Series expansions methods for Feynman integrals and the DiffExp Mathematica package

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Based on: arXiv:2006.05510

Partly based on: arXiv:1907.13156, 1911.06308, in collaboration with:

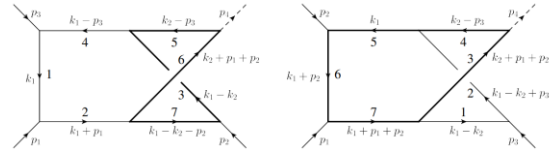
R. Bonciani, V. Del Duca, H. Frellesvig, J. M. Henn, L. Maestri,
F. Moriello, G. Salvatori, V. A. Smirnov

Outline of the talk

Introduction

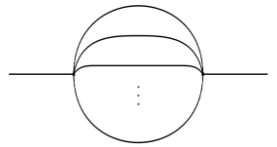
- Overview of analytic vs numerical methods
- The method of differential equations
  - Canonical basis
  - Deriving boundary conditions
- Series solutions methods
  - Finding series solutions
  - Line segmentation
  - Analytic continuation
- The DiffExp Mathematica package
  - Usage and boundary conditions

Applications

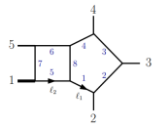


[Bonciani, Del Duca, Frellesvig, Henn, Hidding, Maestri, Moriello, Salvatori, Smirnov] [1907.13156, 1911.06308]

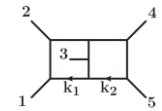
- Higgs plus jet integrals
- Banana graphs



  - 3-Loop equal/unequal mass
  - 4-Loop equal mass
- Examples from the literature



[Abreu et al, 2005.04195]



[Chicherin et al, 1812.11160]

- Special functions
- Final remarks
  - Future prospects

Overview of computational methods

Analytic methods for Feynman integrals

- Typical analytic methods involve expressing Feynman integrals in terms of classes of iterated integrals
  - Multiple polylogarithms
  - Iterated integrals over Eisenstein series
  - Elliptic multiple polylogarithms over the torus or an elliptic curve

- Strengths of analytic methods:
  - Branch-cuts and analytic structure is manifest (through the symbol map)
  - Specialized algorithms can be developed for evaluating the relevant classes of functions

- Drawbacks of analytic methods:
  - The analytic continuation may be difficult to perform
  - Many Feynman integrals lie outside the known classes of functions

Numerical methods

- Prototypical example:
  - Sector decomposition and numerical integration (FIESTA, pySecDec)

- Strengths of numerical methods:
  - Numerical integration is fully algorithmic and general purpose
  - Applicable to integrals with many scales

- Drawbacks of numerical methods:
  - Numbers might not expose symmetries and/or structures underlying the integrals
  - Performance can lack behind analytic methods

Semi-numerical methods

- Semi-numerical methods perform as much as possible of the computation analytically, before resorting to numerical approximations
- We may set up differential equations in analytic form, and then solve these differential equations
  - Numerically using finite difference methods [Mandal, Zhao, 1812.03060]
  - Semi-analytically through one-dimensional series expansions [1907.13156, 1907.13234, 1911.06308, 2006.05510]

- Strengths of series expansion methods:
  - State of the art performance on many types of Feynman integrals
  - Speed improves as more points are computed
  - Analytic continuation of Feynman integrals becomes simple
- Drawbacks:
  - Simplification of the differential equations is not fully algorithmic
  - Derivation of boundary conditions requires some manual effort

Series expansions

- Series expansions have been featured various times in the past literature.
- For single-scale problems, see e.g:

S. Pozzorini and E. Remiddi, *Precise numerical evaluation of the two loop sunrise graph master integrals in the equal mass case*, *Comput. Phys. Commun.* **175** (2006) 381–387, [hep-ph/0505041].

U. Aglietti, R. Bonciani, L. Grassi, and E. Remiddi, *The Two loop crossed ladder vertex diagram with two massive exchanges*, *Nucl. Phys.* **B789** (2008) 45–83, [arXiv:0705.2616].

R. Mueller and D. G. Öztürk, *On the computation of finite bottom-quark mass effects in Higgs boson production*, *JHEP* **08** (2016) 055, [arXiv:1512.08570].

B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, *JHEP* **05** (2018) 028, [arXiv:1802.00833].

R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Solving differential equations for Feynman integrals by expansions near singular points*, *JHEP* **03** (2018) 008, [arXiv:1709.07525].

R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Evaluating elliptic master integrals at special kinematic values: using differential equations and their solutions via expansions near singular points*, *JHEP* **07** (2018) 102, [arXiv:1805.00227].

R. Bonciani, G. Degrassi, P. P. Giardino, and R. Gröber, *A Numerical Routine for the Crossed Vertex Diagram with a Massive-Particle Loop*, *Comput. Phys. Commun.* **241** (2019) 122–131, [arXiv:1812.02698].

- For multi-scale problems, see for example:

K. Melnikov, L. Tancredi, and C. Wever, *Two-loop $gg \to Hg$ amplitude mediated by a nearly massless quark*, *JHEP* **11** (2016) 104, [arXiv:1610.03747].

K. Melnikov, L. Tancredi, and C. Wever, *Two-loop amplitudes for $qg \to Hq$ and $q\bar{q} \to Hg$ mediated by a nearly massless quark*, *Phys. Rev.* **D95** (2017), no. 5 054012, [arXiv:1702.00426].

R. Bonciani, G. Degrassi, P. P. Giardino, and R. Grober, *Analytical Method for Next-to-Leading-Order QCD Corrections to Double-Higgs Production*, *Phys. Rev. Lett.* **121** (2018), no. 16 162003, [arXiv:1806.11564].

R. Bruser, S. Caron-Huot, and J. M. Henn, *Subleading Regge limit from a soft anomalous dimension*, *JHEP* **04** (2018) 047, [arXiv:1802.02524].

J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *Double-Higgs boson production in the high-energy limit: planar master integrals*, *JHEP* **03** (2018) 048, [arXiv:1801.09696].

J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *Double Higgs boson production at NLO in the high-energy limit: complete analytic results*, *JHEP* **01** (2019) 176, [arXiv:1811.05489].

B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, *JHEP* **05** (2018) 028 [1802.00833].

Series expansions

- Some of the previous literature was problem tailored. For example, by:
  - Treating single-scale problems by expanding only at singular points and deriving recurrence relations for the series coefficients [Lee, Smirnov, Smirnov, 1709.07525, 1805.00227]
  - Considering multi-scale cases by expanding in only one parameter [Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 1503.06056]
- An (arguably) more general setup was demonstrated in [F. Moriello, 1907.13234] for the computation of planar integrals relevant to H+j production at NLO
- Simultaneously, in a larger collaboration, we applied these methods to the computation of non-planar H+j integrals

(Figure borrowed from 1907.13234)

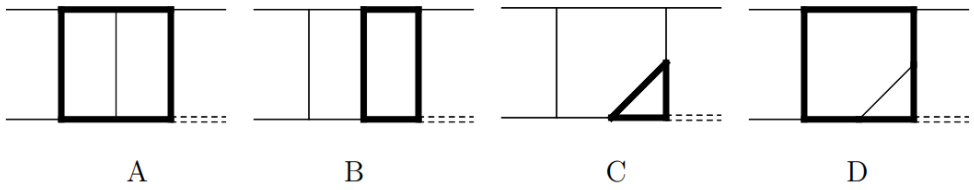
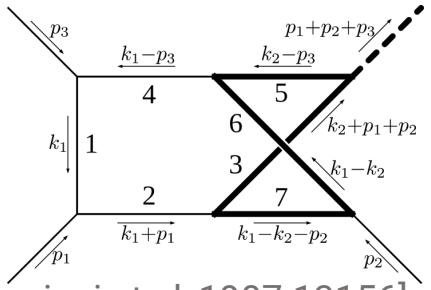
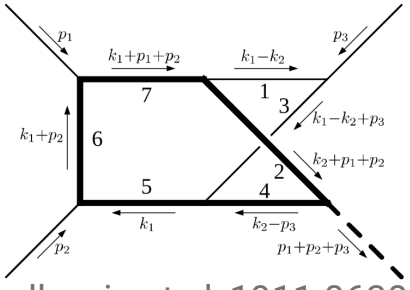


Figure 1: The four planar integral families contributing to two-loop $H+j$-production in QCD.



[Bonciani et al, 1907.13156]



[Frellesvig et al, 1911.06308]

Series expansions

- The main steps of the series expansion method are as follows:

  - Reduce multi-scale problems to a single-scale problem by integrating along a one-dimensional contour
  - Split up the contour into multiple segments such that series expansions converge on each segment
  - Find series solutions of the integrals along each segment, and fix boundary conditions by matching neighbouring segments
  - Cross thresholds by assigning $\pm i\delta$ to logarithms and algebraic roots in the solutions

DiffExp

- A general implementation of these methods was made into the Mathematica package DiffExp, introduced in arXiv:2006.05510, and hosted at https://gitlab.com/hiddingm/diffexp
- DiffExp accepts (any) system of differential equations of the form

$$\frac{\partial}{\partial s}\vec{f}(\{S\},\epsilon) = \mathbf{A}_s\vec{f}(\{S\},\epsilon) \qquad \mathbf{A}_x(x,\epsilon) = \sum_{k=0}^{\infty}\mathbf{A}_x^{(k)}(x)\epsilon^k$$

  for which the matrix entries are combinations of rational and algebraic functions
- It enables one to numerically integrate various multi-scale Feynman integrals at arbitrary points in phase-space, and at precisions of tens of digits (or higher)
- The Feynman integrals do not have to be in canonical form and may also be of “elliptic”-type or associated with more complicated geometries.

The method of differential equations

Differential equations

- Start from a family of scalar Feynman integrals:

$$I_{a_1,\ldots,a_{n+m}} = \int \left(\prod_{i=1}^{l} d^d k_i \right) \frac{\prod_{i=n+1}^{n+m} N_i^{-a_i}}{\prod_{i=1}^{n} D_i^{a_i}}, D_i = -q_i^2 + m_i^2 - i\delta$$

- Derivatives of Feynman integrals can be expressed in the same family.

