Quantum Computing for Quantum Field Theory

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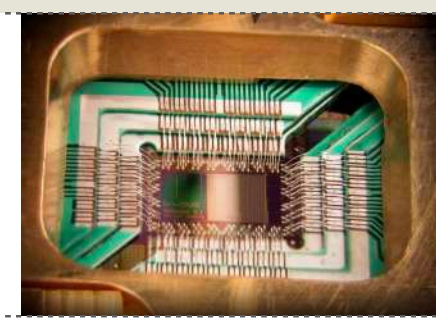
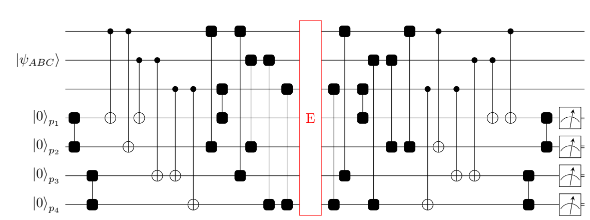
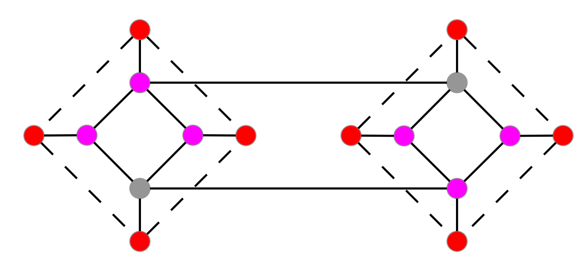
w/ Chancellor and Spannowsky,

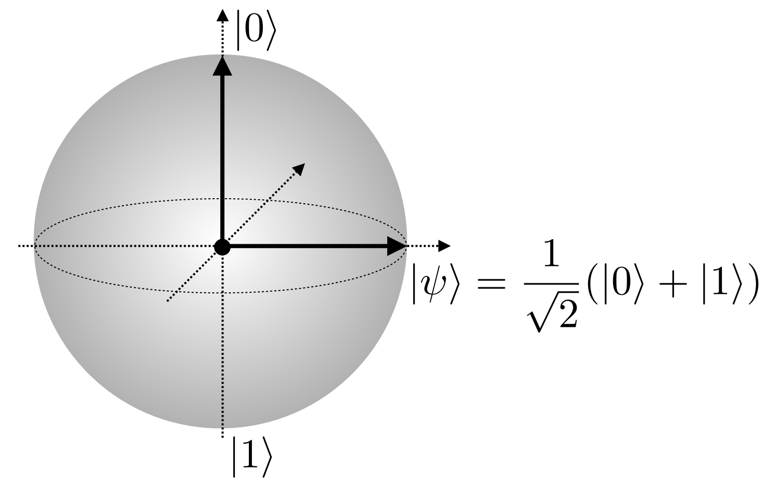
Overview

- Quantum annealers background
- Simple examples of Ising encodings
- Field theory problem: tunnelling in QFT
- Ising encoding of QFT
- Results for thin wall limit
- Thick wall limit and solving PDEs
- Multiple fields and dimensions

Quantum annealers background

Quantum computing has a long and distinguished history but is only now becoming practicable. (Feynman ’81, Zalka '96, Jordan, Lee, Preskill … see Preskill 1811.10085 for review). Two types of Quantum Computer:

Type	Discrete Gate	Quantum Annealer
Property	Universal (any quantum algorithm can be expressed)	Not universal — certain quantum systems
How?	IBM - Qiskit ~50 Qubits	DWave - LEAP ~5000 Qubits
What?		
		



- Both types operate on the Bloch sphere: basically measuring $\sigma_i^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ where $(\sigma^Z|0\rangle = |0\rangle,\, \sigma^Z|1\rangle = -|1\rangle)$ are the possible eigenvector eqns

- Each *i* represents a single qubit

- A discrete quantum gate system is good for looking at things like entanglement, Bell's inequality etc. Also discrete problems, cryptographical problems, Shor's, Grover's algorithms, etc.

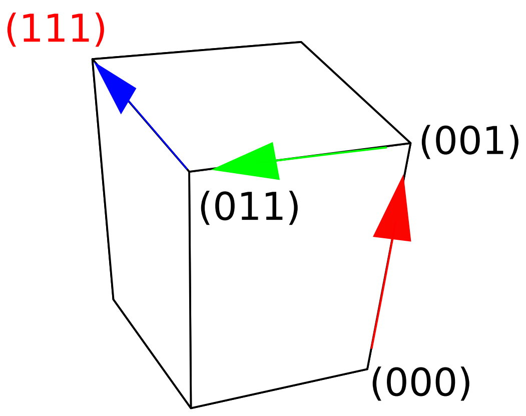
- A quantum annealer is good for looking at network problems but from our perspective it is also a more natural tool for thinking about field theory. It is based on the general transverse field Ising model (Kadowaki, Nishimori):

$$\mathcal{H}_{\mathrm{QA}}(t) = \sum_i \sum_j J_{ij}\sigma_i^Z\sigma_j^Z + \sum_i h_i\sigma_i^Z + \Delta(t)\sum_i \sigma_i^X$$

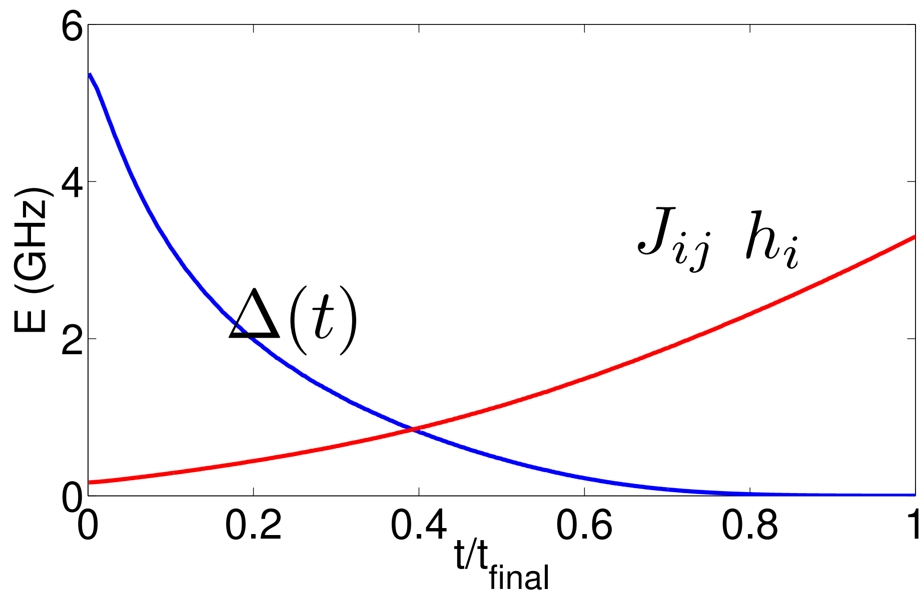
- What does the "anneal" mean?

$$\mathcal{H}_{\mathrm{QA}}(t) = \sum_i \sum_j J_{ij}\sigma_i^Z \sigma_j^Z + \sum_i h_i \sigma_i^Z + \Delta(t) \sum_i \sigma_i^X$$

$\Delta(t)$ induces bit-hopping in the Hamming/Hilbert space



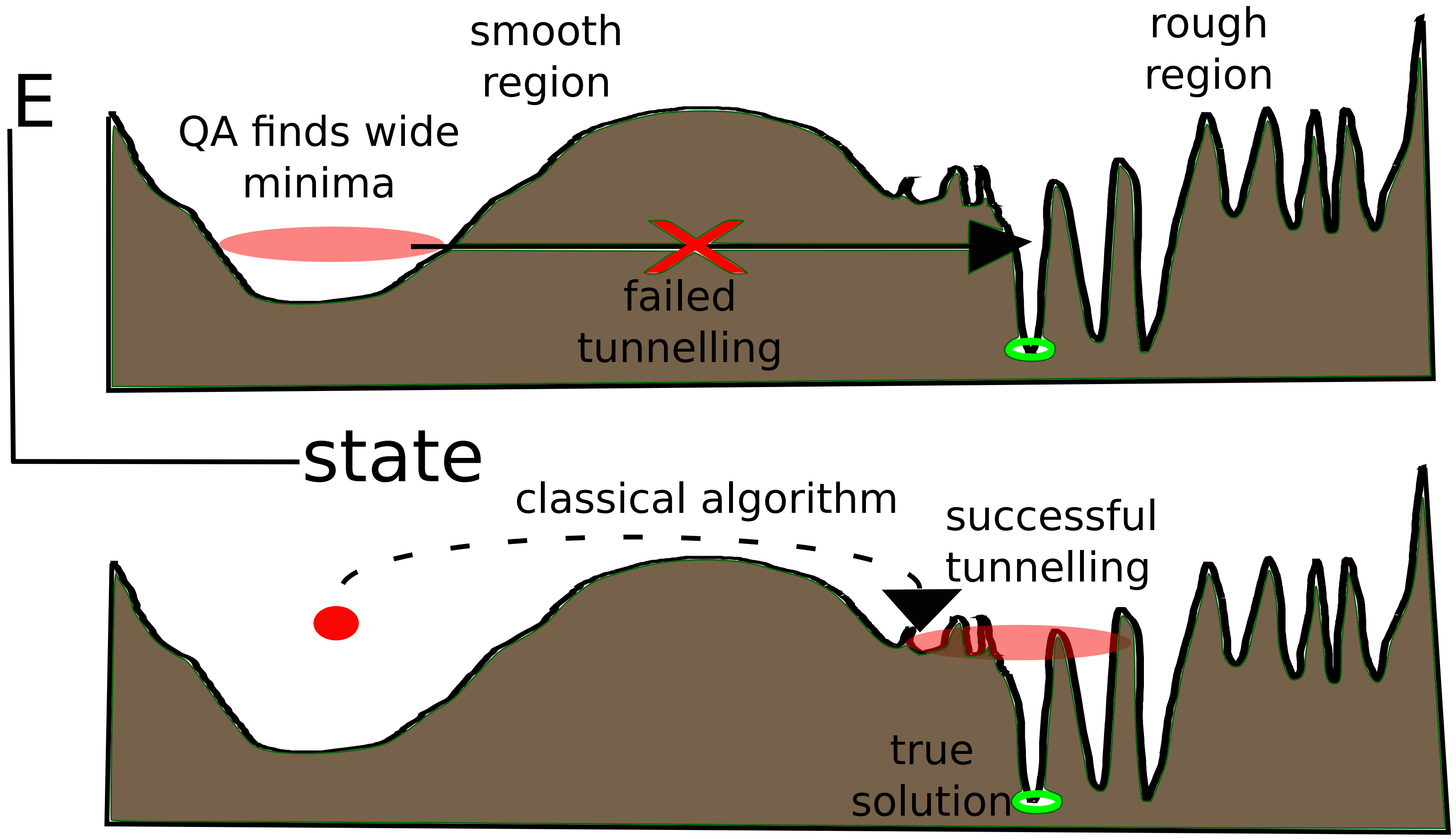
The idea is to dial this parameter to land in the global minimum (i.e. the solution) of some "problem space" described by *J,h*:



