Part II
Nuclear Physics

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Lent Term 2004

12 lectures M,W,F 9:00 a.m.

Contents

Nuclear Physics

Basic Nuclear Properties: Nuclear constituents, binding energy, mass, spin and parity, size, moments, NMR, radioactivity. Cross-sections, units and notation.

The Nucleon Force: General features, the deuteron, nucleon-nucleon scattering. Yukawa potential and meson exchange.

The Nuclear Shell Model: Magic numbers, the Nuclear Shell Model and its predictions, excited states of nuclei (vibrations and rotations).

Nuclear Decay: α decay. β decay, Fermi theory of β decay, parity violation. γ decay, Mössbauer effect.

Nuclear Reactions: Q values, types of reaction, compound nuclei, Breit-Wigner formula. Fission and reactors. Fusion. Nucleosynthesis and the solar ν problem.

Recommended Books

A sample of introductory books to Nuclear Physics is given below:

- *Introductory Nuclear Physics,* Krane KS (Wiley 1988). This covers most of the course material. Recommended.
- *Basic Ideas and Concepts in Nuclear Physics.*, Heyde K (IoP Publishing 1999).
- *Nuclear Physics: Principles and Applications*, Lilley J (Wiley 2002). This is very good for applications.

Introductory books to both Nuclear Physics and Particle Physics:

- *Nuclear and Particle Physics*, Bircham WE and Jobes M (Longman Scientific and Technical 1995)
- *The Physics of Nuclei and Particles*, Dunlop PA (Thomson Brooks/Cole 2003).
- *Introduction to High Energy Physics*, Perkins DH (4th edn CUP 2000). A useful introduction to Particle Physics.

Many older texts are very good, but frequently out of print:

- *Introduction to Nuclear Physics*, Enge HA (Addison-Wesley 1966, 69, 72).
- *The Atomic Nucleus,* Evans RD (McGraw-Hill 1955).
- *Nuclei and Particles,* Segré E (2nd edn Addison-Wesley 1977).
- *Theoretical Nuclear Physics,* Blatt JM & Weisskopf VF (Dover 1991)

Nuclear Data Sheet

Particle Masses

	kg	u	MeV/c^2
Electron	9.11×10^{-31}	5.486×10^{-4}	0.511
Proton	1.67×10^{-27}	1.00728	938.28
Neutron	1.68×10^{-27}	1.00867	939.57
$\pi^{\pm}$	2.49×10^{-28}	0.14983	139.57
π^{0}	2.41×10^{-28}	0.1449	134.98
μ	1.88×10^{-28}	0.1134	105.66

Atomic masses are often given in unified (or atomic) mass units

$$1 \text{ unified mass unit (u)} \equiv \frac{\text{Mass of an atom of } {}^{12}_{6}\text{C}}{12}$$

$$1\text{ u} = 1\text{g}/N_A = 1.66 \times 10^{-27}\text{ kg} = 931.5\text{ MeV}/c^2$$

Constants

electron charge = 1.602×10^{-19} C
$\hbar c$ = 0.197 GeV fm
$\hbar = 6.6 \times 10^{-25}$ GeV s
Fine structure constant $\alpha \approx 1/137$
Bohr magneton $\mu_B = 9.3 \times 10^{-24}$ JT^{-1}
Nuclear magneton $\mu_N = 5.1 \times 10^{-27}$ JT^{-1}

Conversion Factors

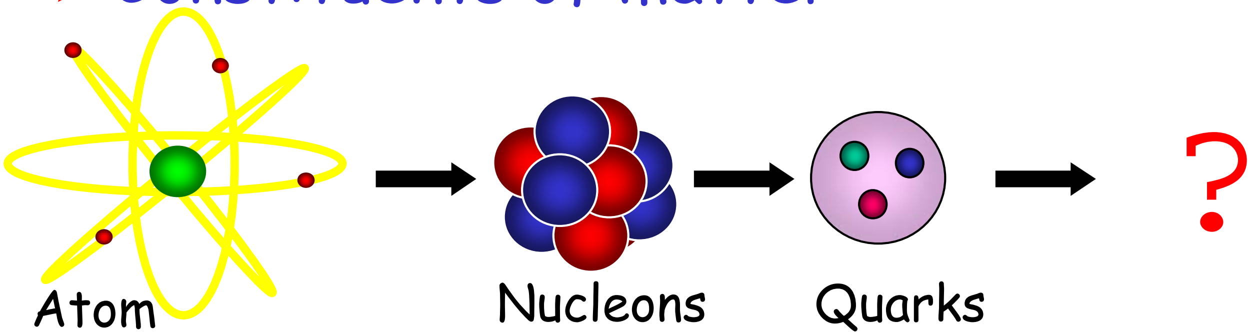
1 eV = 1.602×10^{-19} J, 1 MeV = 10^6 eV, 1 GeV = 10^9 eV
1 fermi (fm) = 10^{-15} m
1 barn (b) = 10^{-28} m^2
1 Curie (Ci) = 3.7×10^{10} decays/s

Section I
Introduction

Why Study Nuclear Physics ?

Nuclear processes play a fundamental role in the physical world:

- Origin of the Universe
- Creation of chemical elements
- Energy of stars
- Constituents of matter



Nuclear processes also have many practical applications:

- Uses of radioactivity in research, health and industry (e.g. NMR, radioactive dating)
- Nuclear power
- Various tools for the study of materials (e.g. Mössbauer, NMR)

Nuclear Force

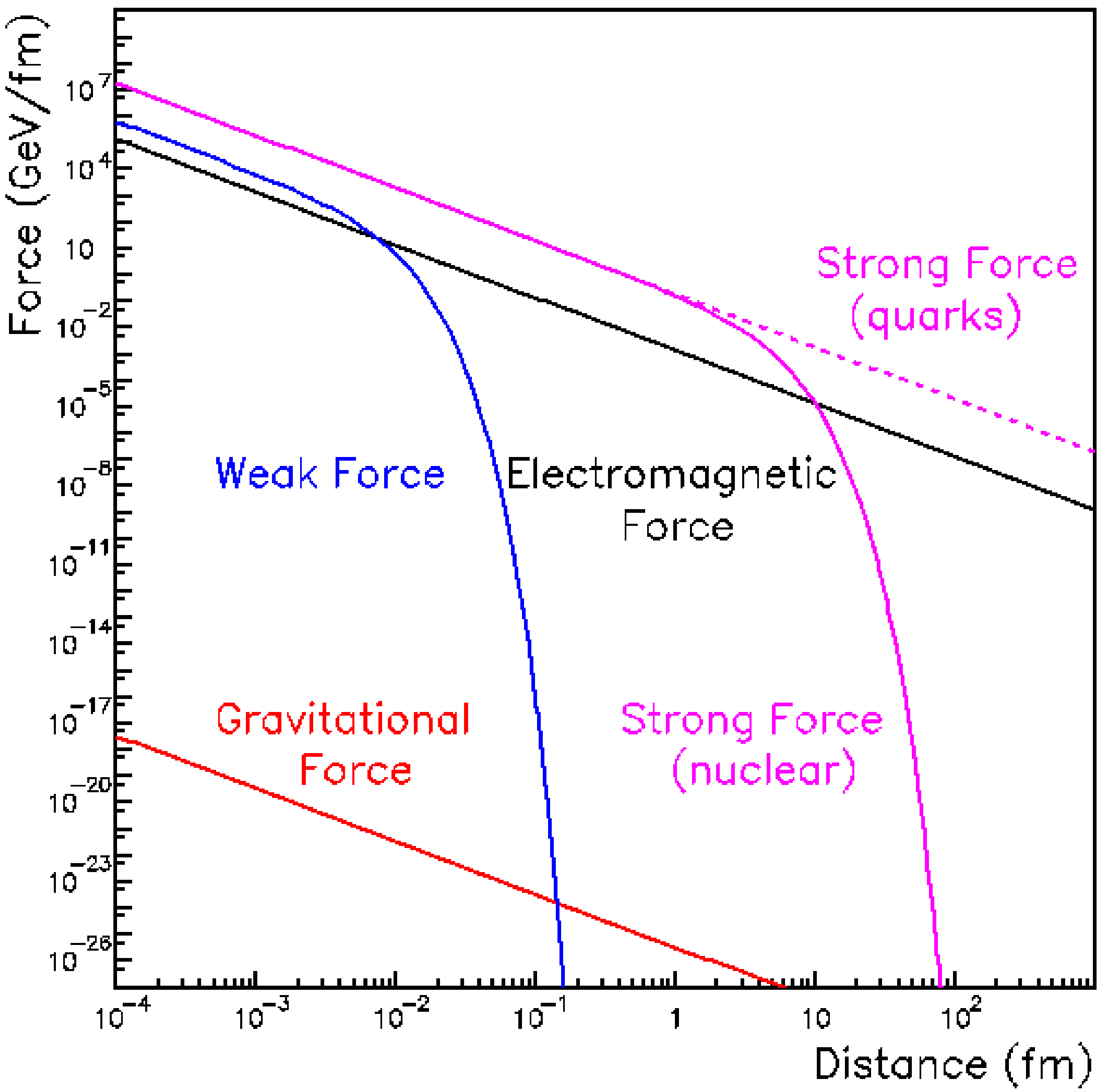
All particle interactions can be explained in terms of 4 basic forces:

Electromagnetic, weak, strong and gravity.

Nucleons experience the strong interaction at short distances (a few fm).

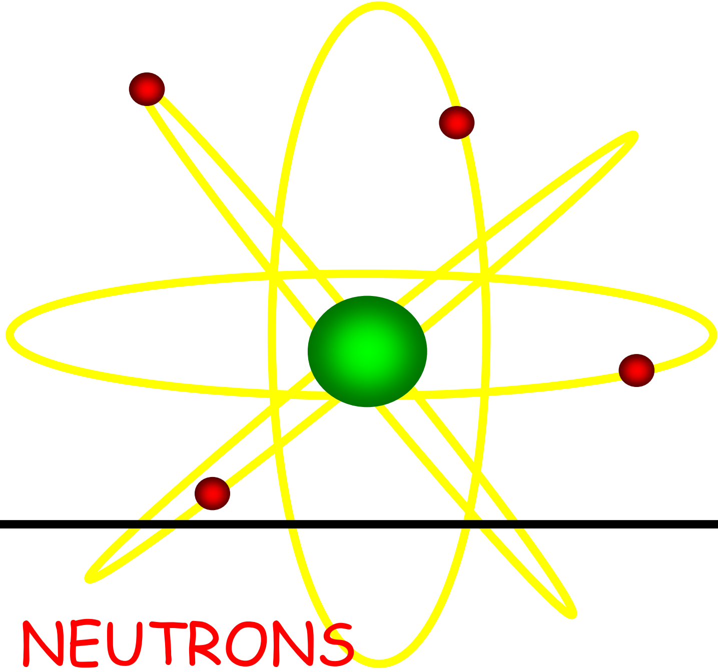
Force	*Strength*	*Quanta*	*Mass [GeV/c²]*	*Range [m]*
Strong	1	gluon	0	10^{-15}
Strong Nuclear		e.g. nucleons, π's	~0.14-2.5	
Electromagnetic	10^{-2}	γ	0	∞
Weak	10^{-5}	$W^{\pm}, Z^0$	~80, 91	10^{-18}
Gravity	10^{-38}	graviton(?)	0	∞

Strength of the fundamental forces



Section II
Basic Nuclear Properties

Constituents



Electron

m_e = 0.511 MeV/c^2

charge = -e

size ≤ 10^{-18} m

Nucleus

Z PROTONS, N NEUTRONS

Proton and neutron are 2 charge states of the nucleon.

A nuclide is a nucleus specified by Z, N.

A (Mass number) = Z (Atomic number) + N

$m_p \approx m_n$= 939.57 MeV/c^2; charge: p = +e, n = 0

size p,n ≈ 1 fm; radius of nucleus (medium A) ≈ 5fm

Atom

Normal state is neutral, Z electrons.

