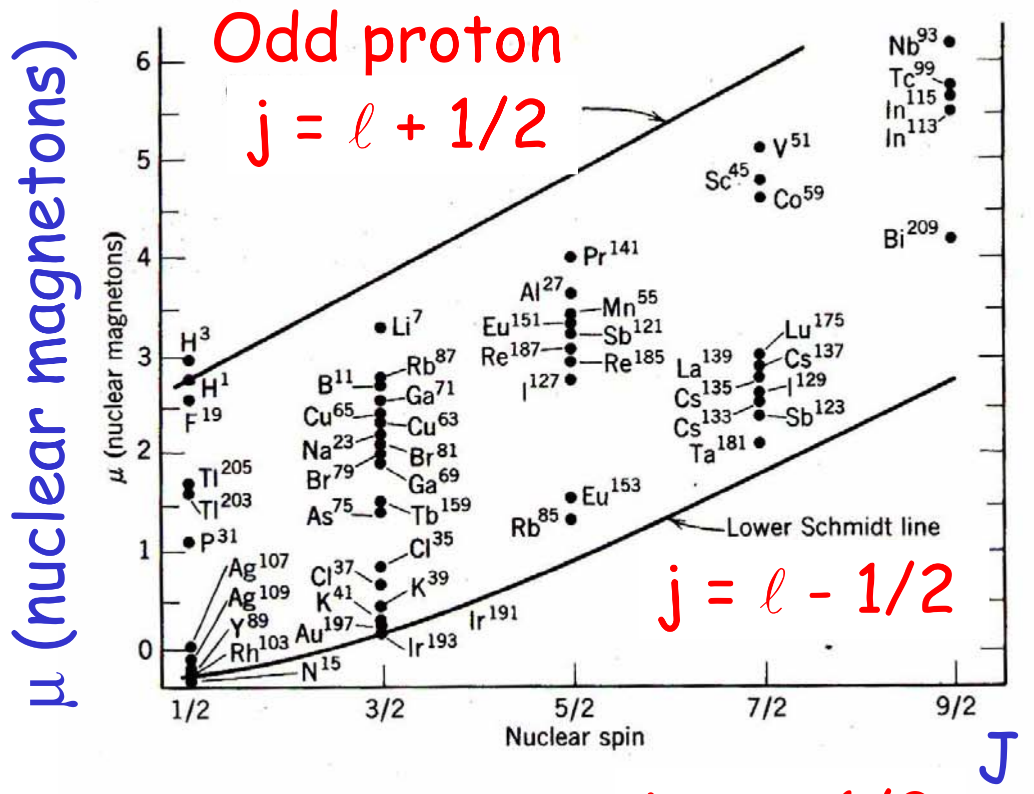
$j = \ell - 1/2$ $\qquad g_J = g_\ell - \frac{g_s - g_\ell}{2\ell + 1}$