- By IBP-reduction we may then obtain a closed system of the form:

$$d\vec{f} = \sum_{s \in S} \mathbf{M}_s \vec{f} ds$$

[Kotikov, 1991], [Remiddi, 1997]
[Gehrmann, Remiddi, 2000]

- For some vector of master integrals $\vec{f}$

Differential equations in a canonical basis

- We may simplify the differential equations by a change of basis
- Let $\vec{B} = \boldsymbol{T}\vec{f}$, then we have: $\frac{\partial}{\partial s_i}\vec{B} = \left[(\partial_{s_i}\mathbf{T})\,\mathbf{T}^{-1} + \mathbf{T}\mathbf{M}_{s_i}\mathbf{T}^{-1}\right]\vec{B}\,.$
- The canonical basis conjecture claims that $\exists\mathbf{T}$: $\boxed{d\vec{B} = \epsilon d\tilde{\mathbf{A}}\vec{B}}$ [Henn, 2013]
- Furthermore, if the integrals are polylogarithmic, we have:

$$d\tilde{\mathbf{A}} = \sum_{i\in\mathcal{A}} \mathbf{C}_i d\log(l_i)$$

Canonical basis

- The formal solution can be written as a path-ordered exponential: [Chen, 1977]

$$d\vec{B} = \epsilon \left(d\tilde{\mathbf{A}}\right) \vec{B} \quad \Rightarrow \quad \vec{B} = \mathbb{P}\exp\left[\epsilon \int_\gamma d\tilde{\mathbf{A}}\right] \vec{B}_{\text{boundary}} \qquad \gamma(x) : [0,1] \to \mathbb{C}^{|S|}$$

- Which can be expanded in terms of Chen's iterated integrals:

$$\vec{B} = \vec{B}^{(0)}(\gamma(0)) + \sum_{k\geq 1} \epsilon^k \sum_{j=1}^{k} \int_0^1 \gamma^*(d\tilde{\mathbf{A}})\,(x_1) \int_0^{x_1} \gamma^*(d\tilde{\mathbf{A}})\,(x_2) \times \ldots \times \int_0^{x_{j-1}} \gamma^*(d\tilde{\mathbf{A}})\,(x_j)\, \vec{B}^{(k-j)}(\gamma(0))$$

- More concisely, consider the ϵ expansion $\vec{B} = \sum_{i\geq 0} \vec{B}^{(i)} \varepsilon^i$, then:

$$\vec{B}^{(i)}(\gamma(1)) = \int_0^1 \mathbf{A}_x \vec{B}^{(i-1)} dx + \vec{B}^{(i)}(\gamma(0))$$

where $\mathbf{A}_x = \sum_{s\in S} \frac{\partial \tilde{\mathbf{A}}}{\partial s}(\gamma(x)) \frac{\partial \gamma_s(x)}{\partial x}$

Canonical basis

- Thus, more compactly, we focus on integrals of the type:

$$\vec{B}^{(i)}(\gamma(1)) = \int_0^1 \mathbf{A}_x \vec{B}^{(i-1)} dx + \vec{B}^{(i)}(\gamma(0))$$

- When $d\tilde{\mathbf{A}} = \sum_{i\in\mathcal{A}} \mathbf{C}_i d\log(l_i)$ and when $\mathcal{A}$ contains only (simultaneously) rationalizable algebraic functions the results are expressible in terms of MPLs:

$$G\left(a_1,\ldots,a_n;z\right) = \int_0^z \frac{dt}{t-a_1} G\left(a_2,\ldots,a_n;t\right), \quad G(;z)\equiv 1$$

- This provides a fully analytic solution of the differential equations, which can be evaluated using GiNaC

[Vollinga, Weinzierl, hep-ph/0410259]

Series expansions - preview

- In the presence of non-rationalizable roots, the results may not be expressible in terms of MPLs at all orders in ϵ. [Brown, Duhr, arXiv:2006.09413]
- In this case, series expansions come to the rescue!
- Starting from $\vec{B}^{(i)}(\gamma(1)) = \int_0^1 \mathbf{A}_x \vec{B}^{(i-1)} dx + \vec{B}^{(i)}(\gamma(0))$, we may perform the expansion:

$$\mathbf{A}_x = x^r \left[\sum_{p=0}^{k} \mathbf{C}_p x^p + \mathcal{O}\left(x^{k+1}\right) \right]$$

- Then integration becomes straightforward: $\int x^m \log(x)^n = x^{m+1} \sum_{j=1}^{n} c_j \log(x)^j$
- E.g. $\int x^{-3/5} \log^2(x) dx = \frac{5}{4} x^{2/5} \left(2\log^2(x) - 10\log(x) + 25\right)$

Boundary conditions

- To solve the system of differential equations, we need to supply boundary conditions at some suitable kinematic point or limit
- One approach is to use sector decomposition to obtain numerical data at some point:
  - Start from a point in the Euclidean region, where `FIESTA` and `(py)SecDec` have favorable run time.
  - Obtain numerics in other regions by solving the differential equations
  - The precision is limited to the precision of the boundary data
- Alternatively, we may obtain boundary conditions analytically:
  - Consider some asymptotic limit where particles go on-shell, or internal masses vanish.
  - Obtain solutions in closed-form in ϵ, ideally in terms of ratio's of gamma functions
  - The precision is unlimited, and we may obtain results at any order in ϵ

Boundary conditions (Analytic form)

- Typically, we consider a limit where most of the external scales vanish, such that the Feynman integrals simplify as much as possible.
- However, we can not in general commute the limit and the integration.
- Let's consider the example of the massive bubble:

$$\frac{e^{\gamma_E \epsilon}}{i\pi^{1-\epsilon}} \int d^d k_1 \frac{1}{\left(-k_1^2+m^2\right)\left(-\left(k_1+p\right)^2+m^2\right)} = \frac{2\log\left(\frac{-\sqrt{-p^2}-\sqrt{4m^2-p^2}}{\sqrt{-p^2}-\sqrt{4m^2-p^2}}\right)}{\sqrt{-p^2}\sqrt{4m^2-p^2}} + \mathcal{O}(\epsilon)$$

- In the limit $m^2 = x$, with $x \downarrow 0$, we obtain: $\sim -\frac{2\left(\log\left(-p^2\right)-\log(x)\right)}{p^2} + \mathcal{O}(x)$

Boundary conditions

$$-\frac{2\left(\log\left(-p^2\right)-\log(x)\right)}{p^2}+\mathcal{O}(x)^1$$

- Now, suppose we had started directly in the massless limit. We'd find:

$$e^{\gamma_E\epsilon}\left(i\pi^{d/2}\right)^{-1}\int d^dk_1\frac{1}{\left(-k_1^2\right)\left(-\left(k_1+p\right)^2\right)}=\frac{2}{p^2\epsilon}-\frac{2\log\left(-p^2\right)}{p^2}+\mathcal{O}(\epsilon)$$

- The kinematic singularity has been transformed into a dimensionally regulated pole, yielding a different result than before!
- How do we obtain boundary conditions without computing the integral in a generic mass configuration first? (which would defeat the purpose)

Boundary conditions

[See works by Beneke and Smirnov]

- The solution is to use the method of expansions by regions.
- There is a particularly simple formulation in the parametric representation, which is implemented in the publicly available Mathematica package `asy.m`

See e.g. [Jantzen, Smirnov, Smirnov, 1206.0546]

- Recall the Feynman parametrization:

$$I_{a_1,\ldots,a_n} = \left(i\pi^{\frac{d}{2}}\right)^l \Gamma\left(a - \frac{ld}{2}\right) \int_{\Delta^{n-1}} \left[d^{n-1}\vec{\alpha}\right] \left(\prod_{i=1}^{n} \frac{\alpha_i^{a_i-1}}{\Gamma(a_i)}\right) \mathcal{U}^{a-\frac{d}{2}(l+1)} \mathcal{F}^{-a+\frac{ld}{2}}$$

$$\left[d^{n-1}\vec{\alpha}\right] \equiv \sum_{j=1}^{n} (-1)^{j-1} \alpha_j d\alpha_1 \wedge \cdots \wedge \widehat{d\alpha_j} \wedge \cdots \wedge d\alpha_n$$

$$\Delta^{n-1} = \left\{[\alpha_1 : \alpha_2 : \ldots : \alpha_n] \in \mathbb{RP}^{n-1} \mid \alpha_i \geq 0, 1 \leq i \leq n\right\}$$

Expansion by regions

Kinematic invariants and masses

- Suppose we are interested in a kinematic limit $s_i \to s_i' = x^{\gamma_i} s_i \text{ for } i = 1, \ldots, |S|$
- Then there exists a set of regions $\{R_i\}$, where $R_i = (r_{i1}, \ldots, r_{im})$ is a vector of rational numbers.
- For each region R_i we rescale the Feynman parametrized integral in the following manner: $\underbrace{\alpha_j \to x^{R_{ij}} \alpha_j\,, \quad d\alpha_j \to x^{R_{ij}} d\alpha_j}\,, \quad \underbrace{s_j \to x^{\gamma_j} s_j}$

  Each Feynman parameter scales according to the given region

  In addition, we take our desired kinematic limit
- The asymptotic limit is then given by summing over the contributions of each region, expanding on x, and integrating.

Expansion by regions

- Let's have another look at the massive bubble. The Feynman parametrization is:

$$\frac{e^{\gamma_E \epsilon}\Gamma(\epsilon+1)}{i\pi^{1-\epsilon}} \int_\Delta d\alpha_1 d\alpha_2 \left(\alpha_1+\alpha_2\right)^{2\epsilon} \left(\alpha_1^2 m^2 + \alpha_2^2 m^2 + 2\alpha_1\alpha_2 m^2 - \alpha_1\alpha_2 p^2\right)^{-1-\epsilon}$$

- We feed `asy.m` the $\mathcal{U}$ and $\mathcal{F}$ polynomials, and obtain the regions:

$$R_1 = \{0,0\}, \quad R_2 = \{0,-1\}, \quad R_3 = \{0,1\}$$

- Leading to:

$$\begin{aligned}\frac{e^{\gamma_E \epsilon}\Gamma(\epsilon+1)}{i\pi^{1-\epsilon}} \int_\Delta d\alpha_1 d\alpha_2 \Big(& x^{-\epsilon}\left(x\alpha_1+\alpha_2\right)^{2\epsilon}\left(x^2\alpha_1^2 - p^2\alpha_1\alpha_2 + 2x\alpha_1\alpha_2 + \alpha_2^2\right)^{-1-\epsilon} \\ & + \left(\alpha_1+\alpha_2\right)^{2\epsilon}\left(x\alpha_1^2 - p^2\alpha_1\alpha_2 + 2x\alpha_1\alpha_2 + x\alpha_2^2\right)^{-1-\epsilon} \\ & + x^{-\epsilon}\left(\alpha_1+x\alpha_2\right)^{2\epsilon}\left(\alpha_1^2 - p^2\alpha_1\alpha_2 + 2x\alpha_1\alpha_2 + x^2\alpha_2^2\right)^{-1-\epsilon}\Big)\end{aligned}$$

- For the purpose of computing boundary conditions, we often only need the leading term of the expansion with respect to the line parameter

Expansion by regions

- At leading order in x, we obtain:

$$\frac{e^{\gamma_E \epsilon}\Gamma(\epsilon+1)}{i\pi^{1-\epsilon}} \int_\Delta d\alpha_1 d\alpha_2 \left(x^{-\epsilon} \alpha_2^{-1+\epsilon} \left(-p^2\alpha_1 + m^2\alpha_2\right)^{-1-\epsilon} + \right.$$
$$\left. + \alpha_1^{-\epsilon-1}\alpha_2^{-\epsilon-1} \left(\alpha_1+\alpha_2\right)^{2\epsilon} \left(-p^2\right)^{-1-\epsilon} + x^{-\epsilon}\alpha_1^{\epsilon-1} \left(\alpha_1 m^2 - \alpha_2 p^2\right)^{-\epsilon-1} \right)$$

- Although we have a sum of terms, each piece is simpler to integrate than the Feynman parametrization of the massive bubble. Performing the integrations yields:

$$\frac{\epsilon \left(-p^2\right)^{-\epsilon-1} \Gamma(-\epsilon)^2 \Gamma(\epsilon)}{\Gamma(-2\epsilon)} - \frac{2x^{-\epsilon}\Gamma(\epsilon)}{p^2} = -\frac{2\left(\log\left(-p^2\right) - \log(x)\right)}{p^2} + \mathcal{O}(\epsilon)$$

- Which agrees with the result we found before!
- Note as well that the boundary conditions are just ratios of gamma functions

Series solutions to differential equations

Solving non-canonical systems

- We saw previously how to find series expansions for a canonical-form system
- Next consider a more general system of partial differential equations of the form:

$$\frac{\partial}{\partial s_i}\vec{f}(\{S\},\epsilon) = \mathbf{A}_{s_i}\vec{f}(\{S\},\epsilon) \overset{\gamma(x):[0,1]\to\mathbb{C}^{|S|}}{\Rightarrow} \frac{\partial}{\partial x}\vec{f}(x,\epsilon) = \mathbf{A}_x\vec{f}(x,\epsilon),$$

- We will restrict ourselves to the following conditions on the ϵ expansion:

$$\mathbf{A}_x(x,\epsilon) = \sum_{k=0}^{\infty}\mathbf{A}_x^{(k)}(x)\epsilon^k, \quad \vec{f}(x,\epsilon) = \sum_{k=0}^{\infty}\vec{f}^{(k)}(x)\epsilon^k$$

- The condition on the basis integrals $\vec{f}$ is trivial, as we can always multiply out the highest pole.
- The finiteness condition on $\mathbf{A}_x(x,\epsilon)$ can typically also be instated by performing appropriate ϵ rescalings of the basis integrals.