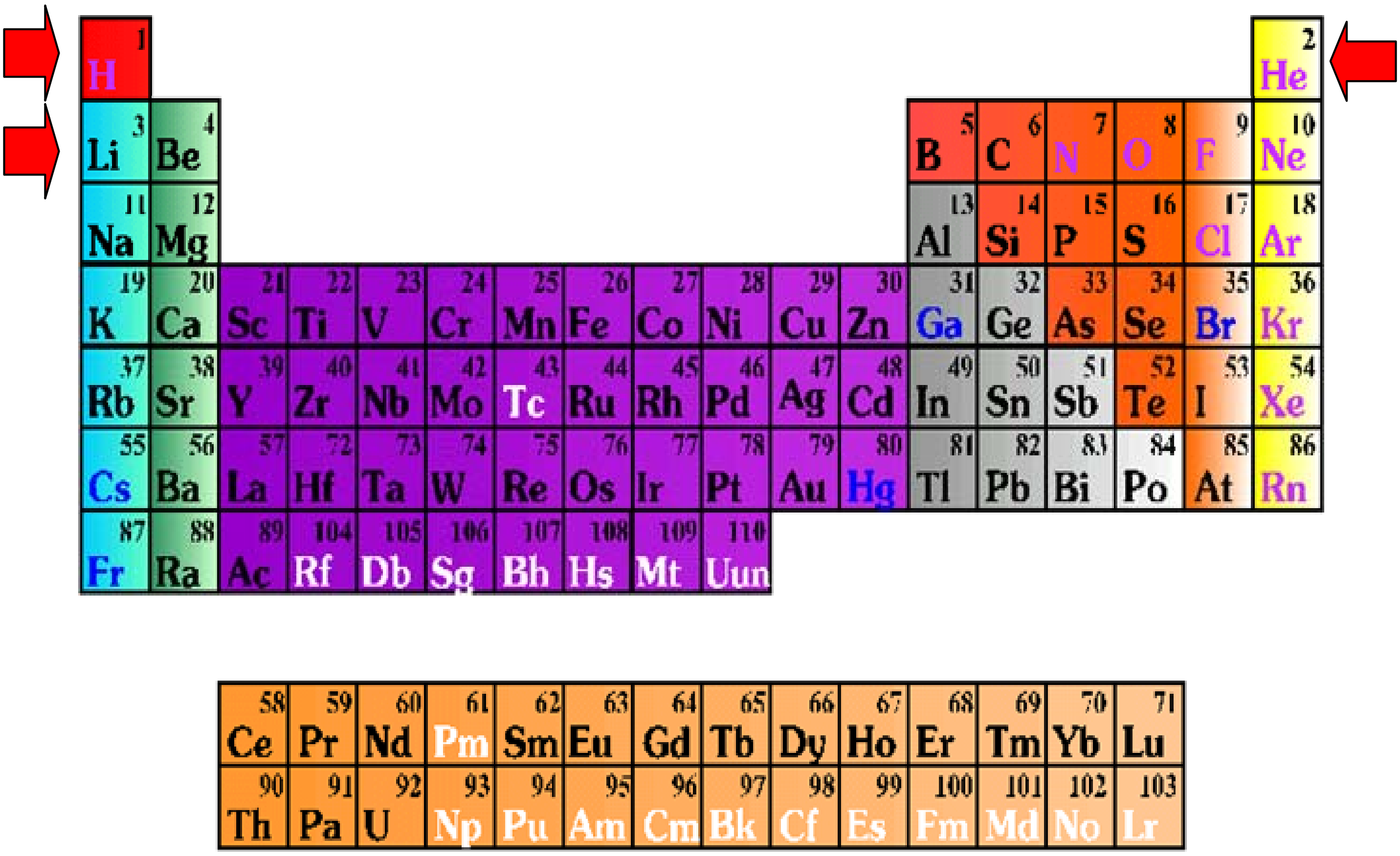
size ≈10^{-10} m

mass (m_p, $m_n \approx 1836\ m_e$) of atom is ≈ all in nucleus.

chemical properties depend on Z.

The Periodic Table of the Elements

Only three elements are formed in the Big Bang. <u>All</u> other elements are formed in stars.



Natural elements : H (Z=1) to U (Z=92)

Notation

NUCLIDE $^{A}_{Z}X$ where X is the chemical symbol

e.g. $^{7}_{3}Li$; Z=3, N=4, A=7

ISOTOPES Nuclides with identical Z

same chemical symbol

e.g. $(^{35}_{17}Cl, ^{37}_{17}Cl)$

ISOBARS Nuclides with identical A

ISOTONES Nuclides with identical N

Mirror Nuclei : Isobars in which N and Z are interchanged $(N = Z \pm 1)$

e.g. $(^{3}_{1}H, ^{3}_{2}He), (^{41}_{20}Ca, ^{41}_{21}Sc)$

Spectroscopic notation: $^{2S+1}L_J$

Binding Energy

Binding energy is the energy required to split a nucleus into its constituents.

$$\text{Mass of nucleus} = Z\,m_p + N\,m_n - B$$

Separation energy of nucleon is the energy required to remove one nucleon from a nucleus.

$$n: \quad B({}^{A}_{Z}X) - B({}^{A-1}_{Z}X) = m({}^{A-1}_{Z}X) + m_n - m({}^{A}_{Z}X)$$

$$p: \quad B({}^{A}_{Z}X) - B({}^{A-1}_{Z-1}X') = m({}^{A-1}_{Z-1}X') + m({}^{1}H) - m({}^{A}_{Z}X)$$

Binding energy is very important: gives information on

- forces between nucleons
- stability of nucleus
- energy released or required in nuclear decays or reactions.

Measurement of B:

- measure masses with mass spectrometer; deduce B
- study reaction or decay e.g. n capture on protons $\rightarrow d+\gamma$

Stable Nuclei

Stable nuclei are those nuclei which do not decay by the strong interaction, although they may transform by β and α emission.

- Tend to have N = Z for light nuclei.
- More have even N or Z, p's and n's tend to form pairs (8/284 have both N and Z odd).
- Certain values of Z - large no. isotopes
  Certain values of N - large no. of isotones

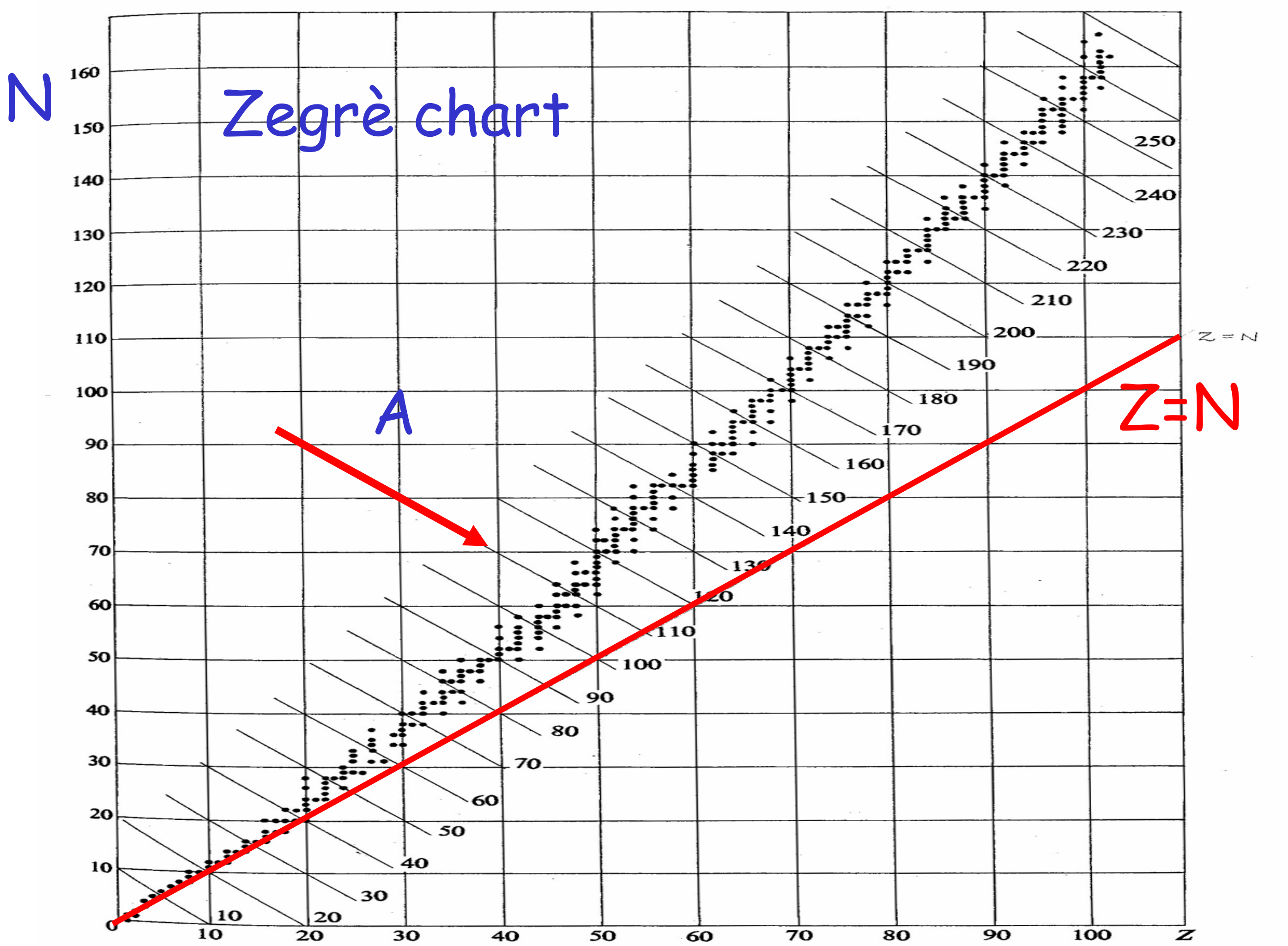
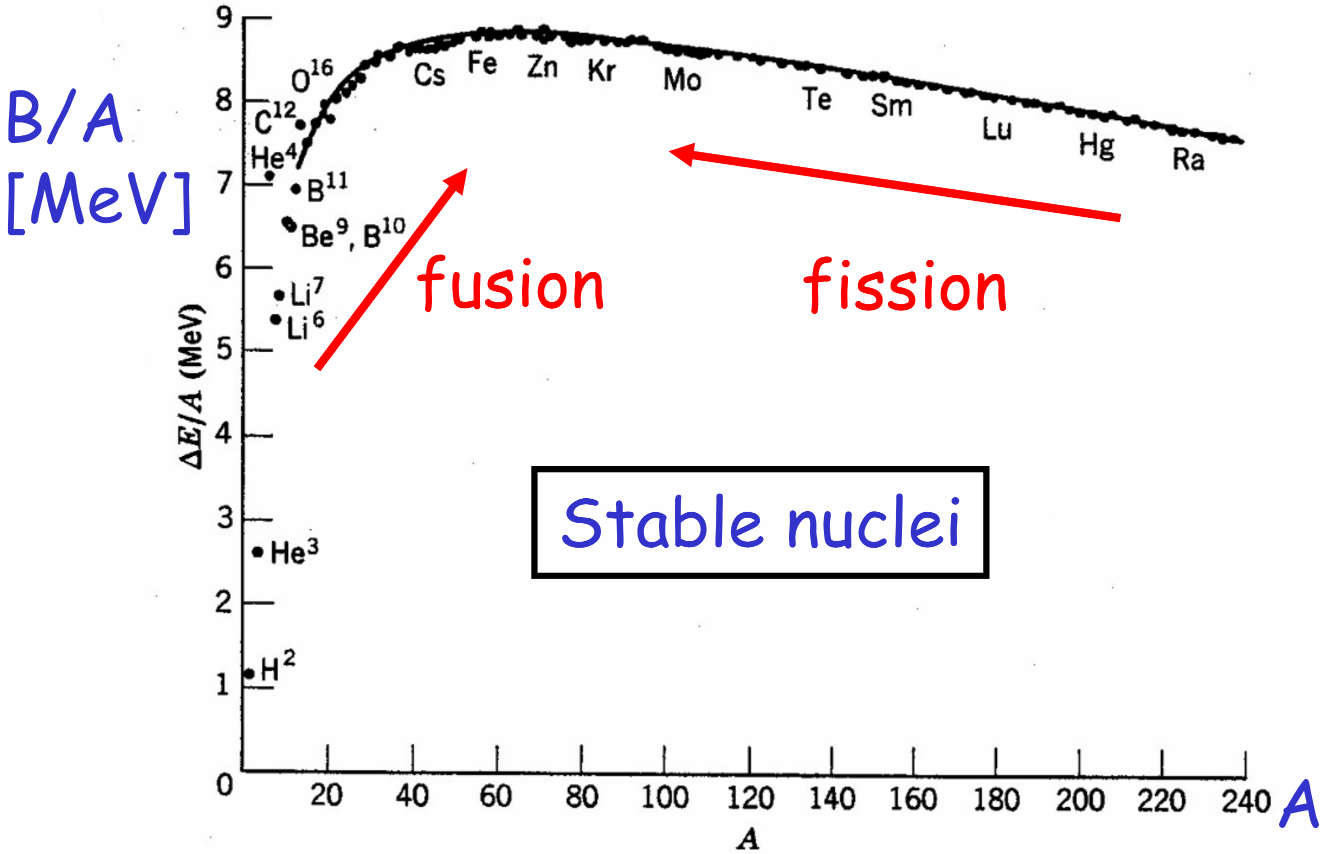


Figure 6-14 Chart of all β-stable nuclei in a Z–N plane.