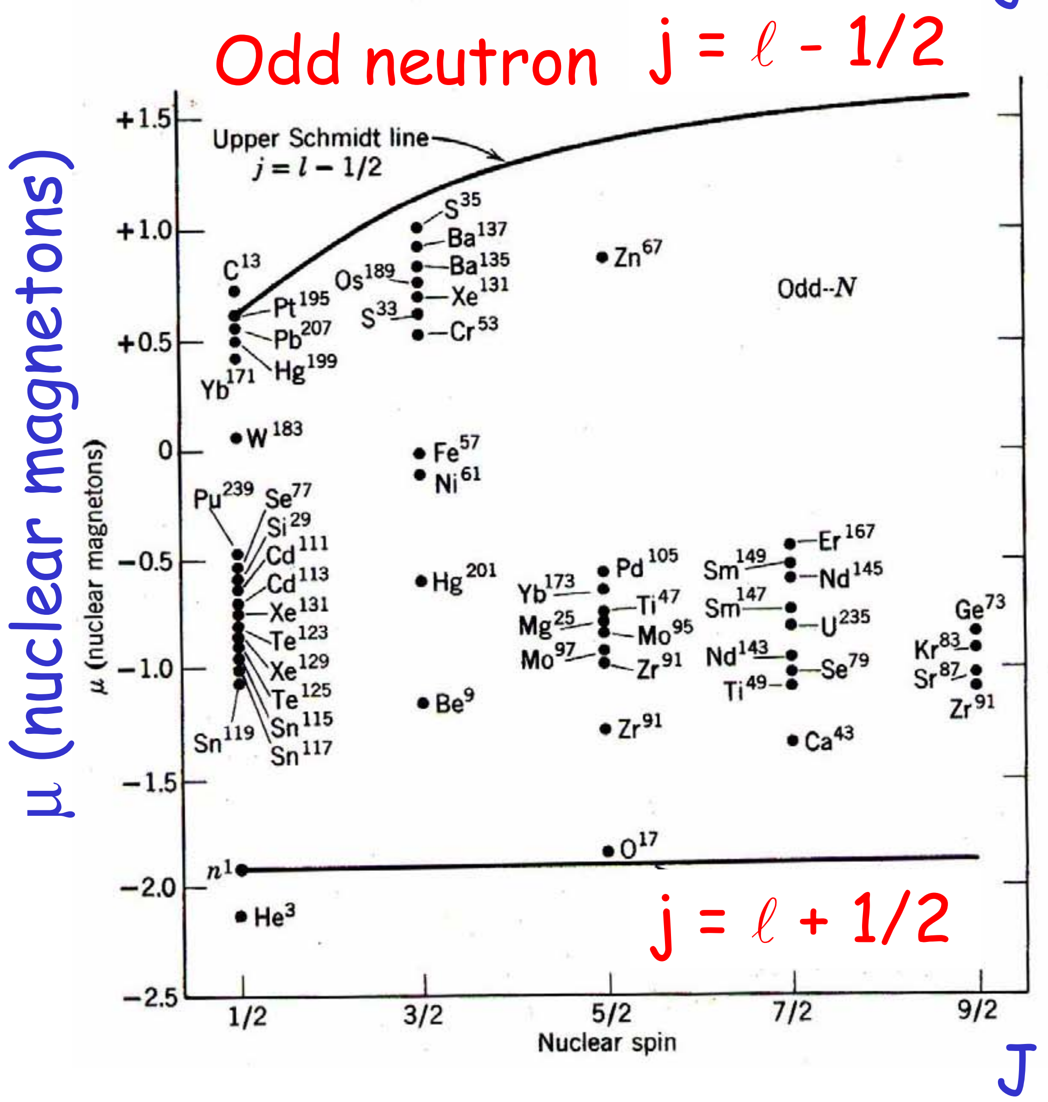
Odd p $\qquad g_\ell = 1 \qquad g_s = +5.586$