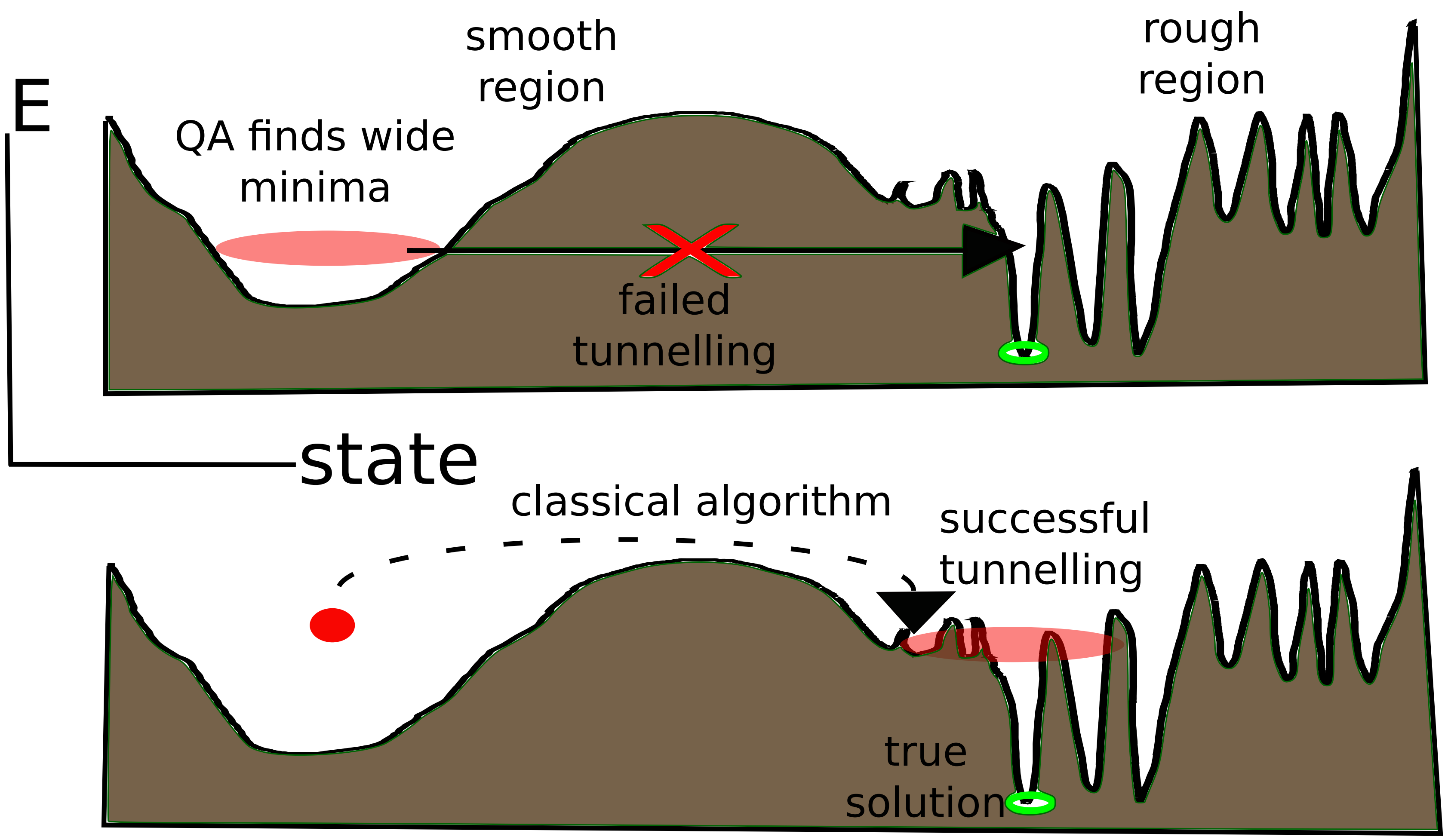
Thermal (classical) and Quantum Annealing are complementary:

- Thermal tunnelling is fast over broad shallow potentials (Quantum "tunnelling" is exponentially slow)

- Quantum Tunnelling is fast through tall thin potentials (Thermal "tunnelling" is exponentially slow — Boltzmann suppression)



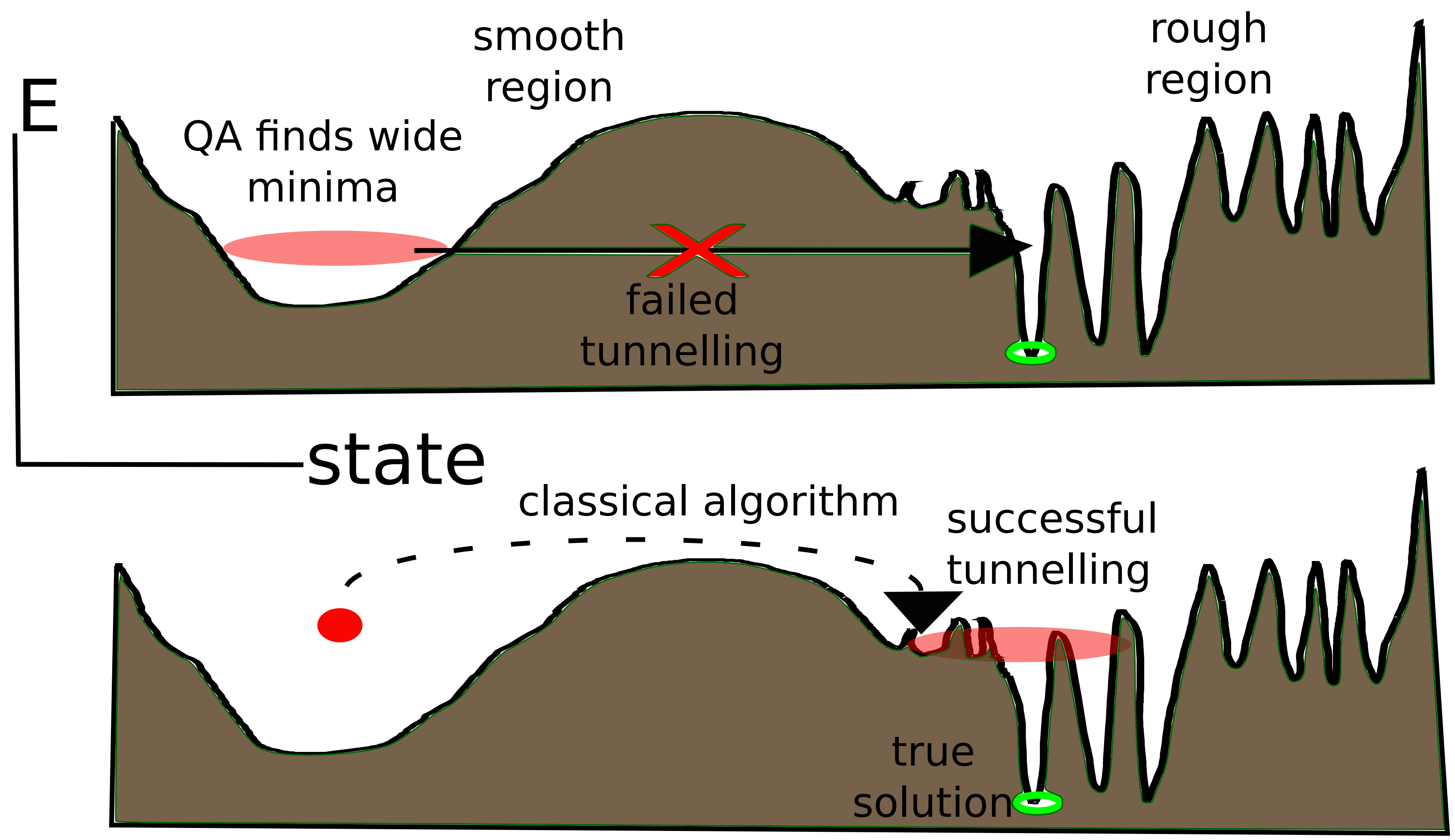
Hence hybrid approach to Quantum Annealing can be useful depending on the solution landscape:



More specifically: thermal annealing uses Metropolis algorithm: accept random σ_i^Z flips with probability

$$P = \begin{cases} 1 & \Delta H \leq 0 \\ e^{-\Delta H/KT} & \Delta H > 0 \end{cases}$$

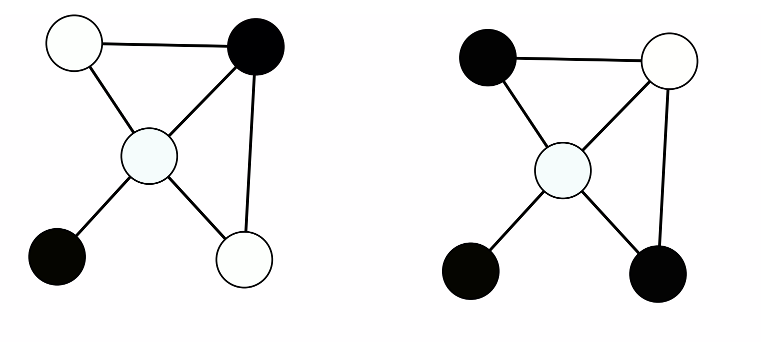
Quantum tunnelling in QFT happens with probability $P \sim e^{-w\sqrt{2m\Delta H}/\hbar}$ ***so by contrast it can be operative for tall barriers if they are made thin***



Simple examples of Ising encodings

Encoding network problems in a general Ising model

- Example 1: how many vertices on a graph can we colour so that none touch? NP-hard problem (from N.Chancellor).



- Let non-coloured vertices have $\sigma_i^Z = -1$ and coloured ones have $\sigma_i^Z = +1$.

- Add a reward for every coloured vertex, and for each link between vertices *i,j* we add a penalty if there are two +1 eigenvalues:

$$\mathcal{H} = -\Lambda \sum_i \sigma_i^Z + \sum_{\text{linked pairs } \{i,j\}} \left[\sigma_i^Z + \sigma_j^Z + \sigma_i^Z \sigma_j^Z\right]$$

- Example 2: N^2 students are to sit an exam in a square room with NxN desks 1.5m apart. half the students (A) have a virus while half of them (B) do not. How can they be arranged to minimise the number of ill students that are less than 2m from healthy students?