Binding Energy/ Nucleon



- B/A ≈ constant ≈ 8 MeV per nucleon, A≥20
  B of electron per nucleon ≤ 3 keV
- Broad maxima at A ≈ 60 (Fe, Co, Ni)
  A ≤ 60 Fusion favoured
  A ≥ 60 Fission favoured
- Light nuclei with A=4n, n=integer, show peaks (α stability)

B/A ≈ constant ⇒ in a nucleus, the nucleons are only attracted by nearby nucleons. Nuclear force is short range and saturated.

Nuclear Mass

Atomic mass

$$M(A,Z) = Z\,(m_p + m_e) + (A-Z)\,m_n \;-\; B$$

Nuclear mass

$$m(A,Z) = Zm_p + (A-Z)m_n \;-\; B$$

Liquid Drop Model : Approximate the nucleus as a sphere with a uniform interior density, that drops to zero at the surface.

Liquid drop	Nucleus
Intermolecular forces short range	Nuclear force short range
Density indep. of drop size	Density indep. of nuclear size
Heat required to evaporate fixed mass indep. of drop size.	$B/A \approx$ constant

$$B = a_V A - a_S A^{2/3} - \frac{a_C Z^2}{A^{1/3}}$$

$+a_V A$ **Volume Term:** Strong force between nucleons increases B and reduces mass by a constant amount per nucleon.
Nuclear volume ~ A

$-a_S A^{2/3}$ **Surface term:** Nucleons on surface are not as strongly bound
$\Rightarrow$ decrease B
Surface area ~ R^2 ~ $A^{2/3}$

$-a_C \frac{Z^2}{A^{1/3}}$ **Coulomb Term:** Protons repel each other $\Rightarrow$ reduce B
Electrostatic P.E. ~ $\frac{Q^2}{R}$ ~ $\frac{Z^2}{A^{1/3}}$

Basic liquid drop model does not account for two other observations:

- $N \approx Z$
- Nucleons tend to pair

Understand these using the Fermi Gas Model in which confined nucleons can only have certain discrete energies in accordance with the Pauli exclusion principle.

$$-a_a \frac{(N-Z)^2}{A}$$

Asymmetry Term: Nuclei tend to have $N \approx Z$.

Kinetic energy of Z protons and N neutrons is minimized if N=Z. The greater the departure from N=Z, the smaller the binding energy. Correction scaled down by 1/A as levels are more closely spaced as A increases.

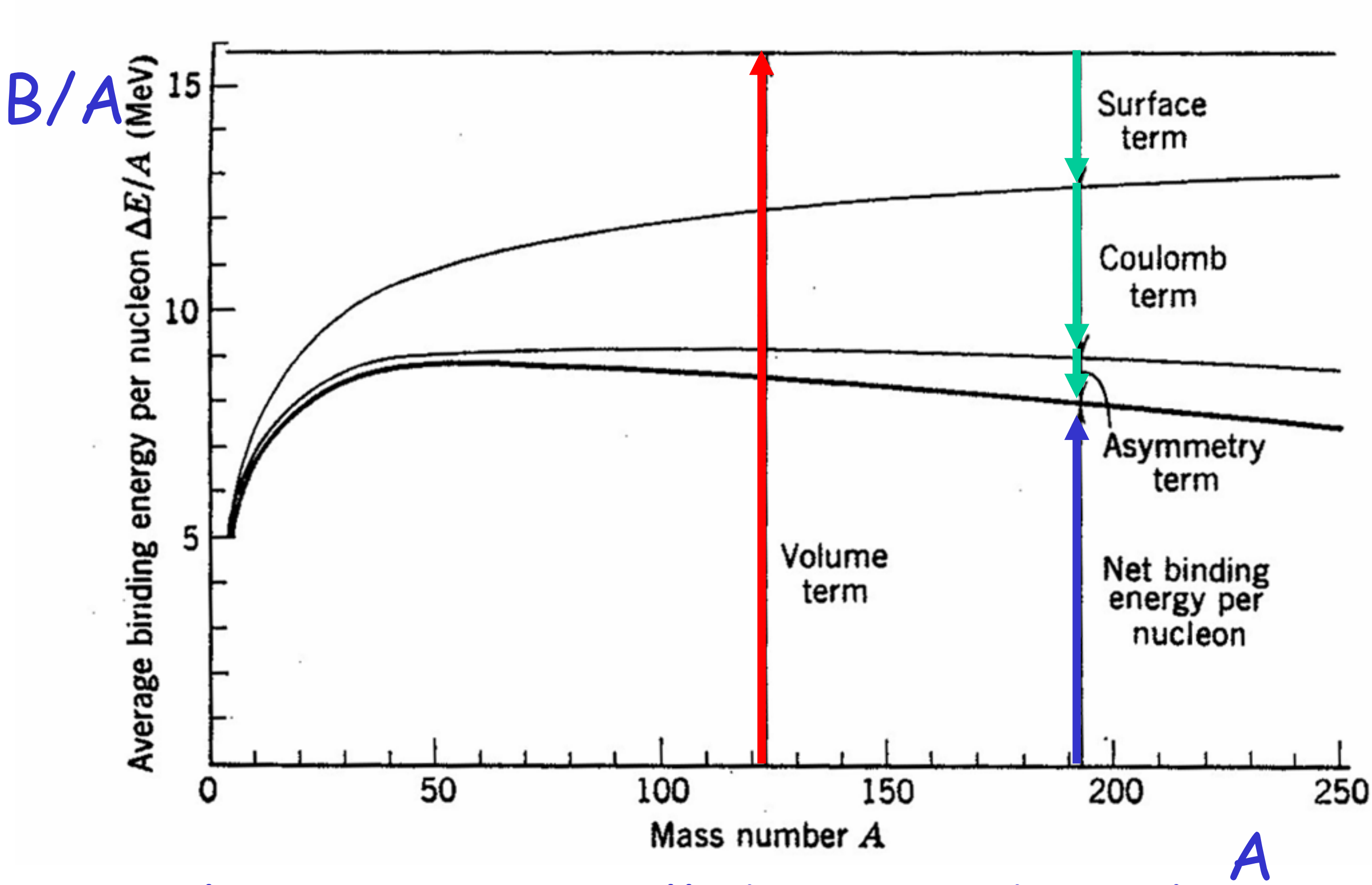
$$+\delta(A)$$

Pairing Term: Nuclei tend to have Z even, N even.

Pairing interaction energetically favours the formation of pairs of like nucleons (pp,nn) with spins ↑↓ and symmetric space wavefunction.

	N/Z
$\delta(A) = +a_p A^{3/4}$	even-even
$= -a_p A^{3/4}$	odd-odd
$= 0$	even-odd

Contributions to B/A



Nuclear mass is well described by the

SEMI-EMPIRICAL MASS FORMULA

(Weizsäcker)

$$m(A,Z) = Z\,m_p + (A-Z)\,m_n - B$$

$$B = a_v A - a_s A^{2/3} - \frac{a_c Z^2}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + \delta(A)$$

with the following coefficients (in MeV)

$a_v = 15.8$, $a_s = 18.0$, $a_c = 0.72$,
$a_a = 23.5$, $a_p = 33.5$

Nuclear Spin and Parity

The nucleus is an isolated system and so has a well defined nuclear spin.

Nuclear spin quantum number = J sometimes called "I"

$$|\vec{J}| = \sqrt{J(J+1)}\,\hbar$$

$$m_J = -J, -(J-1), \ldots, J-1, J.$$

Nuclear spin is the sum of the individual nucleons total angular momentum, j,

$$\vec{J} = \sum_i \vec{j}_i \qquad \text{(jj coupling)}$$

where the total angular momentum of a nucleon is the sum of its intrinsic spin and orbital angular momentum

$$\vec{j} = \vec{\ell} + \vec{s}$$

- intrinsic spin of p or n, $s = 1/2$ units $\hbar$
- orbital angular momentum of nucleon is integer

A even $\rightarrow$ J = integer
A odd $\rightarrow$ J = 1/2 integer
All nuclei with even N and even Z have J=0

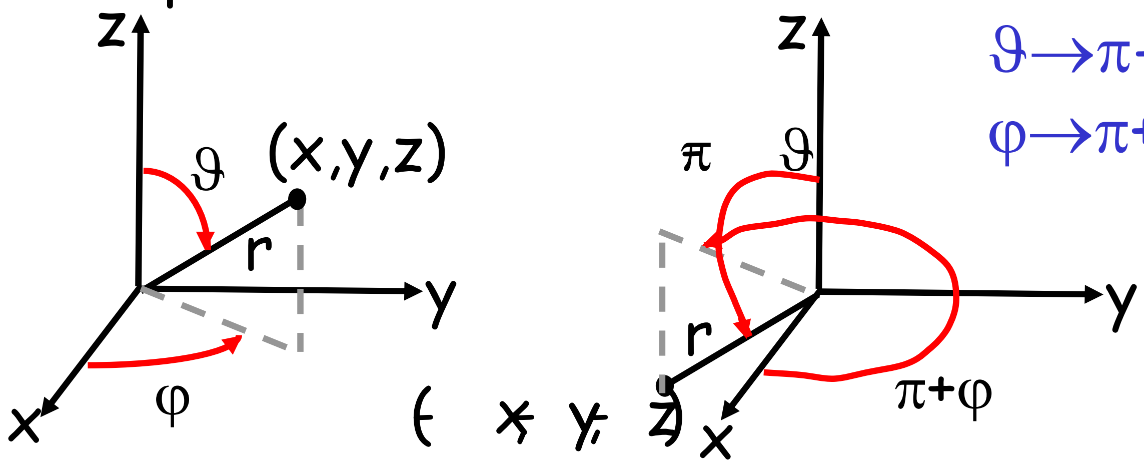
Parity

In a symmetric potential, $V(\vec{r}) = V(-\vec{r})$, wavefunctions have a definite parity

$\psi(\vec{r}) = +\psi(-\vec{r})$ even parity

$\psi(\vec{r}) = -\psi(-\vec{r})$ odd parity

Spherical polar coordinates:



$r \to r$

$\vartheta \to \pi - \vartheta$

$\varphi \to \pi + \varphi$

In general,

$$\psi_{n\ell m_\ell}(r, \pi - \vartheta, \pi + \varphi) = (-1)^\ell \psi_{n\ell m_\ell}(r, \vartheta, \varphi)$$

$[\psi(\vec{r})]^2 = [\psi(-\vec{r})]^2$ is obeyed by all nuclear wavefunctions

Parity is conserved in nuclear processes

Nuclear states are labelled with nuclear spin and parity quantum numbers

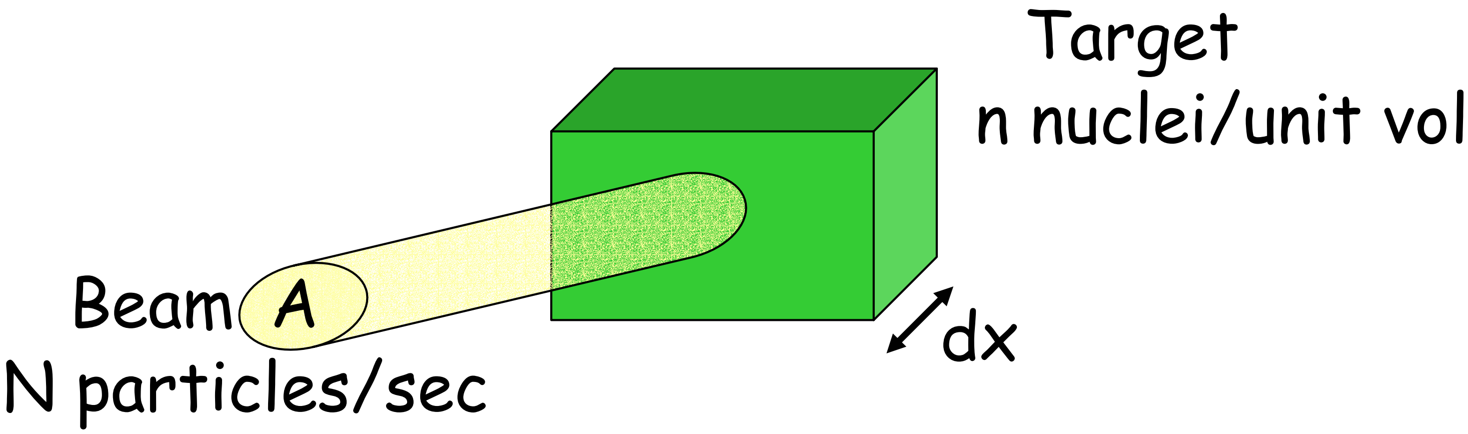
example: 0^+ (J=0, parity even),
2^- (J=2,parity odd)

Cross-section

A cross-section is an area associated with each nucleus through which an incident particle must be considered to pass if it has to cause a specified interaction. It is a property of both particles involved in the collision.

