15. Nuclear Decay
Particle and Nuclear Physics
In this section...

- Radioactive decays
- Radioactive dating
- $\alpha$ decay
- $\beta$ decay
- $\gamma$ decay
Radioactivity

Natural radioactivity: three main types $\alpha, \beta, \gamma$, and in a few cases, spontaneous fission.

**$\alpha$ decay**  
$\alpha$ decay $\alpha$ nuclei emitted.  
$\frac{A}{Z}X \rightarrow \frac{A-4}{Z-2}Y + \frac{4}{2}\text{He}$  
Occurs for $A \geq 210$

For decay to occur, energy must be released $Q > 0$

$Q = m_X - m_Y - m_{\text{He}} = B_Y + B_{\text{He}} - B_X$

**$\beta$ decay**  
emission of electron $e^-$ or positron $e^+$

$n \rightarrow p + e^- + \bar{\nu}_e$  
$\frac{A}{Z}X \rightarrow \frac{A}{Z+1}Y + e^- + \bar{\nu}_e$  
$\beta^-$ decay

$p \rightarrow n + e^+ + \nu_e$  
$\frac{A}{Z}X \rightarrow \frac{A}{Z-1}Y + e^+ + \nu_e$  
$\beta^+$ decay

$p + e^- \rightarrow n + \nu_e$  
$\frac{A}{Z}X + e^- \rightarrow \frac{A}{Z-1}Y + \nu_e$  
Electron capture

n.b. of these processes, only $n \rightarrow p e^- \nu$ can occur outside a nucleus.
Radioactivity

$\gamma$ decay  Nuclei in excited states can decay by emission of a photon $\gamma$. Often follows $\alpha$ or $\beta$ decay.

\[
\begin{array}{c}
\text{Excited states} \\
\downarrow \\
\text{Photons emitted} \\
\downarrow \\
\text{Ground state}
\end{array}
\]

$\Delta E$

<table>
<thead>
<tr>
<th>$\Delta E$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atom $\sim 10$ eV</td>
<td>$\sim 10^{-7}$ m optical</td>
</tr>
<tr>
<td>$\sim 10$ keV</td>
<td>$\sim 10^{-10}$ m X-ray</td>
</tr>
<tr>
<td>Nucleus $\sim$ MeV</td>
<td>$\sim 10^{-12}$ m $\gamma$-ray</td>
</tr>
</tbody>
</table>

A variant of $\gamma$ decay is Internal Conversion:

- an excited nucleus loses energy by emitting a virtual photon,
- the photon is absorbed by an atomic $e^-$, which is then ejected
- n.b. not $\beta$ decay, as nucleus composition is unchanged ($e^-$ not from nucleus)
The **half-life**, $\tau_{1/2}$, is the time over which 50% of the nuclei decay

$$\tau_{1/2} = \frac{\ln 2}{\lambda} = 0.693\tau$$

Some $\tau_{1/2}$ values may be long compared to the age of the Earth.

### 4n series

<table>
<thead>
<tr>
<th>Series Name</th>
<th>Type</th>
<th>Final Nucleus ($^A_X$)</th>
<th>Longest-lived Nucleus</th>
<th>$\tau_{1/2}$ (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thorium</td>
<td>4n</td>
<td>$^{208}$Pb</td>
<td>$^{232}$Th</td>
<td>$1.41 \times 10^{10}$</td>
</tr>
<tr>
<td>Neptunium</td>
<td>4n+1</td>
<td>$^{209}$Bi</td>
<td>$^{237}$Np</td>
<td>$2.14 \times 10^6$</td>
</tr>
<tr>
<td>Uranium</td>
<td>4n+2</td>
<td>$^{206}$Pb</td>
<td>$^{238}$U</td>
<td>$4.47 \times 10^9$</td>
</tr>
<tr>
<td>Actinium</td>
<td>4n+3</td>
<td>$^{207}$Pb</td>
<td>$^{235}$U</td>
<td>$7.04 \times 10^8$</td>
</tr>
</tbody>
</table>

$n$ is an integer
Radioactive Dating  Geological Dating

Can use $\beta^-$ decay to age the Earth, 

$$^{87}\text{Rb} \rightarrow ^{87}\text{Sr} \quad (\tau_{1/2} = 4.8 \times 10^{10} \text{ years})$$

$N_1$ $N_2$

$^{87}\text{Sr}$ is stable $\rightarrow \lambda_2 = 0$

So in this case, we have (using expressions from Chapter 2)

$$N_2(t) = N_1(0) \left[1 - e^{-\lambda_1 t}\right] + N_2(0) = N_1(t) \left[e^{\lambda_1 t} - 1\right] + N_2(0)$$

Assume we know $\lambda_1$, and can measure $N_1(t)$ and $N_2(t)$ e.g. chemically. But we don’t know $N_2(0)$.

