Particle and Nuclear Physics

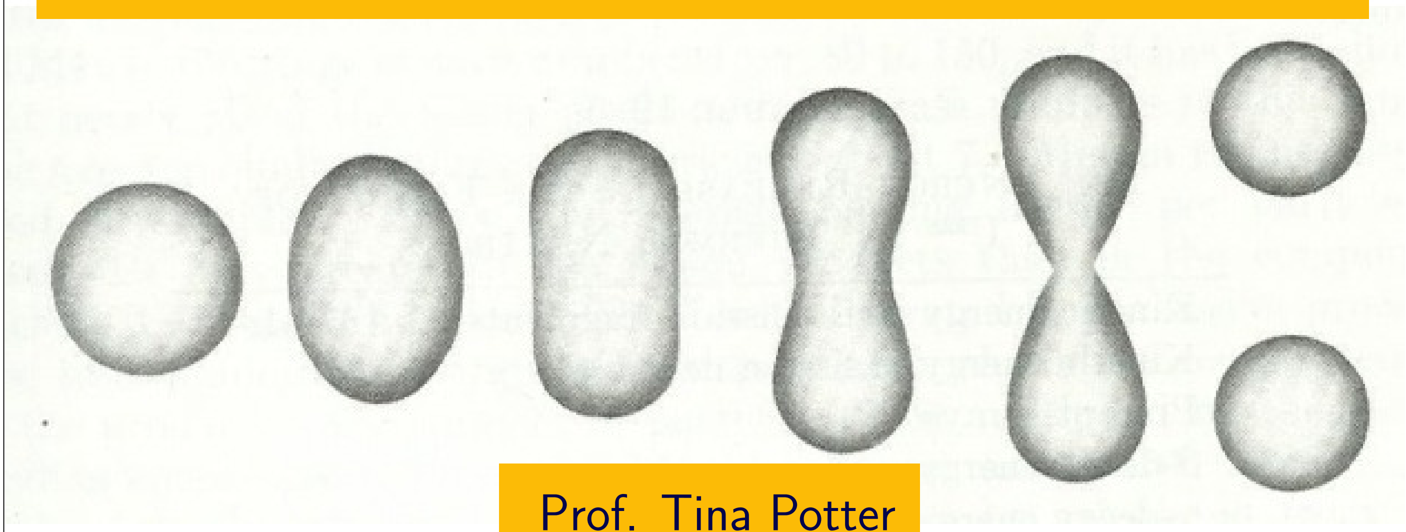
Handout #3

Nuclear Physics

Lent/Easter Terms 2024
Prof. Tina Potter

13. Basic Nuclear Properties

Particle and Nuclear Physics



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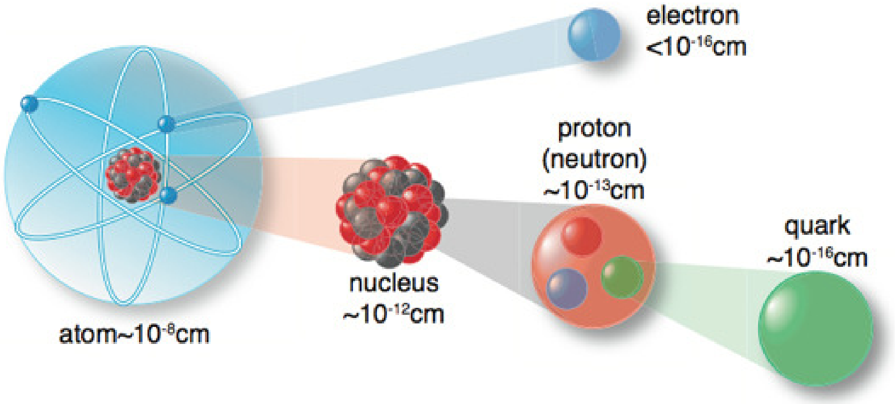
In this section...

- Motivation for study
- The strong nuclear force
- Stable nuclei
- Binding energy & nuclear mass (SEMF)
- Spin & parity
- Nuclear size (scattering, muonic atoms, mirror nuclei)
- Nuclear moments (electric, magnetic)

Introduction

Nuclear processes play a fundamental role in the physical world:

- Origin of the universe
- Creation of chemical elements
- Energy of stars
- Constituents of matter; influence properties of atoms

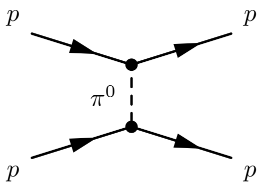


Nuclear processes also have many practical applications:

- Uses of radioactivity in research, health and industry, e.g. NMR, radioactive dating.
- Various tools for the study of materials, e.g. Mössbauer, NMR.
- Nuclear power and weapons.

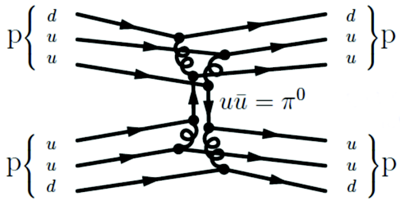
The Nuclear Force

Consider the *pp* interaction, Range $\sim \hbar/m_\pi c \sim 1\text{fm}$



Hadron level

$\equiv$



Quark-gluon level

Pion vs. gluon exchange is similar to the Coulomb potential vs. van der Waals' force in QED.

The treatment of the strong nuclear force between nucleons is a many-body problem in which

- quarks do not behave as if they were completely independent.
- nor do they behave as if they were completely bound.

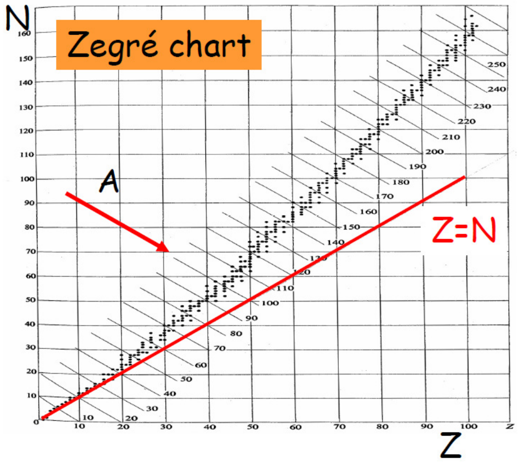
The nuclear force is not yet calculable in detail at the quark level and can only be deduced empirically from nuclear data.

Stable Nuclei

Stable nuclei do not decay by the strong interaction.
They may transform by β and α emission (weak or electromagnetic) with long lifetimes.

Characteristics

- Light nuclei tend to have N=Z.
  Heavy nuclei have more neutrons, $N > Z$.
- Most have even N and/or Z.
  Protons and neutrons tend to form pairs (only 8/284 have odd N *and* Z).
- Certain values of Z and N exhibit larger numbers of isotopes and isotones.



Binding Energy

Binding Energy is the energy required to split a nucleus into its constituents.

$$\text{Mass of nucleus } m(N,Z) = Zm_p + Nm_n - B$$

Binding energy is very important: gives information on

- forces between nucleons
- stability of nucleus
- energy released or required in nuclear decays or reactions

Relies on precise measurement of nuclear masses (mass spectrometry).

Used less in this course, but important nonetheless.
Separation Energy of a nucleon is the energy required to remove a single nucleon from a nucleus.

e.g. n: $B({}^A_Z X) - B({}^{A-1}_{\ \ Z}X) = m({}^{A-1}_{\ \ Z}X) + m(n) - m({}^A_Z X)$

p: $B({}^A_Z X) - B({}^{A-1}_{Z-1}X') = m({}^{A-1}_{Z-1}X') + m({}^1H) - m({}^A_Z X)$

Binding Energy *Binding Energy per nucleon*

Key Observations

Peaks for light nuclei with A = 4n. "α stability"

Broad maximum at A~60

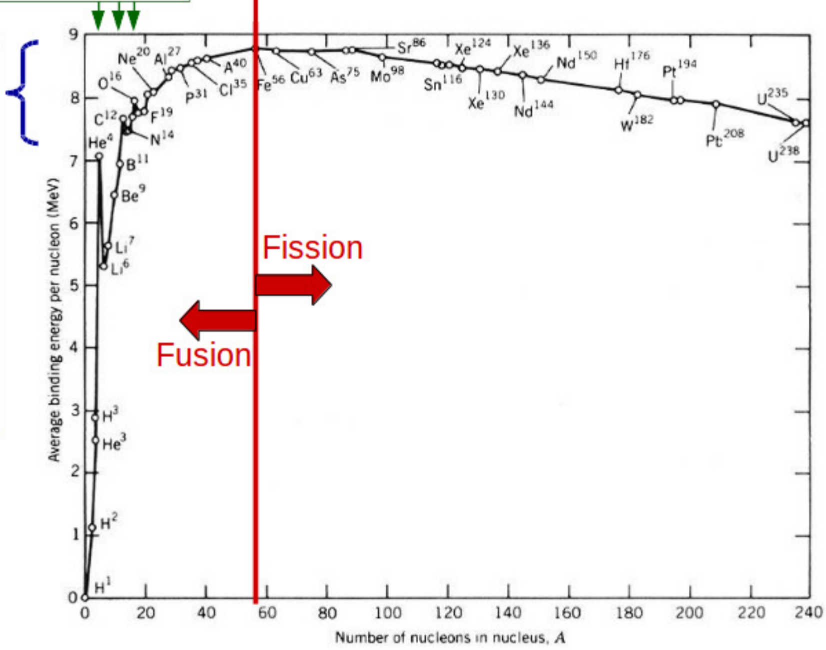
For A>20, B/A ~ constant (~ 8 MeV per nucleon)

Compare to B of atomic electrons per nucleon <3 keV

Implies that nucleons are only attracted by nearby nucleons

→ Nuclear force is short range and saturated

"Saturated" means each nucleus only interacts with a limited number of neighbours; not with all nucleons.



Nuclear mass *The liquid drop model*

Atomic mass: $M(A,Z) = Z(m_p + m_e) \; + (A-Z)m_n \; -B$

Nuclear mass: $m(A,Z) = Zm_p \; + (A-Z)m_n \; -B$

Liquid drop model

Approximate the nucleus as a sphere with a uniform interior density, which drops to zero at the surface.

Liquid Drop

- Short-range intermolecular forces.
- Density independent of drop size.
- Heat required to evaporate fixed mass independent of drop size.

Nucleus

- Nuclear force short range.
- Density independent of nuclear size.
- $B/A \sim$ constant.

Nuclear mass *The liquid drop model*

Predicts the binding energy as: $B = a_V A - a_S A^{2/3} - \frac{a_c Z^2}{A^{1/3}}$

$a_V A$

Volume term

Strong force between nucleons increases B and reduces mass by a constant amount per nucleon.

Nuclear volume $\sim A$

$-a_S A^{2/3}$

Surface term

Nucleons on surface are not as strongly bound $\Rightarrow$ decreases B.

Surface area $\sim R^2 \sim A^{2/3}$

$-\frac{a_c Z^2}{A^{1/3}}$

Coulomb term

Protons repel each other $\Rightarrow$ decreases B.

Electrostatic P.E. $\sim Q^2/R \sim Z^2/A^{1/3}$

But there are problems. Does not account for

- $N \sim Z$
- Nucleons tend to pair up; even N, Z favoured

Nuclear mass *The Fermi gas model*

Fermi gas model: assume the nucleus is a Fermi gas, in which confined nucleons can only assume certain discrete energies in accordance with the Pauli Exclusion Principle.

Addresses problems with the liquid drop model with additional terms:

$-a_A \frac{(N-Z)^2}{A}$

Asymmetry term Nuclei tend to have $N \sim Z$.

Kinetic energy of Z protons and N neutrons is minimised if $N{=}Z$. The greater the departure from $N{=}Z$, the smaller the binding energy. Correction scaled down by $1/A$, as levels are more closely spaced as A increases.

$+\delta(A)$

Pairing term Nuclei tend to have even Z, even N.

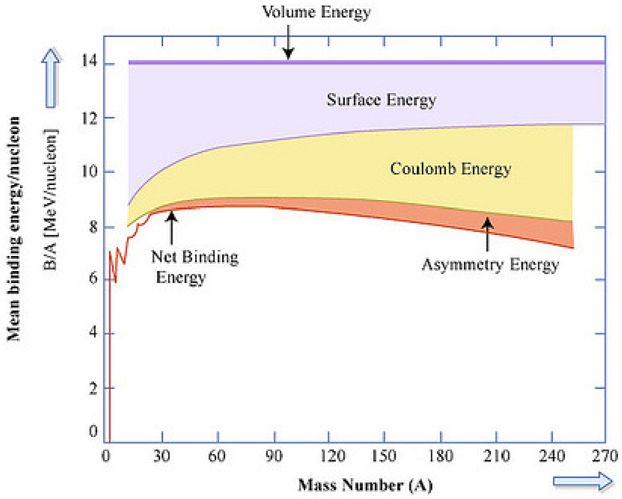
Pairing interaction energetically favours the formation of pairs of like nucleons (pp, nn) with spins $\uparrow\downarrow$ and symmetric spatial wavefunction.

The form is simply empirical.

$$\begin{aligned}\delta(A) &= +a_P A^{-3/4} && N, Z \text{ even-even}\\ &= -a_P A^{-3/4} && N, Z \text{ odd-odd}\\ &= 0 && N, Z \text{ even-odd}\end{aligned}$$

Nuclear mass *The semi-empirical mass formula*

Putting all these terms together, we have various contributions to B/A:



Nuclear mass is well described by the **semi-empirical mass formula**

$$m(A,Z) = Zm_p + (A-Z)m_n - B$$

$$B = a_V A - a_S A^{2/3} - \frac{a_c Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} + \delta(A)$$

with the following coefficients (in MeV) obtained by fitting to data

$$a_V = 15.8,\ a_S = 18.0,\ a_C = 0.72,\ a_A = 23.5,\ a_P = 33.5$$

Nuclear Spin

The nucleus is an isolated system and so has a well defined nuclear spin

Nuclear spin quantum number J

$$|J| = \sqrt{J(J+1)} \qquad \hbar = 1$$

$$m_J = -J,\ \ -(J-1), \ldots,\ J-1,\ J.$$

Nuclear spin is the sum of the individual nucleons total angular momentum, j_i,

$$\vec{J} = \sum_i \vec{j}_i, \qquad \vec{j}_i = \vec{L}_i + \vec{S}_i$$

$j-j$ coupling always applies because of strong spin-orbit interaction (see later)

where the total angular momentum of a nucleon is the sum of its intrinsic spin and orbital angular momentum

- intrinsic spin of p or n is $s = 1/2$
- orbital angular momentum of nucleon is integer

A even $\rightarrow$ J must be integer

A odd $\rightarrow$ J must be $1/2$ integer

All nuclei with even N and even Z have $J = 0$.

Nuclear Parity

- All particles are eigenstates of parity $\hat{P}|\Psi\rangle = P|\Psi\rangle, \quad P = \pm 1$
- Label nuclear states with the nuclear spin and parity quantum numbers.
  Example: 0^+ ($J = 0$, parity even), 2^- ($J = 2$, parity odd)
- The parity of a nucleus is given by the product of the parities of all the neutrons and protons $$P = \left(\prod_i P_i\right)(-1)^L$$ for ground state nucleus, $L = 0$
- The parity of a single proton or neutron is $P = (+1)(-1)^L$
  intrinsic $P = +1$ (3 quarks) nucleon L is important
- For an odd A, the parity is given by the unpaired p or n. *(Nuclear Shell Model)*
- Parity is conserved in nuclear processes (strong interaction).
- Parity of nuclear states can be extracted from experimental measurements, e.g. γ transitions.

Nuclear Size

The size of a nucleus may be determined using two sorts of interaction:

Electromagnetic Interaction gives the **charge** distribution of protons inside the nucleus, e.g.

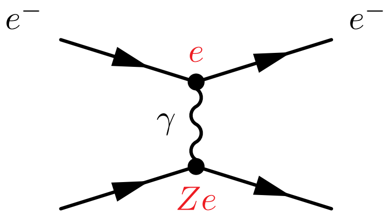
- electron scattering
- muonic atoms
- mirror nuclei

Strong Interaction gives **matter** distribution of protons and neutrons inside the nucleus. Sample nuclear and charge interactions at the same time ⇒ more complex, e.g.

- α particle scattering (Rutherford)
- proton and neutron scattering
- Lifetime of α particle emitters (see later)
- π-mesic X-rays.

⇒ Find charge and matter radii EQUAL for all nuclei.

Nuclear Size *Electron scattering*



Nucleus, Z protons

Use electron as a probe to study deviations from a point-like nucleus.

Electromagnetic Interaction

Coulomb potential $$V(\vec{r}) = -\frac{Z\alpha}{r}$$

Born Approximation $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{E^2}{(2\pi)^2}\left|\int \mathrm{e}^{-i\vec{q}.\vec{r}}V(\vec{r})\,\mathrm{d}^3\vec{r}\right|^2$$

$\vec{q} = \vec{p}_i - \vec{p}_f$ is the momentum transfer

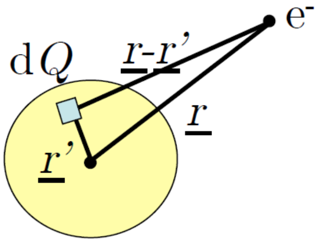
Rutherford Scattering $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2\alpha^2}{4E^2\sin^4\theta/2}$$

To measure a distance of ~1 fm, need large energy (*ultra-relativistic*)

$$E = \frac{1}{\lambda} = 1\ \mathrm{fm}^{-1} \sim 200\ \mathrm{MeV} \qquad \hbar c = 197\ \mathrm{MeV.fm}$$

Nuclear Size *Scattering from an extended nucleus*

But the nucleus is not point-like!
$V(\vec{r})$ depends on the distribution of charge in nucleus.



Potential energy of electron due to charge $\mathrm{d}Q$ $$\mathrm{d}V = -\frac{e\,\mathrm{d}Q}{4\pi\left|\vec{r}-\vec{r'}\right|}$$

where $\mathrm{d}Q = Ze\rho(\vec{r'})\,\mathrm{d}^3\vec{r'}$

$\rho(\vec{r'})$ is the charge distribution (normalised to 1)

$$V(\vec{r}) = \int -\frac{\mathrm{e}^2 Z\rho(\vec{r'})}{4\pi\left|\vec{r}-\vec{r'}\right|} = -Z\alpha\int\frac{\rho(\vec{r'})}{\left|\vec{r}-\vec{r'}\right|}\mathrm{d}^3\vec{r'} \qquad \alpha = \frac{\mathrm{e}^2}{4\pi}$$

This is just a convolution of the pure Coulomb potential $Z\alpha/r$ with the normalised charge distribution $\rho(r)$.
Hence we can use the convolution theorem to help evaluate the matrix element which enters into the Born Approximation.

Nuclear Size *Scattering from an extended nucleus*

Matrix Element $$M_{\rm if} = \int e^{i\vec{q}\vec{r}} V(\vec{r})\, d^3\vec{r} = -Z\alpha \underset{\text{Rutherford scattering}}{\int \frac{e^{i\vec{q}\vec{r}}}{r} d^3\vec{r}} \underset{F(q^2)}{\int \rho(\vec{r}) e^{i\vec{q}\vec{r}}\, d^3\vec{r}}$$

Hence, $$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm point} \left|F(q^2)\right|^2$$

where $F(q^2) = \int \rho(\vec{r}) e^{i\vec{q}\vec{r}}\, d^3\vec{r}$ is called the Form Factor and is the fourier transform of the normalised charge distribution.

Spherical symmetry, $\rho = \rho(r)$, a simple calculation (similar to our treatment of the Yukawa potential) shows that

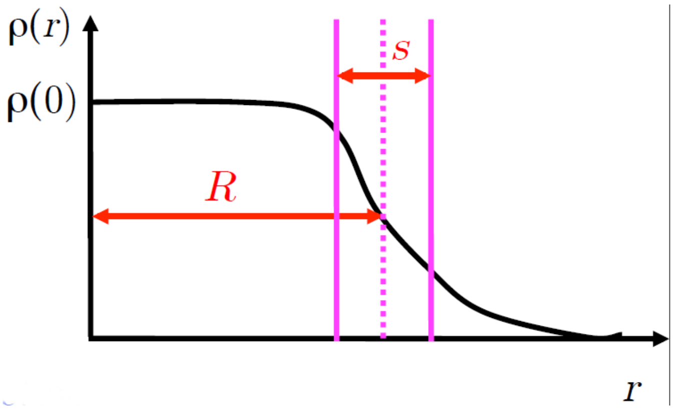
$$F(q^2) = \int_0^\infty \rho(r) \frac{\sin qr}{qr} 4\pi r^2\, dr \quad ; \quad \rho(r) = \frac{1}{2\pi^2} \int_0^\infty F(q^2) \frac{\sin qr}{qr} q^2\, dq$$

So if we measure cross-section, we can infer $F(q^2)$ and get the charge distribution by Fourier transformation.

Nuclear Size *Modelling charge distribution*

Use nuclear diffraction to measure scattering, and find the charge distribution inside a nucleus is well described by the Fermi parametrisation.

$$\rho(r) = \frac{\rho(0)}{1 + e^{(r-R)/s}}$$



Fit this to data to determine parameters R and s.