A cross-section is a measure of a probability of a particular process.

Units: σ 1 barn (b) = 10^{-28} m^2 Area



Number of nuclei in area A $= n\,A\,dx$

Effective area for absorption $= \sigma\, n\, A\, dx$

Rate at which particles are removed from beam $= dN = \frac{N}{A}\,\sigma\, n\, A\, dx$

$$-\frac{dN}{N} = \sigma\, n\, dx$$

$$\sigma = \frac{\text{N}^{\text{o}}\text{ scattered particles /sec}}{N\, n\, dx}$$

Beam attenuation in a target (thickness L)

- thick target ($\sigma n L \gg 1$)

$$\int_{N_i}^{N_f} -\frac{dN}{N} = \int_0^L \sigma\, n\, dx$$

$$N_f = N_i\, e^{-\sigma n L}$$

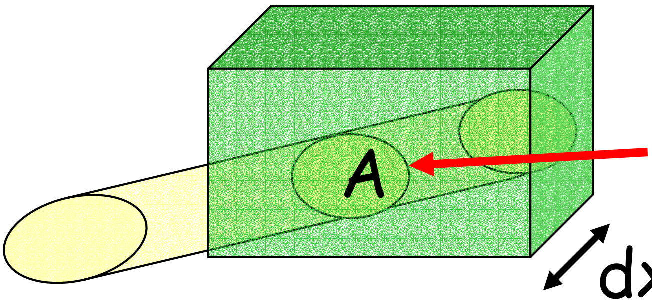
- thin target ($\sigma n L \ll 1$, $e^{-\sigma n L} \approx 1 - \sigma n L$)

$$N_f = N_i (1 - \sigma n L)$$

Mean free path between interaction = $1/n\sigma$

Flux of incident particles is the number passing through unit area per second.

Flux, $F = \frac{N}{A}$



Number of target particles in cylindrical volume

dx $\quad N_T = n * \text{Volume} = n\,A\,dx$

$$\sigma = \frac{\text{N}^o \text{ scattered particles /sec}}{N\, n\, dx}$$

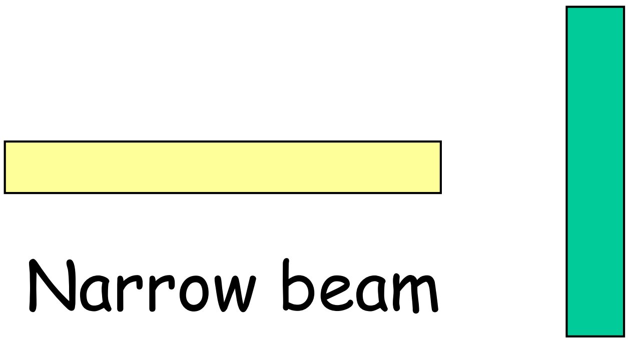
$$= \frac{\text{N}^o \text{ scattered particles /sec}}{(FA)\,(N_T/A\, dx)\, dx}$$

$$\sigma = \frac{\text{N}^o \text{ scattered particles /sec}}{\text{Flux} * \text{Number target particles}}$$

When discussing beam particles have used

- Number beam particles/sec
- Flux = number of beam particles /unit area/sec

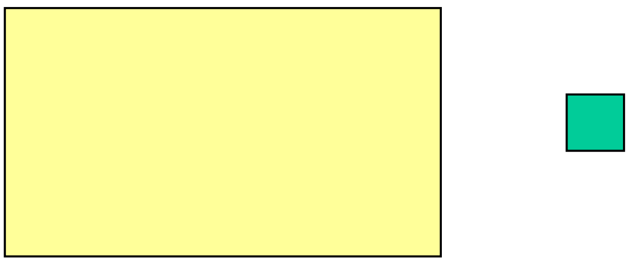
Different terminology appropriate in different physical configurations:



Narrow beam

Large target area

Number beam particles/sec and target density



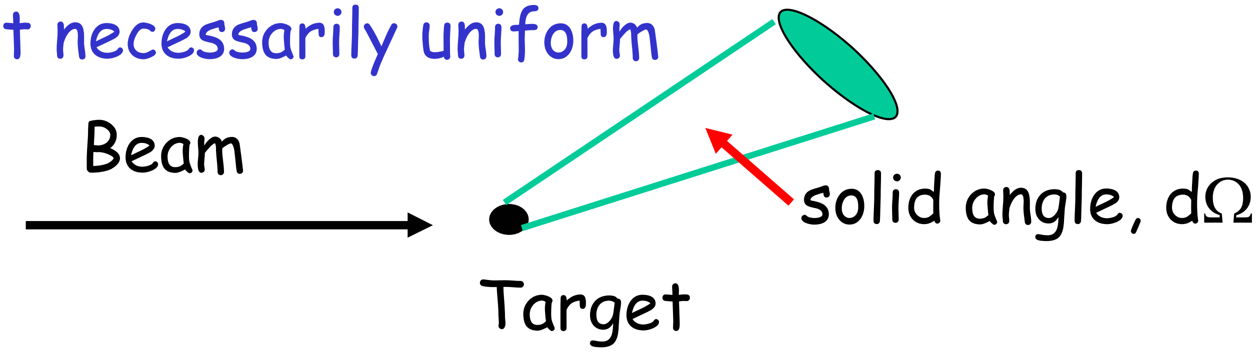
Small target area

Broad beam

Flux of beam and total number of target particles natural terminology
e.g. cosmic ray experiments.

Differential Cross-section

The angular distribution of scattered particles is not necessarily uniform



Number of particles scattered into dΩ = ΔN_Ω

$$\Delta N_\Omega = d\sigma * F * N_T$$

$$\frac{d\sigma}{d\Omega} = \frac{\Delta N_\Omega}{F * N_T * d\Omega}$$

$d\sigma/d\Omega$ is the differential cross-section

Units: area/steradian

The differential cross-section is the number of particles scattered per unit time and solid angle divided by the incident flux and by the number of target nuclei defined by the beam area.

In general, we deal with $d\sigma/d\Omega$. Nuclear physics experiments do not usually cover 4π.

Angular distributions provide more information about the mechanism of interaction.

Many different types of interaction can occur between nuclei.

$$\text{Total} \quad \sigma = \sum_i \sigma_i$$

where the σ_i are called partial cross-sections.

Types of interaction:

Elastic scattering:

$$a + b \to a + b$$

only momenta of a and b change

Inelastic scattering:

$$a + b \to c + d + \ldots\ldots$$

final state not the same as initial state

Units

"Natural Units"

- Set $\hbar=c=1$ and choose energy as basic unit of measurement

Energy	GeV	Time	$(\text{GeV}/\hbar)^{-1}$
Momentum	GeV/c	Length	$(\text{GeV}/\hbar c)^{-1}$
Mass	GeV/c^2	Cross-sect.	$(\text{GeV}/\hbar c)^{-2}$

(often abbreviated to GeV or GeV^{-1})

- Convert back to SI units using

  $\hbar c = 0.197$ GeV fm

  $\hbar = 6.6 \times 10^{-25}$ GeV s

  Cross-sections are usually quoted in barns

  $1\text{ b} = 10^{-28}\text{ m}^2$

- Charge: set $\varepsilon_0=1$ "Heavy-side-Lorentz" units.

  Use dimensionless fine structure constant

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

- Relativistic dynamics

$$E^2 = p^2c^2 + m^2c^4 \rightarrow E^2 = p^2 + m^2$$

  Note: when $E \gg m$, $E \approx p$.

Example

a) Compton wavelength of pion.

b) Rutherford scattering of 200 MeV e^-'s from a gold nucleus (Z=79) at 180 degrees.

Nuclear Size

The "size" of nuclei can be determined using two sorts of interaction:

Electromagnetic interaction gives the charge distribution of protons inside the nucleus.

e.g.

- Electron scattering
- Muonic atoms
- Mirror nuclei

Strong nuclear interaction gives the matter distribution of protons and neutrons inside nucleus. N.B. nuclear and charge interactions at the same time $\rightarrow$ more complex.

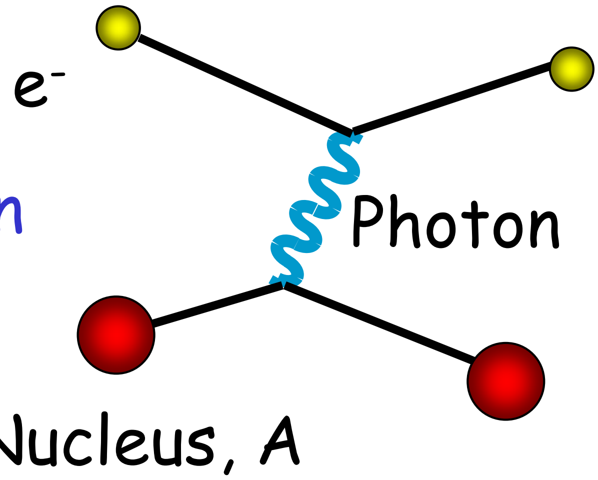
e.g.

- α particle scattering (Rutherford)
- proton scattering
- neutron scattering and absorption
- Lifetime of α particle emitters (see later)
- π^- mesic X-rays.

Find charge and matter radii equal for all nuclei.

Electron Scattering

Use electron as a probe to study deviations from a point-like nucleus.



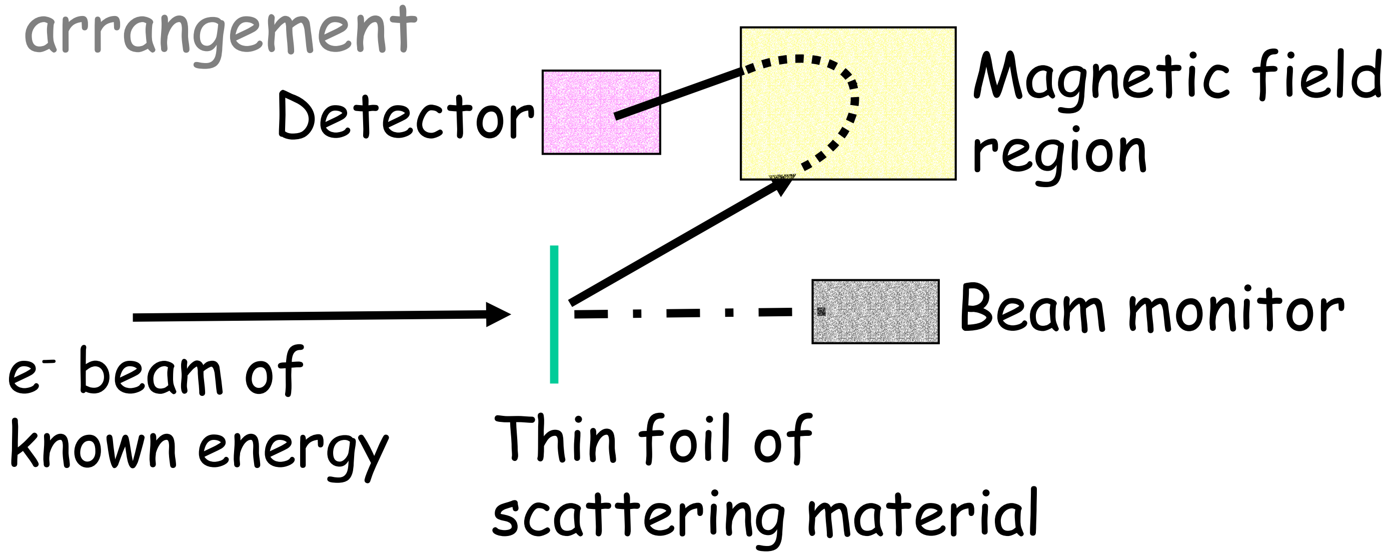
Electromagnetic interaction

To measure distance of ≈ 1 fm need energy

$$E = \frac{1}{\lambda} = 1\ \text{fm}^{-1} \approx 200\ \text{MeV}$$

Measure E, ϑ for scattered e^- → $d\sigma/d\Omega$

Experimental arrangement



$d\sigma/d\Omega$ is calculated using the Born approx in which initial and final states are considered as plane waves and nuclear recoil is ignored.

$$\frac{d\sigma}{d\Omega} = \frac{\text{Number of particles scattered/ unit time into } d\Omega}{\text{Incident Flux} * d\Omega}$$

Transition Rate $\quad \Gamma_{i\to f} = 2\pi |M|^2 \rho(E)$ Fermi Golden Rule $\quad \hbar=1$

$$M = \langle \psi_f | \hat{H} | \psi_i \rangle$$

$\rho(E) =$ density of final states

Wavefunction $\quad \Psi = N e^{i\vec{p}\cdot\vec{r}} \quad$ 1st order P.T.