Solution is to normalise to another (stable) isotope – $^{86}\text{Sr}$ – for which number is $N_0(t) = N_0(0)$.

$$\frac{N_2(t)}{N_0} = \frac{N_1(t)}{N_0} \left[e^{\lambda_1 t} - 1\right] + \frac{N_2(0)}{N_0}$$

**Method:** plot $N_2(t)/N_0$ vs $N_1(t)/N_0$ for lots of minerals. Gradient gives $[e^{\lambda_1 t} - 1]$ and hence $t$.

Intercept = $N_2(0)/N_0$, which should be the same for all minerals (determined by chemistry of formation).
Radioactive Dating  

**Dating the Earth**

\[
\frac{N_2(t)}{N_0} = \frac{N_1(t)}{N_0} [e^{\lambda_1 t} - 1] + \frac{N_2(0)}{N_0}
\]

**Method:** plot \(\frac{N_2(t)}{N_0}\) vs \(\frac{N_1(t)}{N_0}\) for lots of minerals.
Gradient gives \([e^{\lambda_1 t} - 1]\) and hence \(t\).
Intercept = \(N_2(0)/N_0\), which should be the same for all minerals (determined by chemistry of formation).

Using minerals from the Earth, Moon and meteorites.

Intercept gives \(N_2(0)/N_0 = 0.70\)

Slope gives the age of the Earth = \(4.5 \times 10^9\) yrs
Radioactive Dating  Radio-Carbon Dating

For recent organic matter, use $^{14}$C dating

Continuously formed in the upper atmosphere at approx. constant rate.
$^{14}$N + n → $^{14}$C + p

Undergoes $\beta^-$ decay
$^{14}$C → $^{14}$N + e$^-$ + $\bar{\nu}$
$\tau_{1/2} = 5730$ yrs

Atmospheric carbon continuously exchanged with living organisms.
Equilibrium: 1 atom of $^{14}$C to every $10^{12}$ atoms of other carbon isotopes (98.9% $^{12}$C, 1.1% $^{13}$C)

No more $^{14}$C intake for dead organisms.
Fresh organic material
~15 decays/minute/gram of carbon.

Measure the specific activity of material to obtain age, i.e. number of decays per second per unit mass

Complications for the future!
Burning of fossil fuels increases $^{12}$C in atmosphere,
Nuclear bomb testing (adds $^{14}$C to atmosphere)
α Decay

α decay is due to the emission of a $^4_2$He nucleus.

$^4_2$He is doubly magic and very tightly bound.

α decay is energetically favourable for almost all with $A \geq 190$ and for many $A \geq 150$.

Why α rather than any other nucleus?
Consider energy release ($Q$) in various possible decays of $^{232}\text{U}$

<table>
<thead>
<tr>
<th></th>
<th>n</th>
<th>p</th>
<th>$^2\text{H}$</th>
<th>$^3\text{H}$</th>
<th>$^3\text{He}$</th>
<th>$^4\text{He}$</th>
<th>$^5\text{He}$</th>
<th>$^6\text{Li}$</th>
<th>$^7\text{Li}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q$/MeV</td>
<td>-7.26</td>
<td>-6.12</td>
<td>-10.70</td>
<td>-10.24</td>
<td>-9.92</td>
<td>+5.41</td>
<td>-2.59</td>
<td>-3.79</td>
<td>-1.94</td>
</tr>
</tbody>
</table>

α is easy to form inside a nucleus $2p \uparrow\downarrow + 2n \uparrow\downarrow$

(though the extent to which α particles really exist inside a nucleus is still debatable)
A very striking feature of α decay is the strong dependence of lifetime on $E_0$.