Solving non-canonical systems

- After plugging in the ϵ expansions, we have that:

$$\frac{\partial}{\partial x}\vec{f}(x,\epsilon) = \mathbf{A}_x \vec{f}(x,\epsilon) \quad \Rightarrow \quad \boxed{\partial_x \vec{f}^{(k)} = \mathbf{A}_x^{(0)} \vec{f}^{(k)} + \sum_{j=0}^{k-1} \mathbf{A}_x^{(k-j)} f^{(j)}}$$

- We can decompose these differential equation further
- Let $\{f_{\sigma_1}, \ldots, f_{\sigma_p}\}$ be a set of "coupled" integrals, and relabel $f_{\sigma_1} \to g_1, f_{\sigma_2} \to g_2, \ldots$.
- Then $\boxed{\partial_x \vec{g}^{(k)} = \mathbf{M}\vec{g}^{(k)} + \vec{b}^{(k)}}$

Where: $\mathbf{M}_{ij} = \left(\mathbf{A}_x^{(0)}\right)_{\sigma_i,\sigma_j}$, $\vec{b}_i^{(k)} = \sum_{j\notin\Sigma}\left[\left(\mathbf{A}_x^{(0)}\right)_{\sigma_i j} f_j^{(k)} + \sum_{l=0}^{k-1}\left(\mathbf{A}_x^{(k-l)}\right)_{\sigma_i j} f_j^{(l)}\right]$

Homogeneous part: the same at all orders in ϵ

Inhomogeneous part: subtopology terms & lower orders in ϵ

Solving non-canonical systems

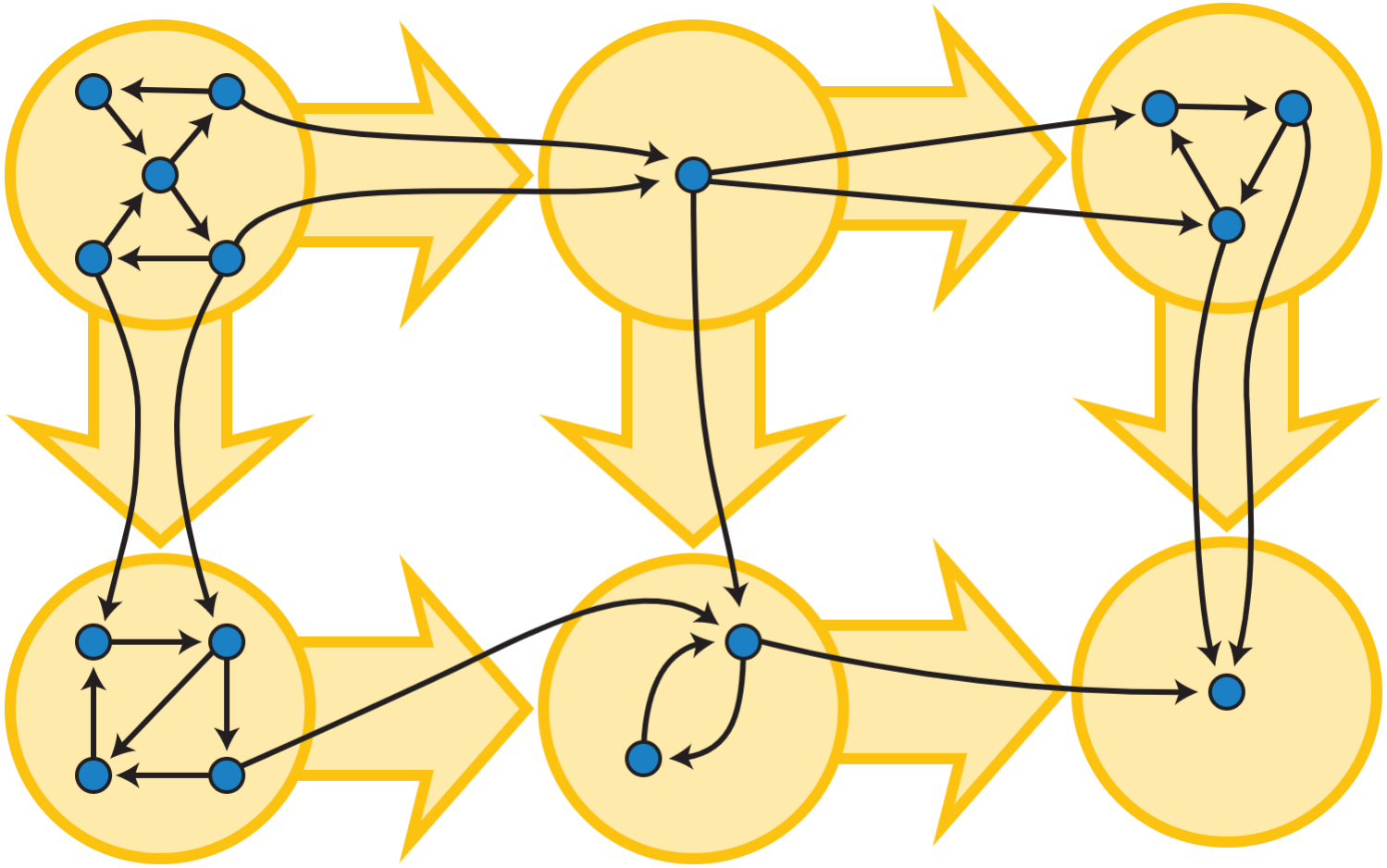
- We consider integrals to be “coupled” when they depend on each other at finite order in ϵ upon repeated differentiation. For example, if $\partial_x f_i$ contains a component of integral f_j, and vice-versa, then f_i and f_j are coupled together.
- To integrate the differential equations, we should proceed at the lowest order in ϵ and integrate from the lowest sectors / topologies, up to the highest. We then move to the next order in ϵ and continue.
- How do we read off an integration order directly from $\mathbf{A}_x^{(0)}$, satisfying these observations?

Integration sequence

- Consider a graph G, which has an edge $f_j \rightarrow f_i$, if the derivative of f_i includes a contribution from f_j at order ϵ^0.
- Next, determine the strongly connected components of G
  - These are sets of vertices for which there is a directed path between every pair of vertices.
  - Note: every vertex is connected to itself by definition
- The strongly connected components are the coupled integrals. Next, we consider the graph of strongly connected components, called the “condensation” of G

Integration sequence

- Example of the condensation of a directed cyclic graph:



- The condensation has edges between two str. conn. components, when there is an edge between its integrals in the original graph G

- The condensation is an acyclic graph defining a partial ordering, from which we read off the integration order.

Picture borrowed from https://commons.wikimedia.org/wiki/File:Graph_Condensation.svg

Solving non-canonical systems

- We have now decomposed the differential equation such that we have to solve for each coupled block $\vec{g}$ at order ϵ^k a system of the form: $\partial_x \vec{g}^{(k)} = \mathbf{M}\vec{g}^{(k)} + \vec{b}^{(k)}$
- Following the previously described integration order, $\vec{b}^{(k)}$ is always available from previously computed data.
- We can split up the task in two parts:
  - 1: Solve the homogeneous diff. eqns: $\partial_x \vec{g}^{(k)} = \mathbf{M}\vec{g}^{(k)}$
  - 2. Obtain solutions to the full system of diff. eqns
- In the following we will drop the subscripts for brevity.

Homogeneous differential equations

- Consider $\partial_x \vec{g} = \mathbf{M}\vec{g}$
- Strategy (based on combination of standard techniques):
  1. Combine the system into a p-th order differential equation for g_i
  2. Find p (homogeneous) solutions for g_i using the Frobenius method and reduction of order
  3. Solve for the remaining g_j in terms of g_i
- Detailed steps can be found in [MH, 2006.05510].
- The result is a matrix of solutions $\mathbf{F}$, satisfying $\partial \mathbf{F} = \mathbf{M}\mathbf{F}$

Inhomogeneous differential equations

- Next, we consider the full system $\partial_x \vec{g} = \mathbf{M}\vec{g} + \vec{b}$
- We leave out the derivation, and provide the solution below:

$$\vec{g} = \sum_{j=1}^{p} \vec{\mathbf{G}}_j, \mathbf{G} = \mathbf{F}\left(\int \mathbf{F}^{-1}\mathbf{B} + \mathbf{E}\right)$$

Where $\mathbf{B} = \frac{1}{p}(\vec{b}, \ldots, \vec{b})$ contains the inhomogeneous terms along the columns, and where $\mathbf{E} = \mathrm{diag}(e_1, \ldots, e_p)$ is a diagonal matrix of integration constants to be determined by boundary values.

Inhomogeneous differential equations

- If we reintroduce a superscript for the order in ϵ we have that:

$$\vec{g}^{(k)} = \sum_{\mathrm{j}=1}^{\mathrm{p}} \vec{\mathbf{G}}_{\mathrm{j}}^{(k)}, \mathbf{G}^{(\mathrm{k})} = \mathbf{F}\left(\int \mathbf{F}^{-1}\mathbf{B}^{(\mathrm{k})} + \mathbf{E}^{(k)}\right)$$

- We need to compute $\mathbf{F}$ and $\mathbf{F}^{-1}$ only once. Higher orders in ϵ are obtained by two matrix multiplications, and a single integration (which is implemented using an efficient replacement rule.)
- This compares favorably to a straightforward “variation of parameters” implementation, which involves computing p determinants of matrices of size $(p-1) \times (p-1)$ for each order in ϵ. (Take into account the matrix entries are themselves series expansion.)

Line segmentation

- The series solutions have a finite radius of convergence.
- By concatenating solutions centered at different line segments we can reach any point in phase-space. How do we choose where to center them?