Normalise wavefunction to 1 particle in a box of side L

$$|\Psi|^2 = N^2 = \frac{1}{L^3}$$

Density of States: is the number of states an e^- can occupy in momentum range $p \to p+dp$.

Particle in a box of side L, periodic boundary conditions

$$k_x = \frac{2\pi n_x}{L} \quad \text{etc} \qquad n_x, n_y, n_z \text{ integers}$$

$$\vec{p} = \left(\frac{2\pi n_x}{L}, \frac{2\pi n_y}{L}, \frac{2\pi n_z}{L} \right) \qquad \hbar=1$$

Each state occupies volume $\left(\frac{2\pi}{L}\right)^3$ in p space.

Number of states between $p \rightarrow p+dp$ in solid angle $d\Omega$

$$dN = p^2\, dp\, d\Omega / (2\pi/L)^3$$

$$\rho(p_f) = \frac{dN}{dp} = p^2\, d\Omega / (2\pi/L)^3$$

Relativistic scattering: $(E \approx p)$

$$\rho(E) = \frac{dN}{dE} = \frac{dN}{dp}\frac{dp}{dE} = \frac{E^2}{(2\pi)^3}\, d\Omega\, L^3$$

Matrix element

$$M = \left\langle \psi_f^* \middle| \hat{H} \middle| \psi_i \right\rangle = \int \psi_f^* \hat{H} \psi_i \, d^3\vec{r}$$

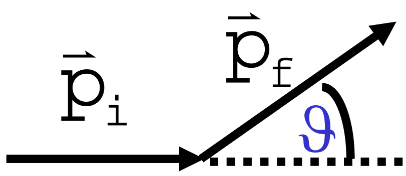
$$= \int N e^{-i\vec{p}_f \cdot \vec{r}} V(\vec{r}) N e^{i\vec{p}_i \cdot \vec{r}} \, d^3\vec{r}$$

$$= \frac{1}{L^3} \int e^{i\vec{q}\cdot\vec{r}} V(\vec{r})\, d^3\vec{r}$$

where $\vec{q} = \vec{p}_i - \vec{p}_f$ is the momentum transfer.

Elastic scattering

$$|\vec{p}_i| = |\vec{p}_f| = p$$



$$|\vec{q}|^2 = |\vec{p}_i - \vec{p}_f|^2$$

$$= p_i^2 + p_f^2 - 2 p_i p_f \cos\vartheta$$

$$= 2p^2 (1 - \cos\vartheta)$$

$$= 4E^2 \sin^2 \vartheta/2$$

Flux = number of incident particles passing unit area per second

Consider target of area A and incident beam of velocity v=c moving towards target.

Any incident particle within volume cA will pass target every second.

$$\text{Flux} = \frac{cA}{A} n_i = c n_i$$

where n_i is the number density of incident particles = 1 per L^3

$$\text{Flux} = \frac{c}{L^3} = \frac{1}{L^3} \qquad (c=1)$$

$$d\sigma = \frac{1}{\text{flux}} 2\pi |M|^2 \rho(E_f)$$

$$= L^3\, 2\pi \left| \frac{1}{L^3} \int e^{i\vec{q}\cdot\vec{r}} V(\vec{r})\, d^3\vec{r} \right|^2 E^2 \left(\frac{L}{2\pi}\right)^3 d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2} \left| \frac{1}{L^3} \int e^{i\vec{q}\cdot\vec{r}} V(\vec{r})\, d^3\vec{r} \right|^2$$

Born Approximation

▸ Scattering from a point-like nucleus

$$V(\vec{r}) = -\frac{Z\alpha}{r}$$

Cancelling normalization factor L^3

$$M = \int e^{i\vec{q}\cdot\vec{r}} V(\vec{r})\, d^3\vec{r}$$

$$= \int_0^{\infty} 2\pi r^2 V(r) \frac{2\sin qr}{qr} dr$$

The integral is ill-defined (keeps oscillating).

Hence, use $V(\vec{r}) = -\frac{Z\alpha}{r} e^{-r/a}$ ($V \to 0$ at $r \to \infty$)

(includes screening) to perform integral.

$$M = \int_0^{\infty} 2\pi r^2 \left(-\frac{Z\alpha}{r} e^{-r/a}\right)\left(\frac{e^{iqr} - e^{-iqr}}{iqr}\right) dr$$

$$= -\frac{4\pi Z\alpha}{1/a^2 + q^2} = -\frac{4\pi Z\alpha}{q^2} \qquad a \to \infty$$

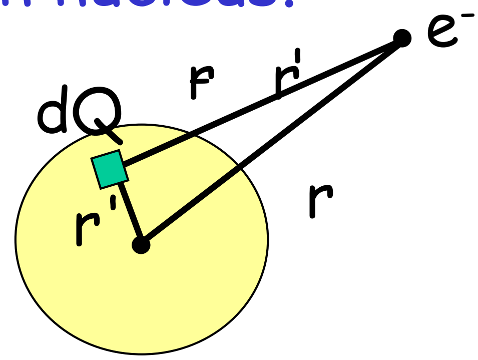
$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2} \frac{(4\pi Z\alpha)^2}{q^4}$$

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4E^2 \sin^4 \vartheta/2}$$

Rutherford scattering

▸ Scattering from an extended nucleus

Let $V(\vec{r})$ depend on the distribution of charge in nucleus.



Potential energy of electron due to charge dQ

$$dV = -\frac{e\,dQ}{4\pi|\vec{r}-\vec{r}\,'|} \qquad \varepsilon_0 = 1$$

$$dQ = Ze\rho(\vec{r}\,')\,d^3\vec{r}\,' \qquad \rho(\vec{r}) \text{ charge distribution}$$

$$V(\vec{r}) = \int -\frac{e^2 Z\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\,d^3\vec{r}\,' = -Z\alpha\int\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3\vec{r}\,'$$

$$M = -Z\alpha\iint e^{i\vec{q}\cdot\vec{r}}\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3\vec{r}\,'\,d^3\vec{r} \qquad \alpha = e^2/4\pi$$

$$= -Z\alpha\iint e^{i\vec{q}\cdot\vec{r}\,'}\rho(\vec{r}\,')\frac{e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|}\,d^3\vec{r}\,'\,d^3\vec{r}$$

Let $\vec{R} = \vec{r} - \vec{r}\,'$ and set $\vec{r}\,'$ constant

i.e. integrate over $\vec{r}$

$$M = -Z\alpha\underbrace{\int\frac{e^{i\vec{q}\cdot\vec{R}}}{\vec{R}}\,d^3\vec{R}}_{\text{Rutherford scattering}}\;\underbrace{\int\rho(\vec{r}\,')e^{i\vec{q}\cdot\vec{r}\,'}\,d^3\vec{r}\,'}_{F(q^2)}$$

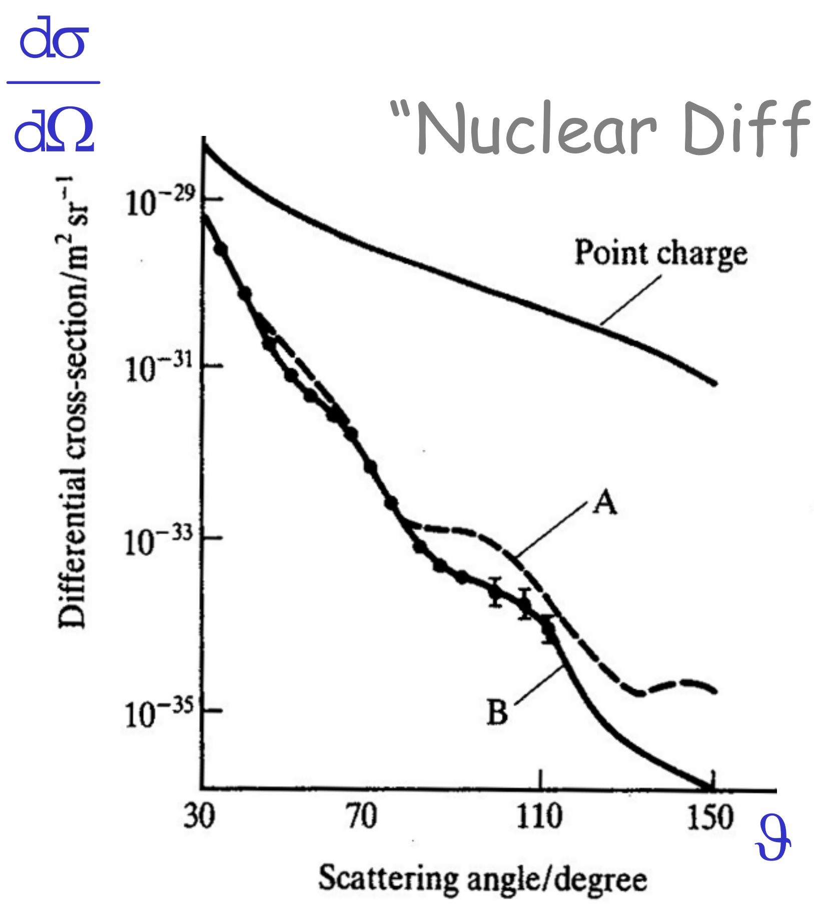
$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{point} \left|F(q^2)\right|^2$$

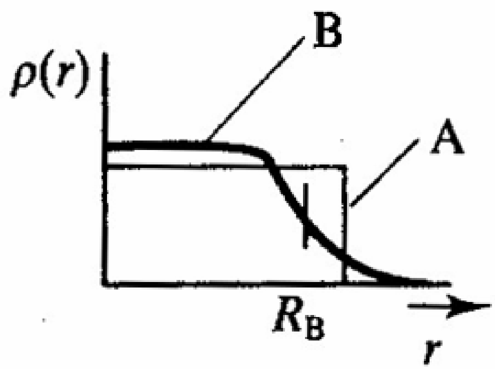
where $F(q^2) = \int \rho(\vec{r}')e^{i\vec{q}\cdot\vec{r}'}d^3\vec{r}'$ is the form factor and the Fourier transform of the charge distribution.

Spherical symmetry, $\rho = \rho(r)$

$$F(q^2) = \int_0^\infty \rho(r')\frac{\sin qr}{qr}4\pi r'^2\,dr \qquad \rho(r) = \frac{1}{2\pi^2}\int_0^\infty F(q^2)\frac{\sin qr}{qr}q^2\,dq$$

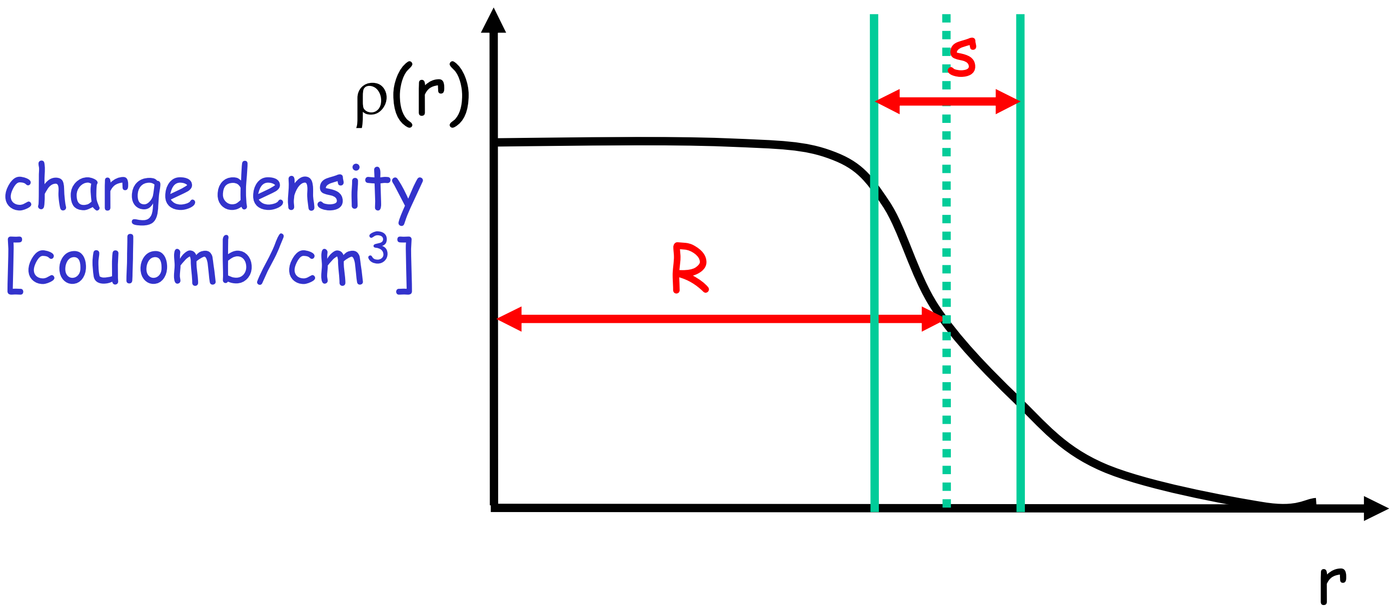
"Nuclear Diffraction"