**Example**

$^{232}\text{Th}$  \hspace{1cm} $E_0 = 4.08$ MeV \hspace{1cm} $\tau_{1/2} = 1.4 \times 10^{10}$ yrs

$^{218}\text{Th}$  \hspace{1cm} $E_0 = 9.85$ MeV \hspace{1cm} $\tau_{1/2} = 1.0 \times 10^{-7}$ s

A factor of $\sim 2.5$ in $E_0$  \Rightarrow  factor $10^{24}$ in $\tau_{1/2}$!

e.g. even $N$, even $Z$ nuclei for a given $Z$ see smooth trend ($\tau_{1/2}$ increases as $Z$ does)
The nuclear potential for the \( \alpha \) particle due to the daughter nucleus includes a Coulomb barrier which inhibits the decay.

\[
V(r) = E_0 - V_0 - \frac{1}{r}
\]

Classically, \( \alpha \) particle cannot enter or escape from nucleus.

Quantum mechanically, \( \alpha \) particle can penetrate the Coulomb barrier

\[ \Rightarrow \text{Quantum Mechanical Tunnelling} \]
**α Decay**  
*Simple Theory*  
*(Gamow, Gurney, Condon 1928)*

Assume $\alpha$ exists inside the nucleus and hits the barrier.

$$\alpha \text{ decay rate}, \quad \lambda = f \ P$$

$f = \text{escape trial frequency}, \ P = \text{probability of tunnelling through barrier}$

semi-classically,  
$$f \sim \frac{v}{2R}$$

$v = \text{velocity of a particle inside nucleus}$, given by:  
$$v^2 = \frac{2E_\alpha}{m_\alpha}$$

and $R = \text{radius of nucleus}$

Typical values: $V_0 \sim 35 \ \text{MeV}, \ E_0 \sim 5 \ \text{MeV} \Rightarrow E_\alpha = 40 \ \text{MeV}$ inside nucleus

$$f \sim \frac{v}{2R} = \frac{1}{2R} \sqrt{\frac{2E_\alpha}{m_\alpha}} \sim 10^{22} \ \text{s}^{-1} \quad m_\alpha = 3.7 \ \text{GeV}$$

$$R \sim 2.1 \ \text{fm}$$

Obtain tunnelling probability, $P$, by solving Schrödinger equation in three regions and using boundary conditions.
Transmission probability (1D square barrier):

\[ P = \left[ 1 + \frac{V_0^2}{4(V_0 - E)E} \sinh^2 ka \right]^{-1} \]

\[ \frac{\hbar^2 k^2}{2m} = V_0 - E \quad m = \text{reduced mass} \]

For \( ka \gg 1 \), \( P \) is dominated by the exp. decay within barrier \( \Rightarrow P \sim e^{-2ka} \).

Coulomb potential, \( V \propto 1/r \), and thus \( k \) varies with \( r \).
Divide into rectangular pieces and multiply together exponentials, i.e. sum exponents.

**Probability to tunnel** through Coulomb barrier

\[ P = \prod_i e^{-2k_i \Delta R} = e^{-2G} \quad k = \frac{[2m\alpha(V(r) - E_0)]^{1/2}}{\hbar} \]

The **Gamow Factor**

\[ G = \int_{R}^{R'} \frac{[2m\alpha(V(r) - E_0)]^{1/2}}{\hbar} \, dr = \int_{R}^{R'} k(r) \, dr \]
\(\alpha\) Decay  \textbf{Simple Theory}  (Gamow, Gurney, Condon 1928)

For \(r > R\), \[V(r) = \frac{Z_\alpha Z'_\alpha e^2}{4\pi \epsilon_0 r} = \frac{B}{r} \quad Z' = Z - Z_\alpha \quad (Z_\alpha = 2)\]

\(\alpha\)-particle escapes at \(r = R'\), \(V(R') = E_0 \Rightarrow R' = \frac{B}{E_0}\)

\[\therefore G = \int_R^{R'} \left(\frac{2m_\alpha}{\hbar^2}\right)^{1/2} \left[\frac{B}{r} - E_0\right]^{1/2} dr = \left(\frac{2m_\alpha B}{\hbar^2}\right)^{1/2} \int_R^{R'} \left[\frac{1}{r} - \frac{1}{R'}\right]^{1/2} dr\]

See Appendix H

\[G = \left(\frac{2m_\alpha}{E_0}\right)^{1/2} \frac{B}{\hbar} \left[\cos^{-1}\left(\frac{R}{R'}\right)^{1/2} - \left\{\left(1 - \frac{R}{R'}\right)\left(\frac{R}{R'}\right)\right\}\right]^{1/2}\]