Odd n $\qquad g_\ell = 0 \qquad g_s = -3.826$

$\Rightarrow$ Schmidt Limits

Schmidt Limits compared to data





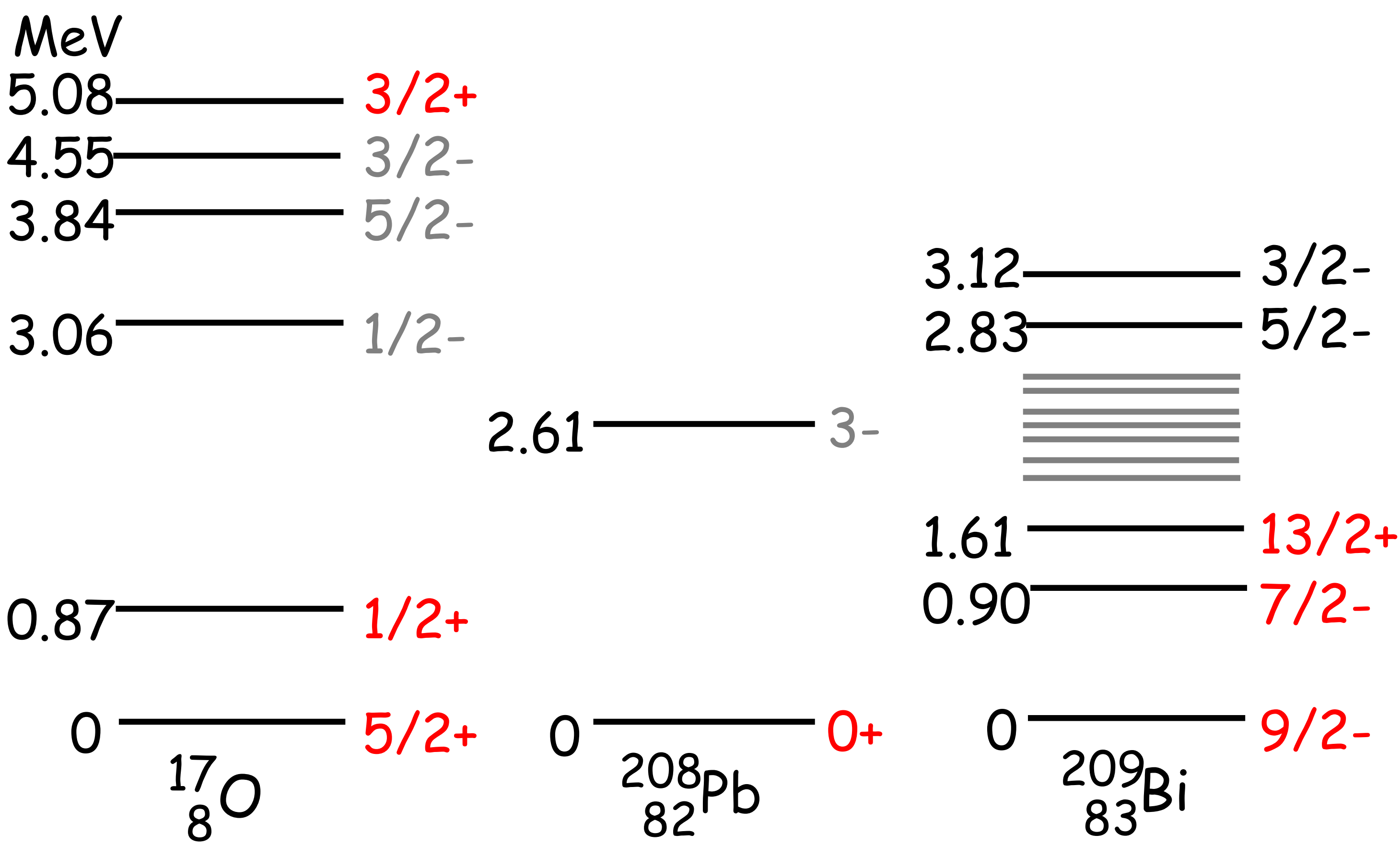
Excited States of Nuclei

In nuclear spectra expect to find 3 kinds of excitations:

- Single nucleon excited states
- Vibrational excited states
- Rotational excited states

Single nucleon excited states can be predicted from the Shell Model. Good for small excitations of odd A nuclei near closed shells.

Example



Vibrational and Rotational motion of nuclei involve the collective motion of nucleons in the core of the nucleus.

Collective motion can be incorporated into the shell model by replacing the static symmetrical potential with a potential that undergoes deformations in shape.

⇒ Collective vibrational and rotational models.

We will only consider even Z, even N nuclei

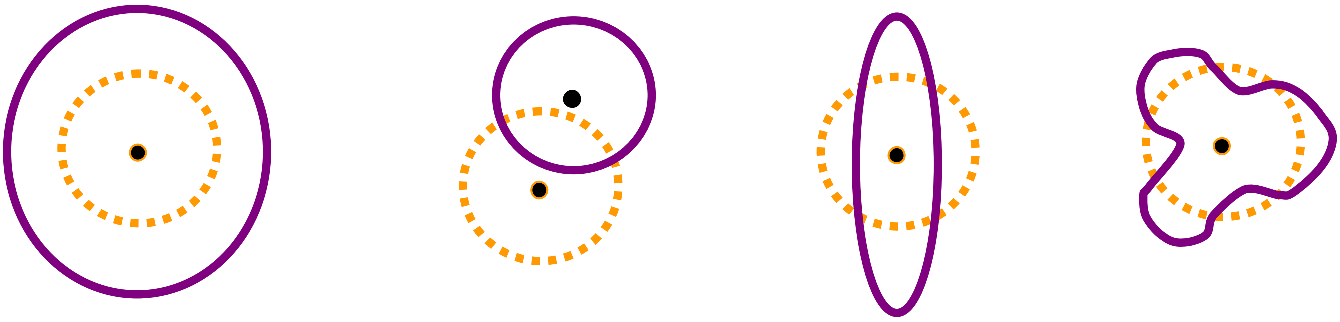
Ground state : $J^P=0^+$
Lowest excited state (nearly always): $J^P=2^+$

Tend to divide into 2 categories:

A	E(2+)	Type
30-150	~ 1 MeV	Vibrational
150-190 *Rare Earth*	~ 0.1 MeV	Rotational
>220 *Actinides*	~ 0.1 MeV	Rotational

Nuclear Vibrations

Vibrational excited states occur when a nucleus oscillates about a spherical equilibrium shape (low energy surface vibrations, near closed shells).



Monopole — Incorporated into the average radius

Dipole — Net displacement of c.of.m ⇒ cannot result from action of nuclear forces.

Quadrupole

Octupole

Quadrupole oscillations are the lowest order nuclear vibrational mode and emit quanta of vibrational energy called phonons.