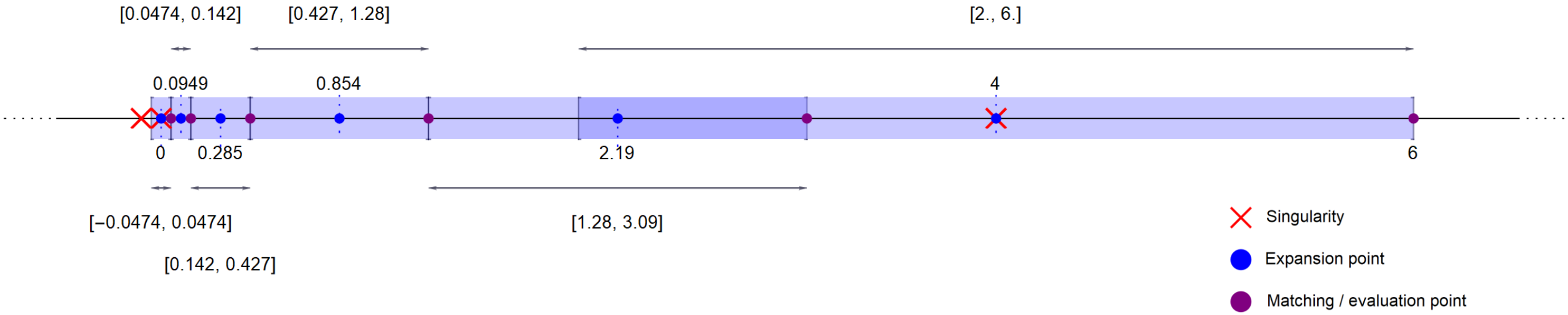
- We may choose the line segments such that each expansion is evaluated at most $1/k$ the distance to the nearest singularity, where $k > 1$.
- To cross singularities, we center an expansion at the singularity

Line segmentation

- For example, suppose: $\mathrm{X}_{\mathrm{sing}} = (\ldots, -0.095, 0, 4, 16, \ldots)$

$$x_{\mathrm{start}} = 0, x_{\mathrm{end}} = 6$$

- Then we may pick the following partitioning into 6 line segments, such that each evaluation happens at most ½ the distance to the nearest singularity:



Analytic continuation

- The series solutions centered at singularities may contain logarithms and square roots.
  - Logarithms appear by integration of poles $1/x$.
  - Square roots can arise from the homogeneous solutions (when the indicial equation has a half-integer root), or from the basis definition.
- By transferring an $i\delta$-prescription to the line parameter, we can perform the analytic continuation of these functions. In particular we can let:

$$\log(x + i\delta) = \log(x), \qquad \sqrt{x + i\delta} = \sqrt{x}$$
$$\log(x - i\delta) = \log(x) - 2\pi i\theta_m, \qquad \sqrt{x - i\delta} = (\theta_p - \theta_m)\sqrt{x}$$

Analytic continuation

- We don't like to carry theta functions around in the series expansions (for performance reasons), so we may instead use replacement rules.
- For example, if x carries $-i\delta$, and we evaluate at a point $x < 0$, we let:

$$\log(x) \to \log(x) - 2\pi i, \quad \sqrt{x} \to -\sqrt{x}$$

- <u>Additional comments:</u>
  - The $i\delta$-prescriptions can be determined from the Feynman prescription
  - Typically, we should avoid crossing two singular regions at the same time

Möbius transformations

- Using Möbius transformations we may improve the convergence of the expansions. For example, consider: $f(x) = \frac{1}{1/10 + x} + \frac{1}{1 - x}$
- Then: $f(x) = 9 - 101x + 999x^2 - 10001x^3 + 99999x^4 - 1000001x^5 + \mathcal{O}(x)^6$
- Next, consider the Möbius transformation: $x = \frac{2y}{11-9y}$, so that for $y \in [-1,1]$, we have $x \in [-1/10,1]$.
- We then have: $f(y) = 9 - \frac{202y}{11} + 18y^2 - \frac{202y^3}{11} + 18y^4 - \frac{202y^5}{11} + \mathcal{O}(y)^6$
- And numerically we find: $S_{100} f(y = 11/13) = -0.335377$

$$f(x = 1/2) = 1/3, \quad S_{100} f(x = 1/2) = -1.31477\ldots \cdot 10^{70},$$

Möbius transformations

- Thus, we may improve the integration strategy in the following way:
  - Find the singularity whose real part is nearest on the left of the origin
  - Find the singularity whose real part is nearest on the right of the origin
  - Map these respective singularities to -1, and 1.
- Disadvantages:
  - Möbius transformations may slow down the series expansions of the matrices, partly negating their speedup. (Perhaps this can be improved in future versions of `DiffExp`.)

Padé approximants

- Lastly, we may use (diagonal) Padé approximants to accelerate the convergence of our series. These are rational functions, whose series expansion matches the original series. For example:

$$\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\frac{7x^5}{256}-\frac{21x^6}{1024}+\frac{33x^7}{2048}-\frac{429x^8}{32768}+\frac{715x^9}{65536}-\frac{2431x^{10}}{262144}+O\left(x^{11}\right)$$

$$S_{10}(\sqrt{1+x})|_{x=1/2} - \sqrt{1+1/2} = -2.72 \cdot 10^{-6}$$

- Padé approximant:

$$\sqrt{1+x} \approx \frac{1+\frac{22x}{9}+\frac{33x^2}{16}+\frac{11x^3}{16}+\frac{55x^4}{768}}{1+\frac{35x}{18}+\frac{175x^2}{144}+\frac{25x^3}{96}+\frac{25x^4}{2304}-\frac{x^5}{4608}}$$

$$(P_{4,5}(\sqrt{1+x})|_{x=1/2} - \sqrt{1+1/2}) = -3.47 \cdot 10^{-10}$$

- Downsides: 1. Requires higher working precision
  2. Computation of the Padé approximants takes time

Examples

Higgs plus jet production @ NLO

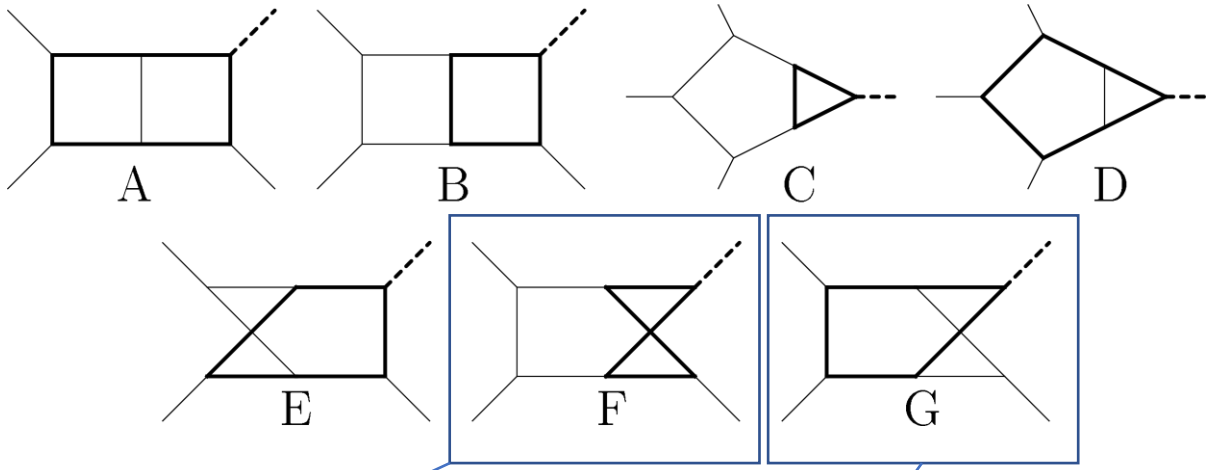
- Main production mode of the Higgs boson @ LHC is via gluon-gluon fusion
- The Higgs particle does not couple directly to gluons. The interaction is mediated by a heavy quark loop, so that next-to-leading order concerns two-loop diagrams
- To this date, no NLO computation is available of the whole p_T-spectrum, including quark-mass effects for all quark flavors
  - An NLO computation including the top-quark mass but neglecting bottom-quark mass has been performed using sector decomposition for the integrals [Jones, Kerner, Luisoni, 2018]
  - Various computations have also been done in HEFT (some up to N^3LO) e.g.: [Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 1503.06056]. [Chen, Gehrmann, Glover, Huss, Mistlberger, Pelloni, 2102.07607]

Higgs + jet integrals

R. Bonciani, V. Del Duca, H. Frellesvig, J. M. Henn, MH, L. Maestri, F. Moriello, G. Salvatori, V. A. Smirnov

[Bonciani et al, 1609.06685]

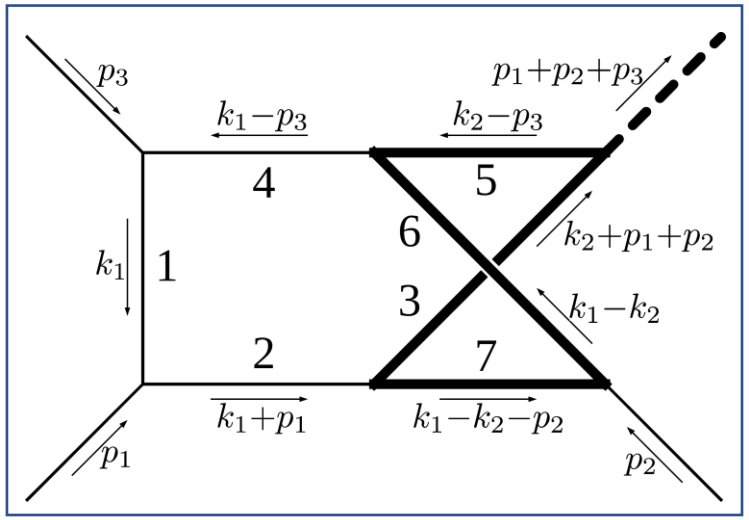
- Integrals relevant for H+j production at NLO with full heavy quark mass dependence
- Dependence on three scales (after normalizing out mass dependence)
- Families A, F, and G contain elliptic sectors



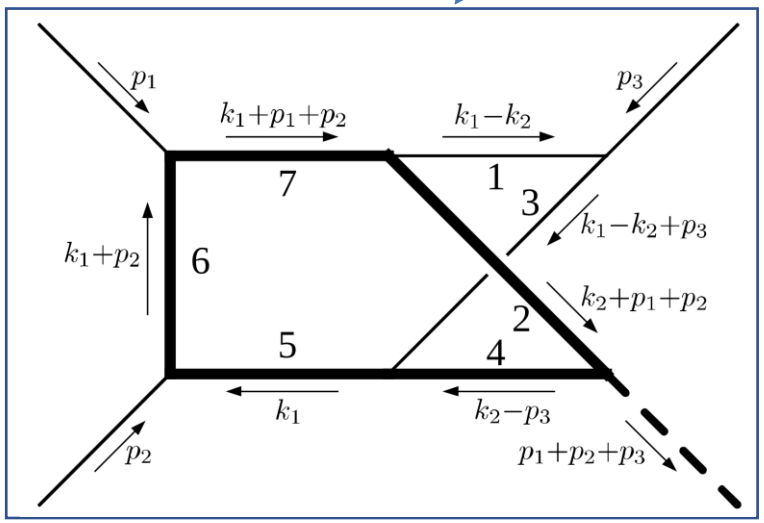
$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2$$

$$p_4^2 = (p_1 + p_2 + p_3)^2 = s + t + u$$

m_q



[Bonciani et al, 1907.13156]



[Frellesvig et al, 1911.06308]