The charge distribution inside a nucleus is described by the Fermi parameterization

$$\rho(r) = \frac{\rho(0)}{\left[1 + e^{(r-R)/s}\right]}$$



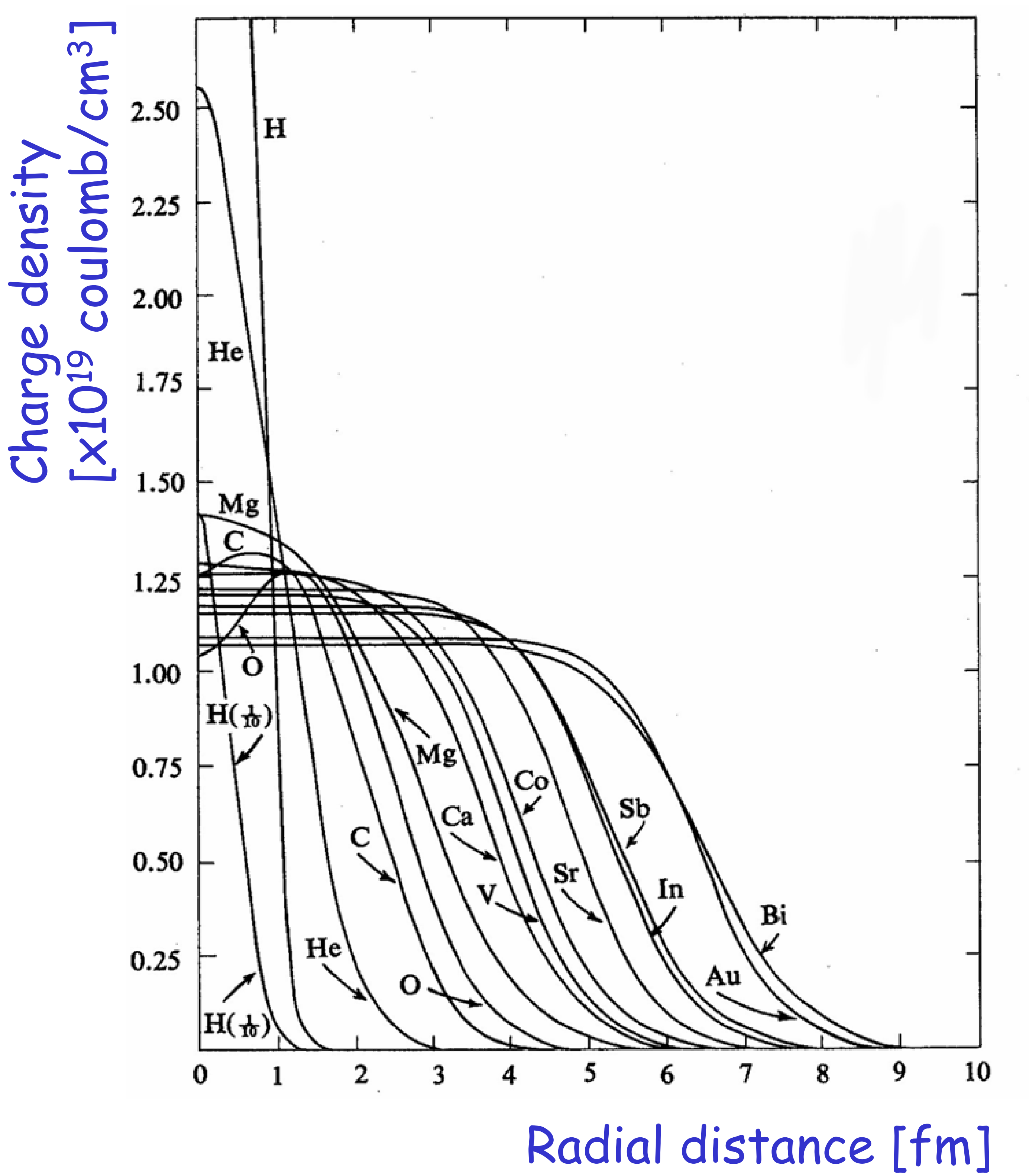
- R is the radius where $\rho(r) = \rho(0)/2$

  R increases with A $R = r_0 A^{1/3}$ $r_0 \approx 1.2$ fm

- s is the surface width or "skin thickness" where $\rho(r)$ falls from 90%→10%

  s is the same for all nuclei $s \approx 2.5$ fm

e⁻ scattering data



Muonic Atoms

Muons (μ^-) brought to rest in matter, get trapped in atomic orbit and have a higher probability than electrons of spending time inside the nucleus.

Bohr radius ~ 1/Zm Energy ~ Z^2m

μ mass ~ 207 m_e μ lifetime~ 2 μs

The muons make transitions to low energy levels, emitting X-rays before decaying

$$\mu^- \rightarrow e^- + \overline{\nu}_e + \nu_\mu$$

- For hydrogen and electrons,

  $r = a_0 = 5 \times 10^4$ fm (Bohr radius)

- For lead and muons,

  $$r = \frac{5 \times 10^4}{82 \times 207} = 3 \text{ fm}$$

Transition energy ($2P_{3/2} \rightarrow 1S_{1/2}$):

16.41 MeV (Bohr theory), 6.02 MeV (measured)

$\therefore Z_{effective}$ and E are changed relative to electrons

Measure X-ray energies → Radius

π^- mesic X rays: π^- can also occupy orbits around the nucleus. X-rays emitted when π^- drops from one orbit to another. Shift in X-ray energy depends on radius.

Mirror Nuclei

Mirror nuclei (e.g. ${}^{11}_{6}C$, ${}^{11}_{5}B$) have different masses due to the p-n difference and the different Coulomb terms in the binding energy:

$$M(A,Z+1)-M(A,Z) = \Delta E_c + m_p + m_e - m_n$$

The mass difference of 2 mirror nuclei can be determined from the β^+ decay spectra of the (A,Z+1) member of the pair

$${}^{11}_{6}C \rightarrow {}^{11}_{5}B + e^+ + \nu_e$$

$$p \rightarrow n + e^+ + \nu_e$$

$$M(A,Z+1)-M(A,Z) = m_e + T_{max} \qquad m_\nu \sim 0$$

where T_{max} is the maximum kinetic energy of the positron.

Example

Use the SEMF to show that the difference in Coulomb energies between mirror nuclei can be written as

$$\Delta E_c = \frac{6}{5}\frac{Z\alpha}{R} \qquad \alpha = e^2/4\pi$$

and predict the radii of the (A,Z+1) member of the pairs

$$\left({}^{11}_{5}B, {}^{11}_{6}C\right) \quad \left({}^{23}_{11}Na, {}^{23}_{12}Mg\right) \quad \left({}^{39}_{19}K, {}^{39}_{20}Ca\right)$$

with T_{max} 0.98, 2.95 and 5.49 MeV respectively.

Nuclear Moments

Static electromagnetic properties of nuclei are specified in terms of electromagnetic moments which give information about the way magnetism and charge is distributed throughout the nucleus.

The two most important moments are

Electric Quadrupole Moment Q
Magnetic Dipole Moment μ

Electric Moments

Depend on the charge distribution inside the nucleus and are a measure of nuclear shape (contours of constant charge density).

Nuclear shape is parameterized by a multipole expansion of the external electric field

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3\vec{r}' \qquad \int \rho(\vec{r}) d^3\vec{r} = Ze$$

r(r')= distance to observer (charge element) from origin.

$$\left|\vec{r}-\vec{r}'\right| = \left[r^2 + r'^2 - 2rr'\cos\vartheta\right]^{1/2}, \quad \left|\vec{r}-\vec{r}'\right|^{-1} = r^{-1}\left[1+\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\vartheta\right]^{-1/2}$$

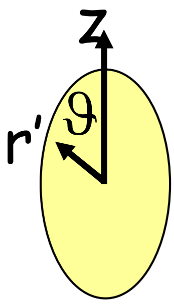
$$\left|\vec{r}-\vec{r}'\right|^{-1} = r^{-1}\left[1-\frac{1}{2}\left(\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\vartheta\right) + \frac{3}{8}\left(\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\vartheta\right)^2 + \ldots\right]^{-1/2}$$

$$\approx r^{-1}\left[1+\frac{r'}{r}\cos\vartheta + \frac{1}{2}\frac{r'^2}{r^2}\left(3\cos^2\vartheta - 1\right) + \ldots\right]^{-1/2} \qquad r' \ll r$$

$$V(r) = \frac{1}{4\pi\varepsilon_0 r}\left[Ze + \frac{1}{r}\int r'\cos\vartheta\,\rho(r')\,d^3r' + \frac{1}{2r^2}\int r'^2\left(3\cos^2\vartheta - 1\right)\rho(r')\,d^3r' + \ldots\right]$$

Quantum limit:

$$\rho(\vec{r}') = \left|\psi(\vec{r}')\right|^2$$



Let r define z axis

$z = r'\cos\vartheta$

E0 moment $= \int \psi^* \psi \, d\vec{r}' = Ze$ — charge

E1 moment $= \int \psi^* z \psi \, d\vec{r}'$ — electric dipole

E2 moment $= \frac{1}{e}\int \psi^* \left(3z^2 - r'^2\right) \psi \, d\vec{r}'$ — electric quadrupole

Nuclear wavefunctions have definite parity

$$\left|\psi(r)\right|^2 = \left|\psi(-r)\right|^2$$

Electric Dipole Moment is zero

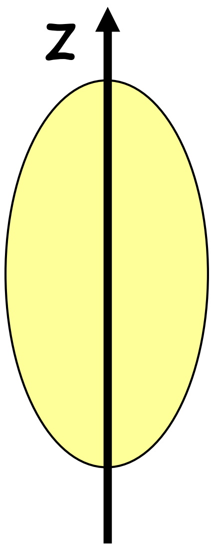
Electric Quadrupole Moment

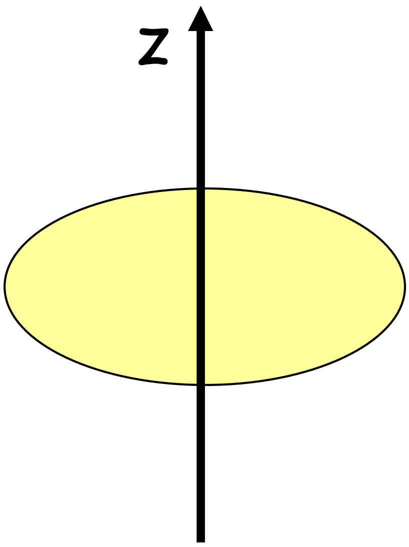
$$Q = \frac{1}{e}\int \psi^* \, (3z^2 - r^2)\, \psi \; d\vec{r}$$

Units: m² or barns Area

Spherical symmetry, $z^2 = \frac{1}{3} r^2 \Rightarrow$ Q=0

All J=0 nuclei have Q=0

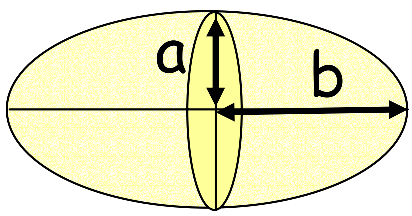




Prolate spheroid
Q +ve

Oblate spheroid
Q -ve

Ellipticity, $\eta = \dfrac{b-a}{\frac{1}{2}(b+a)}$



Experimentally, η is typically $\leq 10\%$

Magnetic Moments

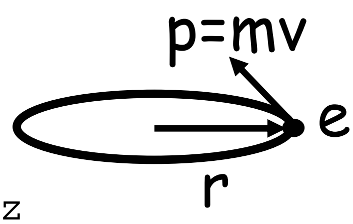
Magnetic dipole moments arise from

- the orbital motion of charged particles and
- the intrinsic spin.

The magnetic dipole moment is the maximum measurable component of the magnetic dipole moment operator $\vec{\mu}$.

Orbital:

Classically, current loop



$$\mu = IA = \frac{ev}{2\pi r}\pi r^2 = \frac{epr}{2m} = \frac{e}{2m}L_z$$

QM, same result

$$\vec{\mu} = g_\ell \frac{e}{2m}\vec{L}_z$$

"g factor"

$g_\ell = 1$ charged particles

$g_\ell = 0$ neutral particles

Intrinsic:

The magnetic moment operator due to intrinsic spin of a particle is

$$\vec{\mu} = g_s \frac{e}{2m}\vec{S}_z$$

g_s is "spin g factor"

Dirac theory (relativistic q.m) for spin 1/2 particle $g_s = 2$.