A phonon carries 2 units of angular momentum and has even parity ($P = (-1)^2 = +1$)

$$\Rightarrow \quad J^P = 2^+$$

Octupole phonons carry 3 units of angular momentum and have odd parity ($P = (-1)^3 = -1$)

Phonons are bosons and must satisfy Bose-Einstein statistics (overall symmetric wavefunction under the interchange of two phonons):

Even-even ground state 0^+ —(1 phonon)→ 2^+

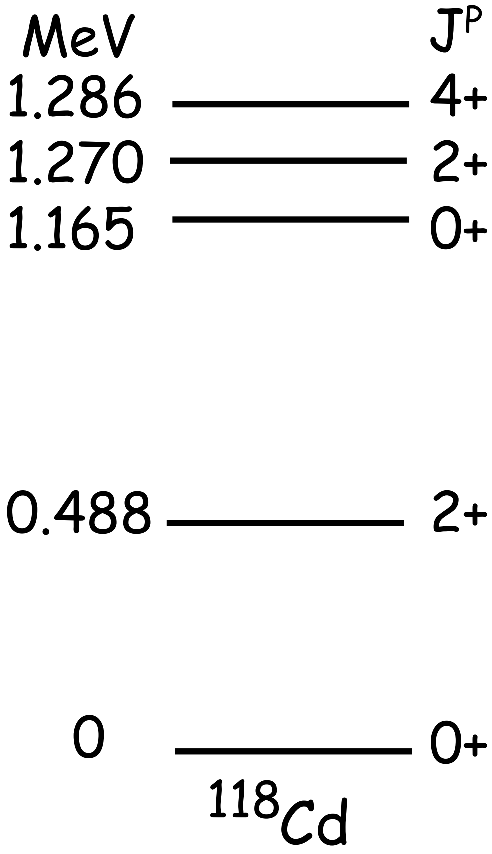
—(2 phonons)→ $0^+, 2^+, 4^+$

In practise not degenerate

Octupole states ($J^P=3^-$) are often seen near the 2 phonon triplet states.

Vibrational states decay rapidly by γ emission (see later).

Example



Energy of phonon is not predicted

Ratio $\dfrac{\text{2nd excited (2 phonons } 0^+, 2^+, 4^+)}{\text{1st excited (1 phonon } 2^+)} \approx 2$

Nuclear Rotations

Collective rotational motion can only be observed in nuclei with non-spherical equilibrium shapes (i.e. far from closed shells, large Q).

A rotating deformed nucleus is a stable equilibrium shape determined by nucleons in rapid internal motion in the nuclear potential with the entire nucleus rotating slowly so as not to effect the nuclear structure.

For even-even nuclei, ground state $J^P=0^+$

The nucleus mirror symmetry restricts the sequence of rotational states to even values of angular momentum.

$$\Rightarrow \boxed{J^P = 0^+ , 2^+ , 4^+, 6^+ \ \ldots\ldots}$$

(Total a.m. = nuclear a.m. + rotational a.m.)

Energy of a rotating nucleus

$$E = \frac{\hbar^2}{2I_{eff}} J(J+1)$$

where I_{eff} is the effective mom. of inertia.

The ratios of energies of rotational excited states are well predicted.