- R is the radius at which $\rho(r) = \rho(0)/2$

  Find R increases with A: $R = r_0 A^{1/3}$ $r_0 \sim 1.2\,\text{fm}$.

- s is the surface width or skin thickness over which $\rho(r)$ falls from 90%→10%.

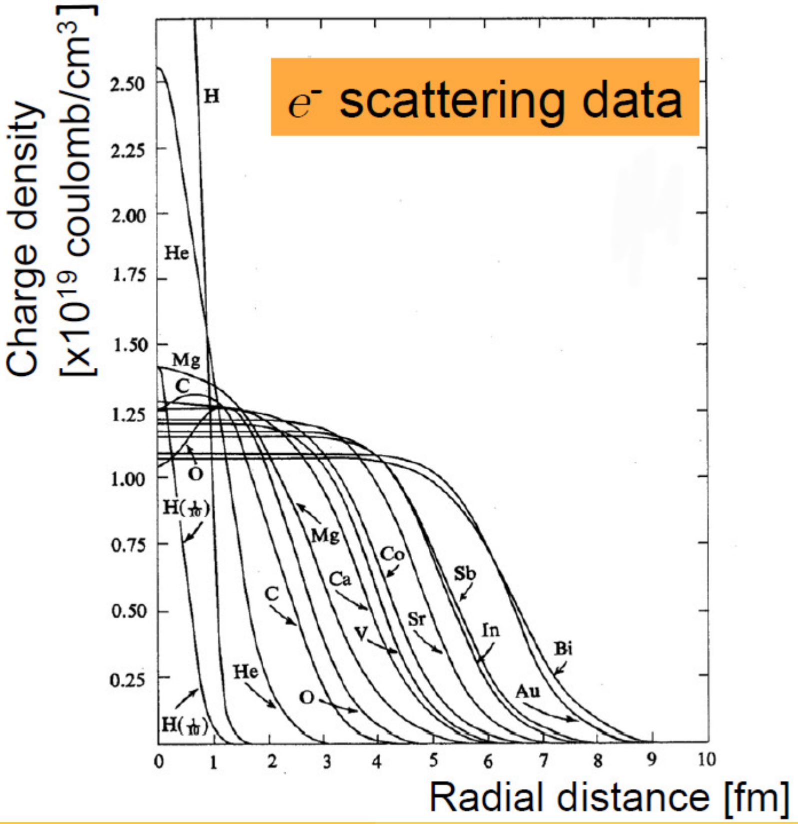
  Find s is is approximately the same for all nuclei ($s \sim 2.5$ fm); governed by the range of the strong nuclear interaction

Nuclear Size *Modelling charge distribution*

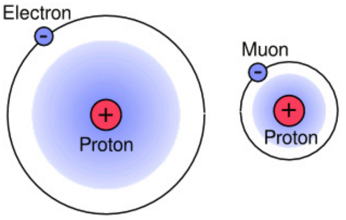
Fits to e^- scattering data show the Fermi parametrisation models nuclear charge distributions well.

Shows that all nuclei have roughly the same density in their interior.

Radius $\sim R_0 A^{1/3}$ with $R_0 \sim 1.2$ fm $\Rightarrow$ consistent with short-range saturated forces.



Nuclear Size *Muonic Atoms*



Muons can be brought to rest in matter and trapped in orbit → probe EM interactions with nucleus.
The large muon mass affects its orbit, $m_\mu \sim 207\ m_e$

Bohr radius, $r \propto 1/Zm$

Hydrogen atom with electrons: $r = a_0 \sim 53,000$ fm

with muons: $r \sim 285$ fm

Lead ($Z = 82$) with muons: $r \sim 3$ fm Inside nucleus!

Energy levels, $E \propto Z^2 m$

Rapid transitions to lower energy levels $\sim 10^{-9}$s

Factor of 2 effect seen from nuclear size in muonic lead

Transition energy ($2P_{3/2} \rightarrow 1S_{1/2}$) : 16.41 MeV (Bohr theory) vs 6.02 MeV (measured)

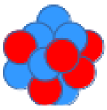
Muon lifetime, $\tau_\mu \sim 2\mu$s

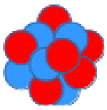
Decays via $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$ – Plenty of time spent in 1s state.

$Z_{\text{effective}}$ and E are changed relative to electrons.
Measure X-ray energies → nuclear radius.

Nuclear Size *Mirror Nuclei*

$^{11}_{5}\mathrm{B}$ $^{11}_{6}\mathrm{C}$





Different nuclear masses from *p*-*n* difference and the different Coulomb terms.

$$m(A,Z) = Zm_p + (A-Z)m_n - \left[a_V A - a_S A^{2/3} - \frac{a_c Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} + \delta(A)\right]$$

For the *atomic* mass difference, don't forget the electrons!

$$M(A,Z+1) - M(A,Z) = \Delta E_c + m_p + m_e - m_n$$

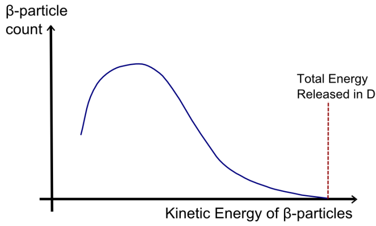
where $\Delta E_c = \frac{3}{5}\frac{A\alpha}{R}$ (see Question 33)

Probe the atomic mass difference between two mirror nuclei by observing β^+ decay spectra (3-body decay).

$$^{11}_{6}\mathrm{C} \rightarrow {}^{11}_{5}\mathrm{B} + e^+ + \nu_e \qquad (p \rightarrow n + e^+ + \nu_e)$$

$$M(A,Z+1) - M(A,Z) = 2m_e + E_{\max} \qquad m_\nu \sim 0$$

where $E_{\max}$ is the maximum kinetic energy of the positron.

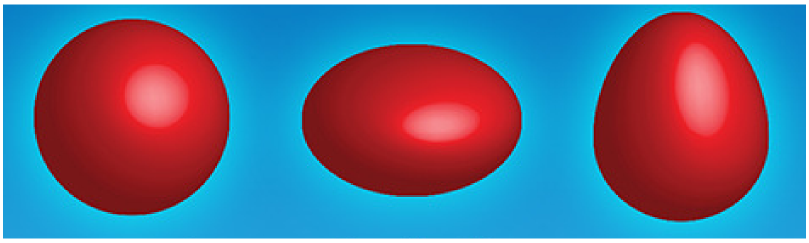


Relate mass difference to ΔE_c and extract the nuclear radius

$$R = \frac{3A\alpha}{5}\left[\frac{1}{E_{\max} - m_p + m_n + m_e}\right]$$

Nuclear Shape

The shape of nuclei can be inferred from measuring their electromagnetic moments.



Nuclear moments give information about the way magnetic moment and charge is distributed throughout the nucleus.

The two most important moments are:

Electric Quadrupole Moment Q

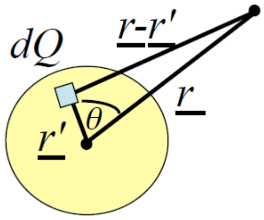
Magnetic Dipole Moment μ

Nuclear Shape *Electric Moments*

Electric moments depend on the charge distribution inside the nucleus.

Parameterise the nuclear shape using a multipole expansion of the external electric field or potential

$$V(r) = \frac{1}{4\pi}\int \frac{\rho(\vec{r'})}{\left|\vec{r}-\vec{r'}\right|}\,\mathrm{d}^3\vec{r'}$$



where $\rho(\vec{r'})\,\mathrm{d}^3\vec{r'} = Ze$ and $r(r') =$ distance to observer (charge element) from origin.

$$\left|\vec{r}-\vec{r'}\right| = \left[r^2 + r'^2 - 2rr'\cos\theta\right]^{1/2} \Rightarrow \left|\vec{r}-\vec{r'}\right|^{-1} = r^{-1}\left[1+\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta\right]^{-1/2}$$

$$\left|\vec{r}-\vec{r'}\right|^{-1} = r^{-1}\left[1 - \frac{1}{2}\left(\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta\right) + \frac{3}{8}\left(\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta\right)^2 + \ldots\right]$$

$$\sim r^{-1}\left[1 + \frac{r'}{r}\cos\theta + \frac{1}{2}\frac{r'^2}{r^2}\left(3\cos^2\theta - 1\right) + \ldots\right]$$

$r' \ll r \Rightarrow$ expansion in powers of $r'r$; or equivalently Legendre polynomials

$$V(r) = \frac{1}{4\pi r}\left[Ze + \frac{1}{r}\int r'\cos\theta\rho(r')\,\mathrm{d}^3\vec{r'} + \frac{1}{2r^2}\int r'^2(3\cos\theta - 1)\rho(r')\,\mathrm{d}^3\vec{r'} + \ldots\right]$$

Nuclear Shape *Electric Moments*

Let r define z-axis, $z = r'\cos\theta$

$$V(r) = \frac{1}{4\pi r}\left[Ze + \frac{1}{r}\int z\rho(r')\,\mathrm{d}^3\vec{r'} + \frac{1}{2r^2}\int (3z^2 - r'^2)\rho(r')\,\mathrm{d}^3\vec{r'} + \ldots\right]$$

Quantum limit: $\rho(r') = Ze.\left|\psi(\vec{r'})\right|^2$

The electric moments are the coefficients of each successive power of $1/r$

E0 moment $\int Ze.\psi^*\psi\,\mathrm{d}^3\vec{r'} = Ze$ *charge*

No shape information

E1 moment $\int \psi^* z\psi\,\mathrm{d}^3\vec{r'}$ *electric dipole*

Always zero since ψ have definite parity

$|\psi(\vec{r})|^2 = |\psi(-\vec{r})|^2$

E2 moment $\int \frac{1}{e}\psi^*(3z^2 - r'^2)\psi\,\mathrm{d}^3\vec{r'}$ *electric quadrupole*

First interesting moment!

Nuclear Shape *Electric Moments*

Electric Quadrupole Moment $$Q = \frac{1}{e}\int (3z^2 - r^2)\rho(\vec{r})\,\mathrm{d}^3\vec{r}$$

Units: m^2 or barns (though sometimes the factor of e is left in)

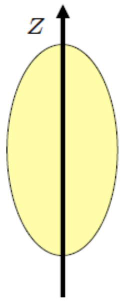
If spherical symmetry, $\overline{z^2} = \frac{1}{3}\overline{r^2} \;\Rightarrow Q = 0$

- $Q = 0$ Spherical nucleus. All $J = 0$ nuclei have $Q = 0$.
- Large Q Highly deformed nucleus. e.g. Na

  Two cases:

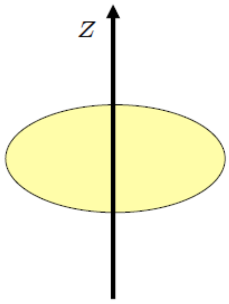
Prolate spheroid

$Q > 0$

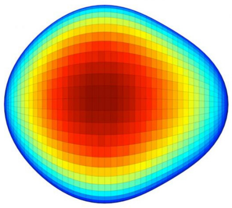


Oblate spheroid

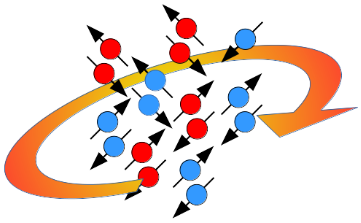
$Q < 0$



Aside: Radium-224 is pear-shaped!
Non-zero quadrupole and octupole moments.
(ISOLDE, CERN, 2013)



Nuclear Shape *Magnetic Moments*



Nuclear magnetic dipole moments arise from

- intrinsic spin magnetic dipole moments of the protons and neutrons
- circulating currents (motion of the protons)

The nuclear magnetic dipole moment can be written as

$$\vec{\mu} = \frac{\mu_N}{\hbar}\sum_i \left[g_L\vec{L} + g_s\vec{s}\right]$$

summed over all p, n

where $\mu_N = e\hbar/2m_p$ is the Nuclear Magneton.

or $\mu = g_J\mu_N J$ where J total nuclear spin quantum number

g_J nuclear g-factor (analogous to Landé g-factor in atoms)

g_J may be predicted using the Nuclear Shell Model (see later), and measured using magnetic resonance (see Advanced Quantum course).

All even-even nuclei have $\mu = 0$ since $J = 0$

Summary

- Nuclear binding energy – short range saturated forces
- Semi-empirical Mass Formula – based on liquid drop model + simple inclusion of quantum effects

$$m(A,Z) = Zm_p + (A-Z)m_n - B$$

$$B = a_V A - a_S A^{2/3} - \frac{a_c Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} + \delta(A)$$

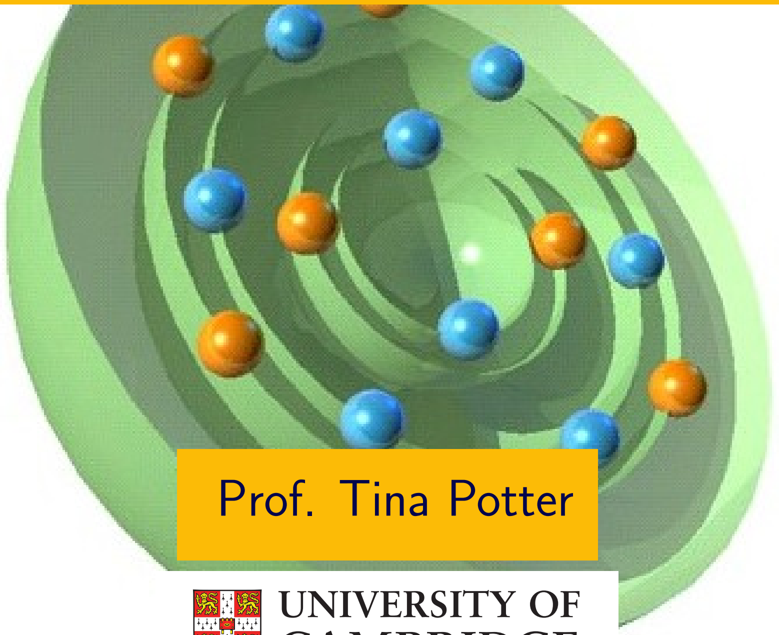
- Nuclear size from electron scattering, muonic atoms, and mirror nuclei. Constant density; radius $\propto A^{1/3}$
- Nuclear spin, parity, electric and magnetic moments.

Problem Sheet: q.31-33

Up next...
Section 14: The Structure of Nuclei

14. Structure of Nuclei

Particle and Nuclear Physics



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In this section...

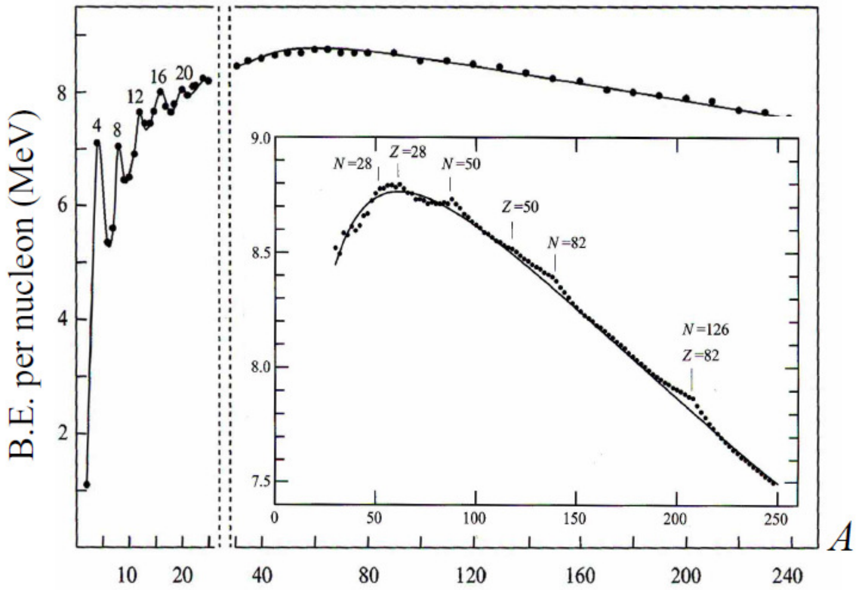
- Magic Numbers
- The Nuclear Shell Model
- Excited States

Magic Numbers

Magic Numbers $= 2, 8, 20, 28, 50, 82, 126...$

Nuclei with a magic number of Z and/or N are particularly stable,

e.g. Binding energy per nucleon is large for magic numbers



Doubly magic nuclei are especially stable.

Magic Numbers

Other notable behaviour includes

- Greater abundance of isotopes and isotones for magic numbers
  e.g. $Z = 20$ has 6 stable isotopes (average $= 2$)
  $Z = 50$ has 10 stable isotopes (average $= 4$)
- Odd A nuclei have small quadrupole moments when magic
- First excited states for magic nuclei higher than neighbours
- Large energy release in α, β decay when the daughter nucleus is magic
- Spontaneous neutron emitters have $N = magic + 1$
- Nuclear radius shows only small change with Z, N at magic numbers.

etc... etc...

Magic Numbers

Analogy with atomic behaviour as electron shells fill.

Atomic case - reminder

- Electrons move independently in central potential $V(r) \sim 1/r$ (Coulomb field of nucleus).
- Shells filled progressively according to Pauli exclusion principle.
- Chemical properties of an atom defined by valence (unpaired) electrons.
- Energy levels can be obtained (to first order) by solving Schrödinger equation for central potential.

$$E_n = \frac{1}{n^2} \qquad n = \text{ principle quantum number}$$

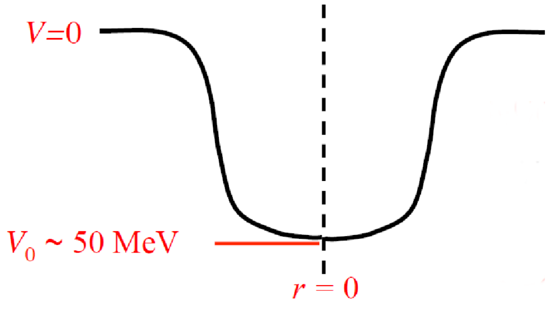
- Shell closure gives noble gas atoms.

Are magic nuclei analogous to the noble gas atoms?

Magic Numbers

Nuclear case (Fermi gas model)

Nucleons move in a net nuclear potential that represents the *average effect* of interactions with the other nucleons in the nucleus.



Nuclear Potential

$$V(r) \sim \frac{-V_0}{\left(1 + e^{(r-R)/s}\right)}$$

"Saxon-Woods potential", i.e. a Fermi function, like the nuclear charge distribution

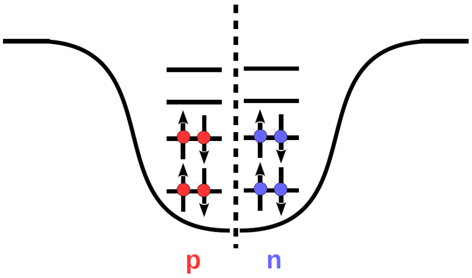
- Nuclear force short range + saturated ⇒ near centre $V(r)$ ~constant.
- Near surface: density and no. of neighbours decreases ⇒ $V(r)$ decreases
- For protons, $V(r)$ is modified by the Coulomb interaction

Magic Numbers

In the ground state, nucleons occupy energy levels of the nuclear potential so as to minimise the total energy without violating the Pauli principle.

The exclusion principle operates independently for protons and neutrons.

Tendency for Z=N
to give the minimum E



Postulate: nucleons move in well-defined orbits with discrete energies.

Objection: nucleons are of similar size to nucleus $\therefore$ expect many collisions. How can there be well-defined orbits?

Pauli principle: if energy is transferred in a collision then nucleons must move up/down to new states. However, all nearby states are occupied $\therefore$ no collision. i.e. almost all nucleons in a nucleus move freely within nucleus if it is in its ground state.

The Nuclear Shell Model

- Treat each nucleon independently and solve Schrödingers equation for nuclear potential to obtain nucleon energy levels.
- Consider spherically symmetric central potential e.g. Saxon-Woods potential
$$V(r) \sim \frac{-V_0}{(1+\mathrm{e}^{(r-R)/s})}$$
- Solution of the form $\psi(\vec{r}) = R_{nL}(r)Y_L^m(\theta,\phi)$
- Obtain 2 equations separately for radial and angular coordinates.

Radial Equation: $\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L(L+1)}{r^2} + 2M(E-V(r))\right]R_{nL}(r) = 0$

Allowed states specified by n, L, m:

n radial quantum number (n.b. different to atomic notation)

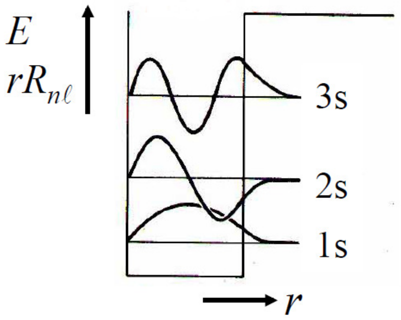
L orbital a.m. quantum no. n.b. any L for given n (c.f. Atomic $L<n$)

m magnetic quantum number $m = -L \ldots + L$

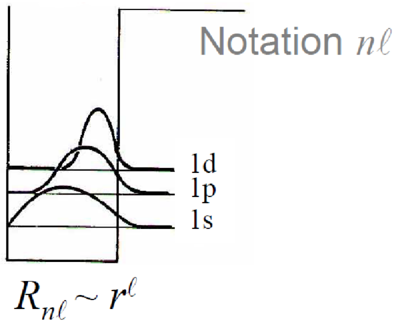
The Nuclear Shell Model

Energy levels increase with n and L (similar to atomic case)

Fix L, increase n



Fix n, increase L



As n increases:
rR_{nL} has more nodes, greater curvature and E increases.