- Call the eigenvalue of A == +1 and that of B == -1. That is if I measure σ^Z at a point to have value +1 then I conclude that I should put an ill person there, and vice-versa.

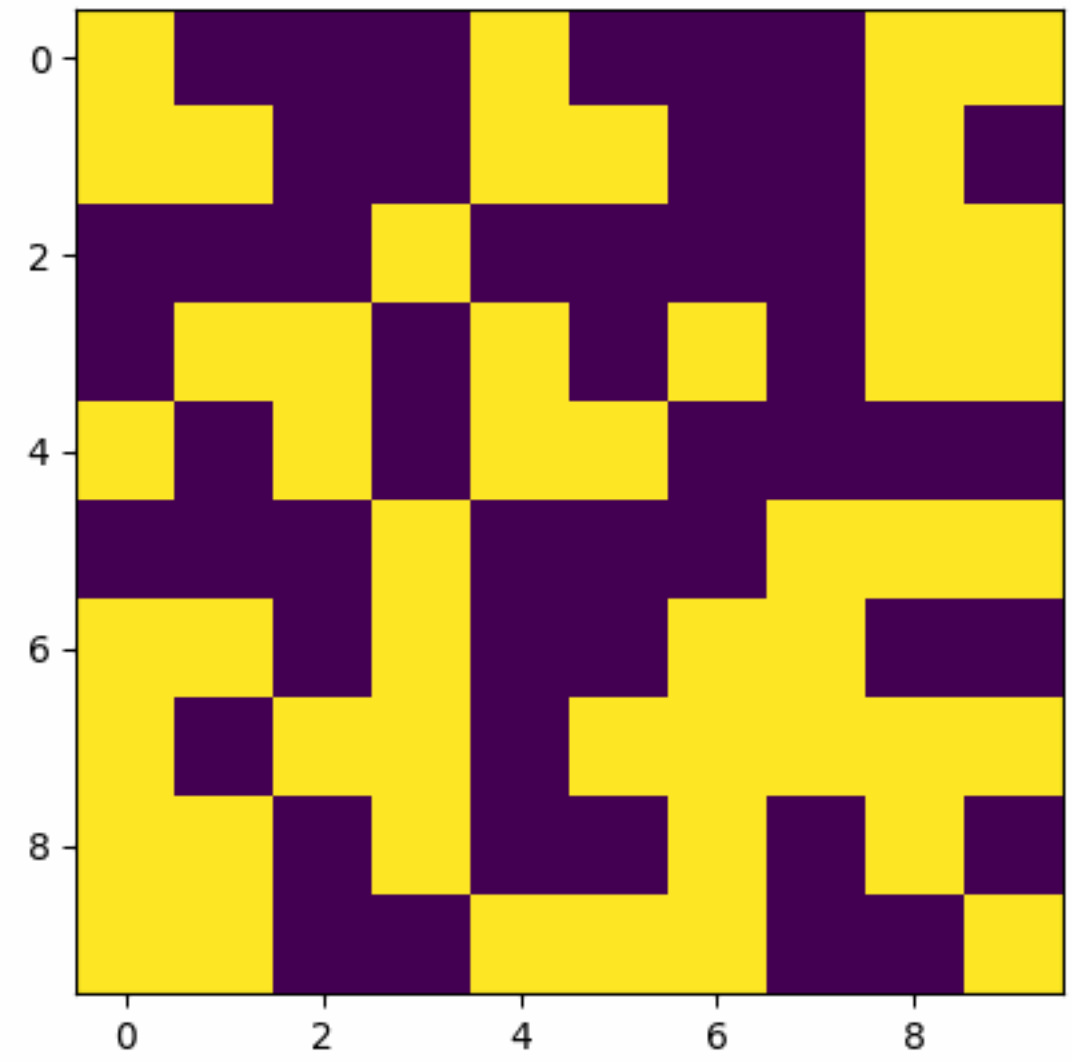
- There are N^2 spins $\sigma^Z_{\ell N+j}$ arranged in rows and columns. I do not care if A>=<A or B>=<B, but if A>=<B then I put a penalty of +2 on the Hamiltonian (ferromagnetic coupling). So …

$$\mathcal{H} = \sum_{\ell m=1}^{N} \sum_{ij=1}^{N} \left(\delta_{\ell m}(\delta_{(i+1)j} + \delta_{(i-1)j}) + \delta_{ij}(\delta_{(\ell+1)m} + \delta_{(\ell-1)m})\right) \left[1 - \sigma^Z_{\ell N+i}\sigma^Z_{mN+j}\right]$$

- Finally I need to apply the constraint that #A = #B:

$$\begin{aligned} \mathcal{H}^{(\mathrm{constr})} &= \Lambda \left(\#A - \#B\right)^2 \\ &= \Lambda \left(\sum_{\ell,i}^{N} \sigma^Z_{\ell N+i}\right)^2 \\ &= \Lambda \sum_{\ell m=1}^{N} \sum_{ij=1}^{N} \sigma^Z_{\ell N+i}\sigma^Z_{mN+j} \end{aligned}$$

- Example 2 done with classical thermal annealing using the Metropolis algorithm. Note this represents a search over ${}_{100}C_{50} \sim 2^{100}$ configurations:



- Importantly the constraint hamiltonian cannot be too big otherwise the hills are too high and it freezes too early. This makes the process require a (polynomial sized) bit of “thermal tuning”.

- In principle this could be done more easily on a quantum annealer as the constraints could be high and it would still work.

- To do this we would simply fill h and J and call the quantum annealer from python as follows:

```
response = sampler.sample_ising(h,J,seed=1234+i,num_reads=3000000, num_sweeps=1)
```

- “response” is a list of [+1,-1,+1,+1 …..] spins ordered by energy

- However the architecture (connectivity of J,h) is limited. (Later)

A field theory problem: Tunnelling in QFT

- We think of the general Ising model as a “universal QFT computer”

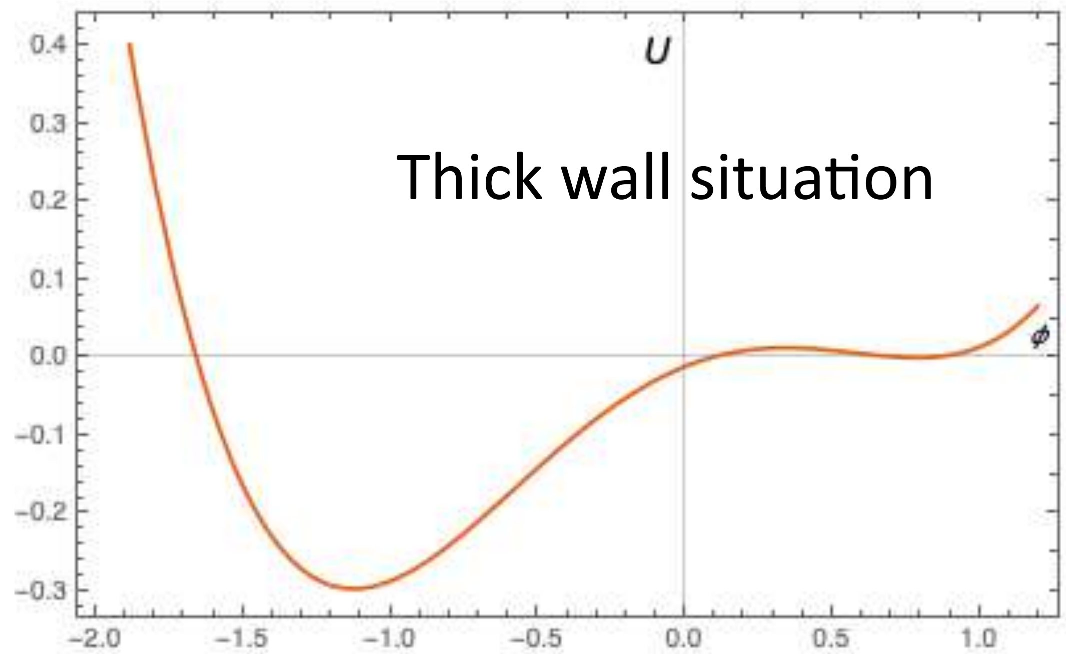
- Simple problem to demonstrate encoding QFT — quantum tunnelling in a scalar theory

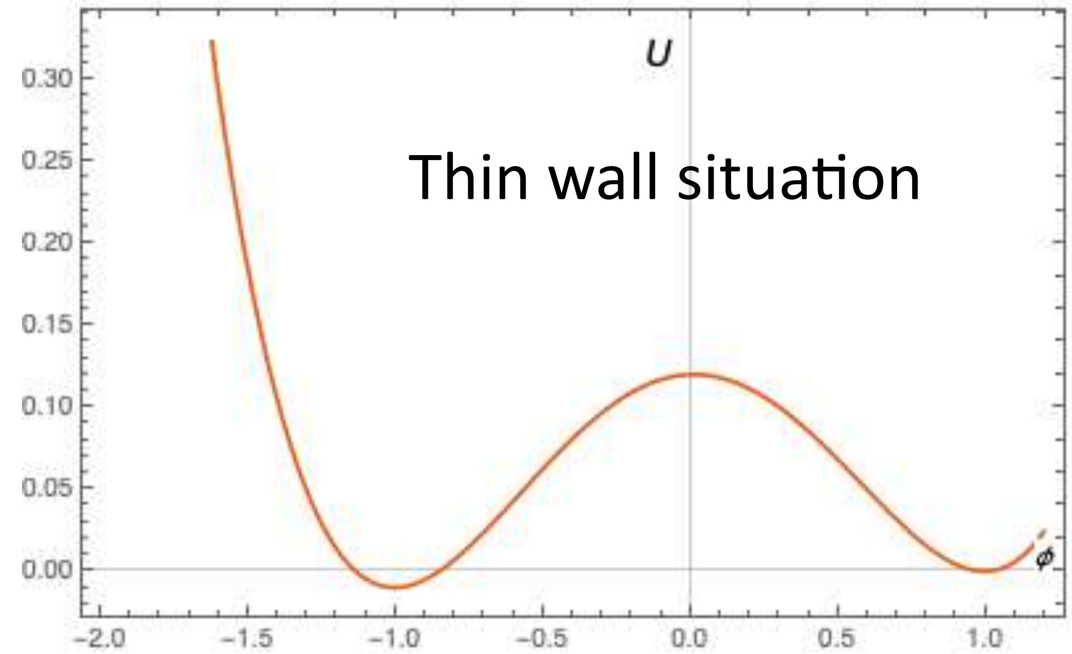
- Advantage 1: easy to prepare the initial state (this non-perturbative process is much easier than preparing scattering states).