Example ^{164}Er

	keV
6^+	614.4
4^+	299.5
2^+	91.4
0^+	0

$$\frac{E(4^+)}{E(2^+)} = \frac{299.5}{91.4} = 3.28$$

Predict $$\frac{4(4+1)}{2(2+1)} = 3.33$$

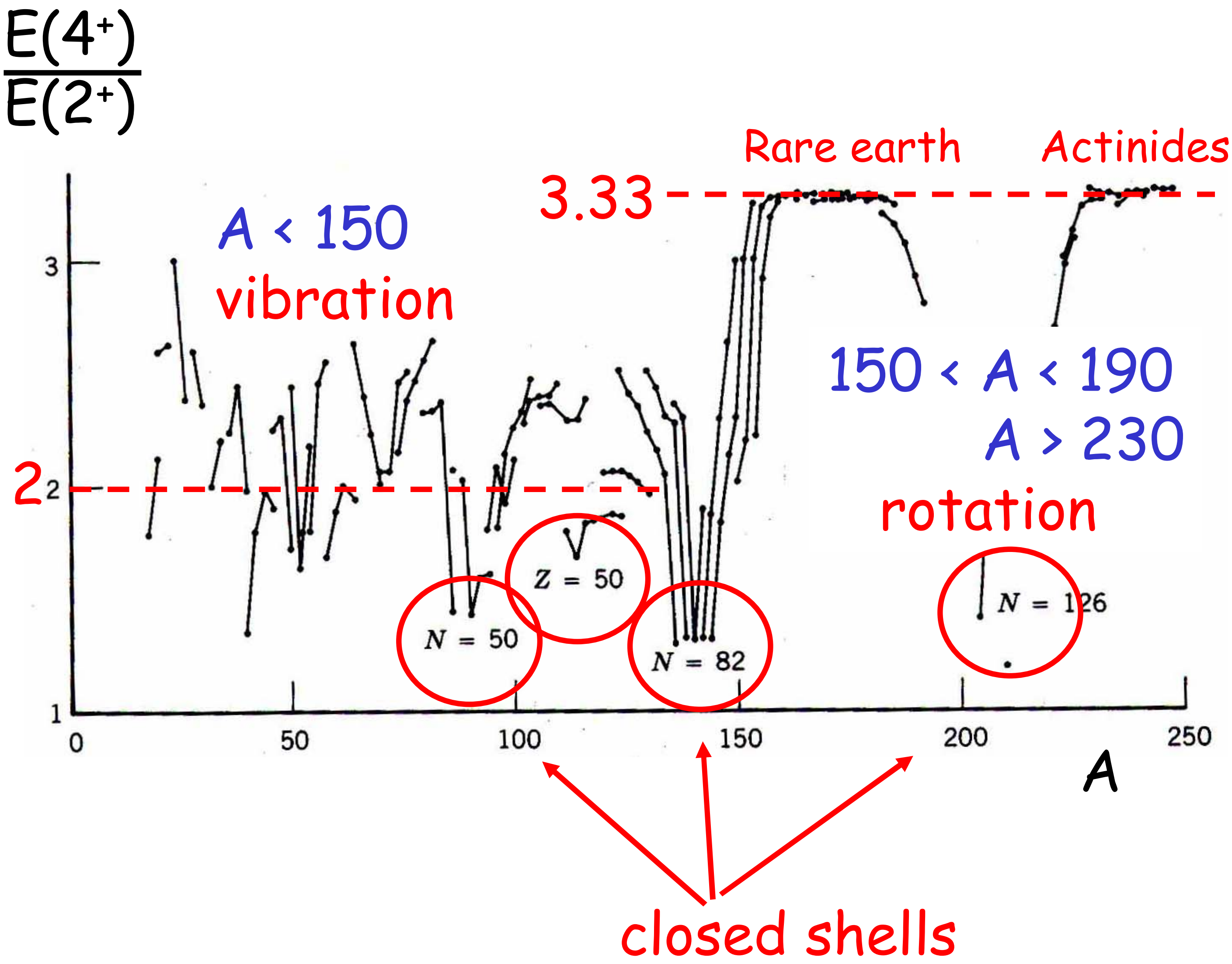
I_{eff} can be deduced from the absolute energies:

$$I_{rigid} > I_{eff} > I_{fluid}$$

Hence, the nucleus does not rotate like a rigid body. Only some of its nucleons are in collective motion.

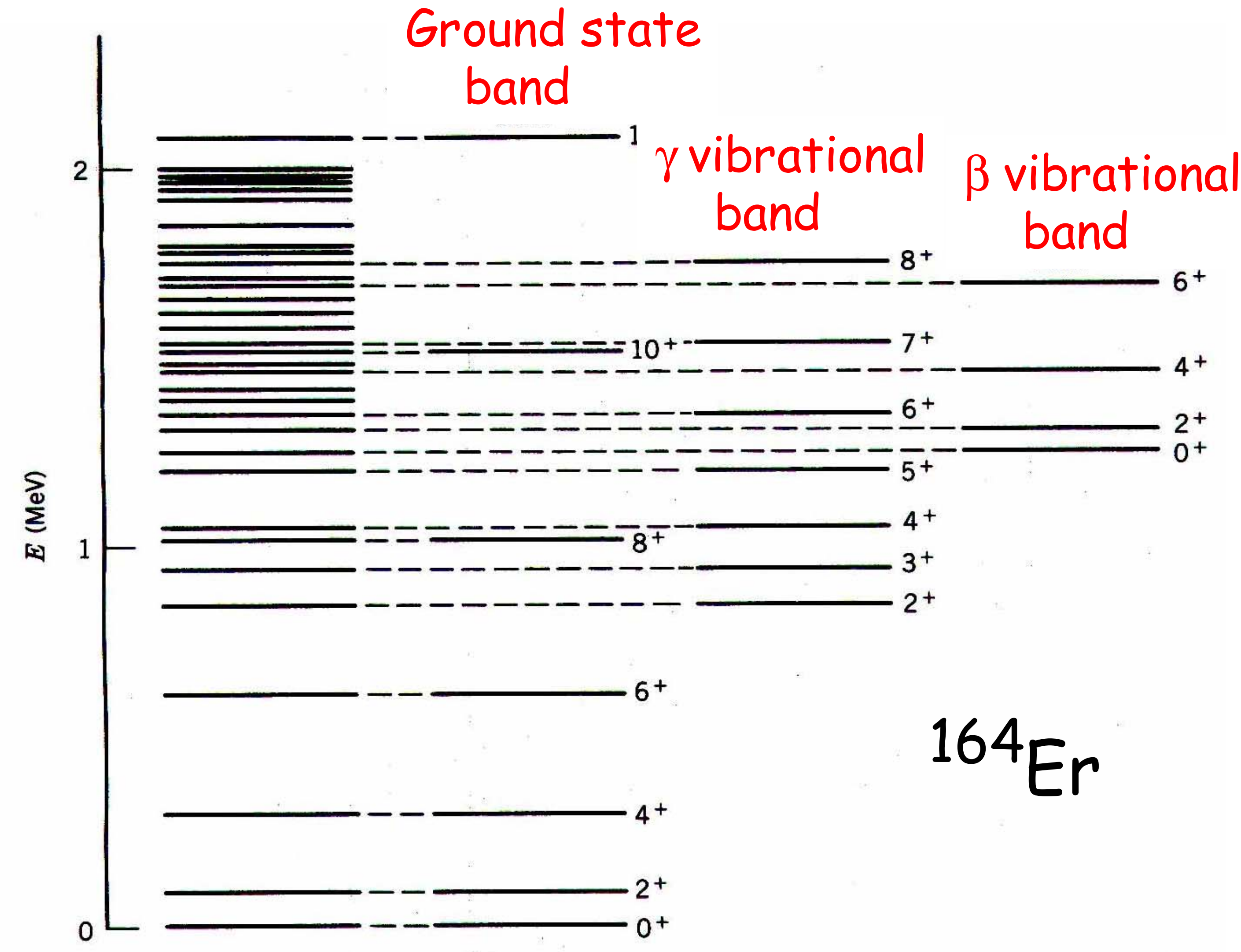
Rotational behaviour is intermediate between the nucleus tightly bonded and weakly bonded i.e. the strong force is not long range.