As L increases:
rR_{nL} has greater curvature and E increases.

Fill shells for both p and n:

$$\text{Degeneracy} = (2s+1)(2L+1) = 2(2L+1) \quad (s = 1/2)$$

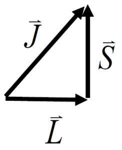
But, this central potential alone cannot reproduce the observed magic numbers. Need to include **spin-orbit interaction**.

Spin-orbit interaction

Mayer and Jensen (1949) included (strong) spin-orbit potential to explain magic numbers.

$$V(r) = V_{\text{central}}(r) + V_{\text{so}}(r)\vec{\hat{L}}.\vec{\hat{S}} \qquad \text{n.b. } V_{\text{so}} \text{ is negative}$$

Spin-orbit interaction splits L levels into their different j values

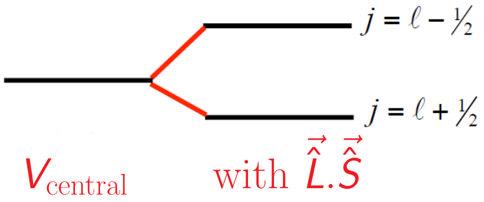


$$\vec{\hat{J}} = \vec{\hat{L}} + \vec{\hat{S}}; \qquad \vec{\hat{J}}^2 = \vec{\hat{L}}^2 + \vec{\hat{S}}^2 + 2\vec{\hat{L}}.\vec{\hat{S}}; \qquad \vec{\hat{L}}.\vec{\hat{S}} = \frac{1}{2}\left[\vec{\hat{J}}^2 - \vec{\hat{L}}^2 - \vec{\hat{S}}^2\right]$$

$$\vec{\hat{L}}.\vec{\hat{S}}|\psi\rangle = \frac{1}{2}\left[j(j+1) - L(L+1) - s(s+1)\right]|\psi\rangle$$

For a single nucleon with $s = \frac{1}{2}$,

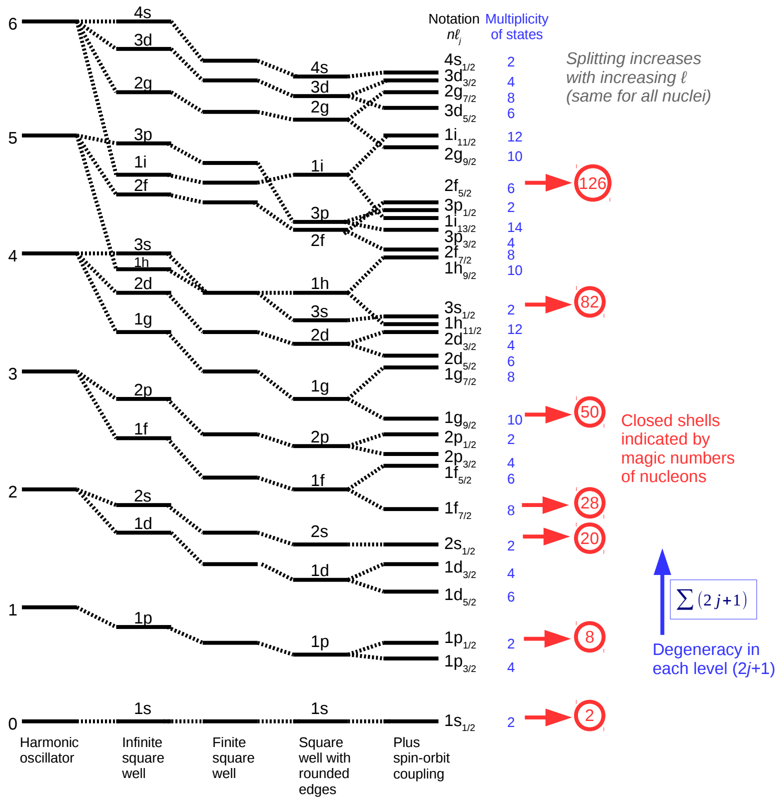
- $j = L - \frac{1}{2}$: $\quad \vec{\hat{L}}.\vec{\hat{S}}|\psi\rangle = -\frac{1}{2}(L+1)|\psi\rangle \quad V = V_{\text{central}} - \frac{1}{2}(L+1)V_{\text{so}}$
- $j = L + \frac{1}{2}$: $\quad \vec{\hat{L}}.\vec{\hat{S}}|\psi\rangle = \frac{1}{2}L|\psi\rangle \quad V = V_{\text{central}} + \frac{1}{2}LV_{\text{so}}$



$$\Delta E = \frac{1}{2}(2L+1)V_{\text{so}}$$

n.b. larger j lies lower

Nuclear Shell Model *Energy Levels*



Nuclear Shell Model Predictions

1. Magic Numbers.
   The Shell Model successfully predicts the origin of the magic numbers. It was constructed to achieve this.
2. Spin & Parity.
3. Magnetic Dipole Moments.

Nuclear Shell Model *Spin and Parity*

The Nuclear Shell Model predicts the spin & parity of ground state nuclei.

Case 1: Near closed shells

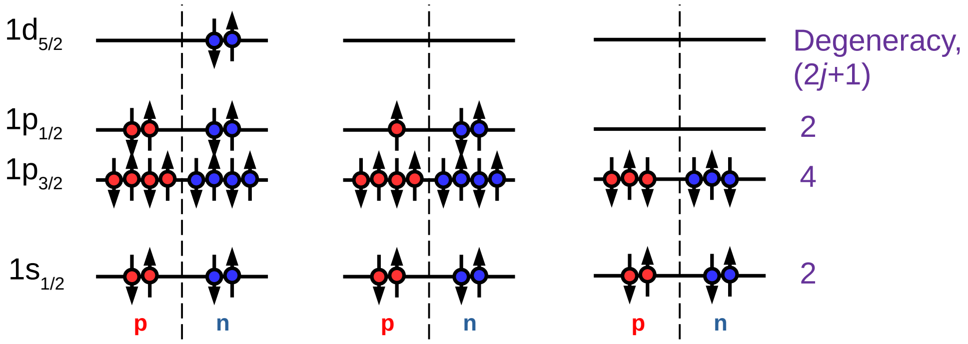
Even-Even Nuclei : $J^P = 0^+$

Even-Odd Nuclei : J^P given by unpaired nucleon or hole; $P = (-1)^L$

Odd-Odd Nuclei : Find J values of unpaired p and n, then apply jj coupling

i.e. $|j_p - j_n| \leq J \leq j_p + j_n$, Parity $= (-1)^{L_p}(-1)^{L_n}$

e.g.



$^{18}_{8}$O J^P=0^+ (obs)

$^{15}_{7}$N J^P=$1/2^-$ (obs)

$^{10}_{5}$B j_p=$3/2^-$, j_n=$3/2^-$

J^P=0^+,1^+,2^+,3^+ ($J^P = 3^+$ observed)

Nuclear Shell Model *Spin and Parity*

The Nuclear Shell Model predicts the spin & parity of ground state nuclei.

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Odd-Odd Nuclei : Find J values of unpaired p and n, then apply jj coupling
i.e. $|j_p - j_n| \leq J \leq j_p + j_n$, Parity $= (-1)^{L_p}(-1)^{L_n}$

There are however cases where this simple prescription fails.

The pairing interaction between identical nucleons is not described by a spherically symmetric potential nor by the spin-orbit interaction.
Lowest energy spin state of pair: $\uparrow\downarrow$ with (j, m) and $(j, -m)$. Total $J = 0$.
Need antisymmetric $\psi_{\text{total}} = \psi_{\text{spin}}\psi_{\text{spatial}}$: ψ_{spin} antisymmetric, thus ψ_{spatial} is symmetric.
This maximises the overlap of their wavefunctions, increasing the binding energy (attractive force). The pairing energy increases with increasing L of nucleons.

Example: $^{207}_{82}$Pb naively expect odd neutron in $2f_{5/2}$ subshell.
But, pairing interaction means it is energetically favourable for the $2f_{5/2}$ neutron and a neutron from nearby $3p_{1/2}$ to pair and leave hole in $3p_{1/2}$. $\Rightarrow J^P = 1/2^-$ (observed)

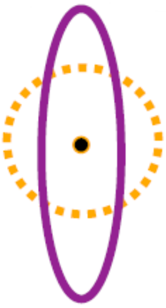
Nuclear Shell Model *Spin and Parity*

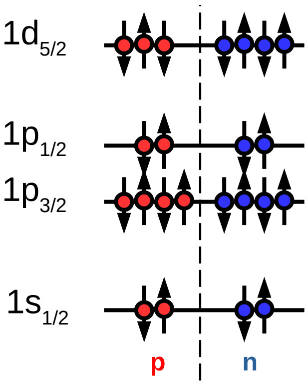
The Nuclear Shell Model predicts the spin & parity of ground state nuclei.

Case 2: Away from closed shells

More than one nucleon can contribute and electric quadrupole moment Q is often large
$\Rightarrow V(r)$ no longer spherically symmetric.

Example: $^{23}_{11}$Na Q is observed to be large, i.e. non-spherical.
Three protons in $1d_{5/2}$; if two were paired up, we expect $J^P = 5/2^+$.





In fact, all three protons must contribute
$\Rightarrow$ can get $J^P = 3/2^+$
(observed)

Nuclear Shell Model *Magnetic Dipole Moments*

The Nuclear Shell Model predicts the magnetic dipole moments of ground state nuclei.

Even-even nuclei : $J = 0 \Rightarrow \mu = 0$
Odd A nuclei: μ corresponds to the unpaired nucleon or hole

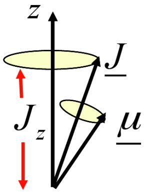
For a single nucleon $\vec{\mu} = \frac{\mu_N}{\hbar}(g_L\vec{L} + g_s\vec{s})$ with $p: \quad g_L = 1, \quad g_s = +5.586,$
$n: \quad g_L = 0, \quad g_s = -3.826,$
where $\mu_N = \frac{e\hbar}{2m_p}$ is the Nuclear Magneton.

$\vec{\mu}$ is not parallel to $\vec{j}$ (since $\vec{j} = \vec{L} + \vec{s}$).
However, the *angle* between $\vec{\mu}$ and $\vec{j}$ is constant, because

$$\cos\theta \;\sim\; \vec{\mu}.\vec{j} \;\sim\; g_L\vec{L}.\vec{j} + g_s\vec{s}.\vec{j} \;=\; \frac{1}{2}\left[g_L(L^2 + j^2 - s^2) + g_s(s^2 + j^2 - L^2)\right]$$

and j^2, L^2 and s^2 are all constants of motion.

Hence, we can calculate the nuclear magnetic moment (projection of $\vec{\mu}$ along the z-axis)



$$\mu_z = \frac{\vec{\mu}.\vec{J}}{|\vec{J}|} \times \frac{J_z}{|\vec{J}|}$$

project $\vec{\mu}$ onto $\vec{J}$ then $\vec{J}$ onto $\vec{z}$

c.f. derivation of Landé g-factor in Quantum course

$$\therefore \mu_z = \mu_N \frac{m_J}{2j(j+1)}\left(\, g_L\left[L(L+1) + j(j+1) - s(s+1)\right] \;+\; g_s\left[s(s+1) + j(j+1) - L(L+1)\right]\,\right)$$

Nuclear Shell Model *Magnetic Dipole Moments*

The Nuclear Shell Model predicts the magnetic dipole moments of ground state nuclei.

Even-even nuclei : $J = 0 \Rightarrow \mu = 0$
Odd A nuclei: μ corresponds to the unpaired nucleon or hole

Thus $\boldsymbol{\mu = g_J \mu_N J}$ for $m_J = J$ and

$$g_J = \frac{1}{2j(j+1)}\left(g_L\left[L(L+1) + j(j+1) - s(s+1)\right] + g_s\left[s(s+1) + j(j+1) - L(L+1)\right]\right)$$

For a single nucleon ($s = 1/2$), there are two possibilities ($j = L + 1/2$ or $L - 1/2$)

$$g_J = g_L \pm \frac{g_s - g_L}{2L+1} \qquad j = L \pm 1/2$$

Odd p: $g_L = 1 \quad g_s = +5.586$
Odd n: $g_L = 0 \quad g_s = -3.826$

called the **"Schmidt Limits"**.

Nuclear Shell Model *Magnetic Dipole Moments*

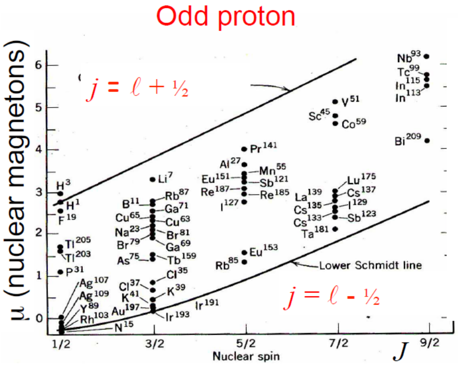
The Nuclear Shell Model predicts the magnetic dipole moments of ground state nuclei.

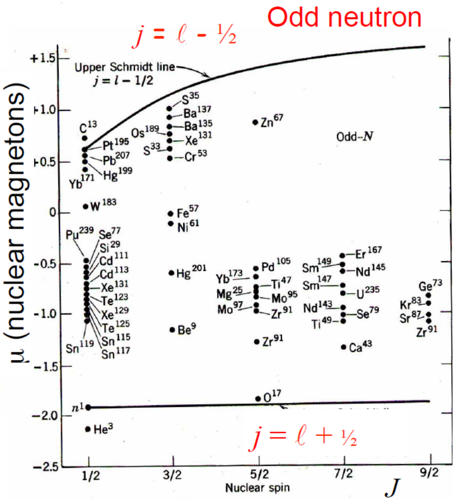
Even-even nuclei : $J = 0 \Rightarrow \mu = 0$

Odd A nuclei: μ corresponds to the unpaired nucleon or hole

Schmidt Limits compared to data: The Nuclear Shell Model predicts the broad trend of the magnetic moments. But not good in detail, except for closed shell ±1 nucleon or so.

$\Rightarrow$ wavefunctions must be more complicated than our simple model.





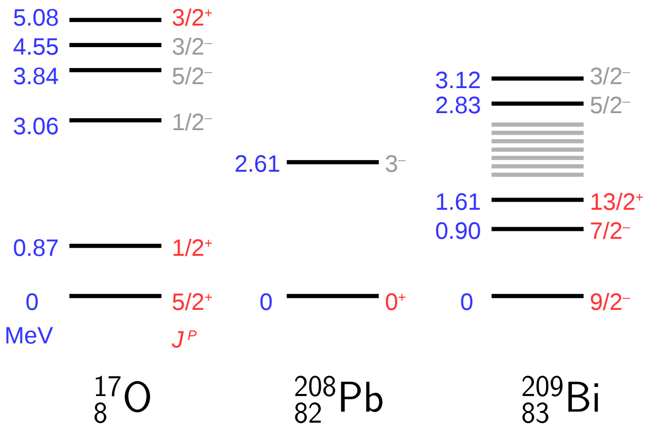
Excited States of Nuclei

In nuclear spectra, we can identify three kinds of excitations:

- Single nucleon excited states
- Vibrational excited states
- Rotational excited states

Single nucleon excited states may, to some extent, be predicted from the simple Shell Model. Most likely to be successful for lowest-lying excitations of odd A nuclei near closed shells.

e.g.



Excited States of Nuclei

Vibrational and **rotational** motion of nuclei involve the collective motion of the nucleons in the nucleus.

Collective motion can be incorporated into the shell model by replacing the static symmetrical potential with a potential that undergoes deformations in shape.

$\Rightarrow$ Collective vibrational and rotational models.

Here we will only consider even Z, even N nuclei

Ground state : $J^P = 0^+$

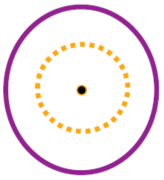
Lowest excited state (nearly always): $J^P = 2^+$

Tend to divide into two categories:

A	$E(2^+)$	**Type**
30–150	~ 1 MeV	Vibrational
150–190 (rare earth) >220 (actinides)	~ 0.1 MeV	Rotational

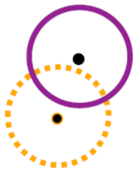
Nuclear Vibrations

Vibrational excited states occur when a nucleus oscillates about a spherical equilibrium shape (low energy surface vibrations, near closed shells). Form of the excitations can be represented by a multipole expansion (just like underlying nuclear shapes).



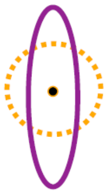
Monopole

Incorporated into the average radius

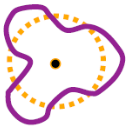


Dipole

Involves a net displacement of centre of mass $\Rightarrow$ cannot result from action of nuclear forces (can be induced by applied e/m field i.e. a photon)



Quadrupole

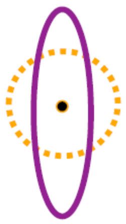


Octupole

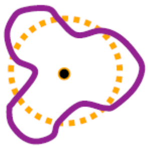
Quadrupole oscillations are the lowest order nuclear vibrational mode.

Similar to SHM – the quanta of vibrational energy are called phonons.

Nuclear Vibrations



A **quadrupole phonon** carries 2 units of angular momentum and has even parity $\Rightarrow J^P = 2^+$



An **octupole phonon** carries 3 units of angular momentum and has odd parity $\Rightarrow J^P = 3^-$

Phonons are bosons and must satisfy Bose-Einstein statistics (overall symmetric wavefunction under the interchange of two phonons).

e.g. for quadrupole phonons:

Even-even ground state $0^+ \xrightarrow{\text{1 phonon}} 2^+$

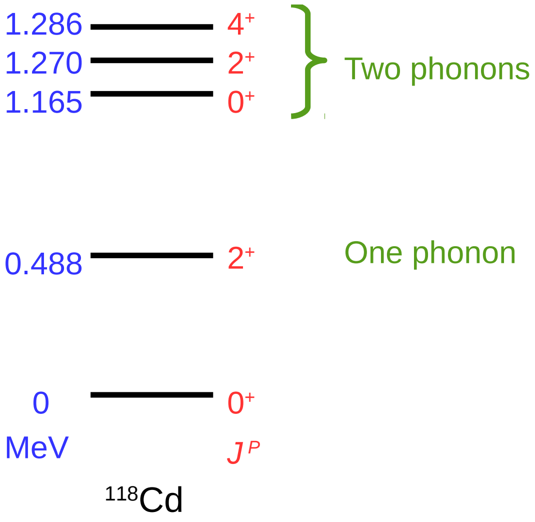
$\xrightarrow{\text{2 phonons}} 0^+,\ 2^+,\ 4^+$ (in practice not degenerate)

Energies of vibrational excitations are not predicted, but we can predict the ratios

$$\frac{\text{Second excited } (2 \text{ phonons};\ 0^+,\ 2^+,\ 4^+)}{\text{First excited } (1 \text{ phonon};\ 2^+)} \sim 2$$

Nuclear Vibrations

Example of vibrational excitations:



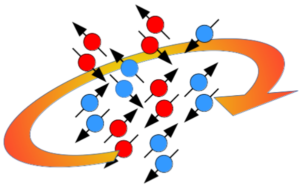
Predict $\frac{\text{2nd excited}}{\text{1st excited}} \sim 2$

Observe $\frac{\text{2nd excited}}{\text{1st excited}} \sim 2.4$

Octupole states ($J^P = 3^-$) are often seen near the triplet of two-phonon quadrupole states.

Vibrational states decay rapidly by γ emission (see later).

Nuclear Rotations



Collective rotational motion can only be observed in nuclei with non-spherical equilibrium shapes (i.e. far from closed shells, large Q).

Rotating deformed nucleus: nucleons in rapid internal motion in the nuclear potential + entire nucleus rotating slowly. Slow to maintain a stable equilibrium shape and not to affect the nuclear structure.

Nuclear mirror symmetry restricts the sequence of rotational states to even values of angular momentum.

Even-even ground state $0^+ \rightarrow 2^+,\ 4^+,\ 6^+$

… (total angular momentum = nuclear a.m. + rotational a.m.)

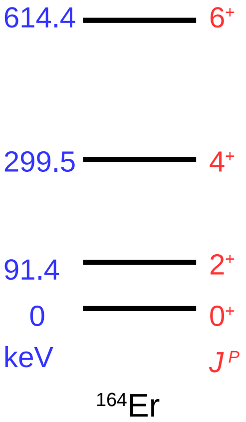
Energy of a rotating nucleus
$$E = \frac{\hbar^2}{2I_{\text{eff}}} J(J+1)$$

where I_{eff} is the effective moment of inertia.

Nuclear Rotations

Energies of rotational excitations are not predicted, but we can predict the ratios

e.g.