Electron:

$$\mu_\ell = -g_\ell \frac{e}{2m_e}\hbar\ell = -\mu_B \ell \qquad \mu_s = -g_s \frac{e}{2m_e}\frac{\hbar}{2} = -\mu_B$$

where $\mu_B = \frac{e\hbar}{2m_e}$ is the Bohr Magneton.

Observed small difference from $g_s = 2$ due to higher order corrections in QED:

$$\mu_s = -\mu_B\left[1 + \frac{\alpha}{2\pi} + O(\alpha^2) + \ldots\right] \qquad \alpha = e^2/4\pi \approx 1/137$$

Experiment and theory agree to ~1 in 10^8 !

Proton and neutron

$$\mu_\ell = g_\ell \frac{e}{2m_p}\hbar\ell \qquad \mu_s = g_s \frac{e}{2m_p}\frac{\hbar}{2} = \frac{1}{2} g_s \mu_N$$

where $\mu_N = \frac{e\hbar}{2m_p}$ is the Nuclear Magneton.

Expect: p spin 1/2, charge +e $\mu_s = \mu_N$
n spin 1/2, charge 0 $\mu_s = 0$

Observe: p $\mu_s = +2.793\,\mu_N \rightarrow g_s = +5.586$
n $\mu_s = -1.913\,\mu_N \rightarrow g_s = -3.826$

p and n not point-like particles and are composed of charged quarks and gluons.

Nucleus:

Nuclear magnetic dipole moments arise from the intrinsic spin magnetic dipole moments of the protons and neutrons in the nucleus and from currents circulating in the nucleus due to the motion of the protons.

$$\vec{\mu} = \frac{\mu_N}{\hbar} \sum_i \left[g_\ell \vec{\ell}_z + g_s \vec{s}_z \right]$$

$\sum_i$ over all p,n

The total nuclear magnetic dipole moment can be written as

$$\mu = g_J \, \mu_N \, J$$

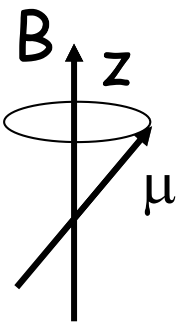
where J total nuclear spin
g_J nuclear g-factor

g_J will be determined using the Nuclear Shell Model (see later)

All even-even nuclei μ=0 as J=0

Nuclear Magnetic Resonance

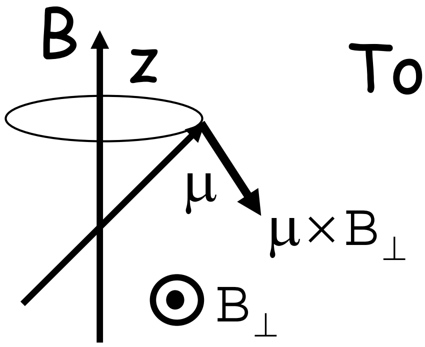
Nuclei with magnetic moment (not J=0 nuclei) in a steady uniform magnetic field, B, exhibit classical Larmor precession



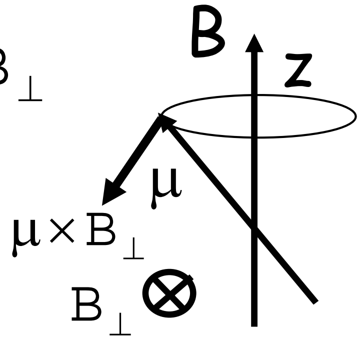
Precession frequency $\omega_L = \frac{\mu B}{\hbar}$

Interaction energy $E = -\vec{\mu} \cdot \vec{B}$

Apply an oscillatory field (frequency ω) $\perp$'r to B



Torque = $\mu \times B_{\perp}$



Maximum energy absorbed at resonance

$$\omega = \omega_L$$

<u>Example</u>: proton in 1T field, $\Delta E \sim 1.8\times10^{-7}$ eV,
ν = 43 MHz (radio frequency)

Observe resonance in r.f. power absorbed

- Measure μ
- If μ known, probe lattice/molecular binding. In nuclear medicine, MRI used to measure distribution of proton-rich tissue.

Radioactivity

Natural radioactivity α, β, γ decay

α decay: ^4_2He nucleus

$A \geq 210$ $\quad {}^A_ZX \rightarrow {}^{A-4}_{Z-2}Y + {}^4_2He$

For decay to occur, energy must be released

$$Q = m_X - m_Y - m_{He} = B_Y + B_{He} - B_X$$

β decay: e^- electron, e^+ positron (antiparticle)

β^- $\quad n \rightarrow p + e^- + \bar{\nu}_e \qquad {}^A_ZX \rightarrow {}^A_{Z+1}Y + e^- + \bar{\nu}_e$

β^+ $\quad p \rightarrow n + e^+ + \nu_e \qquad {}^A_ZX \rightarrow {}^A_{Z-1}Y + e^+ + \nu_e$

electron capture $\quad p + e^- \rightarrow n + \nu_e \qquad {}^A_ZX \rightarrow {}^A_{Z-1}Y + \nu_e$

ν neutrino: mass≈0, charge=0, spin 1/2
interacts weakly with matter
($\sigma_{abs} \approx 10^{-48}\, m^2$)

Only $n \rightarrow pe\nu$ can occur outside nucleus

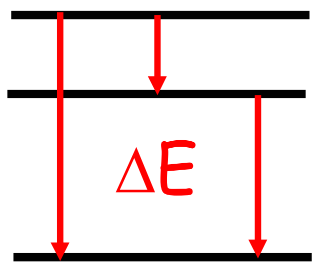
γ decay:

Nuclei with excited states can decay by emission of a γ.

excited states

Photons emitted

ground state

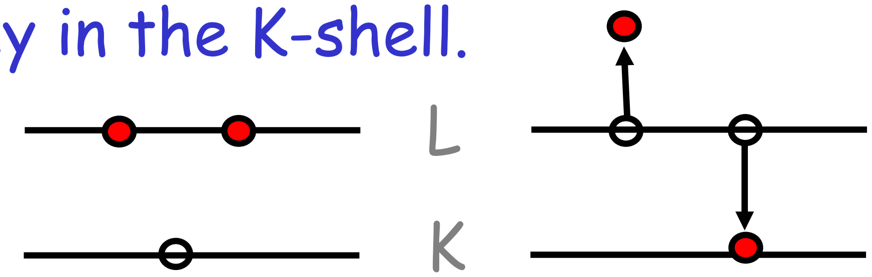


	ΔE	λ	
Atom	~10 eV	~10^{-7} m	optical
	~10 keV	~10^{-10} m	X-ray
Nucleus	~MeV	~10^{-12} m	γ-ray

Internal Conversion: occurs when nuclear excitation energy is lost by the ejection of an atomic e^- (usually from the K-shell).

The vacancy left by the emission of an e^- leads to X-ray or Auger e^- emission as the atom returns to its neutral state.

An Auger e^- is an atomic e^- receiving enough KE to be ejected, usually from the L-shell, when another e^- falls from the same shell to fill a vacancy in the K-shell.



Decay Law

N nuclei at time t
Probability of decay in time dt = λ dt

λ is the decay constant (depends only on nuclide and decay mode.)

Number of nuclei that decay in time dt,

$$dN = -N \lambda \, dt$$

$$\int_{N(0)}^{N(t)} \frac{dN}{N} = \int_0^t -\lambda \, dt$$

$$N(t) = N(0) \; e^{-\lambda t}$$

where N(0) is the number of nuclei at t=0.

The activity at time t, *A*(t), is the number of decays per unit time,

$$A(t) = \left| \frac{dN}{dt} \right| = \lambda \, N(t) = \lambda \, N(0) \, e^{-\lambda t}$$

λ can be measured from a plot of Ln*A*(t) vs t

Units of radioactivity: are defined as the number of decays per unit time.

Becquerel (Bq) = 1 decay per second
Curie (Ci) 1 Ci = 3.7×10^{10} decays per second.

The mean lifetime of a nucleus,

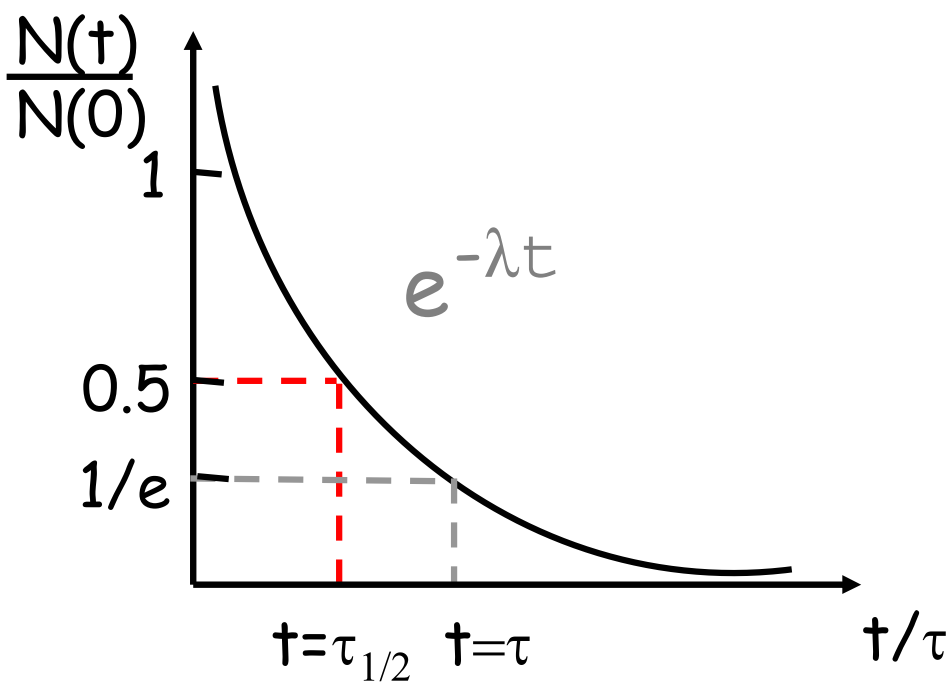
$$\tau = \frac{\sum \text{all lifetimes}}{\text{Total number decays}}$$

$$\tau = \frac{1}{N(0)} \int_{N(0)}^{0} t \, dN = \frac{1}{\lambda}$$

The half-life, $\tau_{1/2}$, is the time over which 50% of the nuclei decay

$$\frac{N(0)}{2} = N(0)\, e^{-\lambda \tau_{1/2}}$$

$$\tau_{1/2} = \frac{\mathrm{Ln}\, 2}{\lambda} = 0.693\, \tau$$



Decay Chain

$$N_1 \xrightarrow{\lambda_1} N_2 \xrightarrow{\lambda_2} N_3 \rightarrow \ldots\ldots$$

Activity of N_2 = $\lambda_2 N_2(t)$

Rate of change of population of N_2

$$\frac{dN_2(t)}{dt} = \lambda_1 N_1(t) - \lambda_2 N_2(t)$$

For more than one decay mode: $N_1 \xrightarrow{\lambda_1} N_2$, $N_1 \xrightarrow{\lambda_2} N_3$

Probability of each decay mode = $\lambda_i \, dt$

Total decay probability = $\sum_i \lambda_i dt$

$$dN = -N \sum_i \lambda_i dt$$

$$N(t) = N(0) e^{-[\lambda_1 + \lambda_2 + \ldots]t}$$

$\lambda = \sum \lambda_i$ and $\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_3} + \ldots$

Measurement of Decay Rates

A variety of techniques can be used to measure a wide range of lifetimes

τ: 10^{-20} s $\rightarrow 10^{10}$ years

τ: days $\rightarrow$ years:

Use "specific" activity (decay rate/gram)
Long lifetime, $A(t) \rightarrow$ constant

$$A(t) = \frac{N(0)}{\tau}$$

Measure $A(t)$ from the number of daughter nuclei and $N(0)$ from chemical analysis or mass spectroscopy.

τ: minutes $\rightarrow$ hours:

Extract τ from plot of $\ln A(t)$ versus t.

τ: 10^{-3} s $\rightarrow 10^{-11}$ s:

Measure time of existence for single nuclides.

X	$\rightarrow$	Y
start clock		stop clock
e.g. X created from $M \rightarrow X+\gamma$		using decay signal

Number decays observed with lifetimes between t and t+δt

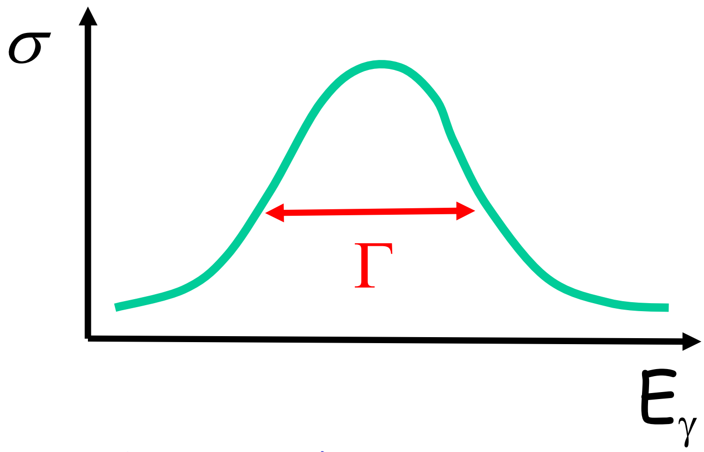
$$\delta N = N(t) - N(t+\delta t)$$
$$= N(0)e^{-\lambda t} - N(0)e^{-\lambda(t+\delta t)}$$
$$= N(0)e^{-\lambda t}\underbrace{(1-e^{-\lambda\delta t})}$$

constant for fixed δt

Extract τ from the binned distribution.