To perform integration, substitute \(r = R' \cos^2 \theta\)

In most practical cases \(R \ll R'\), so term in \([\ldots]\) \(\sim \pi/2\)

\[G \sim \left(\frac{2m_\alpha}{E_0}\right)^{1/2} \frac{B \pi}{\hbar^2}\]

\(B = \frac{Z_\alpha Z'_\alpha e^2}{4\pi \epsilon_0}\)

e.g. typical values: \(Z = 90, E_0 \sim 6\text{ MeV} \Rightarrow R' \sim 40\text{ fm} \gg R\)

\[G \sim Z' \left(\frac{3.9\text{ MeV}}{E_0}\right)^{1/2}\]
\[ \tau = \frac{1}{\lambda} = \frac{1}{fP} \sim \frac{2R}{ve^2} \]

\[ \Rightarrow \ln \tau \sim 2G + \ln \frac{2R}{v} \]

\[ \ln \lambda \sim -\frac{Z'}{E_0^{1/2}} + \text{constant} \]

**Geiger-Nuttall Law**

Not perfect, but provides an explanation of the dominant trend of the data

Simple tunnelling model accounts for

- strong dependence of \( \tau_{1/2} \) on \( E_0 \)
- \( \tau_{1/2} \) increases with \( Z \)
- disfavoured decay to heavier fragments e.g. \(^{12}\text{C}\)

\[ G \propto m^{1/2} \quad \text{and} \quad G \propto \text{charge of fragment} \]
**Deficiencies/complications** with simple tunnelling model:

- Assumed existence of a single $\alpha$ particle in nucleus and have taken no account of probability of formation.
- Assumed “semi-classical” approach to estimate escape trial frequency, $f \sim \nu/2R$, and make absolute prediction of decay rate.
- If $\alpha$ is emitted with some angular momentum, $\ell$, the radial wave equation must include a centrifugal barrier term in Schrödinger equation

$$V' = \frac{\ell(\ell + 1)\hbar^2}{2\mu r^2}$$

where $\ell$ = relative a.m. of $\alpha$ and daughter nucleus, $\mu$ = reduced mass

which raises the barrier and suppresses emission of $\alpha$ in high $\ell$ states.
α Decay  Selection rules

Nuclear Shell Model: $\alpha$ has $J^P = 0^+$

Angular momentum

* e.g. $X \rightarrow Y + \alpha$

Conserving $J$: $J_X = J_Y \oplus J_\alpha = J_Y \oplus \ell_\alpha$

$\ell_\alpha$ can take values from $J_X + J_Y$ to $|J_X - J_Y|$

Parity

Parity is conserved in $\alpha$ decay (strong force).

Orbital wavefunction has $P = (-1)^\ell$

- $X, Y$ same parity $\Rightarrow \ell_\alpha$ must be even
- $X, Y$ opposite parity $\Rightarrow \ell_\alpha$ must be odd

* e.g. if $X, Y$ are both even-even nuclei in their ground states, shell model predicts both have $J^P = 0^+ \Rightarrow \ell_\alpha = 0$.

More generally, if $X$ has $J^P = 0^+$, the states of $Y$ which can be formed in $\alpha$ decay are $J^P = 0^+, 1^-, 2^+, 3^-, 4^+, \ldots$
\( \beta \) Decay

\( \beta^- \)  
\[ n \rightarrow p + e^- + \bar{\nu}_e \quad \]
\[ ^A_Z X \rightarrow ^A_{Z+1} Y + e^- + \bar{\nu}_e \]

\( \beta^+ \)  
\[ p \rightarrow n + e^+ + \nu_e \quad \]
\[ ^A_Z X \rightarrow ^A_{Z-1} Y + e^+ + \nu_e \]

Electron capture  
\[ p + e^- \rightarrow n + \nu_e \quad \]
\[ ^A_Z X + e^- \rightarrow ^A_{Z-1} Y + \nu_e \]