- Advantage 2: we could in principle observe genuine tunnelling in the annealer rather than just simulate it.

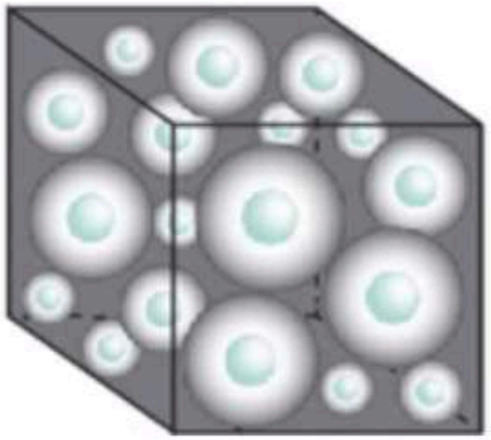
$$V(\phi) = \frac{\lambda}{8}(\phi^2 - v^2)^2 + \frac{\epsilon}{2v}(\phi - v)$$

$$U(\phi) = V(\phi) - V(\phi_+)$$





- A system trapped in the false vacuum will decay by forming bubbles …

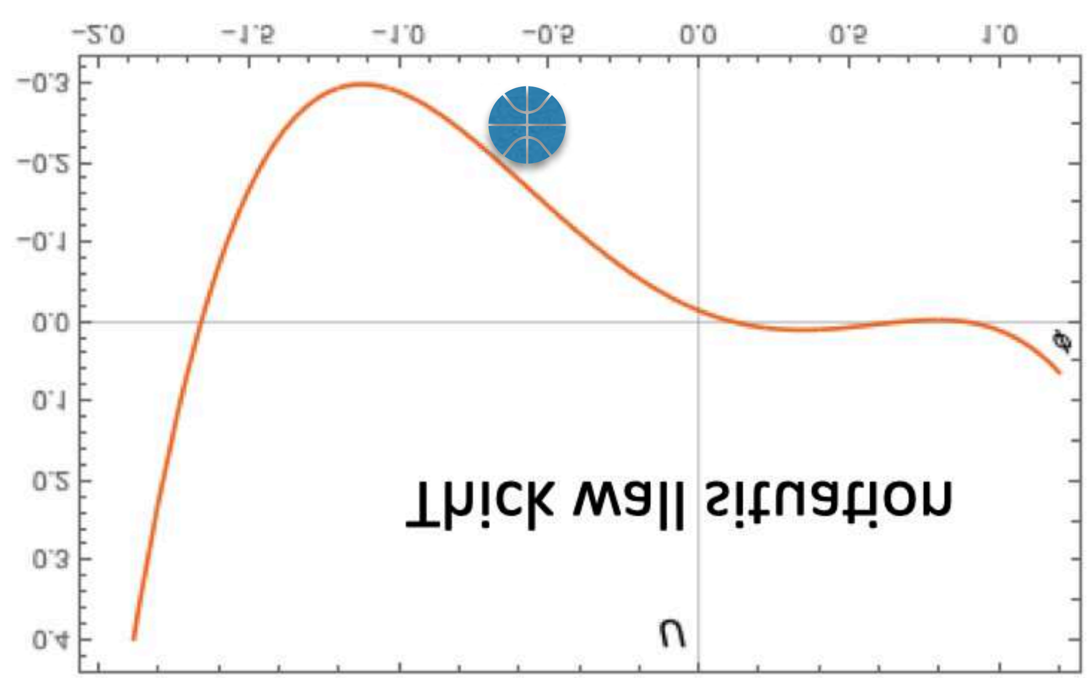


- The analytic result for the tunnelling rate was worked out in several famous papers by Callan, Coleman, de Luccia and Linde

- Decay rate per unit volume is given by the Euclidean actions of the O(4) or O(3) symmetric "bounce" solution (for instanton or thermal resp):

$$\Gamma_4 \;=\; A_4\, e^{-S_4[\phi]}\,,$$
$$\Gamma_3 \;=\; A_3\, T e^{-S_3[\phi]/T}\,,$$

where $$S_{c+1} = \int_0^\infty d\rho \rho^c \left(\frac{\dot{\phi}^2}{2} + U(\phi) \right)$$



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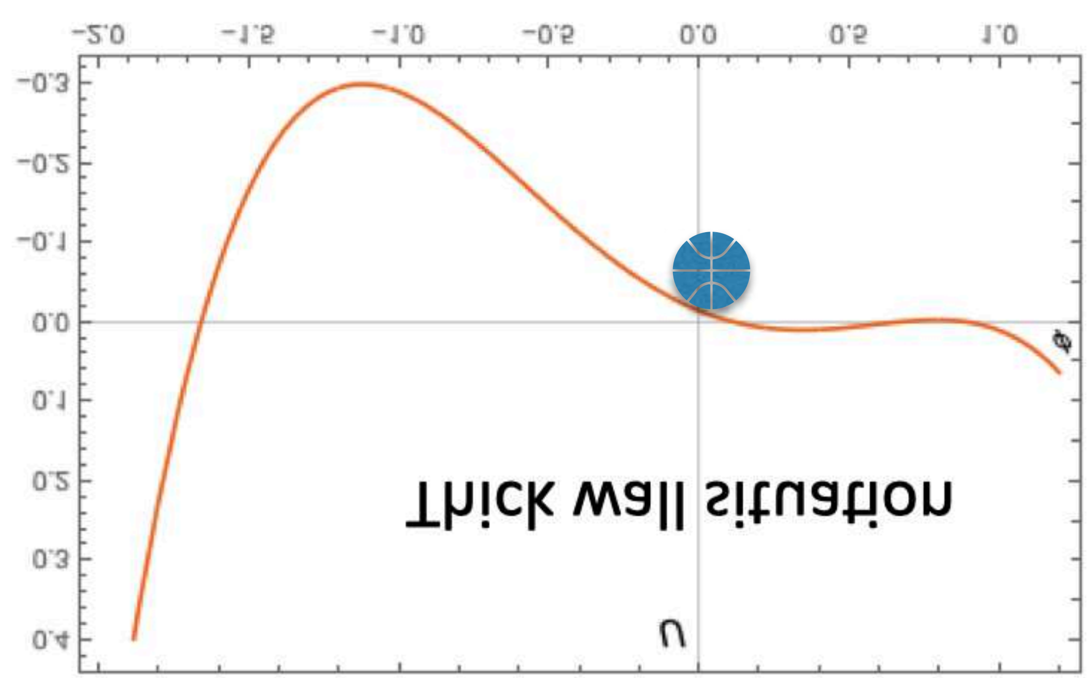

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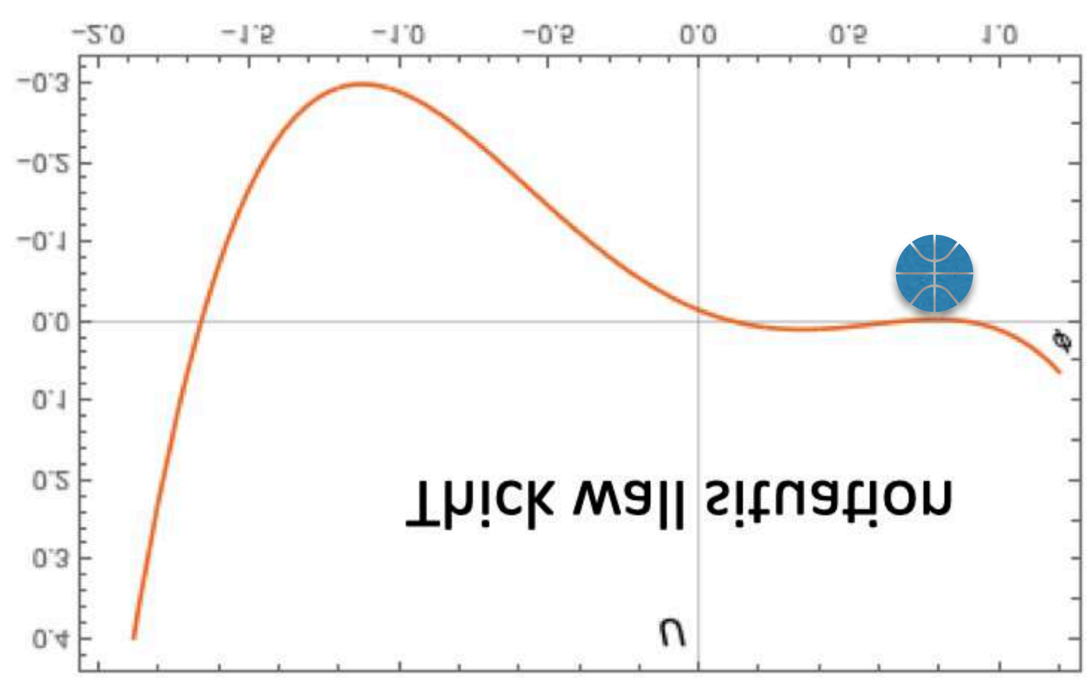
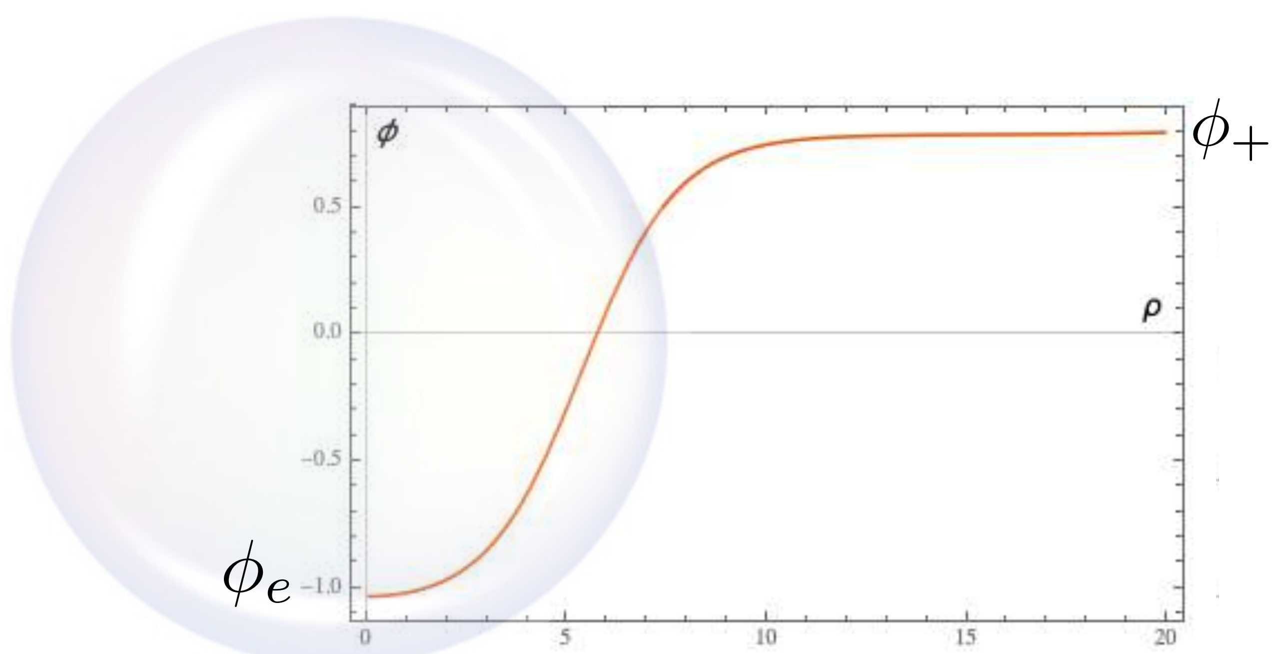