Predict $\dfrac{E(4^+)}{E(2^+)} = \dfrac{4(4+1)}{2(2+1)} = 3.33$

Observe $\dfrac{E(4^+)}{E(2^+)} = \dfrac{299.5}{91.4} = 3.28$

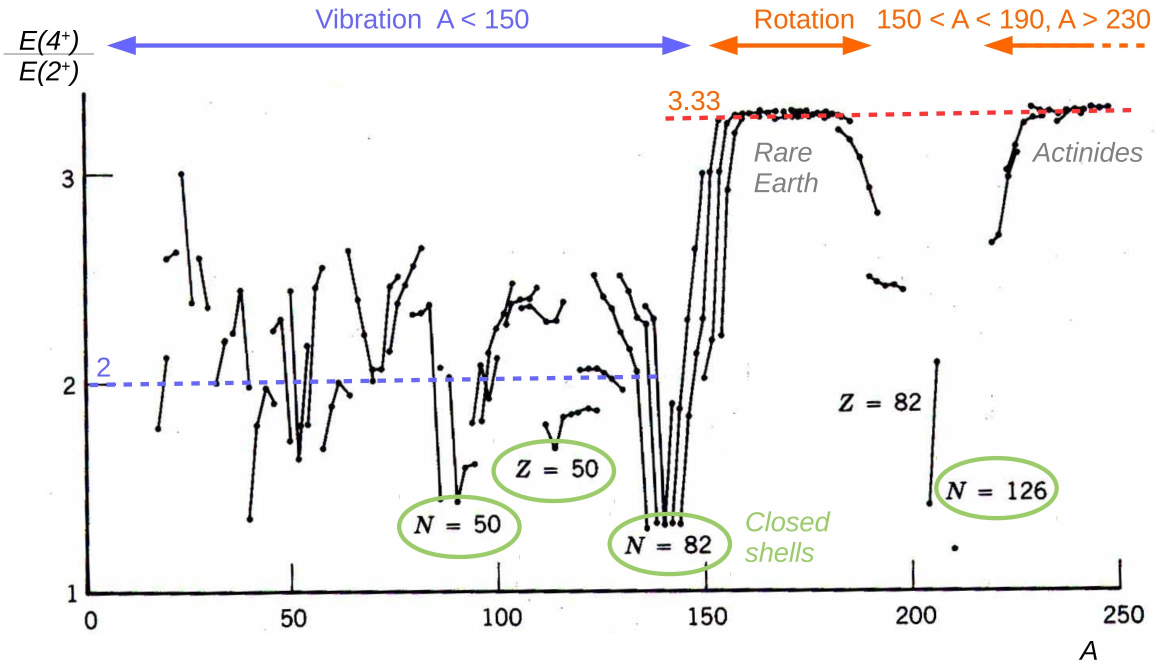
Deduce I_{eff} from the absolute energies; it is found that $I_{\text{rigid}} > I_{\text{eff}} > I_{\text{fluid}}$

→ the nucleus does not rotate like a rigid body. Only some of its nucleons are in collective motion (presumably the outer ones).

Rotational behaviour is intermediate between the nucleus being tightly bonded and weakly bonded i.e. the strong force is not long range.

Nuclear Vibrations and Rotations

For even-even ground state nuclei, the ratio of excitation energies $\frac{E(4^+)}{E(2^+)}$ is a diagnostic of the type of excitation.



Summary

The Nuclear Shell Model is successful in predicting

- Origin of magic numbers
- Spins and parities of ground states
- Trend in magnetic moments
- Some excited states near closed shells, small excitations in odd A nuclei

In general, it is not good far from closed shells and for non-spherically symmetric potentials.

The collective properties of nuclei can be incorporated into the Nuclear Shell Model by replacing the spherically symmetric potential by a deformed potential.
Improved description for

- Even A excited states
- Electric quadrupole and magnetic dipole moments.

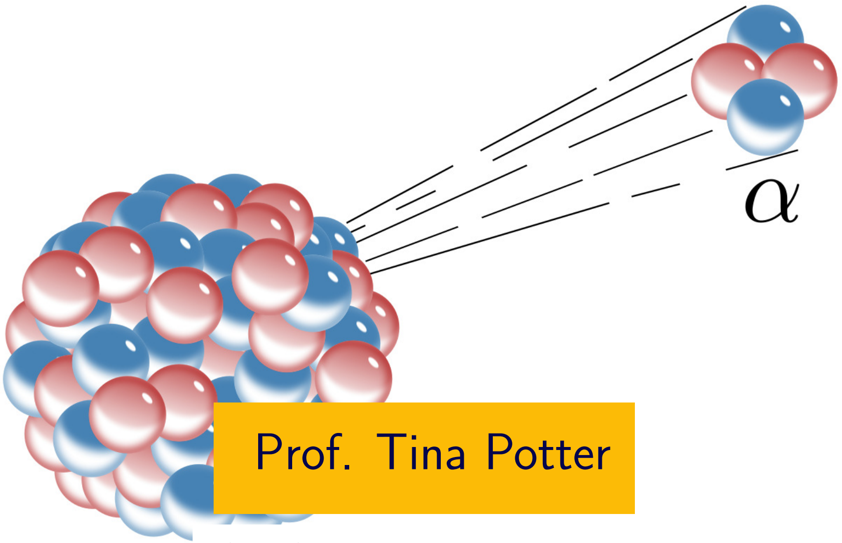
Many more sophisticated models exist (see Cont. Physics 1994 vol. 35 No. 5 329 http://www.tandfonline.com/doi/pdf/10.1080/00107519408222099)

Problem Sheet: q.34-36

Up next... Section 15: Nuclear Decays

15. Nuclear Decay

Particle and Nuclear Physics



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In this section...

- Radioactive decays
- Radioactive dating
- α decay
- β decay
- γ decay

Radioactivity

Natural radioactivity: three main types α, β, γ, and in a few cases, spontaneous fission.

α decay $\quad {}^{4}_{2}\mathrm{He}$ nucleus emitted.

$${}^{A}_{Z}\mathrm{X} \rightarrow {}^{A-4}_{Z-2}\mathrm{Y} + {}^{4}_{2}\mathrm{He} \qquad \text{Occurs for } A \geq 210$$

For decay to occur, energy must be released $Q > 0$

$$Q = m_\mathrm{X} - m_\mathrm{Y} - m_\mathrm{He} = B_\mathrm{Y} + B_\mathrm{He} - B_\mathrm{X}$$

β decay $\quad$ emission of electron e^- or positron e^+

$n \rightarrow p + e^- + \bar{\nu}_e \qquad {}^{A}_{Z}\mathrm{X} \rightarrow {}^{A}_{Z+1}\mathrm{Y} + e^- + \bar{\nu}_e \qquad \beta^-$ decay

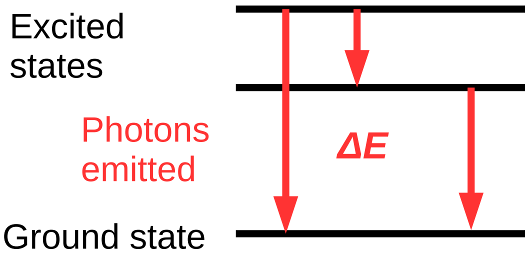
$p \rightarrow n + e^+ + \nu_e \qquad {}^{A}_{Z}\mathrm{X} \rightarrow {}^{A}_{Z-1}\mathrm{Y} + e^+ + \nu_e \qquad \beta^+$ decay

$p + e^- \rightarrow n + \nu_e \qquad {}^{A}_{Z}\mathrm{X} + e^- \rightarrow {}^{A}_{Z-1}\mathrm{Y} + \nu_e \qquad$ Electron capture

n.b. of these processes, only $n \rightarrow pe\nu$ can occur outside a nucleus.

Radioactivity

γ decay $\quad$ Nuclei in excited states can decay by emission of a photon γ. Often follows α or β decay.



	ΔE	λ	
Atom	~ 10 eV	$\sim 10^{-7}$ m	optical
	~ 10 keV	$\sim 10^{-10}$ m	X-ray
Nucleus	$\sim$ MeV	$\sim 10^{-12}$ m	γ-ray

A variant of γ decay is Internal Conversion:

- an excited nucleus loses energy by emitting a virtual photon,
- the photon is absorbed by an atomic e^-, which is then ejected
- n.b. not β decay, as nucleus composition is unchanged (e^- not from nucleus)

Natural Radioactivity

The half-life, $\tau_{1/2}$, is the time over which 50% of the nuclei decay

$$\tau_{1/2} = \frac{\ln 2}{\lambda} = 0.693\tau$$

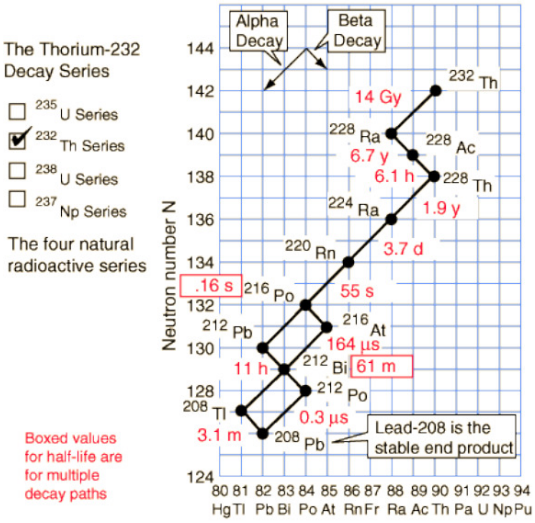
λ Transition rate
τ Average lifetime

Some $\tau_{1/2}$ values may be long compared to the age of the Earth.

Series Name	**Type**	**Final Nucleus (stable)**	**Longest-lived Nucleus**	$\tau_{1/2}$ **(years)**
Thorium	4n	^{208}Pb	^{232}Th	1.41×10^{10}
Neptunium	4n+1	^{209}Bi	^{237}Np	2.14×10^{6}
Uranium	4n+2	^{206}Pb	^{238}U	4.47×10^{9}
Actinium	4n+3	^{207}Pb	^{235}U	7.04×10^{8}

n is an integer

4n series



Radioactive Dating *Geological Dating*

Can use β^- decay to age the Earth, $^{87}\text{Rb} \rightarrow {}^{87}\text{Sr}$ $(\tau_{1/2} = 4.8 \times 10^{10}$ years)

$$\underset{N_1}{^{87}\text{Rb}} \rightarrow \underset{N_2}{^{87}\text{Sr}}$$

^{87}Sr is stable $\rightarrow \lambda_2 = 0$

So in this case, we have (using expressions from Chapter 2)

$$N_2(t) = N_1(0)\left[1 - e^{-\lambda_1 t}\right] + N_2(0) = N_1(t)\left[e^{\lambda_1 t} - 1\right] + N_2(0)$$

Assume we know λ_1, and can measure $N_1(t)$ and $N_2(t)$ e.g. chemically.
But we don't know $N_2(0)$.

Solution is to normalise to another (stable) isotope – ^{86}Sr – for which number is $N_0(t) = N_0(0)$.

$$\frac{N_2(t)}{N_0} = \frac{N_1(t)}{N_0}\left[e^{\lambda_1 t} - 1\right] + \frac{N_2(0)}{N_0}$$

Method: plot $N_2(t)/N_0$ vs $N_1(t)/N_0$ for lots of minerals.
Gradient gives $\left[e^{\lambda_1 t} - 1\right]$ and hence t.
Intercept $= N_2(0)/N_0$, which should be the same for all minerals (determined by chemistry of formation).

Radioactive Dating *Dating the Earth*

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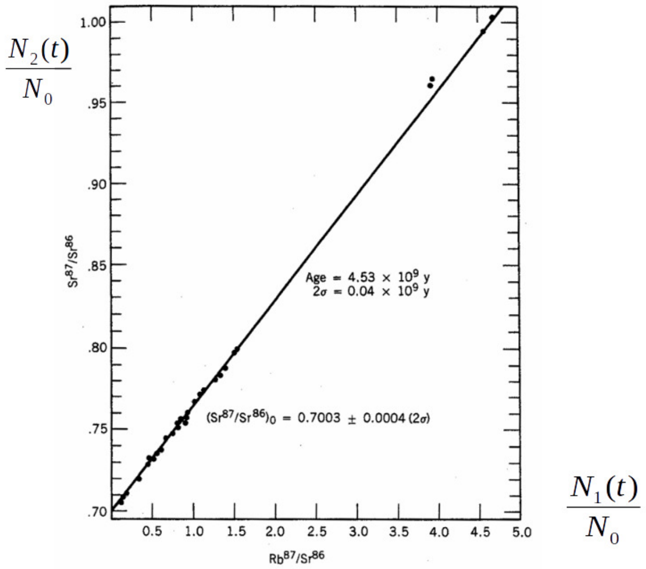


Using minerals from the Earth, Moon and meteorites.

Intercept gives $N_2(0)/N_0 = 0.70$

Slope gives the age of the Earth $= 4.5 \times 10^9$ yrs

Radioactive Dating *Radio-Carbon Dating*

For recent organic matter, use ^{14}C dating

Continuously formed in the upper atmosphere at approx. constant rate.
$^{14}N + n \rightarrow {}^{14}C + p$

Atmospheric carbon continuously exchanged with living organisms.
Equilibrium: 1 atom of ^{14}C to every 10^{12} atoms of other carbon isotopes
(98.9% ^{12}C, 1.1% ^{13}C)

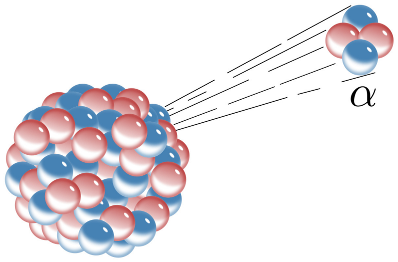
Undergoes β^- decay
$^{14}C \rightarrow {}^{14}N + e^- + \bar{\nu}_e$ $\tau_{1/2} = 5730$ yrs

No more ^{14}C intake for dead organisms.
Fresh organic material
~11 decays/minute/gram of carbon.

Measure the **specific activity** of material to obtain age, i.e. number of decays per second per unit mass

Complications for the future!
Burning of fossil fuels increases ^{12}C in atmosphere,
Nuclear bomb testing (adds ^{14}C to atmosphere)

α Decay



- α decay is due to the emission of a $^4_2$He nucleus.
- $^4_2$He is doubly magic and very tightly bound.
- α decay is energetically favourable for almost all with A≥190 and for many A≥150.

Why α rather than any other nucleus?
Consider energy release (Q) in various possible decays of ^{232}U

	n	p	^{2}H	^{3}H	^{3}He	^{4}He	^{5}He	^{6}Li	^{7}Li
Q/MeV	-7.26	-6.12	-10.70	-10.24	-9.92	+5.41	-2.59	-3.79	-1.94

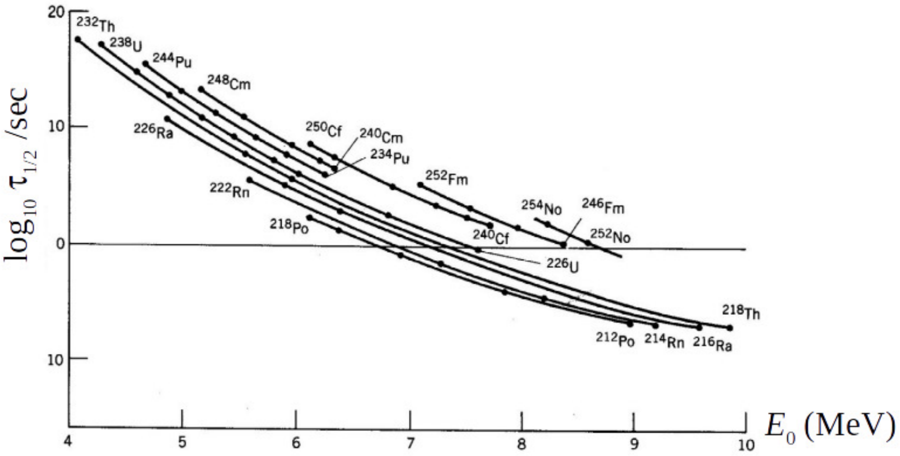
α is easy to form inside a nucleus $2p \uparrow\downarrow + 2n \uparrow\downarrow$
(though the extent to which α particles really exist inside a nucleus is still debatable)

α Decay *Dependence of $\tau_{1/2}$ on E_0*

(Geiger and Nuttall 1911)

A very striking feature of α decay is the strong dependence of lifetime on E_0

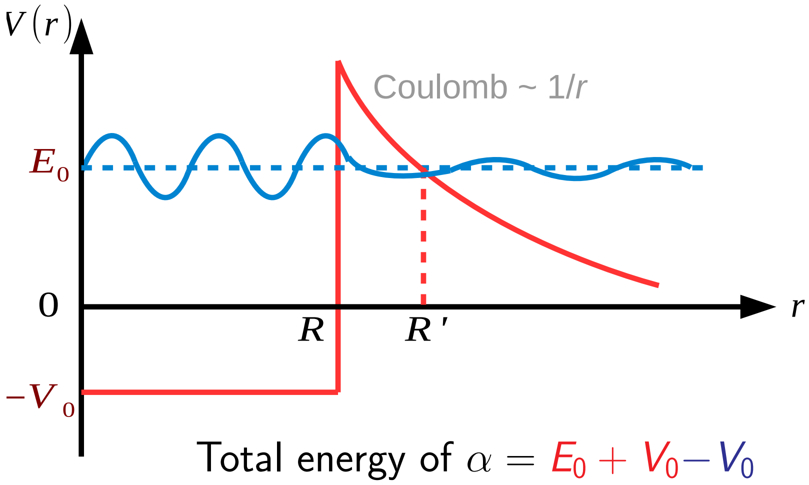
Example ^{232}Th $E_0 = 4.08\ \mathrm{MeV}$ $\tau_{1/2} = 1.4 \times 10^{10}$ yrs
^{218}Th $E_0 = 9.85\ \mathrm{MeV}$ $\tau_{1/2} = 1.0 \times 10^{-7}$ s
A factor of ~2.5 in E_0 ⇒ factor 10^{24} in $\tau_{1/2}$!



e.g. even N, even Z nuclei for a given Z see smooth trend ($\tau_{1/2}$ increases as Z does)

α Decay *Quantum Mechanical Tunnelling*

The nuclear potential for the α particle due to the daughter nucleus includes a Coulomb barrier which inhibits the decay.



Total energy of $\alpha = E_0 + V_0 - V_0$
K.E. P.E.

Classically, α particle cannot enter or escape from nucleus.
Quantum mechanically, α particle can penetrate the Coulomb barrier

$\Rightarrow$ Quantum Mechanical Tunnelling

α Decay *Simple Theory (Gamow, Gurney, Condon 1928)*

Assume α exists inside the nucleus and hits the barrier.

$$\alpha \text{ decay rate}, \quad \lambda = f\,P$$

f = escape trial frequency, P = probability of tunnelling through barrier

$$\text{semi} - \text{classically}, \quad f \sim v/2R$$

v= velocity of a particle inside nucleus, given by: $v^2 = (2E_\alpha/m_\alpha)$
and R = radius of nucleus

Typical values: $V_0 \sim 35\ \text{MeV}$, $E_0 \sim 5\ \text{MeV} \Rightarrow E_\alpha = 40\ \text{MeV}$ inside nucleus

$$f \sim \frac{v}{2R} = \frac{1}{2R}\sqrt{\frac{2E_\alpha}{m_\alpha}} \sim 10^{22}\,\text{s}^{-1}$$

$m_\alpha = 3.7\ \text{GeV}$
$R \sim 2.1$ fm

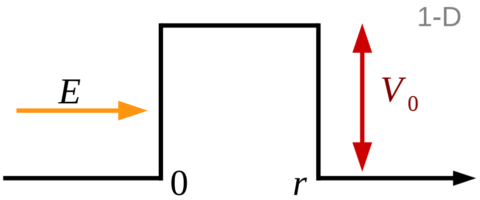
Obtain tunnelling probability, P, by solving Schrödinger equation in three regions and using boundary conditions.

α Decay *Simple Theory* *(Gamow, Gurney, Condon 1928)*

Transmission probability (1D square barrier):

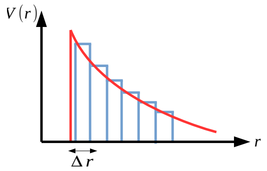
$$P = \left[1 + \frac{V_0^2}{4(V_0 - E)E}\sinh^2 ka\right]^{-1}$$

$$\frac{\hbar^2 k^2}{2m} = V_0 - E \qquad m = \text{reduced mass}$$



For $ka \gg 1$, P is dominated by the exp. decay within barrier $\Rightarrow P \sim e^{-2ka}$.

Coulomb potential, $V \propto 1/r$, and thus k varies with r.
Divide into rectangular pieces and multiply together exponentials, i.e. sum exponents.