- \( \beta \) decay is a weak interaction mediated by the \( W \) boson.
- Parity is violated in \( \beta \) decay.
- Responsible for Fermi postulating the existence of the neutrino.
- Kinematics: Decay is possible if energy release \( E_0 > 0 \)

\[ \begin{align*}
\text{Nuclear Masses} \quad & \quad \text{Atomic Masses} \\
\beta^- & \quad E_0 = m_X - m_Y - m_e - m_\nu \quad & \quad E_0 = M_X - M_Y - m_\nu \\
\beta^+ & \quad E_0 = m_X - m_Y - m_e - m_\nu \quad & \quad E_0 = M_X - M_Y - 2m_e - m_\nu \\
e.c. & \quad E_0 = m_X - m_Y + m_e - m_\nu \quad & \quad E_0 = M_X - M_Y - m_\nu \\
\quad (\text{and note that } m_\nu \sim 0) & \quad \text{using } M(A, Z) = m(A, Z) + Zm_e \\
n.b. \text{ electron capture may be possible even if } \beta^+ \text{ not allowed}
\end{align*} \]
Consider nuclear mass as a function of $N$ and $Z$

$$m(A, Z) = Zm_p + (A - Z)m_n - a_V A + a_S A^{2/3} + \frac{a_C Z^2}{A^{1/3}} + a_A \frac{(N - Z)^2}{A} - \delta(A)$$

For $\beta$ decay, $A$ is constant, but $Z$ changes by $\pm 1$ and $m(A, Z)$ is quadratic in $Z$

Most stable nuclide when

$$\left[ \frac{\partial m(A, Z)}{\partial Z} \right]_{A} = 0$$
Typical situation at constant $A$.

Usually only one isotope table against $\beta$-decay; occasionally two.

Typically two even-even nuclides are stable against $\beta$-decay; almost no odd-odd ones (pairing term).
Fermi Theory of $\beta$-decay

In nuclear decay, weak interaction taken to be a 4-fermion contact interaction:

\[ X \rightarrow Y \ e^- \ \bar{\nu}_e \]

No “propagator” – absorb the effect of the exchanged $W$ boson into an effective coupling strength given by the Fermi constant $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$.

Use Fermi's Golden Rule to get the transition rate

\[ \Gamma = 2\pi |M_{fi}|^2 \rho(\E_f) \]

where $M_{fi}$ is the matrix element and $\rho(\E_f) = \frac{dN}{d\E_f}$ is the density of final states.

\[ \Gamma = \frac{G_F^2 |M_{\text{nuclear}}|^2}{2\pi^3} \int_0^{\E_0} (\E_0 - \E_e)^2 \E_e^2 d\E_e \]

Total decay rate given by Sargent's Rule, $\Gamma \propto \E_0^5$
Fermi Theory of $\beta$-decay

$\beta$ decay spectrum described by

$$\sqrt{\frac{d\Gamma}{dp_e p_e^2}} \propto (E_0 - E_e)$$

Kurie Plot

$^3\text{H} \rightarrow ^3\text{He} + e^- + \bar{\nu}_e$

Endpoint $E_0$
Fermi Theory of $\beta$-decay

BUT, the momentum of the electron is modified by the Coulomb interaction as it moves away from the nucleus (different for $e^-$ and $e^+$).

$\Rightarrow$ Multiply spectrum by Fermi function $F(Z_Y, E_e)$

$$\Gamma = \frac{G_F^2 |M_{\text{nuclear}}|^2}{2\pi^3} \int_0^{E_0} (E_0 - E_e)^2 E_e^2 F(Z_Y, E_e) \, dE_e$$

All the information about the nuclear wavefunctions is contained in the matrix element. Values for the complicated Fermi Integral are tabulated.

$$f(Z_Y, E_0) = \frac{1}{m_e^5} \int_0^{E_0} (E_0 - E_e)^2 E_e^2 F(Z_Y, E_e) \, dE_e$$

Mean lifetime $\tau = 1/\Gamma$, half-life $\tau_{1/2} = \frac{\ln 2}{\Gamma}$

$$f\tau_{1/2} = \ln 2 \frac{2\pi^3}{m_e^5 G_F^2 |M_{\text{nuclear}}|^2}$$

Comparative half-life

this is rather useful because it depends only on the nuclear matrix element
Fermi Theory of $\beta$-decay

Comparative half-lives

In rough terms, decays with

$\log f \tau_{1/2} \sim 3 - 4$ known as super-allowed

$\sim 4 - 7$ known as allowed

$\geq 6$ known as forbidden (i.e. suppressed, small $M_{if}$)

Number of cases

log $f \tau$

Prof. Tina Potter

15. Nuclear Decay
Fermi Theory of $\beta$-decay  

Selection Rules

Fermi theory

$$M_{fi} = G_F \int \psi_p^* e^{-i(\vec{p}_e + \vec{p}_\nu) \cdot \vec{r}} \psi_n \, d^3 \vec{r}$$

$e, \nu$ wavefunctions

Allowed Transitions $\log_{10} f_{\tau_{1/2}} \sim 4 - 7$

Angular momentum of $e\nu$ pair relative to nucleus, $\ell = 0$.