• Normally solution found by solving Euler-Lagrange equations with boundary conditions:

$$\frac{d^2\phi}{d\rho^2} + \frac{c}{\rho}\frac{d\phi}{d\rho} = U' \ , \quad d\phi/d\rho = 0 \quad \text{as} \quad \rho \to 0, \infty$$



• "Escape point" found with overshoot/undershoot method.

- Thick-wall approximation: rescaling arguments give answer in terms of “standard action”

$$S_4 = \frac{3\xi}{\lambda} S_4^0 \quad ; \quad S_4^0 = 91$$

$$S_3 = \frac{3v\xi^{3/2}}{\lambda^{1/2}} S_3^0 \quad ; \quad S_3^0 = 19.4$$

where

$$\xi = \sqrt{2/3(1 - \epsilon/\epsilon_0)}$$

$$\epsilon_0 = 2\lambda v^4/3\sqrt{3}$$

- Thin-wall approximation: action written in terms of c=0 action (Z2 domain wall)

$$S_4 = \frac{27\pi^2 S_1^4}{2\epsilon^3} \quad ; \quad S_3 = \frac{16\pi^3 S_1^3}{3\epsilon^2} \; .$$

In principle if we can encode this field theory on a quantum annealer, we would be able to vary the parameters and perform a tunnelling experiment. As a first step, we will determine S1: finding the extremum of the action is a quasi-convex problem (convex in a finite box).

This means for the $c = 0$ action we will attempt to minimise the Euclidean action holding the endpoints fixed at +/- v:

$$S_1 = 2\pi^2 \int_0^\infty d\rho \; \frac{1}{2}\dot{\phi}^2 + U(\phi)$$

Ising chain encoding of scalar QFT

Consider encoding a continuous field value $\phi(\rho)$ at some point, and discretise into N

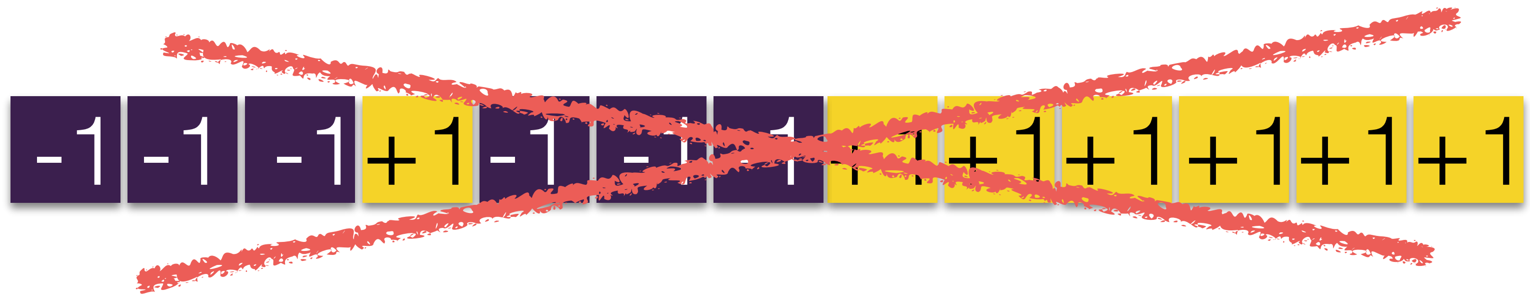
$$\phi(\rho) \;=\; \phi_0 + \alpha_l \xi \quad = \quad \phi_0 + \xi \;\ldots\; \phi_0 + N\xi$$

Wish to represent it as a point on a spin chain == domain wall encoding (Chancellor):

We translate this to a field value using

$$\phi = \phi_0 + \frac{\xi}{2} \sum_{i=1}^{N} (1 - \sigma_i^Z)$$

For this to work as a consistent encoding we have to avoid e.g.



This is the domain-wall encoding. Begin in the Ising model with a ferromagnetic interaction that favours as few flips as possible, but frustrate at least one by having the endpoints pinned at -1 … +1. (Note this is a 1D version of the exam-room example).

$$\mathcal{H}^{(\mathrm{chain})} = \Lambda \left(\sigma_1^Z - \sigma_N^Z - \sum_i^{N-1} \sigma_i^Z \sigma_{i+1}^Z \right)$$

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$$\mathcal{H}^{(\mathrm{chain})} = \Lambda \left(\sigma_1^Z - \sigma_N^Z - \sum_i^{N-1} \sigma_i^Z \sigma_{i+1}^Z \right)$$

Pins the end spins at opposing values

penalty for different adjacent spin

$$h_j^{(\mathrm{chain})} = \Lambda \left(\delta_{j1} - \delta_{jN} \right)$$

$$J_{ij}^{(\mathrm{chain})} = -\frac{\Lambda}{2} \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}_{ij}$$

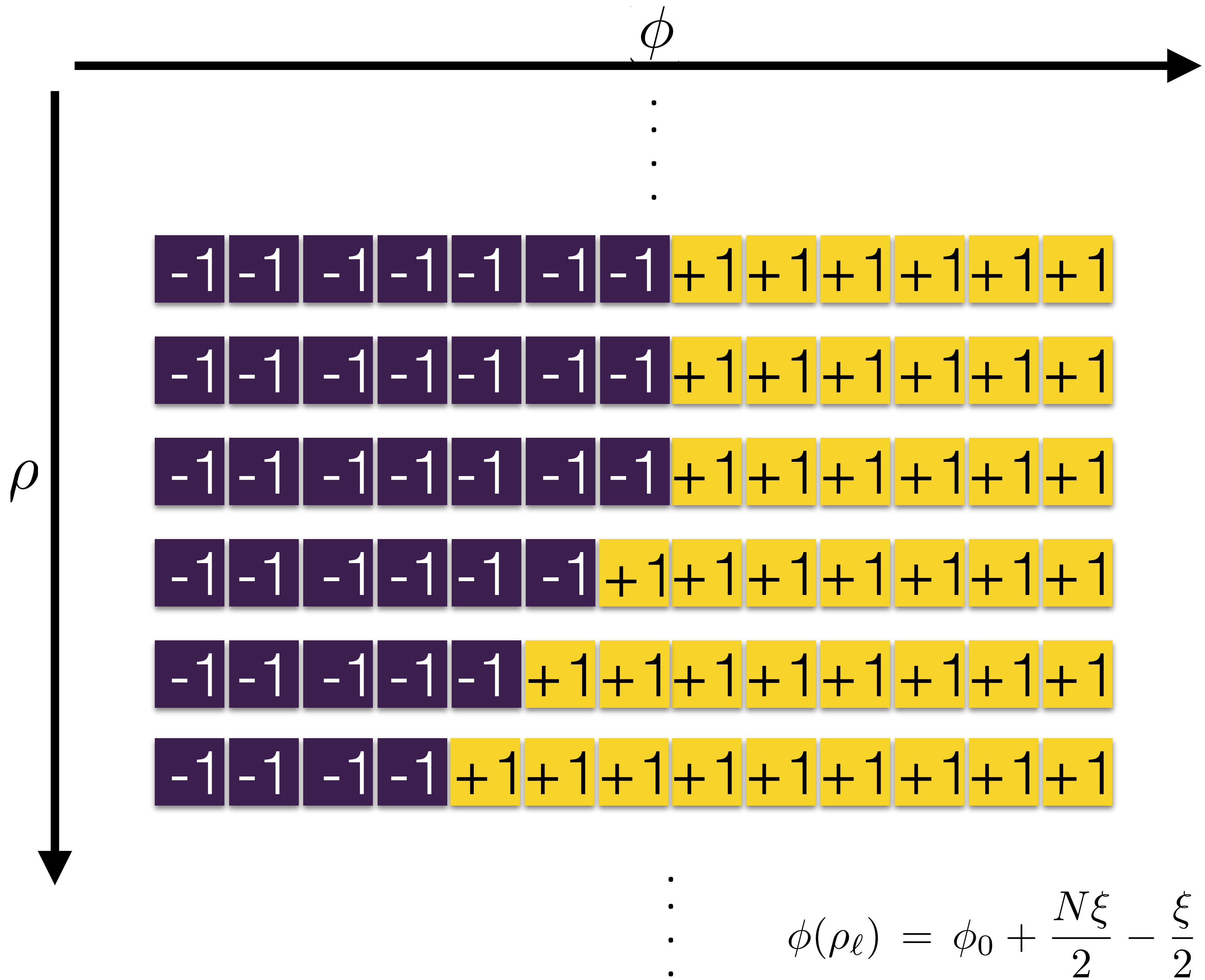
To add a potential we can add a contribution to the linear *h* couplings

only the frustrated link contributes

$$U(\phi) = \frac{1}{2}\sum_{i}^{N-1} U(\phi_0 + j\xi)\left(\sigma^Z_{j+1} - \sigma^Z_j\right)$$

$$\equiv -\frac{1}{2}\sum_{i}^{N-1} U'(\phi_0 + j\xi)\sigma^Z_j$$

Next add the discretised radial spacetime coordinate: $\rho_\ell = \ell\nu = \nu \ldots M\nu$



$$\phi(\rho_\ell) = \phi_0 + \frac{N\xi}{2} - \frac{\xi}{2}\sum_{j=1}^{N}\langle\sigma^Z_{\ell N+j}\rangle$$