$\tau \leq 10^{-11}$ s:

Energy distribution of the γ.



Γ = full width at 1/2 maximum

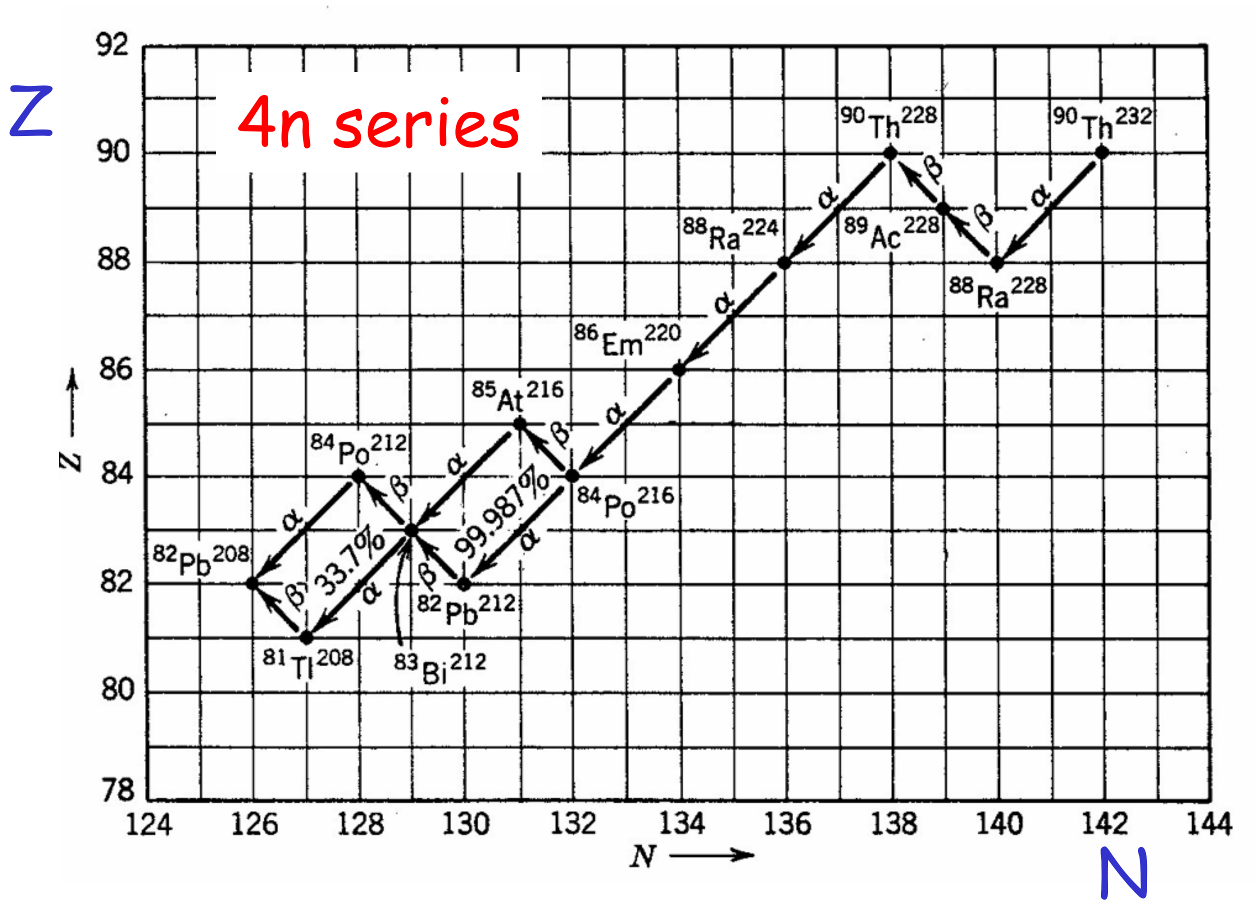
$$\Gamma = \frac{1}{\tau} \qquad \Delta E \Delta t \geq \hbar/2$$

Natural Radioactivity

Some $\tau_{1/2}$ long compared to age of Earth.

Series Name	Type	Final Nucleus (stable)	Longest-Lived Nucleus	$\tau_{1/2}$ (years)
Thorium	4n	^{208}Pb	^{232}Th	1.41×10^{10}
Neptunium	4n+1	^{209}Bi	^{237}Np	2.14×10^{6}
Uranium	4n+2	^{206}Pb	^{238}U	4.47×10^{9}
Actinium	4n+3	^{207}Pb	^{235}U	7.04×10^{8}

n is an integer



Radioactive Dating

Consider a sample of radioactive "parent" nuclei (P) which decay to "daughter" nuclei (D):

Assumptions:

- know τ_P from previous studies
- P trapped when sample came into existence
- no P or D entered or left by other means
- At t=0, N_D=0

$$N_P(t) + N_D(t) = N_P(0)$$

$$N_P(t) = N_P(0)e^{-\lambda\Delta t}$$

$$N_P(t) + N_D(t) = N_P(t)e^{\lambda\Delta t}$$

$$e^{\lambda\Delta t} = 1 + \frac{N_D(t)}{N_P(t)}$$

Age $$\Delta t = \tau_P \, \mathrm{Ln}\left[1 + \frac{N_D(t)}{N_P(t)}\right]$$

Count $N_P(t)$ and $N_D(t)$ e.g. chemically

Complication when $N_D(0) \neq 0$:

$$N_P(t) + N_D(t) = N_P(0) + N_D(0)$$

2 equations, 3 unknowns

If there is another isotope of D, say D′, for which $N_{D'}(t) = N_{D'}(0) = N_{D'}$ (i.e. D′ stable),

$$\frac{N_P(t) + N_D(t)}{N_{D'}} = \frac{N_P(0) + N_D(0)}{N_{D'}}$$

$$\frac{N_D(t)}{N_{D'}} = \frac{N_P(0) + N_D(0)}{N_{D'}} - \frac{N_P(t)}{N_{D'}}$$

$$\frac{N_D(t)}{N_{D'}} = \frac{N_P(t)}{N_{D'}}\left(e^{\lambda \Delta t} - 1\right) + \frac{N_D(0)}{N_{D'}}$$

With several mineral sources from the same source expect

- same age Δt
- same $N_D(0)/N_{D'}(0)$
- different $N_P(0)$

Plot $\frac{N_D(t)}{N_{D'}}$ versus $\frac{N_P(t)}{N_{D'}}$

slope $e^{\lambda \Delta t} - 1$ intercept $\frac{N_D(0)}{N_{D'}}$

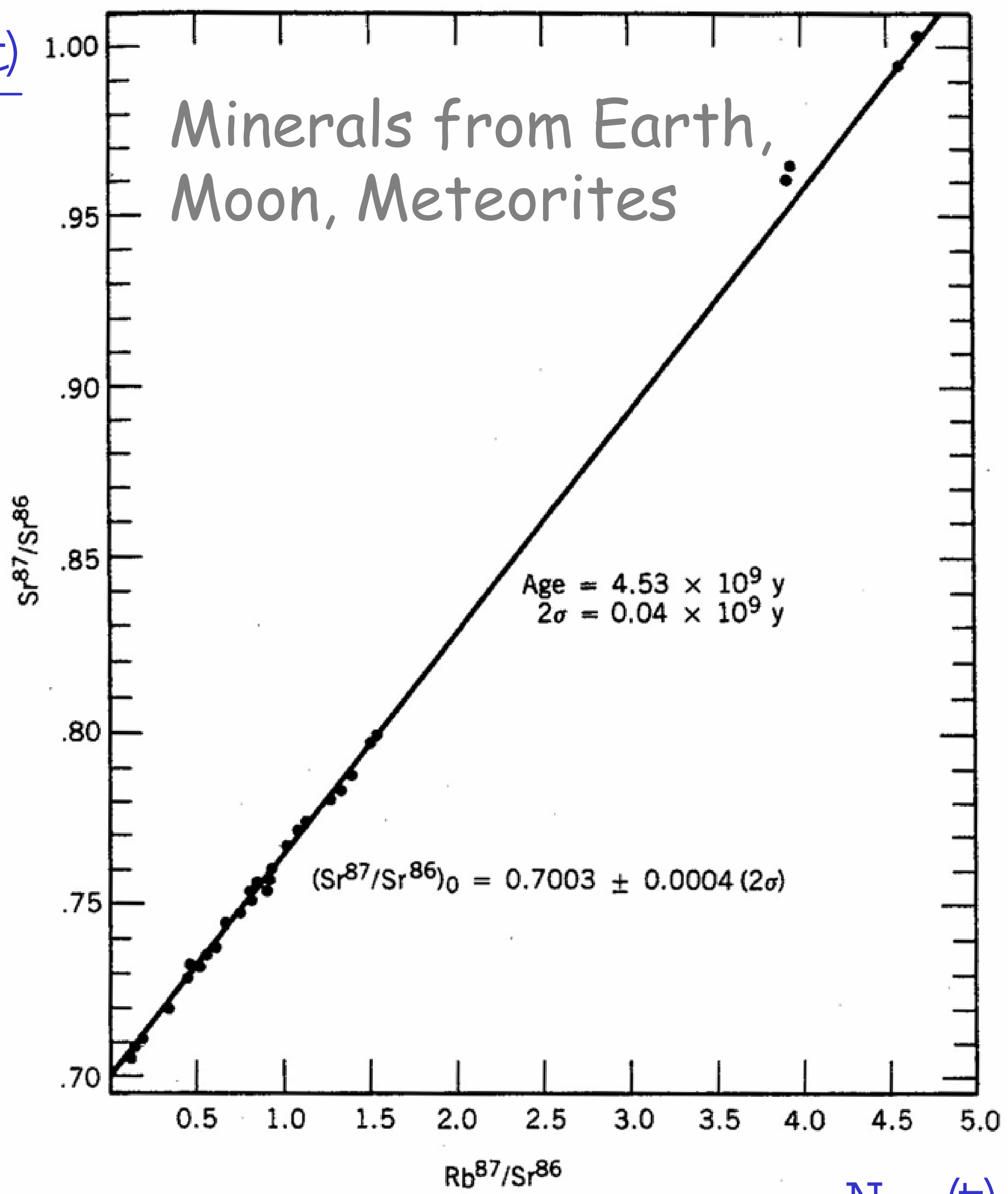
Example: Use β^- decay

$$^{87}Rb \rightarrow {}^{87}Sr \qquad (\tau_{1/2} = 4.8 \times 10^{10} \text{ years})$$

$$D' = {}^{86}Sr \text{ (stable)}$$

Age of Earth from slope = 4.5×10^9 years

$$\frac{N_D(t)}{N_{D'}}$$



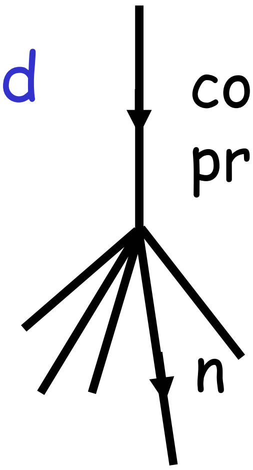
$$\frac{N_P(t)}{N_{D'}}$$

Radio-Carbon Dating

Most recent organic matter; use ^{14}C dating

- ^{14}C is continuously formed in Earths atmosphere

$$^{14}N + n \rightarrow {}^{14}C + p$$



Production rate of ^{14}C is approx constant (e.g. checked by comparison with dating from tree rings)

- The carbon in living organisms is continuously exchanged with atmospheric carbon.

Equilibrium: ~1 atom of ^{14}C to every 10^{12} atoms of other carbon isotopes (98.9% ^{12}C, 1.1% ^{13}C).

- ^{14}C decays in dead organisms. No more ^{14}C from atmosphere.

$$^{14}_{6}C \rightarrow {}^{14}_{7}N + e^{-} + \bar{\nu}_e$$

β^{-} decay

$\tau_{1/2}$=5730 yrs

- Measure the specific activity to obtain age.
- Complications from burning of fossil fuels, nuclear bomb tests etc.

e.g. Turin Shroud [ref: Nature 337 (1989) 611.]