Equivalent to: $e^{-i(\vec{p}_e + \vec{p}_\nu) \cdot \vec{r}} \sim 1$

Superallowed Transitions $\log_{10} f_{\tau_{1/2}} \sim 3 - 4$

subset of Allowed transitions: often mirror nuclei in which $p$ and $n$ have approximately the same wavefunction

$$M_{\text{nuclear}} \sim \int \psi_p^* \psi_n \, d^3 \vec{r} \sim 1$$

$e, \nu$ both have spin $1/2$ $\Rightarrow$ Total spin of $e\nu$ system can be $S_{e\nu} = 0$ or $1$.

There are two types of allowed/superallowed transitions depending on the relative spin states of the emitted $e$ and $\nu$...
Fermi Theory of $\beta$-decay

Selection Rules

For allowed/superallowed transitions, $\ell_{e\nu} = 0$

$S_{e\nu} = 0$ Fermi transitions

$$\begin{align*}
n\uparrow & \rightarrow p\uparrow + \frac{1}{\sqrt{2}} \left[ (e^- \uparrow \bar{\nu}_e \downarrow) - (e^- \downarrow \bar{\nu}_e \uparrow) \right] \\
\Delta J & = 0
\end{align*}$$

$S_{e\nu} = 0, m_s = 0$

$J_X = J_Y$

$S_{e\nu} = 1$ Gamow-Teller transitions

$$\begin{align*}
n\uparrow & \rightarrow p\uparrow + \frac{1}{\sqrt{2}} \left[ (e^- \uparrow \bar{\nu}_e \downarrow) + (e^- \downarrow \bar{\nu}_e \uparrow) \right] \\
\Delta J & = 0
\end{align*}$$

$S_{e\nu} = 1, m_s = 0$

$J_X = J_Y$

$S_{e\nu} = 1, m_s = \pm 1$

$J_X = J_Y \pm 1$

No change in angular momentum of the $e\nu$ pair relative to the nucleus, $\ell_{e\nu} = 0$

$\Rightarrow$ Parity of nucleus unchanged
Fermi Theory of $\beta$-decay 

**Selection Rules**

**Forbidden Transitions** $\log_{10} f_{\tau_{1/2}} \geq 6$

Angular momentum of $e\nu$ pair relative to nucleus, $\ell_{e\nu} > 0$.

$$e^{-i(\vec{p}_e + \vec{p}_\nu) \cdot \vec{r}} = 1 - i(\vec{p}_e + \vec{p}_\nu) \cdot \vec{r} + \frac{1}{2} \left[(\vec{p}_e + \vec{p}_\nu) \cdot \vec{r}\right]^2 - ...$$

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P = (-1)^\ell$</td>
<td>even</td>
<td>odd</td>
<td>even</td>
</tr>
</tbody>
</table>

*Allowed* 1\textsuperscript{st} forbidden 2\textsuperscript{nd} forbidden

Transition probabilities for $\ell > 0$ are small $\Rightarrow$ forbidden transitions (really means “suppressed”).

Forbidden transitions are only competitive if an allowed transition cannot occur (selection rules). Then the lowest permitted order of “forbiddeness” will dominate.

In general, $n^{th}$ forbidden $\Rightarrow$ $e\nu$ system carries orbital angular momentum $\ell = n$, and $S_{e\nu} = 0$ (Fermi) or 1 (G-T). Parity change if $\ell$ is odd.
Fermi Theory of $\beta$-decay

Selection Rules

Examples

$^{34}\text{Cl}(0^+) \rightarrow ^{34}\text{S}(0^+)$

$^{14}\text{C}(0^+) \rightarrow ^{14}\text{N}(1^+)$

$n(1/2^+) \rightarrow p(1/2^+)$

$^{39}\text{Ar}(7/2^-) \rightarrow ^{39}\text{K}(3/2^+)$

$^{87}\text{Rb}(3/2^-) \rightarrow ^{87}\text{Sr}(9/2^+)$
Emission of $\gamma$-rays (EM radiation) occurs when a nucleus is created in an excited state (e.g. following $\alpha$, $\beta$ decay or collision).