Even Z, even N



E(4+) / E(2+)
Rare earth
Actinides
3.33
A < 150
vibration
150 < A < 190
A > 230
rotation
2
N = 50
Z = 50
N = 82
N = 126
1
2
3
0
50
100
150
200
250
A
closed shells

Rotational bands can be based on any intrinsic state e.g. vibrational state in which nucleus vibrates about a deformed equilibrium shape.

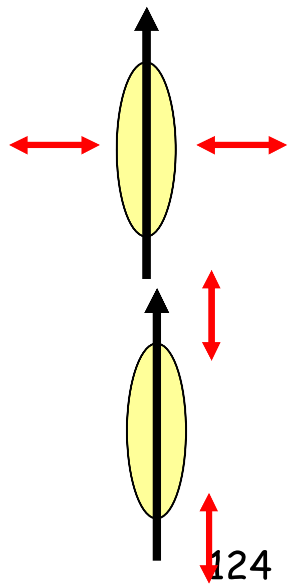


3 rotational bands

Ground state

γ vibration : surface v ⊥'r symmetry axis

β vibration : surface v along symmetry axis



Moments of Excited States

Consider even Z, even N nuclei

Ground state : $J^P = 0^+$, $\mu = 0$

Magnetic Dipole Moments

Vibrational and rotational collective motions give the nucleus a magnetic dipole moment.

Assume that the p's and n's are paired (↑↓) and the collective motion of the n's does not contribute. Then, the magnetic dipole moment of an excited nucleus is due to the angular momentum of the protons.