Family F
Master integrals

- IBP-reduction:
  - 73 master integrals
  - Default FIRE basis: $\mathcal{O}$(1 GB)
  - More suitable (pre-canonical) basis: $\mathcal{O}$(100 MB)
  - Possible using either FIRE or KIRA
- Differential eqns: $\mathcal{O}$(10 MB)

  [Bonciani et al, 1907.13156]

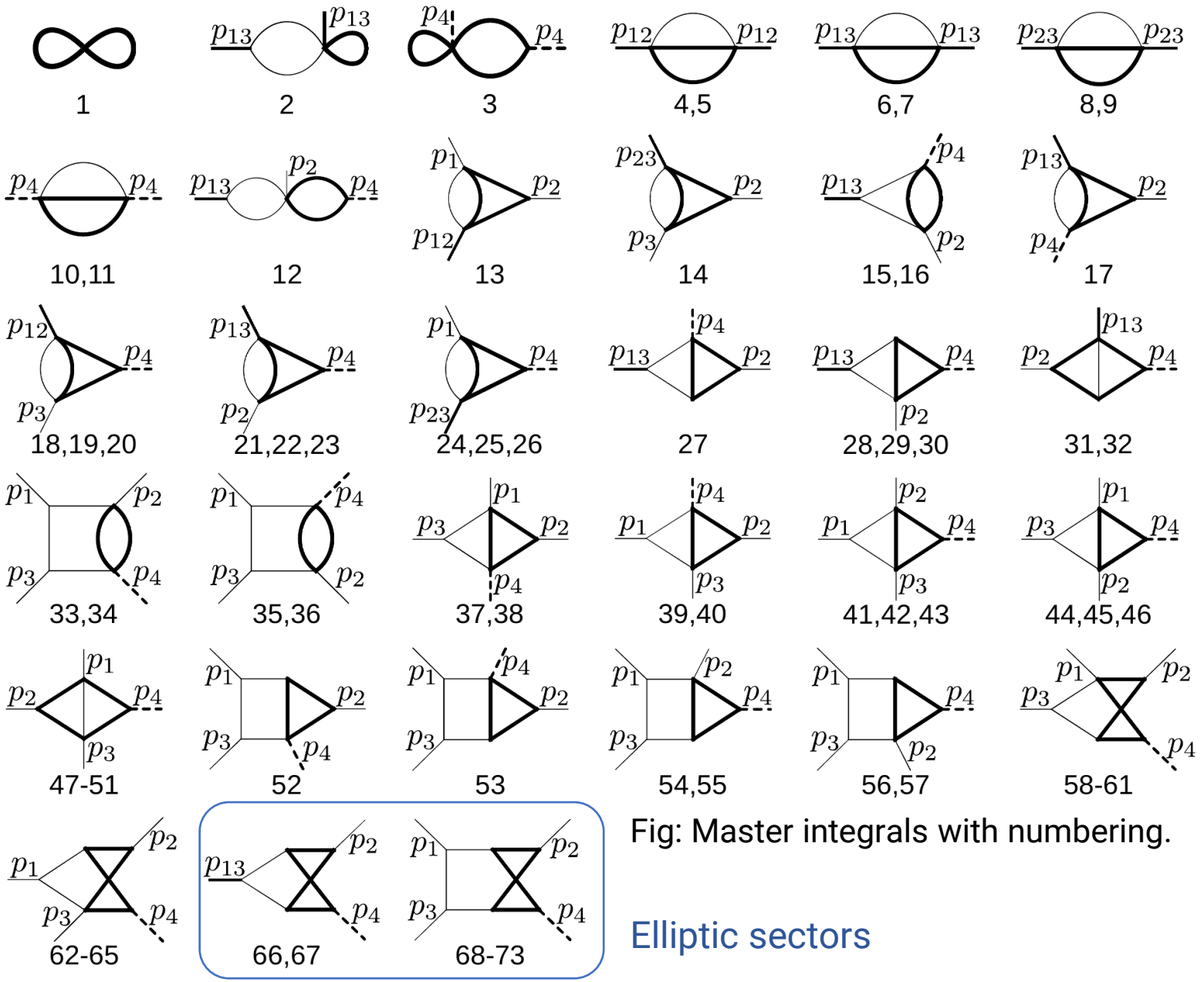


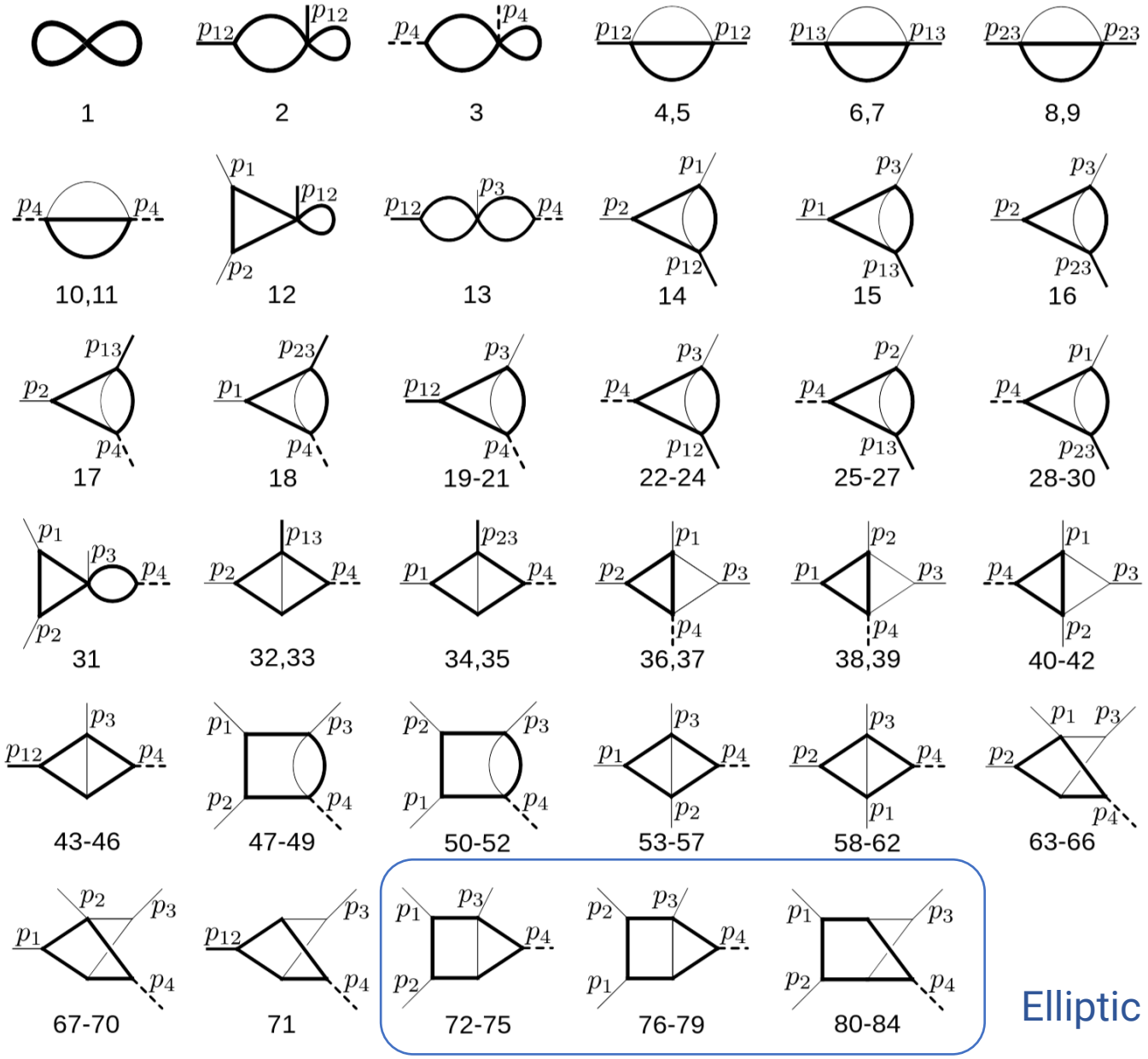
Fig: Master integrals with numbering.

Elliptic sectors

Family G
Master integrals

- IBP-reduction:
  - 84 master integrals
  - Default FIRE basis: $\mathcal{O}$(1 GB)
  - More suitable (pre-canonical) basis: $\mathcal{O}$(100 MB)
  - Possible using either FIRE or KIRA
- Differential eqns: $\mathcal{O}$(10 MB)

[Bonciani et al, 1911.06308]



Elliptic sectors

Boundary conditions of family F

- All boundary conditions for family F: $(s, t, p_4^2, m^2) \to (xs, xt, xp_4^2, m^2)$

$$\lim_{x\to 0} B_1 = e^{2\gamma\epsilon}\Gamma(\epsilon+1)^2 (m^2)^{-2\epsilon}\,,$$

$$\lim_{x\to 0} B_2 \sim x^{-\epsilon}\left(\pi e^{2\gamma\epsilon}\epsilon (m^2)^{-\epsilon}(-t)^{-\epsilon}\Gamma(2\epsilon+1)\cot(\pi\epsilon)\right)\,,$$

$$\lim_{x\to 0} B_i = 0 \quad \text{for } i = 3, \ldots, 72\,.$$

$$\lim_{x\to 0} B_{73} \sim x^{-\epsilon}\left(-4\pi e^{2\gamma\epsilon}\epsilon^3 \frac{(p_4^2 - 4s - t)}{(p_4^2 - 2s - t)} (m^2)^{-\epsilon}(-t)^{-\epsilon}\Gamma(2\epsilon)\cot(\pi\epsilon)\right)\,.$$

- Requires computation of numerous integrals:

Boundary conditions of family F

```
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```

Higgs + jet integrals

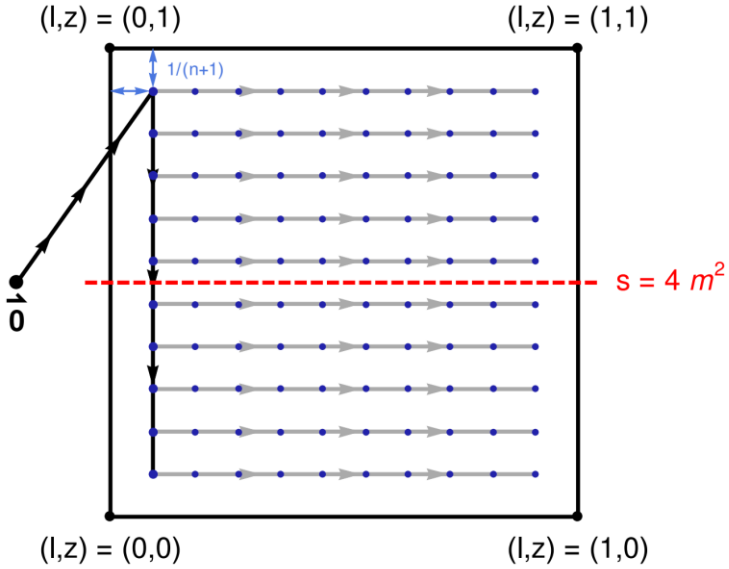
[1907.13156, 1911.06308]

- We can obtain 3-dimensional plots, if we sample enough points. Consider the parametrization:

$$\text{top} \quad (l,z)_t: \quad s = \frac{87-74z}{25z}, \qquad t = \frac{87\,l\,(z-1)}{25z}, \qquad p_4^2 = \frac{13}{25},$$

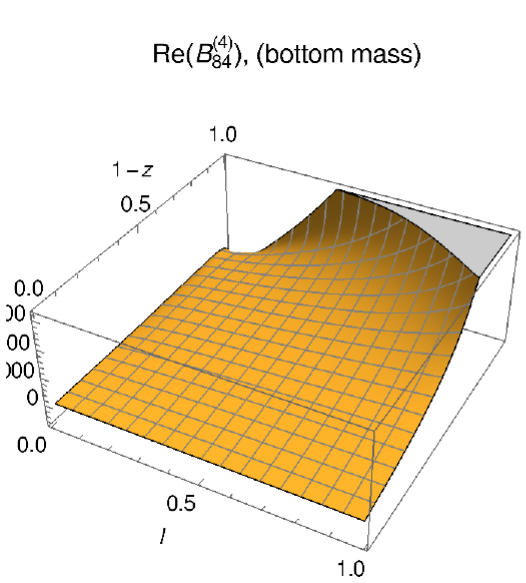
$$\text{bottom} \quad (l,z)_b: \quad s = \frac{323761}{361z}, \qquad t = \frac{323761\,l\,(z-1)}{361z}, \qquad p_4^2 = \frac{323761}{361}.$$

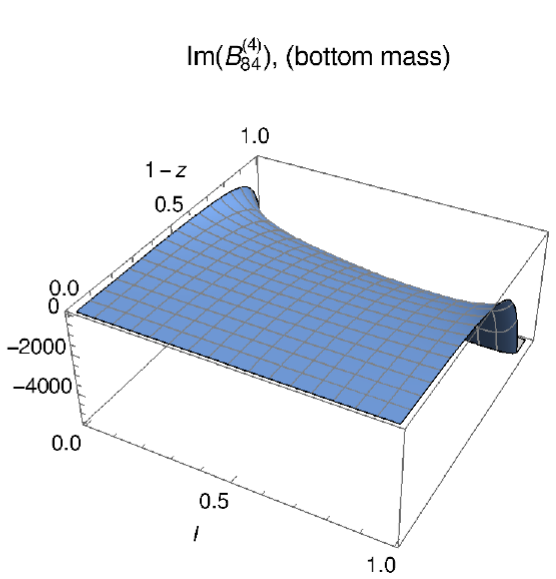
- Which maps the physical regions of the top quark and bottom quark contributions to the unit square:



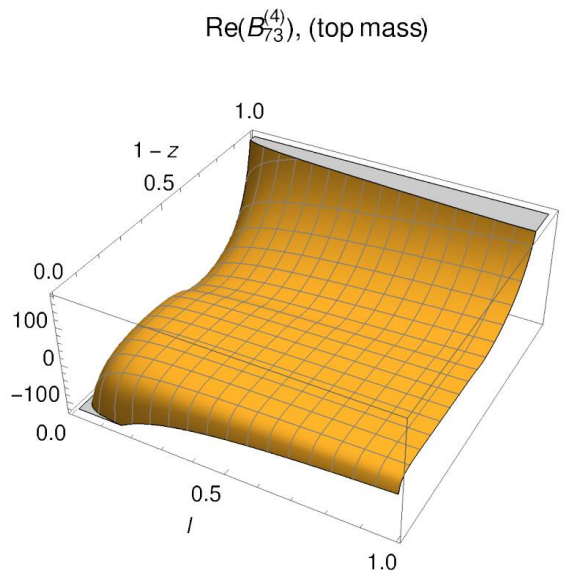
Higgs + jet integrals

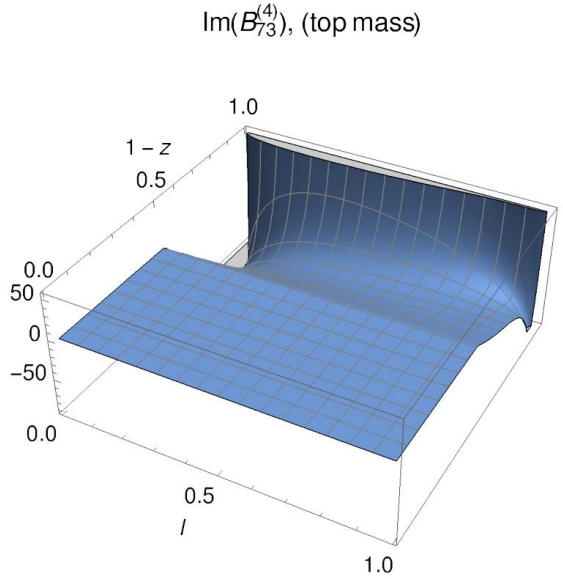
Plots sampled from 10000 points on an evenly spaced grid.





Family G

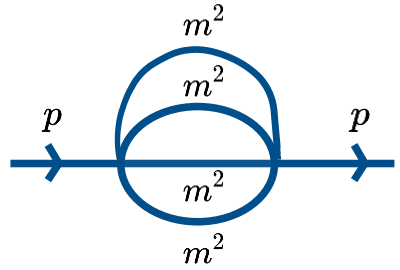




Family F

Example of timing: obtaining 10000 points for family G, on a 4-core laptop CPU took about 19.5 hours for the top quark contributions.

3-loop banana graph



- First, we consider the equal-mass case:

$$I^{\text{banana}}_{a_1 a_2 a_3 a_4} = \left(\frac{e^{\gamma_E \epsilon}}{i\pi^{d/2}}\right)^3 (m^2)^{a-\frac{3}{2}(2-2\epsilon)} \left(\prod_{i=1}^{4} \int d^d k_i\right) D_1^{-a_1} D_2^{-a_2} D_3^{-a_3} D_4^{-a_4}$$

$$D_1 = -k_1^2 + m^2\,, \quad D_2 = -k_2^2 + m^2\,, \quad D_3 = -k_3^2 + m^2\,, \quad D_4 = -(k_1 + k_2 + k_3 + p_1)^2 + m^2$$

- The differential equations are given by:

$$\vec{B}^{\text{banana}} = (\epsilon I^{\text{banana}}_{2211}\,, \epsilon(1+3\epsilon) I^{\text{banana}}_{2111}\,, \epsilon(1+3\epsilon)(1+4\epsilon) I^{\text{banana}}_{1111}\,, \epsilon^3 I^{\text{banana}}_{1110}\,)$$

$$\partial_t \vec{B}^{\text{banana}} = \begin{pmatrix} -\frac{64-2t+t^2+(8+t)^2\epsilon}{t(t-16)(t-4)} & \frac{2(t+20)(2\epsilon+1)}{t(t-16)(t-4)} & -\frac{6(2\epsilon+1)}{t(t-16)(t-4)} & -\frac{2\epsilon}{t(t-16)} \\ \frac{3t(3\epsilon+1)}{t(t-4)} & -\frac{2(t+8)\epsilon+t+4}{t(t-4)} & \frac{3\epsilon+1}{t(t-4)} & 0 \\ 0 & \frac{4(4\epsilon+1)}{t} & \frac{-3\epsilon-1}{t} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{B}^{\text{banana}}$$

- With $t = p_1^2/m^2$

3-loop banana graph

- Next, let us obtain suitable boundary conditions
- The Feynman parametrization is given by:

$$I_{1111}^{\text{banana}} = ie^{3\gamma\epsilon}\Gamma(3\epsilon+1)\left(m^2\right)^{-3\epsilon-1} x^{3\epsilon+1}\int_\Delta d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \left(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_4\alpha_3+\alpha_2\alpha_4\alpha_3+\right.$$
$$\left.\alpha_1\alpha_2\alpha_4\right)^{4\epsilon}\left(\alpha_2\alpha_3\alpha_1^2 x+\alpha_2\alpha_4\alpha_1^2 x+\alpha_3\alpha_4\alpha_1^2 x+\alpha_2\alpha_3^2\alpha_1 x+\alpha_2\alpha_4^2\alpha_1 x+\alpha_3\alpha_4^2\alpha_1 x+\right.$$
$$\alpha_2^2\alpha_3\alpha_1 x+\alpha_2^2\alpha_4\alpha_1 x+\alpha_3^2\alpha_4\alpha_1 x+4\alpha_2\alpha_3\alpha_4\alpha_1 x+\alpha_2\alpha_3\alpha_4^2 x+\alpha_2\alpha_3^2\alpha_4 x+\alpha_2^2\alpha_3\alpha_4 x+$$
$$\left.+\alpha_2\alpha_3\alpha_4\alpha_1\right)^{-3\epsilon-1}$$

- Where we let $t=-1/x$.
- We will compute boundary conditions in the limit $x\to 0$, which is equivalent to the limit where the mass vanishes.

3-loop banana graph

- We use the method of expansions by regions and `asy.m` to obtain the regions.
- They are given by:

$$
\begin{aligned}
&R_1 = \{0,-1,-1,-1\}\,, && R_2 = \{0,-1,-1,0\}\,, && R_3 = \{0,0,0,0\}\,,\\
&R_4 = \{0,0,0,-1\}\,, && R_5 = \{0,1,1,0\}\,, && R_6 = \{0,0,1,0\}\,,\\
&R_7 = \{0,-1,0,-1\}\,, && R_8 = \{0,-1,0,0\}\,, && R_9 = \{0,0,0,1\}\,,\\
&R_{10} = \{0,1,1,1\}\,, && R_{11} = \{0,0,1,1\}\,, && R_{12} = \{0,1,0,0\}\,,\\
&R_{13} = \{0,0,-1,-1\}\,, && R_{14} = \{0,1,0,1\}\,, && R_{15} = \{0,0,-1,0\}\,.
\end{aligned}
$$

- Their contributions work out to:

$$
\begin{aligned}
&\mathrm{I}^{R_1}_{1111} \sim x e^{3\gamma\epsilon}\Gamma(\epsilon)^3\,, && \mathrm{I}^{R_2}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_3}_{1111} \sim \frac{3e^{3\gamma\epsilon}\epsilon x^{3\epsilon+1}\Gamma(-\epsilon)^4\Gamma(3\epsilon)}{\Gamma(-4\epsilon)}\,,\\
&\mathrm{I}^{R_4}_{1111} \sim \frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)}\,, && \mathrm{I}^{R_5}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_6}_{1111} \sim x e^{3\gamma\epsilon}\Gamma(\epsilon)^3\,,\\
&\mathrm{I}^{R_7}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_8}_{1111} \sim \frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)}\,, && \mathrm{I}^{R_9}_{1111} \sim x e^{3\gamma\epsilon}\Gamma(\epsilon)^3\,,\\
&\mathrm{I}^{R_{10}}_{1111} \sim \frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)}\,, && \mathrm{I}^{R_{11}}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_{12}}_{1111} \sim x e^{3\gamma\epsilon}\Gamma(\epsilon)^3\,,\\
&\mathrm{I}^{R_{13}}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_{14}}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_{15}}_{1111} \sim \frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)}
\end{aligned}
$$

3-loop banana graph

- Summing over all contributions, we obtain the following result:

$$I_{1111}^{\text{banana}} \overset{x\downarrow 0}{\sim} \frac{6e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)} + \frac{8e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)} + \frac{3e^{3\gamma\epsilon}\epsilon x^{3\epsilon+1}\Gamma(-\epsilon)^4\Gamma(3\epsilon)}{\Gamma(-4\epsilon)} + 4xe^{3\gamma\epsilon}\Gamma(\epsilon)^3 + \mathcal{O}(x^2)\,.$$

$$I_{1110}^{\text{banana}} = e^{3\gamma\epsilon}\Gamma(\epsilon)^3$$

- Next, we show how to obtain results for any values of p^2 using DiffExp

DiffExp

- Typical usage of the package:
  - Set configuration options using the method `LoadConfiguration[opts_]`
  - Prepare a list of boundary conditions using `PrepareBoundaryConditions[bcs_, line_]`
  - Then we can find series solutions along a line using the function:

```
IntegrateSystem[bcsprepared_, line_]
```

  - Or one can transport the boundary conditions to a new point using:

```
TransportTo[bcsprepared_, point_]
```

3-loop banana graph

- Load DiffExp:

```
Get[FileNameJoin[{NotebookDirectory[], "..", "DiffExp.m"}]];
```

```
Loading DiffExp version 1.0.2
Author: Martijn Hidding. Email: hiddingm@tcd.ie.
```

- Set the configuration options and load the matrices

```
EqualMassConfiguration = {
   DeltaPrescriptions → {t - 16 + I δ},
   MatrixDirectory → NotebookDirectory[] <> "Banana_EqualMass_Matrices/",
   UseMobius → True, UsePade → True
  };

LoadConfiguration[EqualMassConfiguration];
```

```
DiffExp: Loading matrices.
DiffExp: Found files: {dt_0.m, dt_1.m, dt_2.m, dt_3.m, dt_4.m}
DiffExp: Kinematic invariants and masses: {t}
DiffExp: Getting irreducible factors..
DiffExp: Configuration updated.
```

3-loop banana graph

- Prepare the boundary conditions along an asymptotic limit:

```
EqualMassBoundaryConditions = {
    "?",
    "?",
    ϵ (1 + 3 ϵ) (1 + 4 ϵ) (-(4 E^(3 EulerGamma ϵ) Gamma[ϵ]^3)/t + (6 E^(3 EulerGamma ϵ) (-1/t)^(1+ϵ) ϵ Gamma[-ϵ]^2 Gamma[ϵ]^3)/Gamma[-2 ϵ] +
      (8 E^(3 EulerGamma ϵ) (-1/t)^(1+2 ϵ) ϵ Gamma[-ϵ]^3 Gamma[ϵ] Gamma[2 ϵ])/Gamma[-3 ϵ] + (3 E^(3 EulerGamma ϵ) (-1/t)^(1+3 ϵ) ϵ Gamma[-ϵ]^4 Gamma[3 ϵ])/Gamma[-4 ϵ]),
    E^(3 EulerGamma ϵ) ϵ^3 Gamma[ϵ]^3
  } // PrepareBoundaryConditions[#, <|t → -1/x|>] &;
```

```
DiffExp: Integral 1: Ignoring boundary conditions.
DiffExp: Integral 2: Ignoring boundary conditions.
DiffExp: Assuming that integral 3 is exactly zero at epsilon order 0.
DiffExp: Prepared boundary conditions in asymptotic limit, of the form:
```

DiffExp:	?	?	?	?	?
	?	?	?	?	?
	$O[x]^{51}$	$(\ldots)\, x + O[x]^{3/2}$	$(\ldots)\, x + O[x]^{3/2}$	$(\ldots)\, x + O[x]^{3/2}$	$(\ldots)\, x + O[x]^{3/2}$
	$(\ldots) + \sqrt{O[x]}$	$\sqrt{O[x]}$	$(\ldots) + \sqrt{O[x]}$	$(\ldots) + \sqrt{O[x]}$	$(\ldots) + \sqrt{O[x]}$

3-loop banana graph

- Next, we transport the boundary conditions:

```
Transport1 = TransportTo[EqualMassBoundaryConditions, <|t → -1|>];
Transport2 = TransportTo[Transport1, <|t → x|>, 32, True];
```

DiffExp: Transporting boundary conditions along $\left\langle \left| t \to -\frac{1.}{x} \right| \right\rangle$ from x = 0. to x = 1.

DiffExp: Preparing partial derivative matrices along current line..

DiffExp: Determining positions of singularities and branch-cuts.

DiffExp: Possible singularities along line at positions {0.}.

DiffExp: Analyzing integration segments.

DiffExp: Segments to integrate: 3.

DiffExp: Integrating segment: $\left\langle \left| t \to \frac{8.\ (-1.+1.\ x)}{x} \right| \right\rangle$.

DiffExp: Integrated segment 1 out of 3 in 20.8565 seconds.

DiffExp: Evaluating at x = 0.0625

DiffExp: Current segment error estimate: 5.14483×10^{-31}

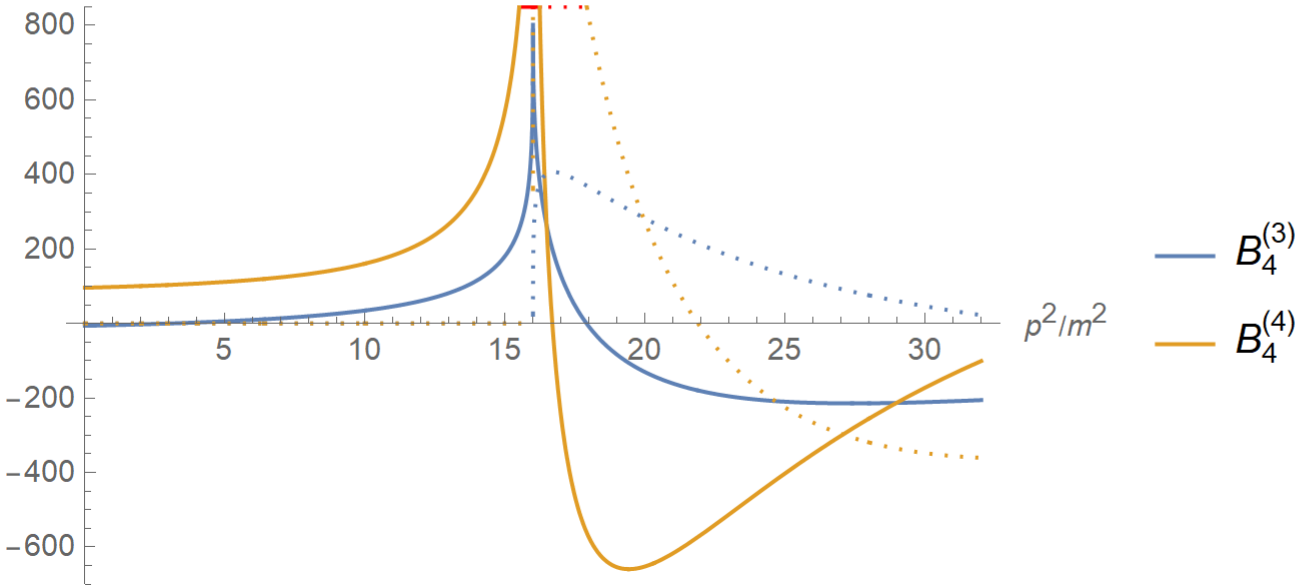
DiffExp: Total error estimate: 5.14483×10^{-31}

DiffExp: Integrating segment: $\left\langle \left| t \to \frac{-1.+1.\ x}{\ } \right| \right\rangle$

3-loop banana graph

- Lastly, we plot the result:

```
ResultsForPlotting = ToPiecewise[Transport2];
Quiet[ReImPlot[{ResultsForPlotting[[3, 4]][x], ResultsForPlotting[[3, 5]][x]}, {x, 0, 32},
  ClippingStyle → Red, PlotLegends → {"B4(3)", "B4(4)"}, AxesLabel → {"p²/m²"}, PlotRange → {-700, 850},
  MaxRecursion → 15, WorkingPrecision → 100]]
```



3-loop banana graph

- Timing:
  - Moving from $p^2 = -\infty$ to $p^2 = 30$ at a precision of 25 digits takes about 90 sec, where we computed the top sector integrals up to and including order ϵ^3.
  - Moving from $p^2 = -\infty$ to $p^2 = 30$ at a precision of 100 digits takes a bit under 20 min, where we computed the top sector integrals up to and including order ϵ^3.
  - Obtaining 100+ digits at $p^2 = -100$ up to and including order ϵ^3 takes about 2.5 min.
- $B_3^{(k)}$:

```
0
4.0824132027040596078019914610450973398555012537742224344965637983148482839073301994896032486421781 29
-0.7713150915227857546258559692543676298350939151980774607908277236769934490973612004866036340787026038
-15.52268532416518855576696548019433617730937578226039207428302008586262767404183548619606743796239099
78.12509728148001692986790482079302619114776011817121195506011258285334682242128391076363566162968586
```

3-Loop banana graph

- In the unequal mass case, we may choose the “precanonical” basis:

$$\vec{B}^{\text{banana}} = \left\{ \begin{array}{l} \epsilon I^{\text{banana}}_{1122}, \epsilon I^{\text{banana}}_{1212}, \epsilon I^{\text{banana}}_{1221}, \epsilon I^{\text{banana}}_{2112}, \epsilon I^{\text{banana}}_{2121}, \epsilon I^{\text{banana}}_{2211}, \\ \epsilon(1+3\epsilon) I^{\text{banana}}_{1112}, \epsilon(1+3\epsilon) I^{\text{banana}}_{1121}, \epsilon(1+3\epsilon) I^{\text{banana}}_{1211}, \\ \epsilon(1+3\epsilon) I^{\text{banana}}_{2111}, \epsilon(1+3\epsilon)(1+4\epsilon) I^{\text{banana}}_{1111}, \\ \epsilon^3 I^{\text{banana}}_{0111}, \epsilon^3 I^{\text{banana}}_{1011}, \epsilon^3 I^{\text{banana}}_{1101}, \epsilon^3 I^{\text{banana}}_{1110} \end{array} \right\}$$

- The unequal mass case is significantly more difficult to compute for DiffExp, due to the presence of 11 coupled master integrals.
- The series expansions grow wildly at intermediate stages of the calculations, which puts the linear algebra routines off track.
- Therefore, we must work at a high working precision (1000+), and rescale the line parameters in a way that the series coefficients don’t become too large at high orders

3-Loop banana graph

- We provide 55 digits of basis integral B_{11} below, in the point

$$(p^2 = 50, m_1^2 = 2, m_2^2 = 3/2, m_3^2 = 4/3, m_4^2 = 1)$$

$$B_{11}^{(0)} = 0$$

$$B_{11}^{(1)} = 5.19725211369650431701295785385636524056189391223890786 45 + i\ 6.87551695353902075013706856455389022995590245518309565 94$$

$$B_{11}^{(2)} = -17.95801081120940608995233616989284789487806870538990757 33 + i\ 31.74367036336930909084029322990119719135089506494942310 47$$

$$B_{11}^{(3)} = -121.51011520681775652033928075412160849628807729083063706 68 - i\ 40.76907623602027664537759999171722265374282585291457547 46$$

$$B_{11}^{(4)} = 125.61133880236055347455937640047989582321186326812570739 23 - i\ 229.92002571723885899520627575712151768344717834951127550 27$$

- This point can be obtained in about 23 min.

4-Loop banana graph

- We can also compute higher loop banana graphs.

$$I^{\text{banana}}_{a_1 a_2 a_3 a_4} = \left(\frac{e^{\gamma_E \epsilon}}{i\pi^{d/2}}\right)^4 \left(m^2\right)^{a-2(2-2\epsilon)} \left(\prod_{i=1}^{5} \int d^d k_i\right) D_1^{-a_1} D_2^{-a_2} D_3^{-a_3} D_4^{-a_4} D_5^{-a_5}$$

$$D_1 = -k_1^2 + m^2, \quad D_2 = -k_2^2 + m^2, \quad D_3 = -k_3^2 + m^2$$
$$D_4 = -k_4^2 + m^2, \quad D_5 = -\left(k_1 + k_2 + k_3 + k_4 + p_1\right)^2 + m^2$$

- We consider the following basis of “precanonical” masters:

$$B_1 = \epsilon \mathrm{I}^{\text{banana}}_{11222}$$
$$B_2 = \epsilon(2\epsilon+1) \mathrm{I}^{\text{banana}}_{11122}$$
$$B_3 = \epsilon(2\epsilon+1)(4\epsilon+1) \mathrm{I}^{\text{banana}}_{11112}$$
$$B_4 = \epsilon(2\epsilon+1)(4\epsilon+1)(5\epsilon+1) \mathrm{I}^{\text{banana}}_{11111}$$
$$B_5 = \epsilon^4 \mathrm{I}^{\text{banana}}_{11110}$$

4-Loop banana graph

- We may find boundary conditions by imposing the vanishing of non-physical singularities.
- This allows a determination of the integrals completely from the differential equations, without any need for asymptotic expansions.
- This follows the approach of:
  - [Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser, 1809.06240]
  - [Abreu, Ita, Moriello, Page, Tschernow, Zeng, 2005.04195]
- First, we need to provide an overall normalization for the basis. This is provided by the tadpole integral which is equal to: $I^{\text{banana}}_{11110} = e^{4\gamma_E \epsilon} \epsilon^4 \Gamma(\epsilon)^4$

4-Loop banana graph

- Then we do the following:
  - We compute the general solution of the top sector integrals at $t = 0$.
  - The expansions contain powers of logarithms, we set their coefficients to zero, which solves some of the indeterminate constants.
  - Next, we transport and center an expansion at $t = 1$. There are again logarithms in the expansions, and we set their coefficients to zero.
  - We repeat this a final time and get rid of a non-physical singularity at $t = 9$.

- Only the physical singularity at $t = 25$ remains at the end and all coefficients are fixed

4-Loop banana graph

- Our original expansion was centered at $t = 0$, where we now find the results:

0	0.5626161626035411	0.3475481638835365	1.911555944481455	0.2718352134528369
0	1.923605373745244	0.6752648394943755	6.876325052991839	−1.339280364786555
0	7.989117602399249	10.23095239518146	7.206853721986161	86.05832181407076
0	39.94558801199625	91.10034998790354	−72.59332146214688	900.8362379685953
1.000000000000000	0	3.289868133696453	−1.602742537546126	6.493939402266829

- Or, moving to the point $t = 50$, we have:

0	−0.127301395 + 0.060055594 i	0.44510796 + 0.05196919 i	−3.6356361 + 2.3160967 i	−3.337360 − 16.053530 i
0	−0.6629555 + 1.5090835 i	−4.6446551 − 6.2746996 i	24.255982 − 2.263457 i	−9.971355 + 57.386664 i
0	4.6398127 + 9.9513277 i	−41.998524 + 31.366660 i	−130.68313 − 120.45461 i	327.35602 − 310.64441 i
0	50.934939 + 28.927728 i	77.63780 + 297.73512 i	−952.72123 + 736.44670 i	−1977.1722 − 1153.1457 i
1.0000000	0	3.2898681	−1.6027425	6.4939394

Special functions (extra)

- As a simple exercise, we can also use DiffExp for evaluating MPLs. For example:

$$\partial_z \begin{pmatrix} G(1,2;z) \\ G(2;z) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{t-1} & 0 \\ 0 & 0 & \frac{1}{t-2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} G(1,2;z) \\ G(2;z) \\ 1 \end{pmatrix}$$

- For which the boundary conditions are $(0,0,1)$ at $z = 0$.

- After building a wrapper function, we can evaluate any MPL:

```
G[1, 2, 3] /. G → GEvaluate // AbsoluteTiming
```

$$\{0.210704,\ (-3.77032114761465429761193 3 + 0. \times 10^{-27}\ i) + 9.59146 \times 10^{-25}\ \text{pm}\}$$

```
G[-1 + I, 1 / 2, 1 / 4] /. G → GEvaluate // AbsoluteTiming
```

$$\{0.224892,\ (-0.037843655542722548767317976280272 - 0.032401313158193018998614285553716\ i) + 2.91955 \times 10^{-34}\ \text{pm}\}$$

Special functions (extra)

- Under normal circumstances, the timing lacks (considerably) behind GiNaC.

- But, in certain edge cases, we can beat GiNaC:

```
G[1, 2, 3, 4, 5] /. G → GEvaluate // AbsoluteTiming
G[1, 2, 3, 4, 5, 6] /. G → GEvaluate // AbsoluteTiming
G[1, 2, 3, 4, 5, 6, 7] /. G → GEvaluate // AbsoluteTiming
G[1, 2, 3, 4, 5, 6, 7, 8] /. G → GEvaluate // AbsoluteTiming
G[1, 2, 3, 4, 5, 6, 7, 8, 9] /. G → GEvaluate // AbsoluteTiming
G[1, 2, 3, 4, 5, 6, 7, 8, 9, 10,
   11, 12, 13, 14, 15, 16, 17, 18, 19, 20] /. G → GEvaluate // AbsoluteTiming
```

$\{1.08976,\ (1.60952262244033111588101496665449269230020513047 + 0.\times 10^{-49}\,\mathrm{i}) + 3.70371\times 10^{-48}\ \mathrm{pm}\}$

$\{1.67828,\ (0.\times 10^{-49} + 0.71789987161399442910474431842108605646469682529\,\mathrm{i}) + 5.19787\times 10^{-48}\ \mathrm{pm}\}$

$\{2.41852,\ (-0.26582341298336027219930343877387480773417067112 + 0.\times 10^{-50}\,\mathrm{i}) + 4.73081\times 10^{-48}\ \mathrm{pm}\}$

$\{3.32486,\ (0.\times 10^{-49} - 0.08417238229887542016854461030494632174559284663 1\,\mathrm{i}) + 2.6084\times 10^{-48}\ \mathrm{pm}\}$

$\{4.45558,\ (0.023286104182601022207577211044712620080786210006 + 0.\times 10^{-51}\,\mathrm{i}) + 6.62625\times 10^{-49}\ \mathrm{pm}\}$

$\{26.3489,\ (0.\times 10^{-57} - 4.656054613250180920446716454085413397136538 1\times 10^{-11}\,\mathrm{i}) + 3.59355\times 10^{-55}\ \mathrm{pm}\}$

```
Ginsh[G[1, 2, 3, 4, 5], {x → x}] // AbsoluteTiming
Ginsh[G[1, 2, 3, 4, 5, 6], {x → x}] // AbsoluteTiming
Ginsh[G[1, 2, 3, 4, 5, 6, 7], {x → x}] // AbsoluteTiming
Ginsh[G[1, 2, 3, 4, 5, 6, 7, 8], {x → x}] // AbsoluteTiming
Ginsh[G[1, 2, 3, 4, 5, 6, 7, 8, 9], {x → x}] // AbsoluteTiming
```

{0.091646, 1.60952262244033111588101496665449269230}

{0.390622, 0.71789987161399442910474431842108605647 i}

{2.40738, -0.26582341298336027219930343877387480812}

{14.3798, -0.08417238229887542016854461030494632151 16 i}

{84.4906, 0.02328610418260102220757721104471262053 0}

Special functions (extra)

- We can also evaluate generalized hypergeometric functions, such as the Appell functions. For example, we have with $F_1(x,y) \equiv F_1(a,b_1,b_2,c;x,y)$

$$x(1-x)\frac{\partial^2 F_1(x,y)}{\partial x^2} + y(1-x)\frac{\partial^2 F_1(x,y)}{\partial x \partial y} + [c-(a+b_1+1)x]\frac{\partial F_1(x,y)}{\partial x} - b_1 y \frac{\partial F_1(x,y)}{\partial y} - ab_1 F_1(x,y) = 0$$

$$y(1-y)\frac{\partial^2 F_1(x,y)}{\partial y^2} + x(1-y)\frac{\partial^2 F_1(x,y)}{\partial x \partial y} + [c-(a+b_2+1)y]\frac{\partial F_1(x,y)}{\partial y} - b_2 x \frac{\partial F_1(x,y)}{\partial x} - ab_2 F_1(x,y) = 0$$

- This can be combined into:

$$\partial_x \begin{pmatrix} F_1(x,y) \\ \partial_y F_1(x,y) \\ \partial_x F_1(x,y) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{b_1}{x-y} & \frac{b_2}{x-y} \\ \frac{ab_1}{x-x^2} & \frac{(-1+y)yb_1}{(-1+x)x(x-y)} & -\frac{-c+x+ax+xb_1+\frac{(-1+x)yb_2}{x-y}}{(-1+x)x} \end{pmatrix} \begin{pmatrix} F_1(x,y) \\ \partial_y F_1(x,y) \\ \partial_x F_1(x,y) \end{pmatrix}$$

$$\partial_y \begin{pmatrix} F_1(x,y) \\ \partial_y F_1(x,y) \\ \partial_x F_1(x,y) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{ab_2}{y-y^2} & \frac{(x-xy)b_1+(x-y)(-c+y+ay+yb_2)}{(-1+y)y(-x+y)} & \frac{(-1+x)xb_2}{(-1+y)y(-x+y)} \\ 0 & -\frac{b_1}{x-y} & \frac{b_2}{x-y} \end{pmatrix} \begin{pmatrix} F_1(x,y) \\ \partial_y F_1(x,y) \\ \partial_x F_1(x,y) \end{pmatrix}$$

Special functions (extra)

- Using the boundary conditions $(1, ab_2/c, ab_1/c)$ at $x = y = 0$, we may use DiffExp to evaluate the Appell F1 for arbitrary (real) x, y.
- For example, $F_1(1, 1/2, -3/2, 5; 3/20, 1/2)$:

```
F1BoundaryConditions = {1, a b2 / c, a b1 / c} /. {a → 1, b1 → 1 / 2, b2 → -3 / 2, c → 5} //
   PrepareBoundaryConditions[#, {x1 → 3 / 20 x, y1 → 1 / 2 x}] &;

Res = TransportTo[F1BoundaryConditions, F1BoundaryConditions[[1]]]; // EchoTiming
Res[[2, 1, 1]] + pm Res[[3, 1, 1]] // N[#, 40] &
```

```
0.437219
```

$$0.8683725567150101477163534326556218507347 + 1.78536 \times 10^{-34}\ \text{pm}$$

- Although the timing is not competitive with other methods, this approach might be straightforward to generalize to other hypergeometric functions. In addition, we can do ϵ expansions of HGFs. Lastly, the analytic continuation is straightforward in this approach.

Conclusion

- Series expansion methods provide an efficient way to evaluate Feynman integrals
- Series expansion methods allow for obtaining high-precision numerical results for beyond elliptic type integrals
- The Mathematica package DiffExp can be used for computing user-provided systems of differential equations

Thank you for listening!