The photon carries away net angular momentum $\ell_\gamma$ when a proton in the nucleus makes a transition from its initial a.m. state $J_i$ to its final a.m. state $J_f$.

$$\vec{J}_i = \ell_\gamma \oplus \vec{J}_f \quad \text{and} \quad |\vec{J}_i - \vec{J}_f| \leq \ell_\gamma \leq |\vec{J}_i + \vec{J}_f|$$

The photon carries $J^P = 1^-$ $\Rightarrow \ell_\gamma \geq 1$.

$\Rightarrow$ Single $\gamma$ emission is forbidden for a transition between two $J = 0$ states. (0 $\rightarrow$ 0 transitions can only occur via internal conversion (emitting an electron) or via the emission of more than one $\gamma$.)
Radiative transitions in nuclei are generally the same as for atoms, except

**Atom** \( E_\gamma \sim \text{eV} ; \quad \lambda \sim 10^8 \text{ fm} \sim 10^3 \times r_{\text{atom}} ; \quad \Gamma \sim 10^9 \text{s}^{-1} \)

Only dipole transitions are important.

**Nuclei** \( E_\gamma \sim \text{MeV} ; \quad \lambda \sim 10^2 \text{ fm} \sim 25 \times r_{\text{nucl}} ; \quad \Gamma \sim 10^{16} \text{s}^{-1} \)

Collective motion of many protons lead to higher transition rates.

\[ \Rightarrow \] Higher order transitions are also important.

Two types of transitions:

**Electric (E) transitions** arise from an oscillating charge which causes an oscillation in the external electric field.

**Magnetic (M) transitions** arise from a varying current or magnetic moment which sets up a varying magnetic field.

Obtain transition probabilities using Fermi’s Golden Rule

\[ \Gamma = 2\pi |M_{if}|^2 \rho(E_f) \]
\[ \Gamma_{i \rightarrow f} = \frac{\omega^3}{3\pi\varepsilon_0 c^3 \hbar} |\langle \psi_f | e \hat{r} | \psi_i \rangle|^2 \]

see Adv. Quantum Physics; after averaging over initial and summing over final states

Order of magnitude estimate of this rate,

\[ |\langle \psi_f | e \hat{r} | \psi_i \rangle|^2 \sim |eR|^2 \Rightarrow \Gamma \sim \frac{4}{3} \alpha E_\gamma^3 R^2 \]

\( R = \text{radius of nucleus}, \quad \alpha = \frac{e^2}{4\pi\varepsilon_0 c\hbar}, \quad E_\gamma = \hbar\omega, \quad \hbar = c = 1. \)

e.g. \( E_\gamma = 1 \text{ MeV}, \ R = 5 \text{ fm} \) \((\hbar c = 197 \text{ MeVfm}, \ \hbar = 6.6 \times 10^{-22} \text{ MeVs})\)

\[ \Gamma(E1) = 0.24 \text{ MeV}^3\text{fm}^2 = \frac{0.24}{(197)^2 \times 6.6 \times 10^{-22}} \text{s}^{-1} = 10^{16} \text{s}^{-1} \] \( \text{c.f. atoms } \Gamma \sim 10^9 \text{s}^{-1} \)

As nuclear wavefunctions have definite parity, the matrix element can only be non-zero if the initial and final states have opposite parity.

\[ e \hat{r} \xrightarrow{\hat{P}} - e \hat{r} \quad \text{ODD} \]

E1 transition \( \Rightarrow \) parity change of nucleus
Magnetic dipole matrix element
\[ |\langle \psi_f | \mu \vec{\sigma} | \psi_i \rangle |^2 \]

\( \mu = \) magnetic moment, \( \vec{\sigma} = \) Pauli spin matrices

Typically
\[ \langle \mu \sigma \rangle \sim \frac{e\hbar}{2m_p} = \mu_N \]  
Nuclear magneton

For a proton
\[ \frac{\hbar}{m_p} \sim 0.2 \text{fm} \sim \frac{R}{25} \]  
for \( R = 5 \text{ fm} \)