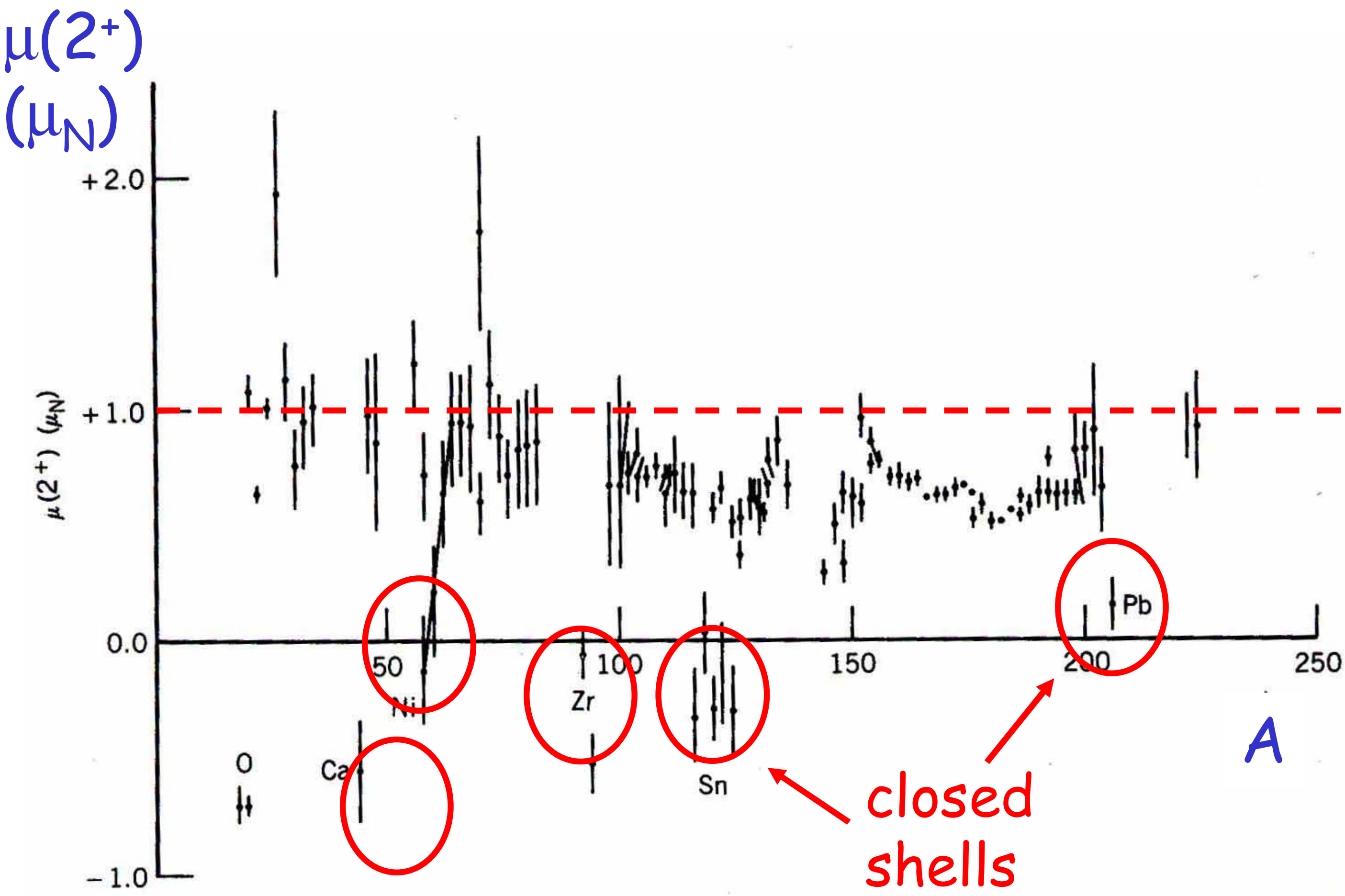
Single proton $\mu = \ell \mu_N$

All protons

$$\mu = \frac{Z}{A} J \mu_N$$

J = angular momentum of nuclear state

Magnetic moments for 1st excited state of even-even nuclei



Low A	$Z/A \sim 0.5$	$\mu \approx 1\ \mu_N$
High A	$Z/A \sim 0.4$	$\mu \approx 0.8\ \mu_N$

Reasonable agreement

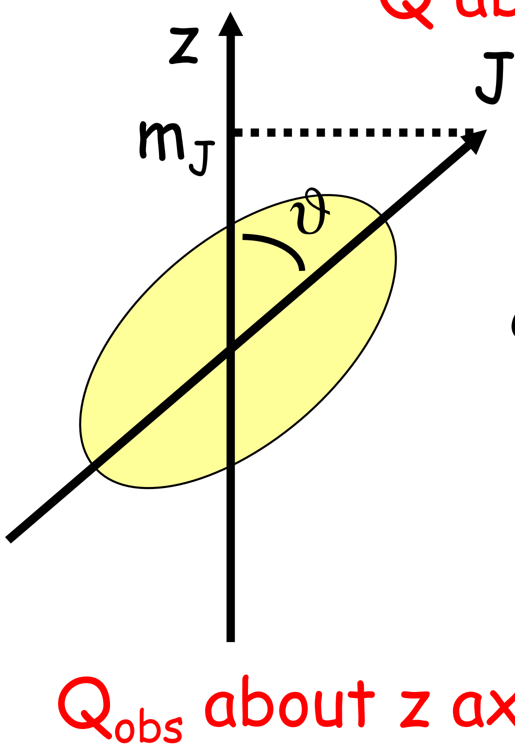
Electric Quadrupole Moments

Observation of rotational spectra for even-even nuclei ⇒ nucleus has intrinsic Q.

J=0 nuclei: have no particular orientation in space ∴ averaged over time the are observed as spherical i.e. $Q_{obs} = 0$.

J≠0 nuclei: rotational symmetry about J. Observability of Q depends on m_J.