Probability to tunnel through Coulomb barrier

$$P = \prod_i e^{-2k_i \Delta R} = e^{-2G} \qquad k = \frac{[2m_\alpha(V(r) - E_0)]^{1/2}}{\hbar}$$

The Gamow Factor $\quad G = \int_R^{R'} \frac{[2m_\alpha(V(r) - E_0)]^{1/2}}{\hbar}\,\mathrm{d}r = \int_R^{R'} k(r)\,\mathrm{d}r$

α Decay *Simple Theory* *(Gamow, Gurney, Condon 1928)*

For $r > R$, $\quad V(r) = \frac{Z_\alpha Z' e^2}{4\pi\epsilon_0 r} = \frac{B}{r} \qquad Z' = Z - Z_\alpha \quad (Z_\alpha = 2)$

α-particle escapes at $r = R'$, $V(R') = E_0 \Rightarrow R' = B/E_0$

$$\therefore G = \int_R^{R'} \left(\frac{2m_\alpha}{\hbar^2}\right)^{1/2} \left[\frac{B}{r} - E_0\right]^{1/2} \mathrm{d}r = \left(\frac{2m_\alpha B}{\hbar^2}\right)^{1/2} \int_R^{R'} \left[\frac{1}{r} - \frac{1}{R'}\right]^{1/2} \mathrm{d}r$$

See Appendix H

$$G = \left(\frac{2m_\alpha}{E_0}\right)^{1/2} \frac{B}{\hbar} \left[\cos^{-1}\left(\frac{R}{R'}\right)^{1/2} - \left\{\left(1 - \frac{R}{R'}\right)\left(\frac{R}{R'}\right)\right\}^{1/2}\right]$$

To perform integration, substitute $r = R' \cos^2\theta$

In most practical cases $R \ll R'$, so term in [...] $\sim \pi/2$

$$G \sim \left(\frac{2m_\alpha}{E_0}\right)^{1/2} \frac{B}{\hbar}\frac{\pi}{2} \qquad B = \frac{Z_\alpha Z' e^2}{4\pi\epsilon_0}$$

e.g. typical values: $Z = 90$, $E_0 \sim 6\ \mathrm{MeV} \Rightarrow R' \sim 40$ fm $\gg R$

$$G \sim Z' \left(\frac{3.9\ \mathrm{MeV}}{E_0}\right)^{1/2}$$

α Decay *Simple Theory (Gamow, Gurney, Condon 1928)*

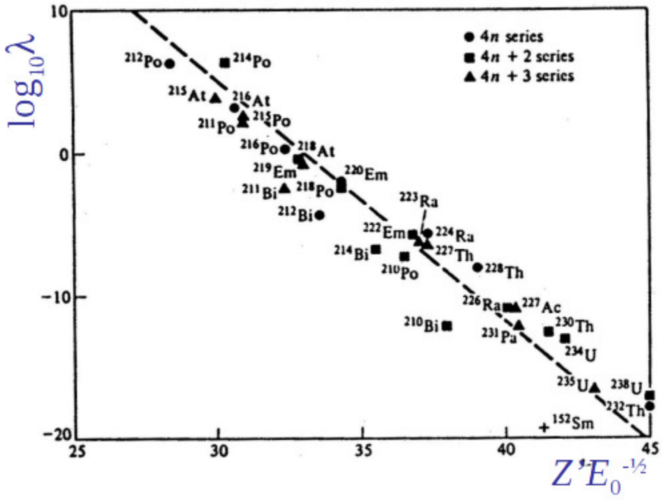
Lifetime $\tau = \frac{1}{\lambda} = \frac{1}{fP} \sim \frac{2R}{v} e^{2G}$

$$\Rightarrow \quad \ln\tau \sim 2G + \ln\frac{2R}{v}$$

$$\ln\lambda \sim -\frac{Z'}{E_0^{1/2}} + \text{constant}$$

Geiger-Nuttall Law

Not perfect, but provides an explanation of the dominant trend of the data



Simple tunnelling model accounts for

- strong dependence of $\tau_{1/2}$ on E_0
- $\tau_{1/2}$ increases with Z
- disfavoured decay to heavier fragments e.g. ^{12}C

$$G \propto m^{1/2} \quad \text{and} \quad G \propto \text{charge of fragment}$$

α Decay *Simple Theory (Gamow, Gurney, Condon 1928)*

Deficiencies/complications with simple tunnelling model:

- Assumed existence of a single α particle in nucleus and have taken no account of probability of formation.
- Assumed "semi-classical" approach to estimate escape trial frequency, $f \sim v/2R$, and make absolute prediction of decay rate.
- If α is emitted with some angular momentum, L, the radial wave equation must include a centrifugal barrier term in Schrödinger equation

$$V' = \frac{L(L+1)\hbar^2}{2\mu r^2}$$

L = relative a.m. of α and daughter nucleus
μ = reduced mass

which raises the barrier and suppresses emission of α in in high L states.

α Decay *Selection rules*

Nuclear Shell Model: α has $J^P = 0^+$

Angular momentum

e.g. $X \rightarrow Y + \alpha$ Conserve J: $J_X = J_Y \oplus J_\alpha = J_Y \oplus L_\alpha$

L_α can take values from $J_X + J_Y$ to $|J_X - J_Y|$

Parity

Parity is conserved in α decay (strong force).

Orbital wavefunction has $P = (-1)^L$

X, Y same parity $\Rightarrow L_\alpha$ must be even

X, Y opposite parity $\Rightarrow L_\alpha$ must be odd

e.g. if X, Y are both even-even nuclei in their ground states, shell model predicts both have $J^P = 0^+ \;\Rightarrow\; L_\alpha = 0$.

More generally, if X has $J^P = 0^+$, the states of Y which can be formed in α decay are $J^P = 0^+, 1^-, 2^+, 3^-, 4^+$, ...

β Decay

β^-	$n \rightarrow p + e^- + \bar{\nu}_e$	${}^A_Z\mathrm{X} \rightarrow {}^{A}_{Z+1}\mathrm{Y} + e^- + \bar{\nu}_e$
β^+	$p \rightarrow n + e^+ + \nu_e$	${}^A_Z\mathrm{X} \rightarrow {}^{A}_{Z-1}\mathrm{Y} + e^+ + \nu_e$
electron capture	$p + e^- \rightarrow n + \nu_e$	${}^A_Z\mathrm{X} + e^- \rightarrow {}^{A}_{Z-1}\mathrm{Y} + \nu_e$

- β decay is a weak interaction mediated by the W boson.
- Parity is violated in β decay.
- Responsible for Fermi postulating the existence of the neutrino.
- Kinematics: Decay is possible if energy release $E_0 > 0$

	Nuclear Masses	*Atomic Masses*
β^-	$E_0 = m_X - m_Y - m_e - m_\nu$	$E_0 = M_X - M_Y - m_\nu$
β^+	$E_0 = m_X - m_Y - m_e - m_\nu$	$E_0 = M_X - M_Y - 2m_e - m_\nu$
e.c.	$E_0 = m_X - m_Y + m_e - m_\nu$	$E_0 = M_X - M_Y - m_\nu$
	(and note that $m_\nu \sim 0$)	using $M(A,Z) = m(A,Z) + Zm_e$

n.b. electron capture may be possible even if β^+ not allowed

β Decay *Nuclear stability against β decay*

Consider nuclear mass as a function of N and Z

$$m(A,Z) = Zm_p + (A-Z)m_n - a_V A + a_S A^{2/3} + \frac{a_C Z^2}{A^{1/3}} + a_A \frac{(N-Z)^2}{A} - \delta(A)$$

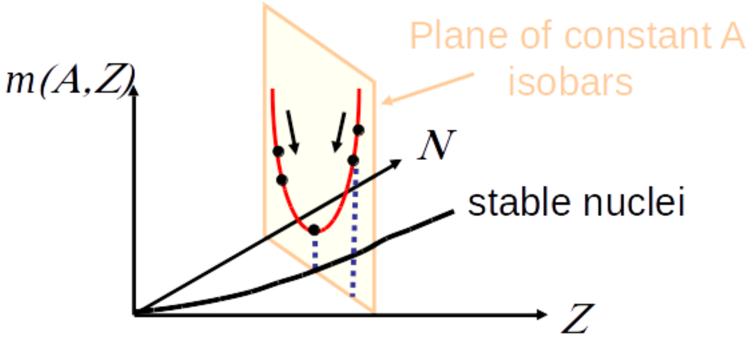
using SEMF

For β decay, A is constant,

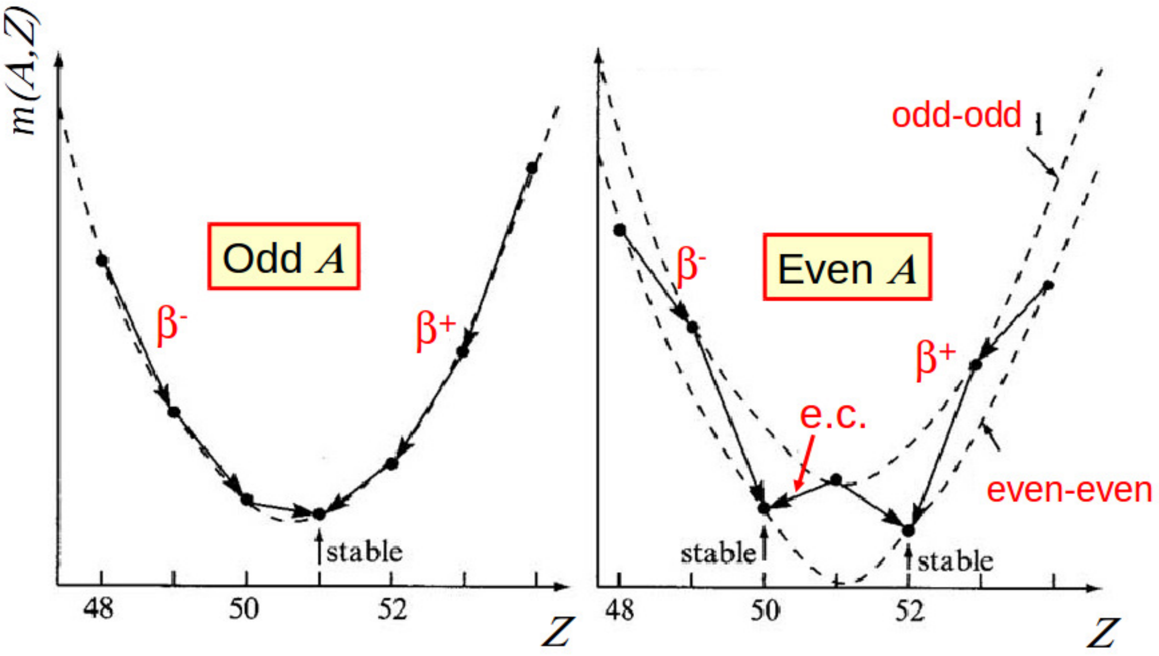
but Z changes by ±1 and $m(A,Z)$ is quadratic in Z

Most stable nuclide when

$$\left[\frac{\partial m(A,Z)}{\partial Z}\right]_A = 0$$



β Decay *Typical situation at constant A*



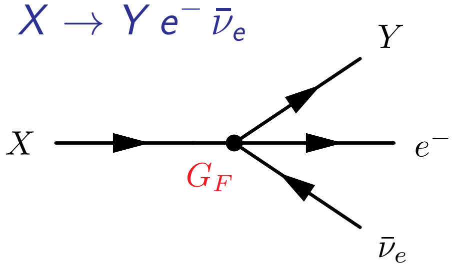
Usually only one isotope table against β-decay; occasionally two.

Typically two even-even nuclides are stable against β-decay; almost no odd-odd ones (pairing term).

Fermi Theory of β-decay

In nuclear decay, weak interaction taken to be a 4-fermion contact interaction:

$X \to Y e^- \bar{\nu}_e$



No "propagator" – absorb the effect of the exchanged W boson into an effective coupling strength given by the Fermi constant $G_F = 1.166 \times 10^{-5}\ \mathrm{GeV}^{-2}$.

Use Fermi's Golden Rule to get the transition rate $\Gamma = 2\pi |M_{\mathrm{fi}}|^2 \rho(E_{\mathrm{f}})$

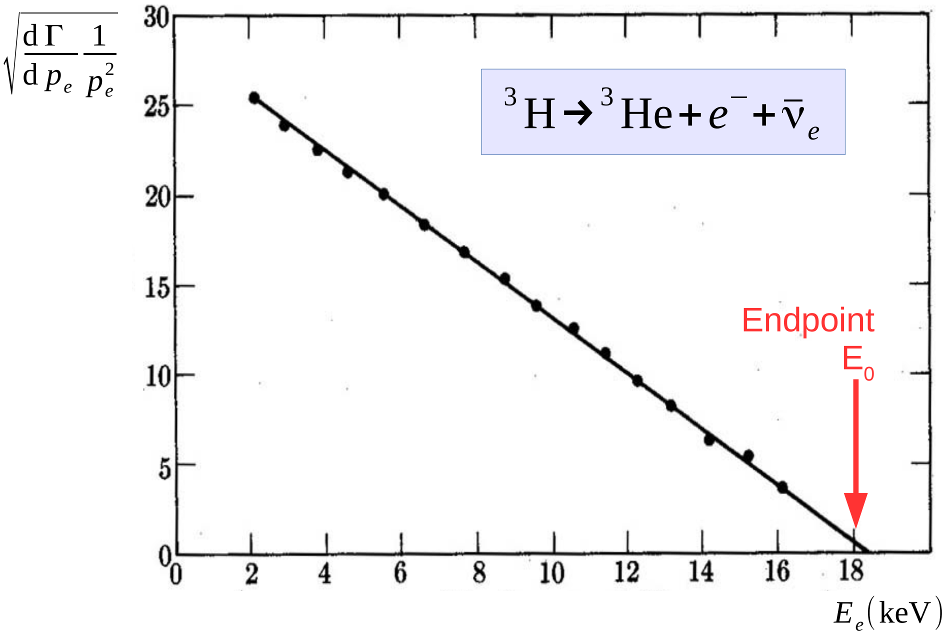
where M_{fi} is the matrix element and $\rho(E_{\mathrm{f}}) = \frac{\mathrm{d}N}{\mathrm{d}E_{\mathrm{f}}}$ is the density of final states.

$$\Gamma = \frac{G_F^2\, |M_{\mathrm{nuclear}}|^2}{2\pi^3} \int_0^{E_0} (E_0 - E_e)^2 E_e^2\, \mathrm{d}E_e$$

Total decay rate given by Sargent's Rule, $\Gamma \propto E_0^5$

Fermi Theory of β-decay

β decay spectrum described by $\sqrt{\frac{\mathrm{d}\Gamma}{\mathrm{d}p_e}\frac{1}{p_e^2}} \propto (E_0 - E_e)$ Kurie Plot



Fermi Theory of β-decay

BUT, the momentum of the electron is modified by the Coulomb interaction as it moves away from the nucleus (different for e^- and e^+).

$\Rightarrow$ Multiply spectrum by Fermi function $F(Z_Y, E_e)$

$$\Gamma = \frac{G_F^2\,|M_{\text{nuclear}}|^2}{2\pi^3}\int_0^{E_0}(E_0 - E_e)^2 E_e^2\, F(Z_Y, E_e)\,\mathrm{d}E_e$$

All the information about the nuclear wavefunctions is contained in the matrix element. Values for the complicated Fermi Integral are tabulated.

$$f(Z_Y, E_0) = \frac{1}{m_e^5}\int_0^{E_0}(E_0 - E_e)^2 E_e^2\, F(Z_Y, E_e)\,\mathrm{d}E_e$$

Mean lifetime $\tau = 1/\Gamma$, half-life $\tau_{1/2} = \frac{\ln 2}{\Gamma}$

$$f\tau_{1/2} = \ln 2 \frac{2\pi^3}{m_e^5 G_F^2\,|M_{\text{nuclear}}|^2}$$

Comparative half-life

this is rather useful because it depends only on the nuclear matrix element

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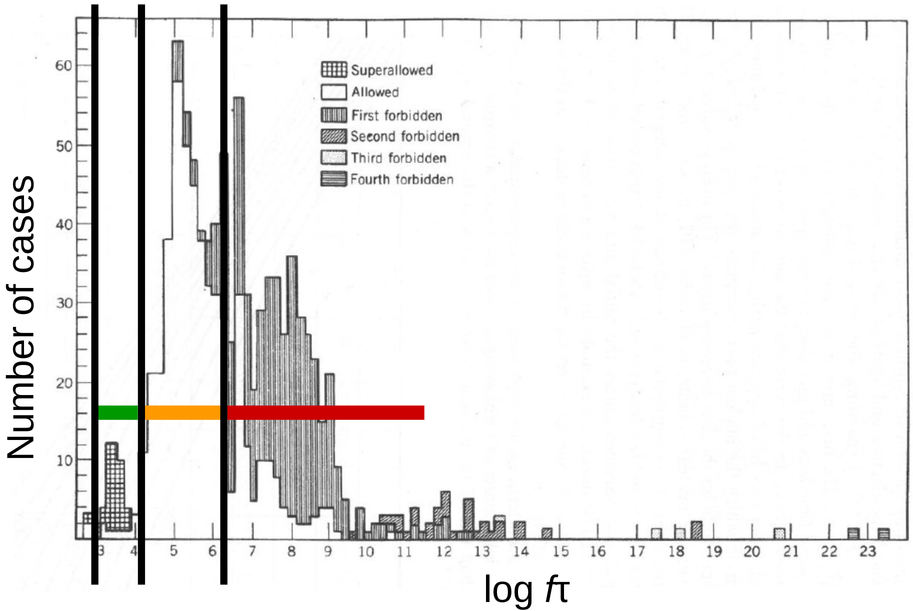
Fermi Theory of β-decay *Comparative half-lives*



In rough terms, decays with

$\log f\tau_{1/2}$ $\sim 3-4$ known as **super-allowed**

$\sim 4-7$ known as **allowed**

≥ 6 known as **forbidden** (i.e. suppressed, small M_{if})

Fermi Theory of β-decay *Selection Rules*

Fermi theory
$$M_{\mathrm{fi}} = G_F \int \psi_p^* \, \mathrm{e}^{-\mathrm{i}(\vec{p}_e+\vec{p}_\nu).\vec{r}} \, \psi_n \, \mathrm{d}^3\vec{r}$$

e, ν wavefunctions

Allowed Transitions $\log_{10} f\tau_{1/2} \sim 4-7$

Angular momentum of $e\nu$ pair relative to nucleus, $L = 0$.

Equivalent to: $\mathrm{e}^{-\mathrm{i}(\vec{p}_e+\vec{p}_\nu).\vec{r}} \sim 1$

Superallowed Transitions $\log_{10} f\tau_{1/2} \sim 3-4$

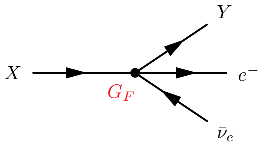
subset of Allowed transitions: often **mirror nuclei** in which p and n have approximately the same wavefunction
$$M_{\mathrm{nuclear}} \sim \int \psi_p^* \psi_n \, \mathrm{d}^3\vec{r} \sim 1$$

e, ν both have spin 1/2 $\Rightarrow$ Total spin of $e\nu$ system can be $S_{e\nu} = 0$ or 1.

There are two types of allowed/superallowed transitions depending on the relative spin states of the emitted e and ν...

Fermi Theory of β-decay *Selection Rules*

For allowed/superallowed transitions, $L_{e\nu} = 0$



$X \to Y + e + \nu$

$J_X = J_Y \oplus S_{e\nu} \oplus L_{e\nu}$

e.g. $n \to pe^-\bar{\nu}_e$
4 spin states of $e\nu$
(3 G-T, 1 Fermi)

$S_{e\nu} = 0$ Fermi transitions

$$n\uparrow \;\to\; p\uparrow \;+\; \frac{1}{\sqrt{2}}\left[(e^-\uparrow\bar{\nu}_e\downarrow) - (e^-\downarrow\bar{\nu}_e\uparrow)\right] \qquad \Delta J = 0$$

$S_{e\nu} = 0, m_s = 0$ $\qquad J_X = J_Y$

$S_{e\nu} = 1$ Gamow-Teller transitions

$$n\uparrow \;\to\; p\uparrow \;+\; \frac{1}{\sqrt{2}}\left[(e^-\uparrow\bar{\nu}_e\downarrow) + (e^-\downarrow\bar{\nu}_e\uparrow)\right] \qquad \Delta J = 0$$

$0 \to 0$ forbidden

$S_{e\nu} = 1, m_s = 0$ $\qquad J_X = J_Y$

$$n\uparrow \;\to\; p\downarrow \;+\; e^-\uparrow \;+\; \bar{\nu}_e\uparrow \qquad \Delta J = \pm 1$$

$S_{e\nu} = 1, m_s = \pm 1$ $\qquad J_X = J_Y \pm 1$

No change in angular momentum of the $e\nu$ pair relative to the nucleus, $L_{e\nu} = 0$

$\Rightarrow$ Parity of nucleus unchanged

Fermi Theory of β-decay *Selection Rules*

Forbidden Transitions $\log_{10} f\tau_{1/2} \geq 6$

Angular momentum of $e\nu$ pair relative to nucleus, $L_{e\nu} > 0$.

$$e^{-i(\vec{p}_e+\vec{p}_\nu).\vec{r}} = 1 - i(\vec{p}_e+\vec{p}_\nu).\vec{r} + \frac{1}{2}[(\vec{p}_e+\vec{p}_\nu).\vec{r}]^2 - \ldots$$

$L =$	0	1	2
$P = (-1)^L =$	even	odd	even
	Allowed	1st forbidden	2nd forbidden

Transition probabilities for $L > 0$ are small ⇒ forbidden transitions (really means "suppressed").