Everything done so far is then trivially extended in the *l* spacetime index:

$$h^{(\mathrm{chain})}_{\ell N+j} = \Lambda\left(\delta_{j1} - \delta_{jN}\right) \qquad J^{(\mathrm{chain})}_{\ell N+i, mN+j} = -\frac{\Lambda}{2}\,\delta_{\ell m}\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}_{ij}$$

$$h^{(\mathrm{QFT})}_{N\ell+j} = \begin{cases} -\frac{\nu\xi}{2} U'(\phi_0 + j\xi)\ ; & j < N \\ \frac{\nu}{2} U(\phi_0 + (N-1)\xi)\ ; & j = N \end{cases}$$

Then kinetic terms are as follows:

$$S_{KE} \equiv \int_0^{\Delta\rho} d\rho \frac{1}{2}\dot{\phi}^2 = \lim_{M\to\infty} \sum_{\ell=1}^{M-1} \frac{1}{2\nu}\left(\phi(\rho_{\ell+1}) - \phi(\rho_\ell)\right)^2$$

$$= \sum_{\ell=1}^{M-1} \sum_{ij}^{N} \frac{\xi^2}{8\nu}\left[\sigma^Z_{(\ell+1)N+i} - \sigma^Z_{\ell N+i}\right] \times \left[\sigma^Z_{(\ell+1)N+j} - \sigma^Z_{\ell N+j}\right]$$

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$$J^{(\mathrm{QFT})}_{\ell N+i,mN+j} = \frac{\xi^2}{8\nu}\,\left(2\delta_{\ell m} - \delta_{\ell(m+1)} - \delta_{(\ell+1)m}\right)$$

Next we need to impose the physical boundary condition with:

$$\mathcal{H}^{(BC)} = \frac{\Lambda'}{2}(\phi(0)+v)^2 + \frac{\Lambda'}{2}(\phi(\rho_M)-v)^2$$

We can think of these as just boundary mass-term potentials in *U*:

$$h^{(\mathrm{BC})}_{N\ell+j} = \begin{cases} -\Lambda'(\phi_0 + j\xi + v)\ ;\ \ \ell = 1, \forall j \\ -\Lambda'(\phi_0 + j\xi - v)\ ;\ \ \ell = M-1, \forall j \end{cases}$$

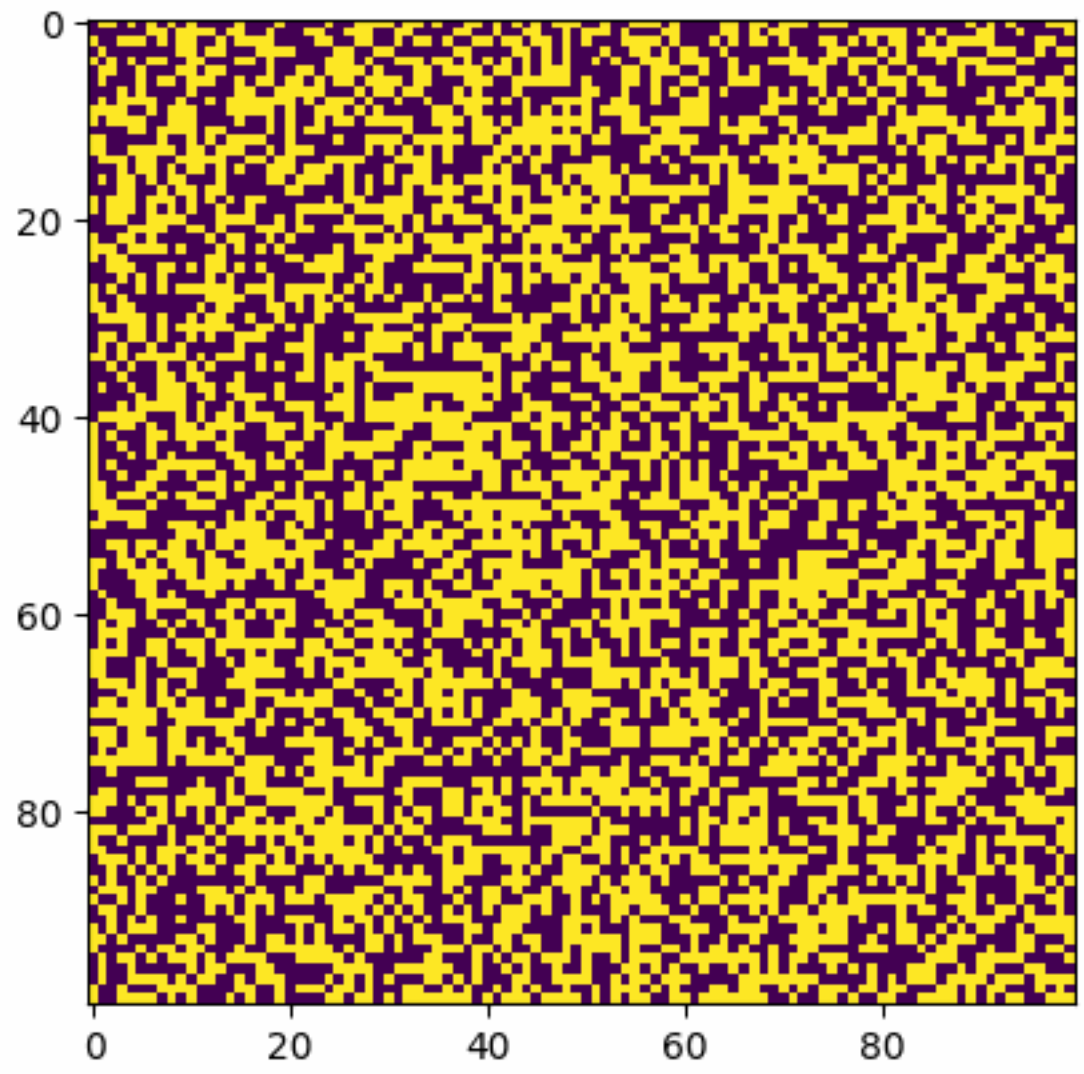
Finally add everything together!

$$\mathcal{H} \ = \ \mathcal{H}^{(\mathrm{chain})} + \mathcal{H}^{(\mathrm{QFT})} + \mathcal{H}^{(\mathrm{BC})}.$$

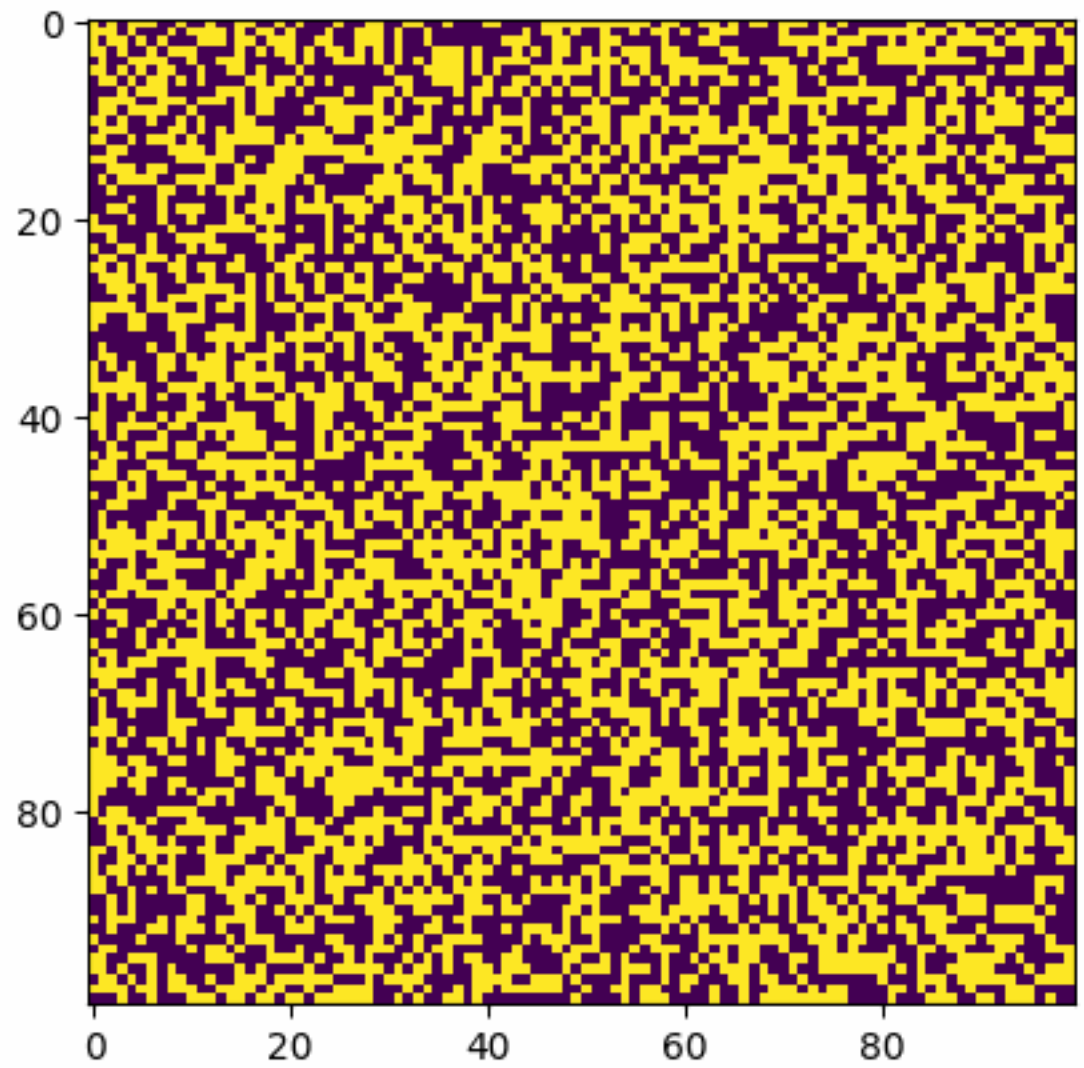
Results for thin wall limit

Can solve classical simulated annealing with the Metropolis algorithm. Again have to be careful how we set the temperatures and parameters:

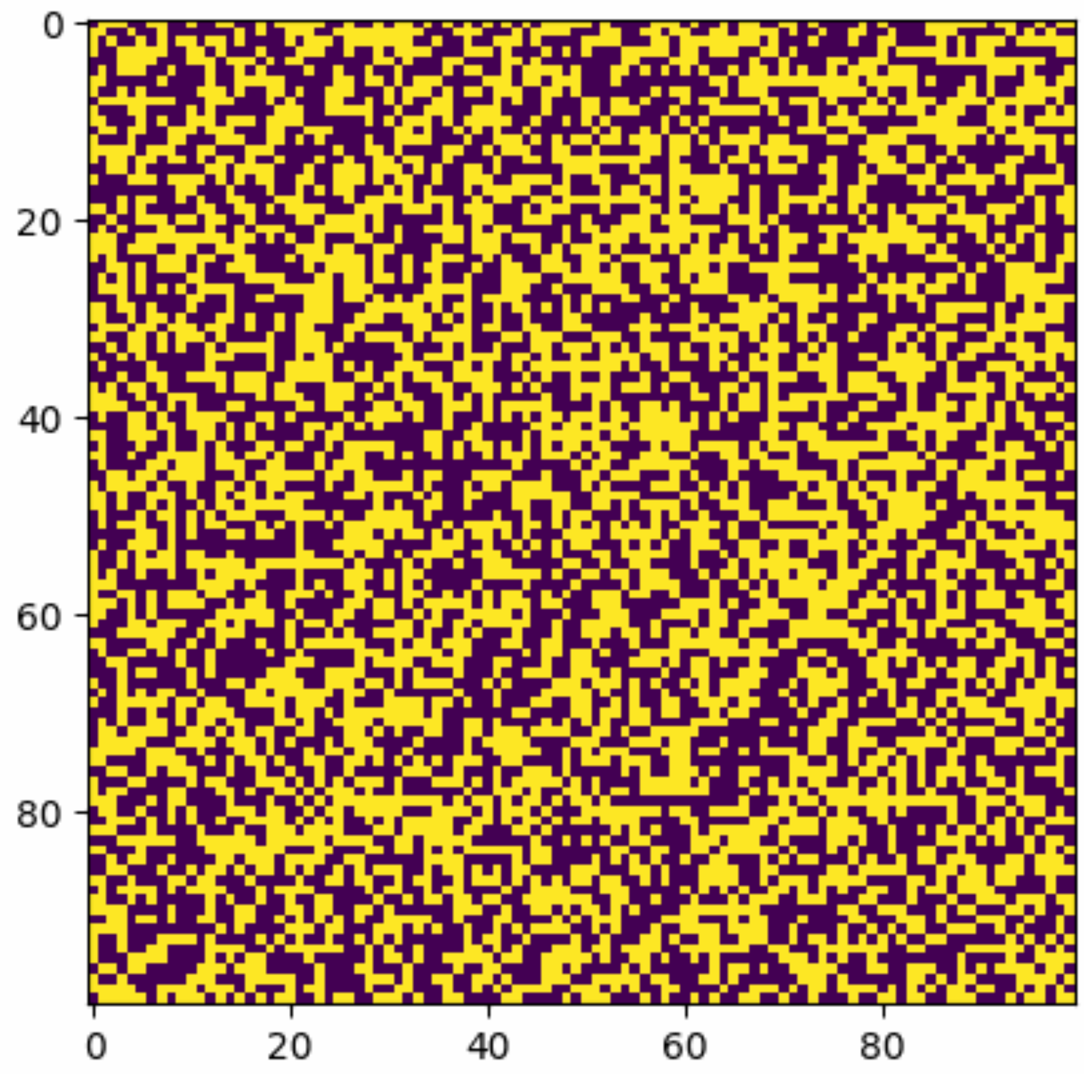
Too hot



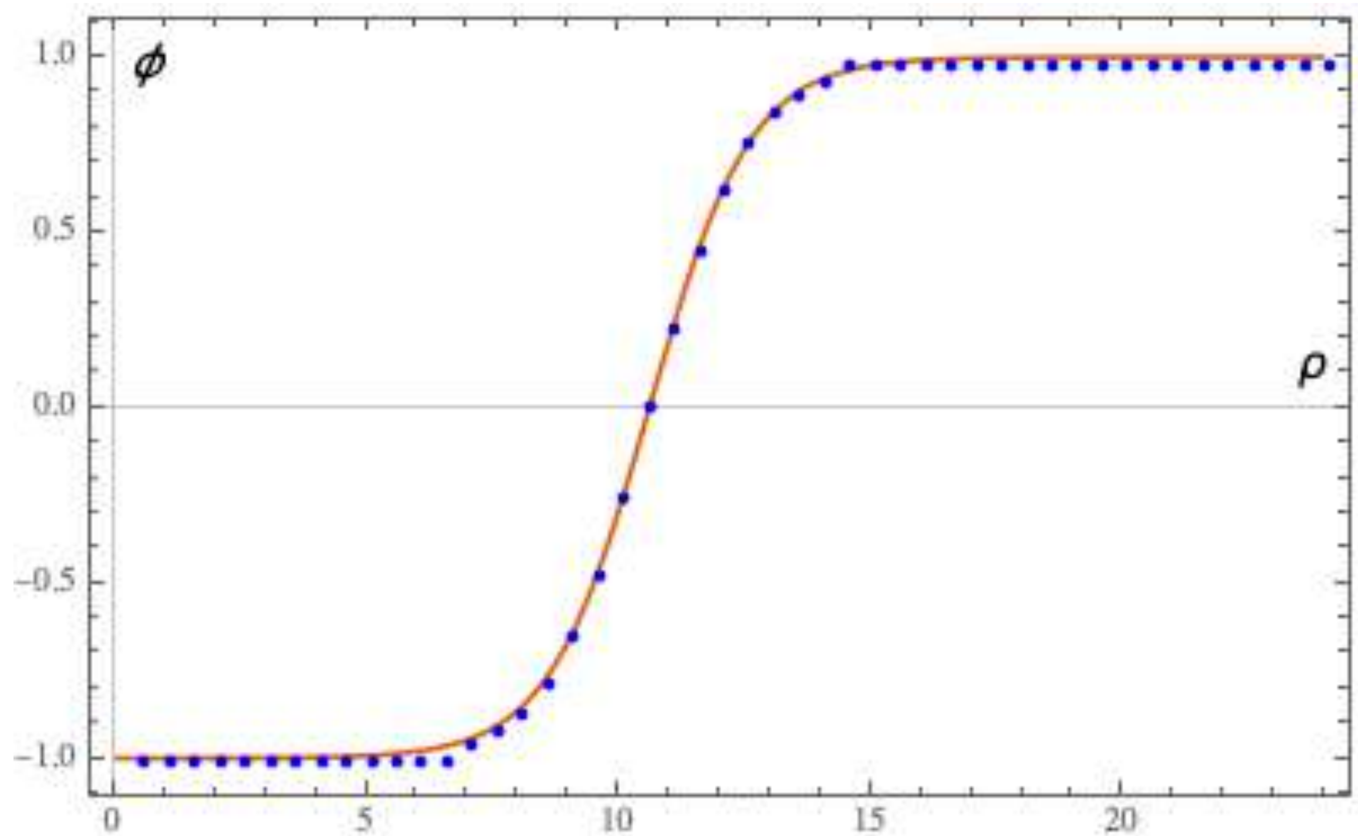
Too cold



Just right (two stage annealing process)

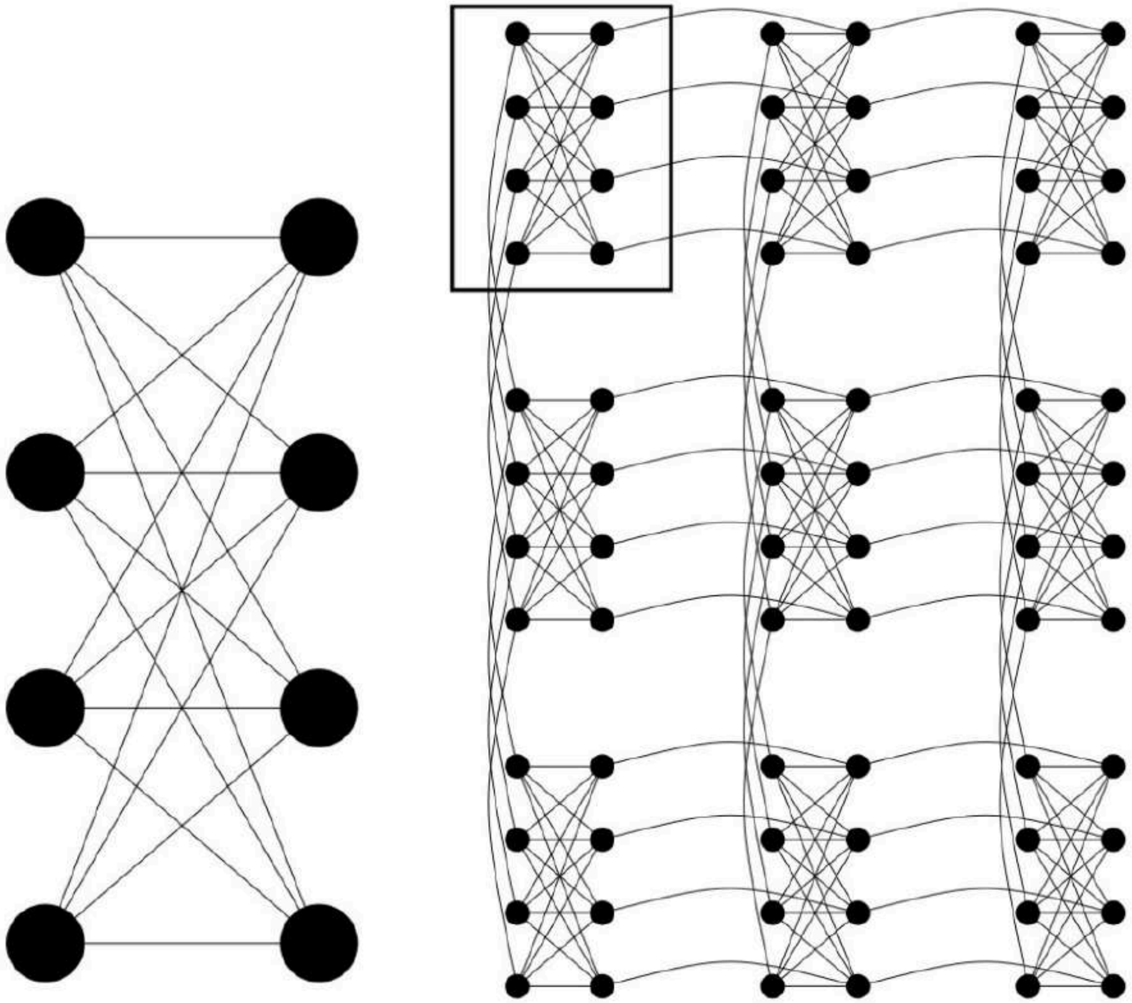


Same result on Dwave using hybrid quantum/classical Kerberos annealer (It finds best samples of parallelised tabu search + simulated annealing + D-Wave subproblem sampling)



Notably the Kerberos sampler is much more robust than pure simulated annealing.