Q about body axis



$$Q_{obs} = \frac{1}{2}\left(3\cos^2\vartheta - 1\right)Q$$

transform to lab frame

$$\cos\vartheta\Big|_{\vartheta=\min} = \frac{J}{\sqrt{J(J+1)}}$$

cannot align along z-axis

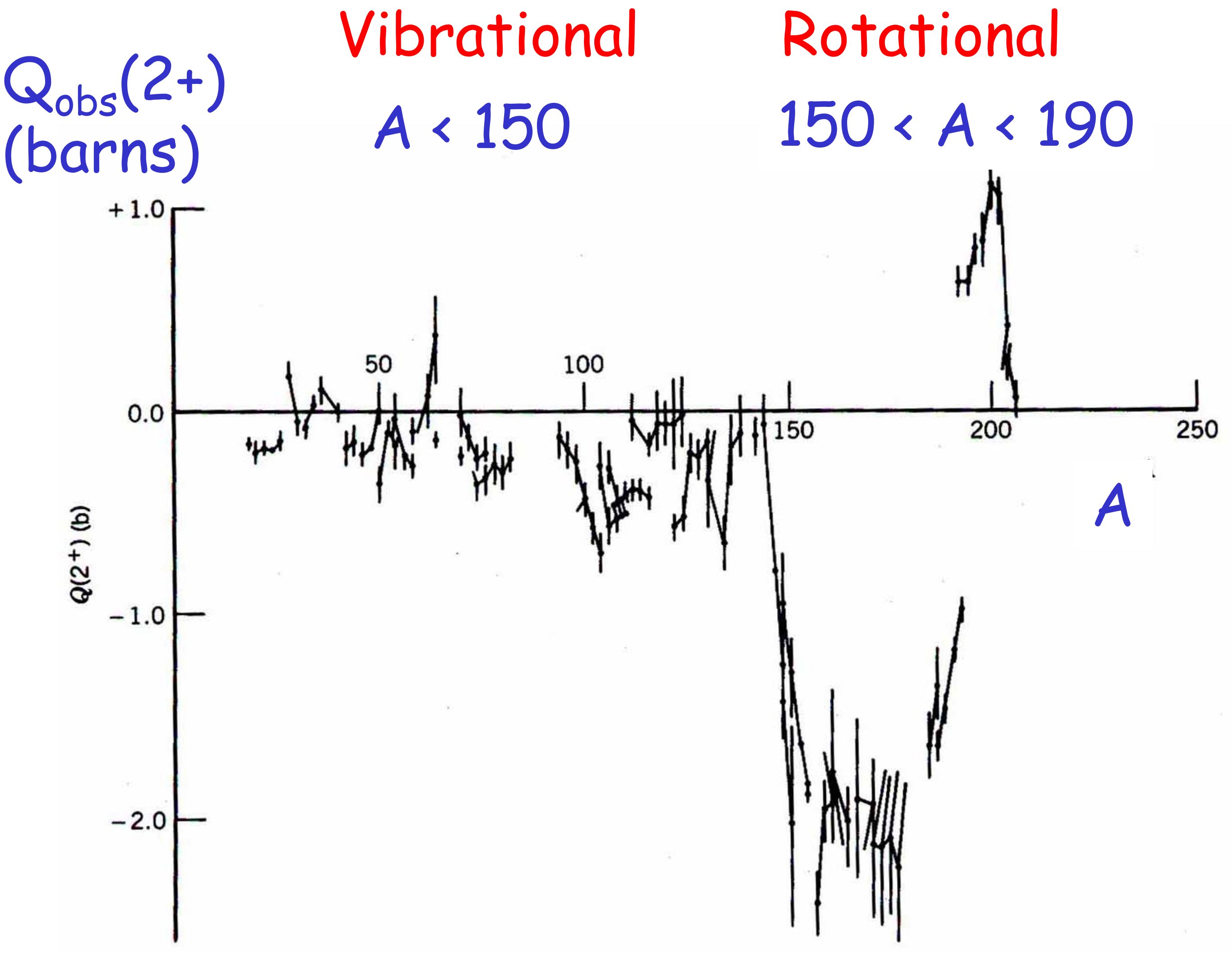
Q_{obs} about z axis

$$Q_{obs} = \left(\frac{3J^2}{2J(J+1)} - \frac{1}{2}\right)Q$$

$$= \frac{2J-1}{2(J+1)}Q$$

$J=1/2 \Rightarrow Q_{obs} = 0$, $J \geq 1$ observable Q

Electric quadrupole moments for 1^{st} excited states of even-even nuclei



Summary

The Nuclear Shell Model is successful in predicting

- Origin of magic numbers
- Spins and parities of ground states
- Trend in magnetic moments
- Some excited states near closed shells and for small excitations in odd A nuclei

In general, it is not good far from closed shells and non-spherically symmetric potentials

The collective properties of nuclei can be incorporated into the Nuclear Shell Model by replacing the spherically symmetric potential by a deformed potential.

Improved description for

- Even A excited states
- Electric quadrupole moments.

There are many more sophisticated models that work towards a theoretical understanding for all nuclear properties.

(see Cont. Physics 1994 vol. 35 No. 5 329-346)