Forbidden transitions are only competitive if an allowed transition cannot occur (selection rules). Then the lowest permitted order of "forbiddeness" will dominate.

In general, n^{th} forbidden ⇒ $e\nu$ system carries orbital angular momentum $L = n$, and $S_{e\nu} = 0$ (Fermi) or 1 (G-T). Parity change if L is odd.

Fermi Theory of β-decay *Selection Rules*

Examples

$^{34}\text{Cl}(0^+) \rightarrow {}^{34}\text{S}(0^+)$

$^{14}\text{C}(0^+) \rightarrow {}^{14}\text{N}(1^+)$

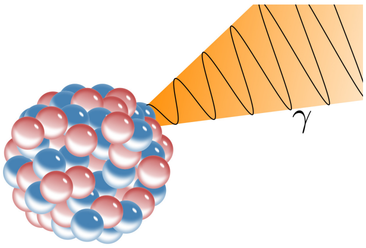
$n(1/2^+) \rightarrow p(1/2^+)$

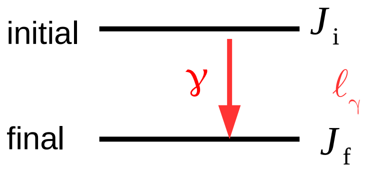
$^{39}\text{Ar}(7/2^-) \rightarrow {}^{39}\text{K}(3/2^+)$

$^{87}\text{Rb}(3/2^-) \rightarrow {}^{87}\text{Sr}(9/2^+)$

γ Decay

Emission of γ-rays (EM radiation) occurs when a nucleus is created in an excited state (e.g. following α, β decay or collision).





The photon carries away net angular momentum L_γ when a proton in the nucleus makes a transition from its initial a.m. state J_{i} to its final a.m. state J_{f}.

$$\vec{J_{\text{i}}} = \vec{L_\gamma} \oplus \vec{J_{\text{f}}} \quad \text{and} \quad |\vec{J_{\text{i}}} - \vec{J_{\text{f}}}| \;\leq\; L_\gamma \;\leq\; |\vec{J_{\text{i}}} + \vec{J_{\text{f}}}|$$

The photon carries $J^P = 1^-$ $\quad \Rightarrow L_\gamma \geq 1$.

$\Rightarrow$ Single γ emission is forbidden for a transition between two $J = 0$ states.

(0 $\rightarrow$ 0 transitions can only occur via internal conversion (emitting an electron) or via the emission of more than one γ.)

γ Decay

Radiative transitions in nuclei are generally the same as for atoms, except

Atom $\quad E_\gamma \sim \text{ eV}$; $\quad \lambda \sim 10^8\,\text{fm} \sim 10^3 \times r_{\text{atom}}$; $\quad \Gamma \sim 10^9\,\text{s}^{-1}$

Only dipole transitions are important.

Nuclei $\quad E_\gamma \sim \text{ MeV}$; $\quad \lambda \sim 10^2\,\text{fm} \sim 25 \times r_{\text{nucl}}$; $\quad \Gamma \sim 10^{16}\,\text{s}^{-1}$

Collective motion of many protons lead to higher transition rates.

$\Rightarrow$ Higher order transitions are also important.

Two types of transitions:

Electric (E) transitions arise from an oscillating charge which causes an oscillation in the external electric field.

Magnetic (M) transitions arise from a varying current or magnetic moment which sets up a varying magnetic field.

Obtain transition probabilities using Fermi's Golden Rule

$$\Gamma = 2\pi |M_{\text{if}}|^2 \rho(E_{\text{f}})$$

γ Decay *Electric Dipole Transitions (E1)* $L = 1$

Insert dipole matrix element into FGR $\Gamma_{\mathrm{i}\to\mathrm{f}} = \dfrac{\omega^3}{3\pi\epsilon_0 c^3\hbar} \, | \, \langle\psi_\mathrm{f}|e\vec{r}|\psi_\mathrm{i}\rangle \, |^2$

see Adv. Quantum Physics; after averaging over initial and summing over final states

Order of magnitude estimate of this rate,

$$| \, \langle\psi_\mathrm{f}|e\vec{r}|\psi_\mathrm{i}\rangle \, |^2 \sim |eR|^2 \Rightarrow \Gamma \sim \frac{4}{3}\alpha E_\gamma^3 R^2$$

R = radius of nucleus,
$\alpha = \frac{e^2}{4\pi\epsilon_0 c\hbar}$, $E_\gamma = \hbar\omega$, $\hbar = c = 1$.

e.g. $E_\gamma = 1$ MeV, $R = 5$ fm ($\hbar c = 197$ MeVfm, $\hbar = 6.6\times 10^{-22}$ MeVs)

$$\Gamma(E1) = 0.24\ \mathrm{MeV^3 fm^2} = \frac{0.24}{(197)^2 \times 6.6\times 10^{-22}}\,\mathrm{s^{-1}} = 10^{16}\,\mathrm{s^{-1}}$$ (c.f. atoms $\Gamma \sim 10^9\mathrm{s^{-1}}$)

As nuclear wavefunctions have definite parity, the matrix element can only be non-zero if the initial and final states have opposite parity.

$$e\vec{r} \overset{\hat{P}}{\rightarrow} -e\vec{r} \qquad \text{ODD}$$

E1 transition ⇒ parity change of nucleus

γ Decay *Magnetic Dipole Transitions (M1)* $L = 1$

Magnetic dipole matrix element $| \, \langle\psi_\mathrm{f}|\mu\vec{\sigma}|\psi_\mathrm{i}\rangle \, |^2$

μ = magnetic moment, $\vec{\sigma}$ = Pauli spin matrices

Typically $\langle\mu\sigma\rangle \sim \dfrac{e\hbar}{2m_p} = \mu_N$ Nuclear magneton

For a proton $\dfrac{\hbar}{m_p} \sim 0.2\mathrm{fm} \sim \dfrac{R}{25}$ for $R = 5\,\mathrm{fm}$

Compare to E1 transition rate $\dfrac{\Gamma(M1)}{\Gamma(E1)} = \left(\dfrac{e\hbar}{2m_p}\right)^2 \dfrac{1}{(eR)^2} = 10^{-3}$

Magnetic moment transforms the same way as angular momentum

$$e\vec{r}\times\vec{p} \overset{\hat{P}}{\rightarrow} e(-\vec{r})\times(-\vec{p}) = e\vec{r}\times\vec{p} \qquad \text{EVEN}$$

M1 transition ⇒ no parity change of nucleus

γ Decay *Higher Order Transitions (EL, ML, where $L > 1$)*

If the initial and final nuclear states differ by more than 1 unit of angular momentum

$\Rightarrow$ higher multipole radiation

The perturbing Hamiltonian is a function of electric and magnetic fields and hence of the vector potential $\langle\psi_f|H'(\vec{A})|\psi_i\rangle$

$\vec{A}$ for a photon is taken to have the form of a plane wave

$$\vec{A}e^{i\vec{p}.\vec{r}} = 1 \qquad -i\vec{p}.\vec{r} \qquad +\frac{1}{2}(\vec{p}.\vec{r})^2 + \qquad \ldots\frac{(-i\vec{p}.\vec{r})^n}{n!}$$

	Dipole	Quadrupole	Octupole
$L =$	1	2	3
	E1,M1	E2,M2	E3,M3

Each successive term in the expansion of $\vec{A}$ is reduced from the previous one by a factor of roughly $\vec{p}.\vec{r}$.

e.g. Compare E1 to E2 for $p \sim 1$ MeV, $R \sim 5$fm

$\Rightarrow$ $pR \sim 5$ MeVfm ~ 0.025, $|pR|^2 \sim 10^{-3}$

$$\frac{\Gamma(E2)}{\Gamma(E1)} \sim 10^{-3} \sim \frac{\Gamma(M1)}{\Gamma(E1)}$$

The matrix element for E2 transitions $\sim r^2$ i.e. even under a parity transformation.

γ Decay *Transitions*

In general, EL transitions Parity $= (-1)^L$

ML transitions Parity $= (-1)^{L+1}$

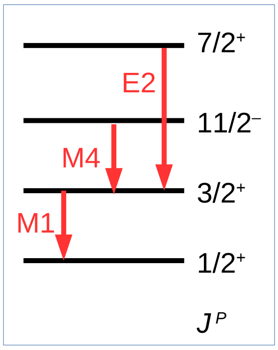
Rate	1	10^{-3}	10^{-6}	10^{-9} ...
	E1	E2	E3	E4 ...
		M1	M2	M3 ...
Parity change	✓	✗	✓	✗
J^P of γ E:	1^-	2^+	3^-	4^+
M:		1^+	2^-	3^+

In general, a decay will proceed dominantly by the lowest order (i.e. fastest) process permitted by angular momentum and parity.

e.g. if a process has $\Delta J = 2$, no parity change, it will go by the E2, even though M3, E4 are also allowed.

γ Decay *Transitions*

e.g. $^{117}_{50}$Sn



$3/2^+ \rightarrow 1/2^+$ M1 (E2 also allowed)

$11/2^- \rightarrow 3/2^+$ M4
More likely than $11/2^- \rightarrow 1/2^+$ (E5)

$7/2^+ \rightarrow 3/2^+$ E2
M2 } less likely $7/2^+ \rightarrow 11/2^-$
M3 } less likely $7/2^+ \rightarrow 1/2^+$

Information about the nature of transitions (based on rates and angular distributions) is very useful in inferring the J^P values of states.

Please note: this discussion of rates is fairly naïve. More complete formulae can be found in textbooks.
Also collective effects may be important if

- many nucleons participate in transitions,
- nucleus has a large electric quadrupole moment, Q, $\rightarrow$ rotational excited states enhance E2 transitions.

Summary

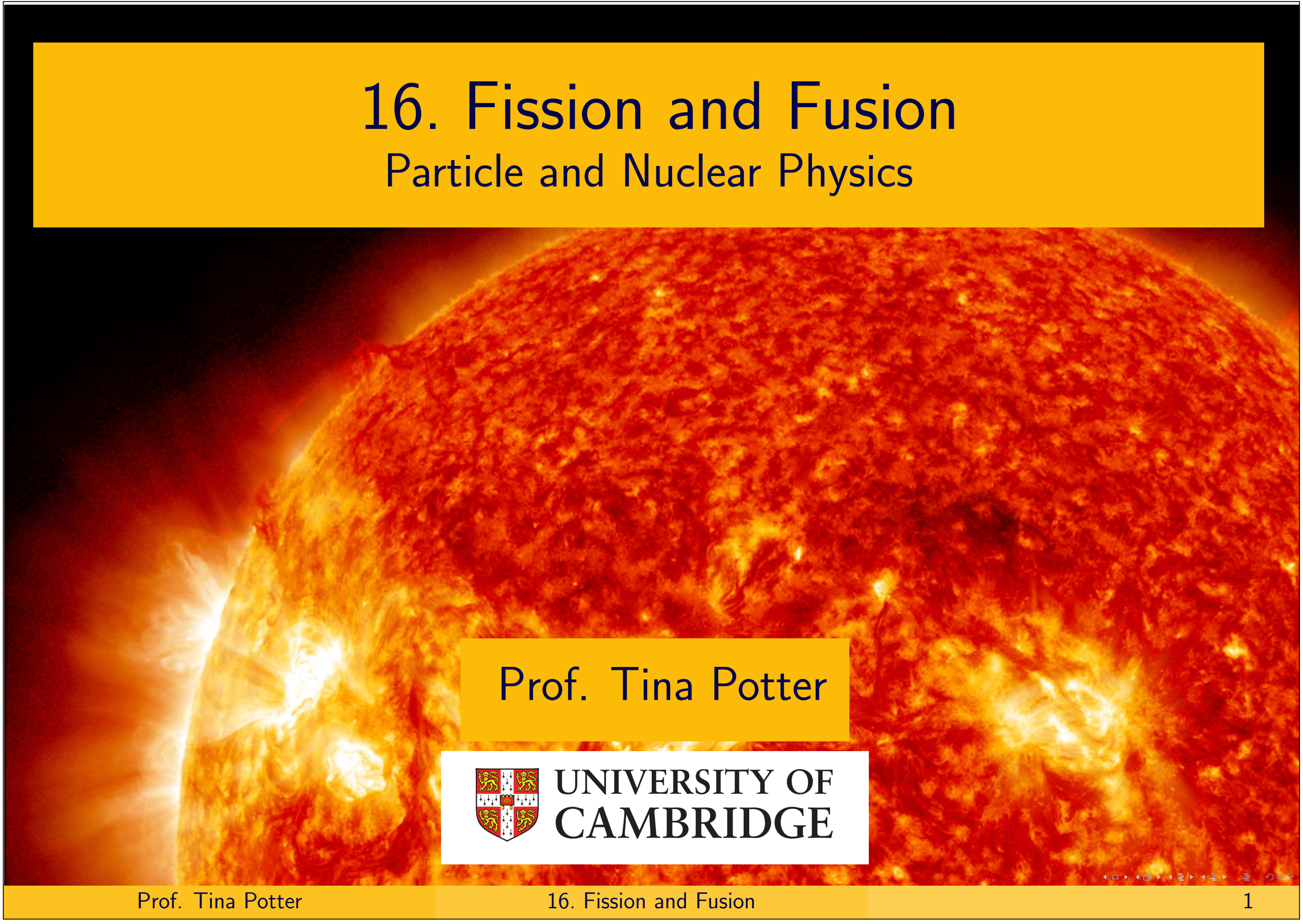
- Radioactive decays and dating.
- α-decay Strong dependence on E, Z
  Tunnelling model (Gamow) – Geiger-Nuttall law $\ln \tau_{1/2} \sim \frac{Z'}{E_0^{1/2}} + \text{const.}$
- β-decay β^+, β^-, electron capture; energetics, stability
  Fermi theory – 4-fermion interaction plus 3-body phase space.

$$\Gamma = \frac{G_F^2 \left|M_{\text{nuclear}}\right|^2}{2\pi^3} \int_0^{E_0} (E_0 - E_e)^2 p_e^2 \, \mathrm{d}p_e$$

  Electron energy spectrum; Kurie plot.
  Comparative half-lives.
  Selection rules; Fermi, Gamow-Teller; allowed, forbidden.
- γ-decay Dipole, quadrupole; electric, magnetic transitions.
  Selection rules.

Problem Sheet: q.37-41

Up next... Section 16: Fission and Fusion



16. Fission and Fusion

Particle and Nuclear Physics

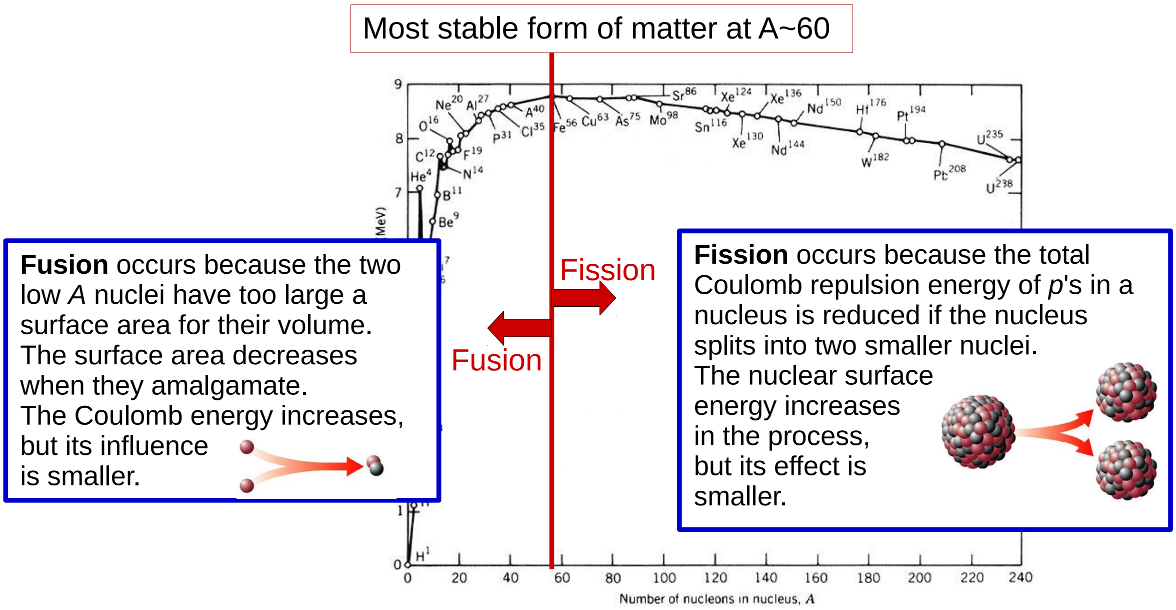
Prof. Tina Potter

UNIVERSITY OF CAMBRIDGE

In this section...

- Fission
- Reactors
- Fusion
- Nucleosynthesis
- Solar neutrinos

Fission and Fusion



Fusion occurs because the two low *A* nuclei have too large a surface area for their volume. The surface area decreases when they amalgamate. The Coulomb energy increases, but its influence is smaller.

Fission occurs because the total Coulomb repulsion energy of *p*'s in a nucleus is reduced if the nucleus splits into two smaller nuclei. The nuclear surface energy increases in the process, but its effect is smaller.

Expect a large amount of energy released in the fission of a heavy nucleus into two medium-sized nuclei or in the fusion of two light nuclei into a single medium nucleus.

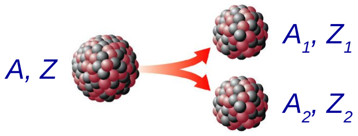
SEMF $$B(A,Z) = a_V A - a_S A^{2/3} - \frac{a_c Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} + \delta(A)$$

Spontaneous Fission

Expect spontaneous fission to occur if energy released

$$E_0 = B(A_1, Z_1) + B(A_2, Z_2) - B(A,Z) > 0$$

Assume nucleus divides as



where $\frac{A_1}{A} = \frac{Z_1}{Z} = y$ and $\frac{A_2}{A} = \frac{Z_2}{Z} = 1 - y$

from SEMF $$E_0 = a_S A^{2/3}(1 - y^{2/3} - (1-y)^{2/3}) + a_C \frac{Z^2}{A^{1/3}}(1 - y^{5/3} - (1-y)^{5/3})$$

maximum energy released when $\frac{\partial E_0}{\partial y} = 0$

$$\frac{\partial E_0}{\partial y} = a_S A^{2/3}(-\frac{2}{3}y^{-1/3} + \frac{2}{3}(1-y)^{-1/3}) + a_C \frac{Z^2}{A^{1/3}}(-\frac{5}{3}y^{2/3} + \frac{5}{3}(1-y)^{2/3}) = 0$$

solution $y = 1/2$ $\Rightarrow$ Symmetric fission

max. $E_0 = 0.37 a_C \frac{Z^2}{A^{1/3}} - 0.26 a_S A^{2/3}$

e.g. $^{238}_{92}$U: maximum $E_0 \sim 200$ MeV ($a_S = 18.0$ MeV, $a_C = 0.72$ MeV)

$\sim 10^6\times$ energy released in chemical reaction!

Fission Barrier

In the fission process, nuclei have to pass through an intermediate state where the surface energy is increased, but where the Coulomb energy is not yet much reduced.

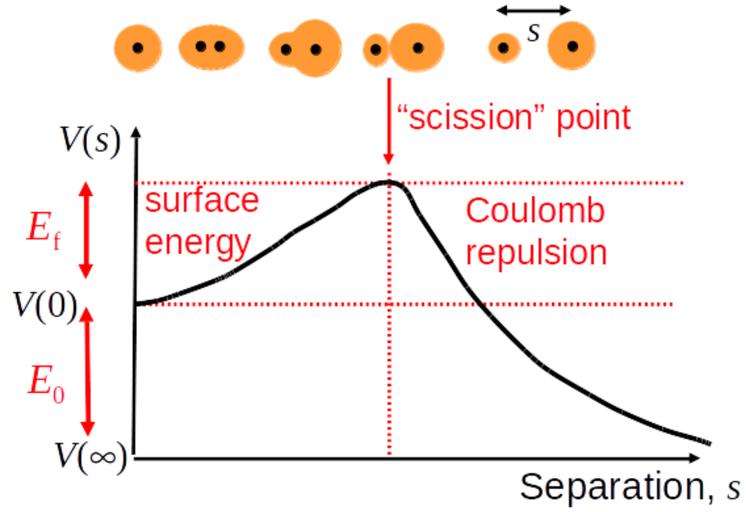
This is a tunnelling problem, similar to α decay.

E_f = fission activation energy

$E_f \sim 6$ MeV $^{236}_{92}$U

E_0 = energy released

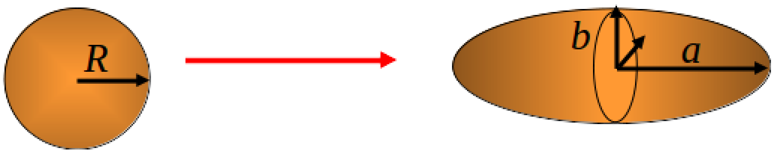
→ K.E. of fragments.



Although E_0 is maximal for symmetric fission, so is the Coulomb barrier.
In fact, asymmetric fission is the norm.