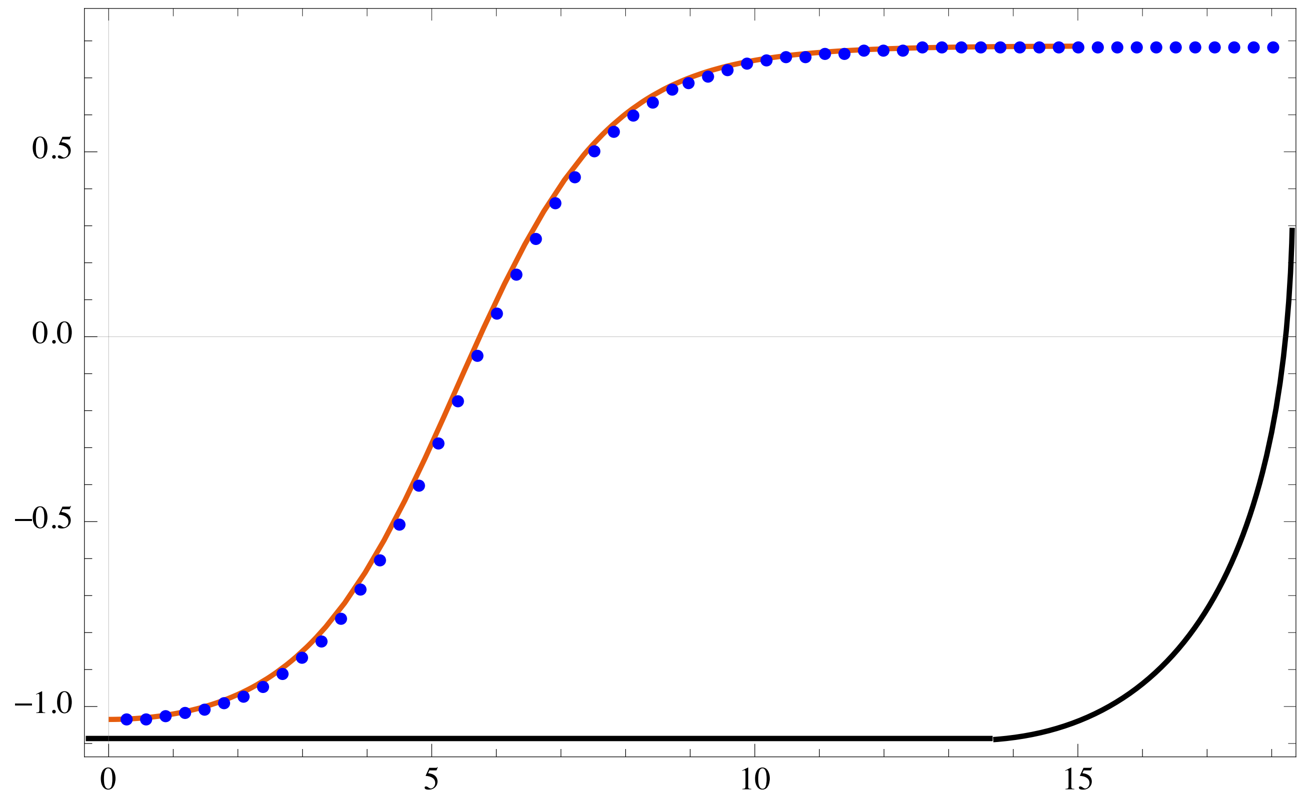
Why not pure Quantum annealer? The connectivity is not general enough for this problem (in particular encoding the kinetic terms): it has a Chimera structure …



But the principle has been proven: we can encode a pure field theory potential on the chimera structure, so we can experiment with QFT tunnelling (c.f. Johnson 2011)

Thick wall limit: solving PDEs

To find the c =3 solution shown here is less easy because just using the action tends to give the black line:



This is because the critical point of $S_{c+1} = 2\pi^2 \int_0^\infty d\rho\, \rho^c \left(\frac{1}{2}\dot{\phi}^2 + U(\phi) \right)$ is a saddle.

Instead the correct bubble profile is found by solving the E-L PDE by minimising

$$\tilde{S}_{c+1} = \int_0^\infty d\rho \left(\frac{d^2\phi}{d\rho^2} + \frac{c}{\rho}\frac{d\phi}{d\rho} - U' \right)^2$$

It is squared in derivatives, so it can be written mostly as adjustments in *J* …

$$
\begin{aligned}
\frac{4\nu^3}{\xi^2} J^{(\mathrm{QFT})}_{\ell N+i, mN+j} = \; & \frac{c^2}{\ell m} \Big(2\delta_{\ell m} - \delta_{\ell(m+1)} - \delta_{(\ell+1)m} \Big) + \Big(6\delta_{\ell m} - 4\delta_{\ell(m+1)} - 4\delta_{(\ell+1)m} + \delta_{\ell(m+2)} + \delta_{(\ell+2)m} \Big) \\
& + \frac{c}{m} \Big(3\delta_{(\ell+1)m} + \delta_{\ell(m+1)} - \delta_{(\ell+2)m} - 3\delta_{\ell m} \Big) \; + \; \frac{c}{\ell} \Big(3\delta_{(m+1)\ell} + \delta_{m(\ell+1)} - \delta_{(m+2)\ell} - 3\delta_{\ell m} \Big) \\
& - \nu^2 U''(\phi_0 + i\xi) \Big(\delta_{\ell(m+1)} - \delta_{\ell(m+2)} + \Big(1 - \frac{c}{m}\Big)(\delta_{\ell(m+1)} - \delta_{\ell m}) \Big) \\
& - \nu^2 U''(\phi_0 + j\xi) \Big(\delta_{m(\ell+1)} - \delta_{m(\ell+2)} + \Big(1 - \frac{c}{\ell}\Big)(\delta_{m(\ell+1)} - \delta_{m \ell}) \Big) \; + \; \nu^4 \, U''(\phi_0 + i\xi) \, U''(\phi_0 + j\xi) \, \delta_{\ell m} \; ,
\end{aligned}
$$

together with …

$$
h^{(QFT)}_{N\ell+j} = \frac{\epsilon}{v} \frac{\xi c}{2\nu} \left(\frac{1}{\ell - 1} - \frac{1}{\ell} \right)
$$

and boundary condition terms for

$$
\begin{aligned}
\dot{\phi}(0) \; = \; \dot{\phi}(\infty) \; = \; \ddot{\phi}(0) \; = \; \ddot{\phi}(\infty) \; = \; 0 \, , \\
\phi(\infty) \; = \; \phi_+ \; ,
\end{aligned}
$$

Multiple fields and dimensions: the U(1) string

Consider 2D system with 2 fields:

$$H_{U(1)} = \int_0^\infty d^2x \; \frac{1}{2}\nabla\phi_a \cdot \nabla\phi^a + U(\phi_a)$$

$$U(\phi_a) = \frac{\lambda}{8}(\phi_0^2 + \phi_1^2 - v^2)^2$$

U(1) vortex is again a convex problem: can be discretised as before,

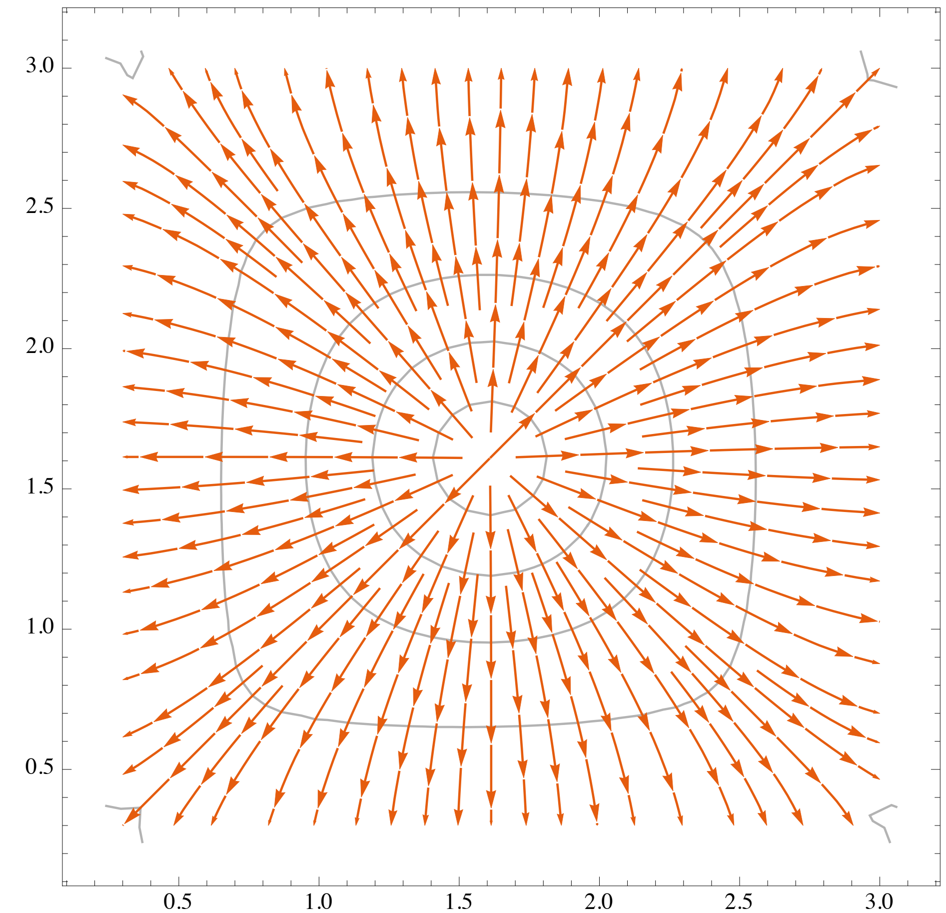
$$\ell \to \ell_{\mu=0\ldots d-1}$$

$$i \to i_{a=0\ldots n-1}$$

Flatten the Ising model indices as:

$$\{i_a\} \equiv (n\ell_\mu M^{\mu-1} + a)N + i_a \; ; \quad a = 0\ldots n-1$$

Ising model is $nM^dN \times nM^dN$



Future directions

- We have seen how the general Ising model can be used to encode QFT
- Genuine tunnelling of metastable nontrivial (d=0 system)?
- Deduce quantum prefactors as well as classical actions
- GPU encoding for finite temperature (simulated annealling) (c.f. Parisi et al)
- Soliton dynamics?