Fission Barrier

Estimate mass at which nuclei become unstable to fission (i.e. point at which energy change due to ellipsoidal deformation gives a change in binding energy, $\Delta B > 0$)



$$a = R(1+\epsilon) \qquad \epsilon \ll 1$$
$$b = R(1+\epsilon)^{-1/2}$$

SEMF Volume term unchanged: $\text{Volume} = \text{const} = \frac{4}{3}\pi ab^2 = \frac{4}{3}\pi R^3$

Change in Surface term: $a_S A^{2/3} \longrightarrow a_S A^{2/3}(1 + \frac{2}{5}\epsilon^2)$

Change in Coulomb term: $a_C \frac{Z^2}{A^{1/3}} \longrightarrow a_C \frac{Z^2}{A^{1/3}}(1 - \frac{\epsilon^2}{5})$

Not proved, just geometry

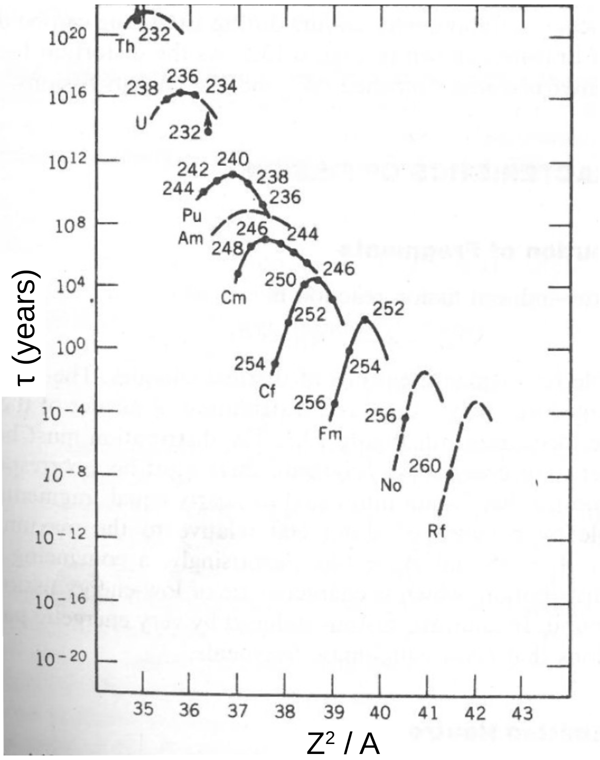
Change in Binding Energy: $\Delta B = B(\epsilon) - B(0) = a_C A^{2/3}\left(\frac{Z^2}{A} - \frac{2a_S}{a_C}\right)\frac{\epsilon^2}{5}$

i.e. if $\frac{Z^2}{A} > \frac{2a_S}{a_C}$, then $\Delta B > 0$ and the nucleus unstable under deformation

$\Rightarrow \frac{Z^2}{A} > 47$ predicted point (roughly) at which the fission barrier vanishes.

Fission Barrier

And indeed we observe that spontaneous fission lifetimes fall rapidly as Z^2/A increases.



Fission Barrier

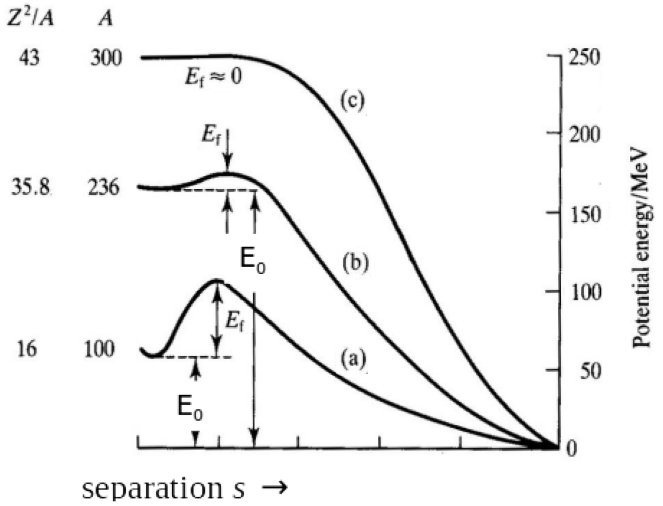
Spontaneous fission is possible if tunnelling through fission barrier occurs (c.f. α decay).

Tunnelling probability depends on height of barrier

$$E_{\mathrm{f}} \propto \left(\frac{Z^2}{A}\right)^{-1}$$

and on the mass of fragment

$$P \propto \mathrm{e}^{-2G}; \quad G \propto m^{1/2}$$



Large mass fragment → low probability for tunnelling

e.g. fission is $\sim 10^6$ less probable than α decay for $^{238}_{92}$U

So there are naturally occurring spontaneously fissile nuclides, but it tends to be a rare decay.

Neutron Induced Fission *Low energy neutron capture*

Use neutrons to excite nuclei and overcome fission barrier.

Important for the design of thermonuclear reactors.

Low energy neutrons are easily absorbed by nuclei (no Coulomb barrier) → excited state.

Excited state may undergo $n + {}^{A}\mathrm{U} \to {}^{A+1}\mathrm{U}^* \to {}^{A+1}\mathrm{U} + \gamma$ or $\mathrm{X}^* + \mathrm{Y}^*$

γ **decay** (most likely): (n,γ) reaction

Fission (less likely): excitation energy may help to overcome E_f

(n,γ) reaction:

Breit-Wigner cross-section

$$\sigma(n,\gamma) = \frac{g\pi\bar{\lambda}^2\Gamma_n\Gamma_\gamma}{(E-E_0)^2+\Gamma^2/4}, \qquad \Gamma_n \ll \Gamma_\gamma \sim \Gamma$$

At resonance

$$\sigma(n,\gamma) = 4\pi\bar{\lambda}^2 g\frac{\Gamma_n\Gamma_\gamma}{\Gamma^2} \sim 4\pi\bar{\lambda}^2 g\frac{\Gamma_n}{\Gamma}$$

Typically, $\Gamma_n \sim 10^{-1}$ eV, $\Gamma \sim 1$ eV; for 1 eV neutron, $\sigma \sim 10^3$ b (largest: ^{135}Xe $\sigma \sim 10^6$ b)

Far below resonance, $(E \ll E_0)$

$$\sigma(n,\gamma) = \bar{\lambda}^2\Gamma_n\left[\frac{g\pi\Gamma_\gamma}{E_0^2+\Gamma^2/4}\right] = \bar{\lambda}^2\Gamma_n \times \text{constant}$$

Γ_n dominated by phase space

$$\Gamma_n \sim \frac{p^2}{v} \sim v; \quad \bar{\lambda} = \frac{\hbar}{p} \quad \to \quad \bar{\lambda}^2 \sim \frac{1}{v^2}$$

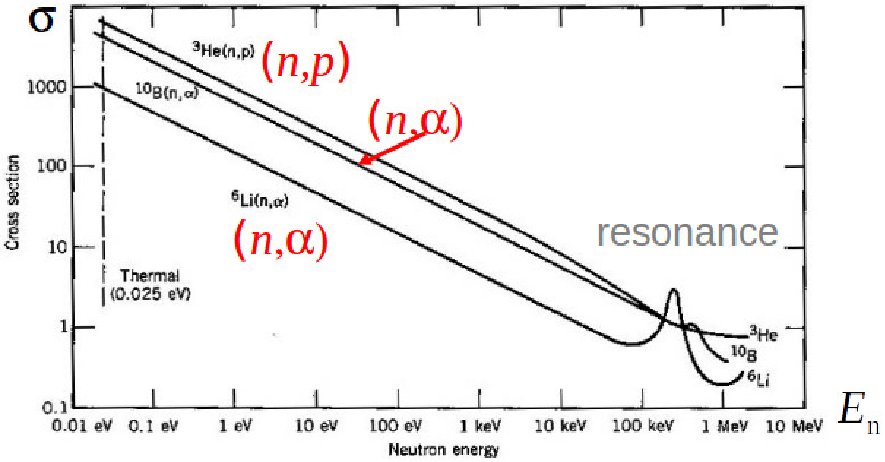
$\therefore\ \sigma(n,\gamma) \sim 1/v$

"1/v law" (for low energy neutron reactions)

Neutron Induced Fission *Low energy neutron capture*

$\sigma \sim 1/v$ dependence far below resonances

$E \propto v^2 \ \Rightarrow \ln\sigma \propto -1/2\ln E + \text{constant}.$



Low energy neutrons can have very large absorption cross-sections.

Neutron Induced Fission *Induced Fission*

Induced fission occurs when a nucleus captures a low energy neutron receiving enough energy to climb the fission barrier.

e.g. $^{235}_{92}$U $\quad n + {}^{235}_{92}\mathrm{U} \rightarrow {}^{236}_{92}\mathbf{U}^* \rightarrow \mathbf{X}^* + \mathbf{Y}^* \rightarrow \mathrm{X} + \mathrm{Y} + \kappa n$

$\kappa \sim 2.4$ prompt neutrons

Excitation energy of $^{236}\mathrm{U}^* > E_\mathrm{f}$ fission activation energy, hence fission occurs rapidly, even for zero energy neutrons

$\rightarrow$ thermal neutrons will induce fission.

Otherwise need to supply energy using K.E. of neutron.

e.g. $^{238}_{92}$U $\quad n + {}^{238}\mathrm{U} \rightarrow {}^{239}\mathbf{U}^* \qquad E_\mathrm{f} \sim 6$ MeV

$E_n = 0 \qquad E^* \sim 5$ MeV $\qquad$ no thermal fission

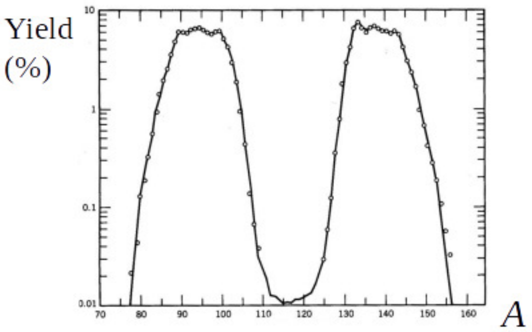
$E_n = 1.4$ MeV $\qquad E^* \sim 6.4$ MeV $\qquad$ rapid fission

but neutron absorption cross-section decreases rapidly with energy.

^{235}U is the more interesting isotope for fission reactor (or bombs).

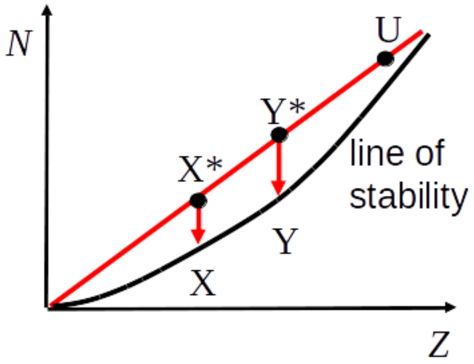
Neutron Induced Fission *Induced Fission*

$n + {}^{A}\mathrm{U} \rightarrow {}^{A+1}\mathbf{U}^* \rightarrow \mathrm{X}^* + \mathrm{Y}^*$



Masses of fragments are unequal (in general). Tend to have Z, N near magic numbers.

Fragments X*, Y* tend to have same Z/N ratio as parent $\rightarrow$ neutron rich nuclei which emit prompt neutrons (10^{-16}s).



X and Y undergo β decay more slowly; may also undergo neutron emission

$\rightarrow$ delayed neutron emission

($\sim$1 delayed neutron per 100 fissions).

Note wide variety of (usually radioactive) nuclei are produced in fission; can be very useful, but potentially very nasty.

Neutron Induced Fission *Chain Reaction*

Neutrons from fission process can be used to induce further fission

→ chain reaction, can be sustained if at least one neutron per fission induces another fission process.

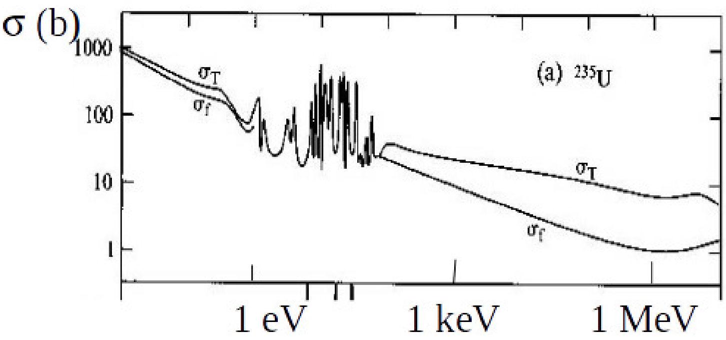
k = number of neutrons from one fission which induce another fission

$k < 1$ sub-critical,

$k = 1$ critical, ← For reactors want a steady energy release, exactly critical

$k > 1$ super-critical.

Prompt neutrons are fast, $\langle E \rangle \sim 2\ \mathrm{MeV}$ and their absorption σ is small.
Need to slow down fast neutrons before they escape or get absorbed by (n,γ) process
→ achieve a chain reaction.



Fission Reactors

Power reactor
e.g. Sizewell in Suffolk
KE of fission products → heat → electric power

Research reactor
e.g. ISIS at RAL in Oxfordshire
Beams of neutrons for (e.g.) condensed matter research

Breeder reactor
e.g. Springfields in Lanarkshire
Converts non-fissile to fissile isotopes, e.g.

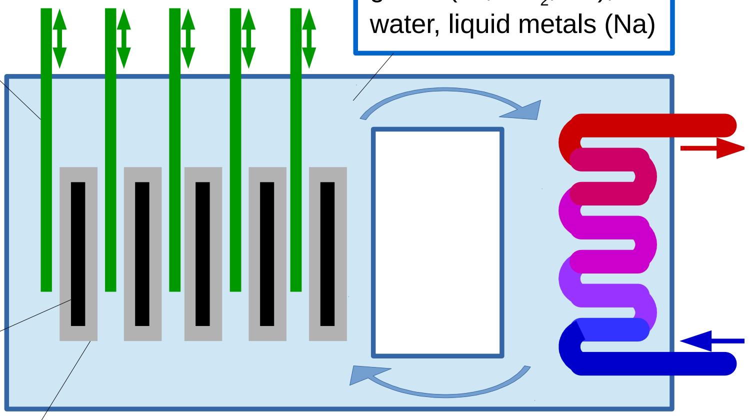
Plutonium: $n + {}^{238}\mathrm{U} \rightarrow {}^{239}\mathrm{U} \rightarrow {}^{239}\mathrm{Np} \rightarrow {}^{239}\mathrm{Pu}$

Uranium: $n + {}^{232}\mathrm{Th} \rightarrow {}^{233}\mathrm{Th} \rightarrow {}^{233}\mathrm{Pa} \rightarrow {}^{233}\mathrm{U}$

Can separate fissile isotopes chemically

Fission Reactors

A simple reactor needs fuel, moderators, control rods, and a cooling system.



Cooling System:
gases (air, CO_2, He),
water, liquid metals (Na)

Control Rods: control number of neutrons by absorption (^{113}Cd)

Fuel:
Natural U (0.72% ^{235}U),
enriched U (2-3% ^{235}U),
^{239}Pu, ^{233}U

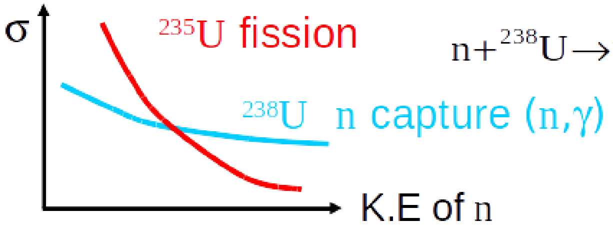
Moderator: slows neutrons via elastic collisions.
Large energy transfer requires use of a light nucleus.
H_2O – cheap but absorbs neutrons through n+p → d+γ
D_2O – extractable from seawater, but forms nasty radioactive tritium ^{3}H
^{13}C – graphite, larger mass → less energy transfer per collison → need more of it.

UK reactors are mainly graphite moderated, gas cooled.

Fission Reactors

The problem

Natural U is (99.3% ^{238}U, 0.7% ^{235}U) and n capture cross-section large for ^{238}U



$$n + {}^{238}U \rightarrow {}^{239}U^* \rightarrow {}^{239}U + \gamma$$

Need to

1. thermalise fast neutrons away from ^{238}U to avoid capture (moderators)
2. control number of neutrons by absorption (control rods).

But
typical time between fission and daughter inducing another fission $\sim 10^{-3}$s
$\rightarrow$ mechanical control of rods in times $\ll$ seconds not possible!

Fission Reactors

The consequence – what happens if we fail to control the neutrons?

$$N(t+\mathrm{d}t) = N(t) + (k-1)N(t)\frac{\mathrm{d}t}{\tau}$$

$N(t)$ number of neutrons at time t

$(k-1)$ fractional change in number of neutrons in 1 cycle

τ mean time for one cycle $\sim 10^{-3}$s (fission → fission)

$$\mathrm{d}N = (k-1)N\frac{\mathrm{d}t}{\tau} \quad \Rightarrow \quad \int_{N(0)}^{N(t)} \frac{\mathrm{d}N}{N} = \int_0^t (k-1)\frac{\mathrm{d}t}{\tau} \quad \Rightarrow \quad N(t) = N(0)\mathrm{e}^{(k-1)t/\tau}$$

for $k > 1$ → exponential growth – bad!

e.g. $k = 1.01$, $\tau = 0.001\mathrm{s}$, $t = 1\mathrm{s}$

$$\frac{N(t)}{N(0)} = \mathrm{e}^{0.01/0.001} = \mathrm{e}^{10} \qquad (\times 22,000 \text{ in } 1\mathrm{s})$$

Note: Uranium reactor will not explode if it goes super-critical. As it heats up, K.E. of neutrons increases and fission cross-section drops. Reactor stabilises at a very high temperature ⇒ MELTDOWN

Fission Reactors

The solution

Make use of delayed neutron emission (delay ∼13s).

Design reactor to be subcritical to prompt neutrons and use the delayed neutrons to take it to critical.

Thermal reactors require the following steps:

fission → fast neutrons

↓

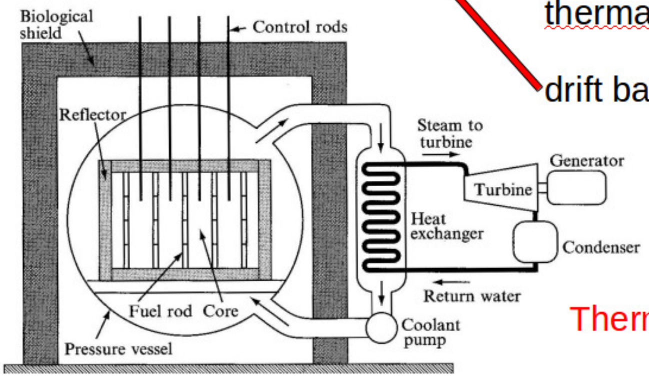
remove from fissile material

↓

thermalize them (gives heat)

↓

drift back into fuel → fission



Thermal Reactor

Nuclear Fusion

Energetically favourable for light nuclei to fuse and release energy.

However, nuclei need energy to overcome Coulomb barrier

e.g. most basic process: $p+p \rightarrow d + e^+ + \nu_e$, $E_0 = 0.42\ \mathrm{MeV}$

but Coulomb barrier $V = \frac{e^2}{4\pi\epsilon_0 R} = \frac{\alpha\hbar c}{R} = \frac{197}{137 \times 1.2} = 1.2\ \mathrm{MeV}$

Overcoming the Coulomb barrier

Accelerators: Energies above barrier easy to achieve. However, high particle densities for long periods of time very difficult. These would be required to get a useful rate of fusion reactions for power generation.

Stars: Large proton density $10^{32}\ \mathrm{m}^{-3}$. Particle K.E. due to thermal motion.

To achieve $kT \sim 1\ \mathrm{MeV}$, require $T \sim 10^{10}\mathrm{K}$

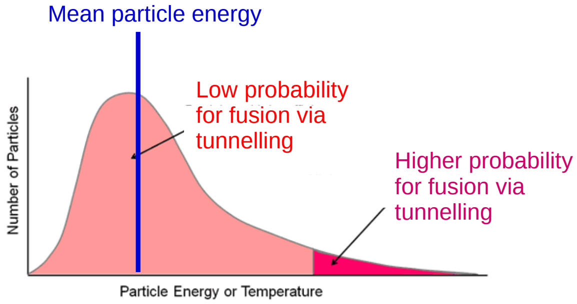
Interior of Sun: $T \sim 10^7\mathrm{K}$, i.e. $kT \sim 1\ \mathrm{keV}$

$\Rightarrow$ Quantum Mechanical tunnelling required.

Fusion in the Sun

Fusion rate in the Sun

Particles in the Sun have Maxwell-Boltzmann velocity distribution with long tails – very important because tunnelling probability is a strong function of energy.



Reaction rate in unit volume for particles of velocity v: $\Gamma = \sigma(v)\Phi N$, where flux $\Phi = Nv$

σ is dominated by the tunnelling probability $P = e^{-2G(v)}$

and a factor $1/v^2$ arising from the $\bar{\lambda}^2$ in the Breit-Wigner formula.

reminder, Gamow Factor $G(v) \sim \left(\frac{2m}{E_0}\right)^{1/2} \frac{e^2}{4\pi\epsilon_0}\frac{Z_1Z_2}{\hbar}\frac{\pi}{2} = \frac{e^2}{4\pi\epsilon_0}\frac{\pi Z_1Z_2}{\hbar v}$

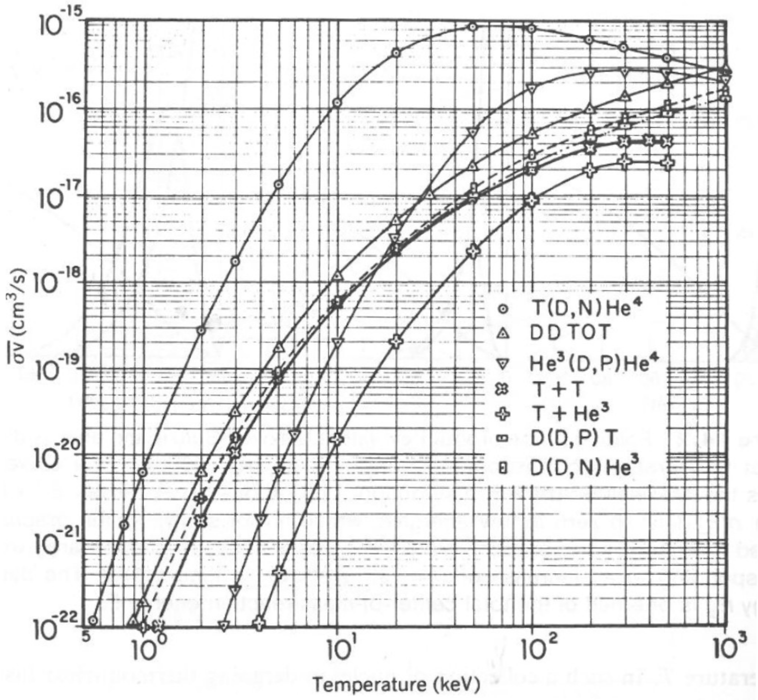
Averaged over the Maxwell-Boltzmann velocity distribution $\Gamma \sim N^2\langle\sigma v\rangle$

Probability velocity between v and $v + \mathrm{d}v$ $= f(v)\,\mathrm{d}v \propto v^2 e^{-mv^2/2kT}\,\mathrm{d}v$

$$\Rightarrow \Gamma \propto \int N.Nv.\frac{1}{v^2}e^{-2G}f(v)\,\mathrm{d}v \propto \int ve^{-2G}e^{-mv^2/2kT}\,\mathrm{d}v \propto \int e^{-2G}e^{-E/kT}\,\mathrm{d}E$$

Fusion in the Sun *Fusion rate in the Sun*

Typical fusion reactions peak at $kT \sim 100\ \mathrm{keV} \Rightarrow T \sim 10^9\mathrm{K}$



e.g. for $\mathrm{p}+\mathrm{p} \rightarrow \mathrm{d} + e^+ + \nu_e$

$\sigma \sim 10^{-32}\mathrm{b}$ – tiny! weak!

but there are an awful lot of protons...

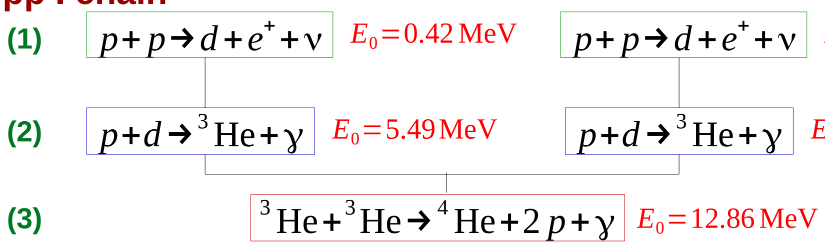
per proton, $\Gamma \sim 5 \times 10^{-18}\mathrm{s}^{-1}$

$\Rightarrow$ Mean life, $\tau = 10^{10}$ yrs.

This defines the burning rate in the Sun.

Fusion in the Sun *Fusion processes in the Sun*

pp I chain



(1) $p+p \rightarrow d+e^+ +\nu$ $E_0=0.42\,\mathrm{MeV}$ $p+p \rightarrow d+e^+ +\nu$ $E_0=0.42\,\mathrm{MeV}$

(2) $p+d \rightarrow {}^3\mathrm{He}+\gamma$ $E_0=5.49\,\mathrm{MeV}$ $p+d \rightarrow {}^3\mathrm{He}+\gamma$ $E_0=5.49\,\mathrm{MeV}$

(3) ${}^3\mathrm{He}+{}^3\mathrm{He} \rightarrow {}^4\mathrm{He}+2p+\gamma$ $E_0=12.86\,\mathrm{MeV}$

Net reaction ($2e^+$ annihilate with $2e^-$): $4p \rightarrow^4 \mathrm{He} + 2e^+ + 2\nu$ $E_0 = 4m_e = 2.04\ \mathrm{MeV}$

Total energy release in fusion cycle $= 26.7\ \mathrm{MeV}$ (per proton $= 26.7/4 = 6.7\ \mathrm{MeV}$)

ν's emerge without further interaction with $\sim 2\%$ of the energy. The rest of the energy (γ-rays; KE of fission products) heats the core of the star.

Observed luminosity $\sim 4{\times}10^{26}$ J/s $(1\ \mathrm{MeV} = 1.6 \times 10^{-13}\ \mathrm{J})$

$\Rightarrow$ Number of protons consumed $= \frac{4{\times}10^{26}}{1.6{\times}10^{-13}}\frac{1}{6.7} = 4{\times}10^{38}\,\mathrm{s}^{-1}$

Fusion in the Sun *Fusion processes in the Sun*

pp I chain

(1) $p+p\rightarrow d+e^{+}+\nu$ $E_0=0.42\,\text{MeV}$ $p+p\rightarrow d+e^{+}+\nu$ $E_0=0.42\,\text{MeV}$

(2) $p+d\rightarrow {}^{3}\text{He}+\gamma$ $E_0=5.49\,\text{MeV}$ $p+d\rightarrow {}^{3}\text{He}+\gamma$ $E_0=5.49\,\text{MeV}$

(3) ${}^{3}\text{He}+{}^{3}\text{He}\rightarrow {}^{4}\text{He}+2p+\gamma$ $E_0=12.86\,\text{MeV}$

pp II chain

(1) ${}^{3}\text{He}+{}^{4}\text{He}\rightarrow {}^{7}\text{Be}+\gamma$

(2) ${}^{7}\text{Be}+e^{-}\rightarrow {}^{7}\text{Li}+\nu$ $E_\nu=0.861/0.383\,\text{MeV}$ Li vs Li*

(3) ${}^{7}\text{Li}+p\rightarrow 2\,{}^{4}\text{He}$

pp III chain

(1) ${}^{3}\text{He}+{}^{4}\text{He}\rightarrow {}^{7}\text{Be}+\gamma$

(2) ${}^{7}\text{Be}+p\rightarrow {}^{8}\text{B}+\gamma$

(3) ${}^{8}\text{B}\rightarrow {}^{8}\text{Be}+e^{+}+\nu$ $E_\nu=14.06\,\text{MeV}$

(4) ${}^{8}\text{Be}\rightarrow 2\,{}^{4}\text{He}$

Other fusion cycles also possible e.g. C-N-O cycle.

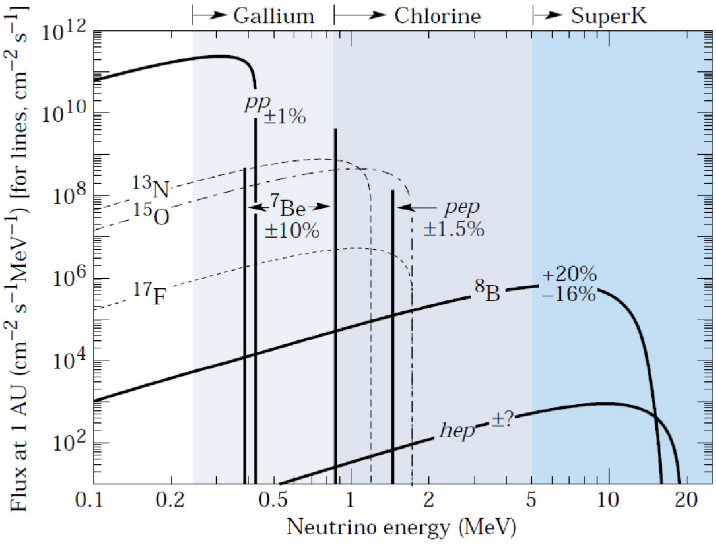
Observation of solar neutrinos from the various sources directly addresses the theory of stellar structure and evolution (Standard Solar Model). Probes the core of the Sun where the nuclear reactions are taking place.

The Sun also provides an opportunity to investigate ν properties e.g. mass, oscillations...

Also, the rare pp IV (Hep) chain: ${}^{3}_{2}\text{He}+{}^{1}_{1}\text{H}\rightarrow {}^{4}_{2}\text{He}+e^{+}+\nu_e$ $(E_\nu=18.8\,\text{MeV})$

Solar Neutrinos

Many experiments have studied the solar neutrino flux



Expected flux depends on

- Standard Solar Model (temperature, density, composition vs r)
- Nuclear reaction cross-sections

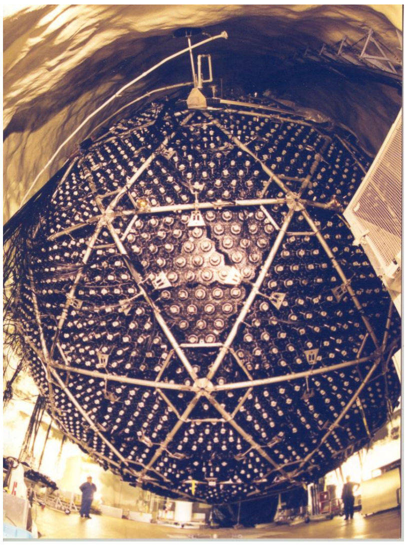
Observed ν flux $\sim$ 1/3 expected ν flux "Solar ν problem"

Solar Neutrinos

The Solar ν problem has recently been resolved by the Sudbury Neutrino Observatory (SNO) collaboration. They have reported evidence for a non-ν_e neutrino component in the solar ν flux

$\rightarrow$ Neutrino Oscillations

SNO (1000 tons D_2O in spherical vessel) measures the ^8B solar ν flux using three reactions:



Measure ν_e flux

$$\nu_e + d \rightarrow e^- + p + p$$

Measure total flux for all ν species

$$\nu_X + d \rightarrow \nu_X + p + n$$
$$\nu_X + e^- \rightarrow \nu_X + e^-$$

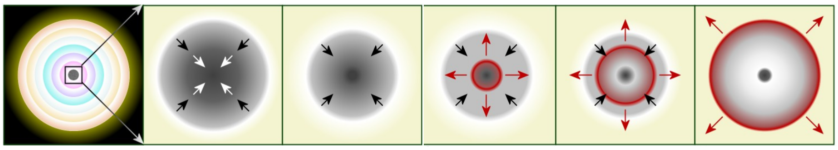
Observe a depletion in the ν_e flux, while the flux summed over all neutrino flavours agrees with expected solar flux.

Evidence for $\nu_e \Leftrightarrow \nu_X$ at 5σ

Further nuclear processes in astrophysics

Creating the heavy elements

Once the hydrogen is exhausted in a star, further gravitational collapse occurs and the temperature rises.



Eventually, it is hot enough to "burn" ^{4}He via fusion:

$$^4He + {}^4He \rightarrow {}^8Be + \gamma$$
$$^4He + {}^8Be \rightarrow {}^{12}C + \gamma$$
$$^4He + {}^{12}C \rightarrow {}^{16}O + \gamma$$

When the ^{4}He is exhausted, star undergoes further collapse

$\rightarrow$ further fusion reactions (and repeat)

Until we have the most tightly bound nuclei ^{56}Fe, ^{56}Co, ^{56}Ni.

Heavier elements are formed in supernova explosions:

$$n + {}^{56}Fe \rightarrow {}^{56}Fe + \gamma$$
$$n + {}^{57}Fe \rightarrow {}^{58}Fe + \gamma$$
$$n + {}^{58}Fe \rightarrow {}^{59}Fe + \gamma$$
$$^{59}Fe \rightarrow {}^{59}Co + e^- + \bar{\nu}_e$$

etc etc

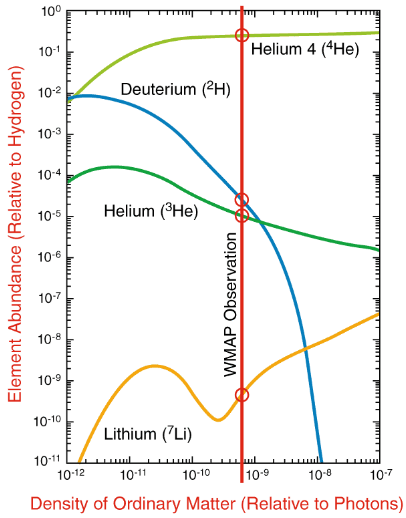
Further nuclear processes in astrophysics

Big bang nucleosynthesis

Fusion processes are also important in the Big Bang.
Both p and n present, at $T \gg 10^9$K.
Typical reactions:

$$n + p \to d + \gamma \qquad d + p \to {}^3\text{He} + \gamma$$
$$d + n \to {}^3\text{H} + \gamma \qquad d + d \to {}^3\text{H} + p$$
$${}^3\text{H} + p \to {}^4\text{He} + \gamma \qquad d + d \to {}^3\text{He} + n$$
$${}^3\text{He} + n \to {}^4\text{He} + \gamma$$

Observed abundances of these light elements provide a sensitive test of the Big Bang model.



NASA/WMAP Science Team
WMAP101087

Element Abundance graphs: Steigman, Encyclopedia of Astronomy and Astrophysics (Institute of Physics) December, 2000

In particular, they depend on aspects of particle physics which determine the n/p ratio, which depends on the temperature at which the reactions

$$p + \bar{\nu}_e \to n + e^+ \qquad n + \nu_e \to p + e^-$$

"freeze out", which in turn depends on the number of neutrino species.

Fusion in the lab

Fusion in the laboratory was first demonstrated in 1932, here at the Cavendish (Oliphant).

For fusion we need sufficiently high temperatures and controlled conditions.
The challenge now is to generate more power than expended.

Possible fusion reactions:

$$d + d \to {}^3\text{He} + n \qquad Q = 3.3 \text{ MeV}$$
$$d + d \to {}^3\text{H} + p \qquad Q = 4.0 \text{ MeV}$$
$$d + {}^3\text{H} \to {}^4\text{He} + n \qquad Q = 17.6 \text{ MeV}$$

The $d + {}^3$H (aka DT) reaction is especially attractive

✓ largest energy release (α particle very stable)

✓ lowest Coulomb barrier

✗ 80% of the energy is released in the neutron – less easy to use, and doesn't help to heat the plasma.

✗ ^{3}H (tritium) unstable ($\tau_{1/2} \sim 12$ yr); need to produce it via $n + {}^6\text{Li} \to {}^4\text{He} + {}^3\text{H}$ or $n + {}^7\text{Li} \to {}^4\text{He} + {}^3\text{H} + n$ using some of the neutrons formed in the fusion reaction.

A recipe for controlled fusion

- Need $T \sim 10^8\,\mathrm{K}$ i.e. $E \sim 10\ \mathrm{keV} \gg$ ionisation energy $\Rightarrow$ **plasma**
  reminder: plasmas are electrically conductive and can be controlled with magnetic fields.
- Heat plasma by applying r.f. energy.
  Declare **Ignition** when the process is self-sustaining: the heating from 3.5 MeV α-particles produced in fusion exceeds the losses (due to bremsstrahlung, for example).
- **Break even** achieved when there is more power out (incl. losses) than in.

Fusion rate $= n_D n_T \langle\sigma v\rangle = \frac{1}{4}n^2\langle\sigma v\rangle$ (assumes $n_D = n_T = \frac{1}{2}n$, where n is the electron density).
Rate of generation of energy $= \frac{1}{4}n^2\langle\sigma v\rangle Q$
Rate of energy loss $= W/\tau$ where $W = 3nkT$ is the energy density in the plasma ($3kT/2$ for electrons and the same for the ions) and τ is the lifetime of the plasma due to losses.
Break even if $\frac{1}{4}n^2\langle\sigma v\rangle Q \ > \ 3nkT/\tau$, i.e.

Lawson criterion $$n\tau > \frac{12kT}{Q\langle\sigma v\rangle}$$

For DT, this is $n\tau > 10^{20}\,\mathrm{m}^{-3}\mathrm{s}$ at $kT \gg 10\ \mathrm{keV}$.

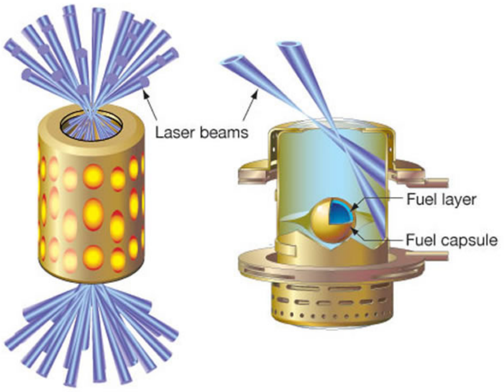
People commonly look at the "triple product" $n\tau T$ for fusion processes.

Controlled fusion – confinement

Need $T \sim 10^8\,\mathrm{K}$ i.e. $E \sim 10\ \mathrm{keV} \gg$ ionisation energy $\Rightarrow$
need to **control** a plasma

Inertial confinement

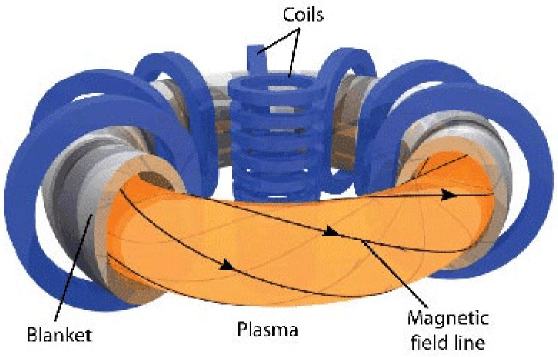
Use a pellet containing d+^{3}H zapped from all sides with lasers or particle beams to heat it. Need very high power lasers + repeated feeding of fuel.



e.g. National Ignition Facility, LBNL, US

Magnetic confinement

Use a configuration of magnetic fields to control the plasma (Tokamak) and keep it away from walls.



e.g. International Thermonuclear Experimental Reactor, France

Controlled fusion – the status today

JET (Joint European Torus),
TFTR (Tokamak Fusion Test Reactor), both achieved appropriate values of plasma density (n) and lifetime (τ), but not simultaneously $\Rightarrow$ yet to break even.

NIF (National Ignition Facility) closing in on ignition.

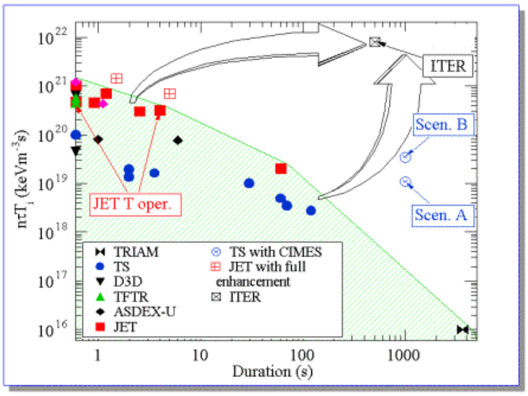
ITER (International Thermonuclear Experimental Reactor) should break even. Build time ∼10 years; then ∼20 years of experimentation starting 2025.

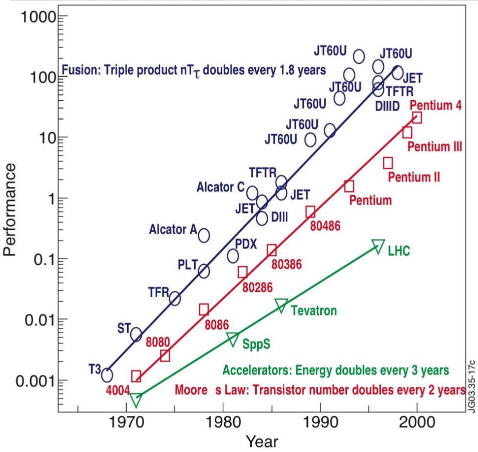
Commercial fusion power can't realistically be expected before 2050.

Recent progress

Aug 2021: NIF produced 1.3 MJ – 70% of delivered laser energy.
Dec 2022: NIF produced 3.15 MJ – 150% of delivered laser energy. *Breakeven!*
Feb 2022: JET broke 23 year old energy record – 59 MJ over 5 s
Feb 2024: JET – 69 MJ over 5 s

This could be *your* work!





Summary

- Spontaneous fission – energetically possible for many nuclei, but tunnelling needed – rate only competitive for a few heavy elements.
- Neutron induced fission – neutron absorption into a fissile excited state. Practical importance in power generation and bombs.
- Asymmetric fission; neutrons liberated
- Chain reaction. Use of delayed neutron component for control.
- Fusion – again a tunnelling problem. Needs very high temperatures for useful rates.
- Fusion processes in the sun (solar neutrinos).
- Nucleosynthesis in the big bang.
- Controlled fusion.

Problem Sheet: q.42-44

Thank you for being a great class! Farewell!