Compare to E1 transition rate
\[ \frac{\Gamma(M1)}{\Gamma(E1)} = \left( \frac{e\hbar}{2m_p} \right)^2 \frac{1}{(eR)^2} = 10^{-3} \]

Magnetic moment transforms the same way as angular momentum
\[ e\vec{r} \times \vec{p} \rightarrow e(-\vec{r}) \times (-\vec{p}) = e\vec{r} \times \vec{p} \]  
EVEN

M1 transition \( \Rightarrow \) no parity change of nucleus
**γ Decay**  

**Higher Order Transitions (\(E_\ell, M_\ell\), where \(\ell > 1\))**

If the initial and final nuclear states differ by more than 1 unit of angular momentum  
\[ \Rightarrow \text{higher multipole radiation} \]

The perturbing Hamiltonian is a function of electric and magnetic fields and hence of the vector potential \[ \langle \psi_f | H'(\vec{A}) | \psi_i \rangle \]

\(\vec{A}\) for a photon is taken to have the form of a plane wave  
\[ \vec{A}e^{i\vec{p}.\vec{r}} = 1 - i\vec{p}.\vec{r} + \frac{1}{2}(\vec{p}.\vec{r})^2 + \frac{(-i\vec{p}.\vec{r})^n}{n!} \]

\(\ell = 1\)  \(\ell = 2\)  \(\ell = 3\)  
Dipole  Quadrupole  Octupole  
\(E_1,M_1\)  \(E_2,M_2\)  \(E_3,M_3\)

Each successive term in the expansion of \(\vec{A}\) is reduced from the previous one by a factor of roughly \(\vec{p}.\vec{r}\).

**e.g.** Compare \(E_1\) to \(E_2\) for \(p \sim 1\) MeV, \(R \sim 5\) fm  
\[ pR \sim 5\text{ MeVfm} \sim 0.025, \ \ |pR|^2 \sim 10^{-3} \]

\[ \frac{\Gamma(E_2)}{\Gamma(E_1)} \sim 10^{-3} \sim \frac{\Gamma(M_1)}{\Gamma(E_1)} \]

The matrix element for \(E_2\) transitions \(\sim r^2\) i.e. even under a parity transformation.
In general, $E^\ell$ transitions Parity $= (-1)^\ell$

$M^\ell$ transitions Parity $= (-1)^{\ell+1}$

<table>
<thead>
<tr>
<th>Parity change</th>
<th>$\gamma$</th>
<th>$1^-$</th>
<th>$2^+$</th>
<th>$3^-$</th>
<th>$4^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate</td>
<td>1</td>
<td>$10^{-3}$</td>
<td>$10^{-6}$</td>
<td>$10^{-9}$</td>
<td>...</td>
</tr>
<tr>
<td>$E^1$</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>$E^2$</td>
<td>✔</td>
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<tr>
<td>$E^3$</td>
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<td></td>
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<td>✔</td>
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<tr>
<td>$E^4$</td>
<td>✔</td>
<td></td>
<td></td>
<td></td>
<td>✔</td>
</tr>
<tr>
<td>$M^1$</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>$M^2$</td>
<td>✔</td>
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<td>✔</td>
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<td>✔</td>
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<tr>
<td>$M^3$</td>
<td>✔</td>
<td></td>
<td></td>
<td>✔</td>
<td></td>
</tr>
</tbody>
</table>

In general, a decay will proceed dominantly by the lowest order (i.e. fastest) process permitted by angular momentum and parity.

**e.g.** if a process has $\Delta J = 2$, no parity change, it will go by the $E2$, even though $M3$, $E4$ are also allowed.
Information about the nature of transitions (based on rates and angular distributions) is very useful in inferring the $J^P$ values of states.

Please note: this discussion of rates is fairly naïve. More complete formulae can be found in textbooks.

Also collective effects may be important if
- many nucleons participate in transitions,
- nucleus has a large electric quadrupole moment, $Q$, $\rightarrow$ rotational excited states enhance E2 transitions.
Summary

- **Radioactive decays and dating.**

  - **α-decay** Strong dependence on $E$, $Z$
    Tunnelling model (Gamow) – Geiger-Nuttall law $\ln \frac{\tau_1}{2} \sim \frac{Z'}{E_0^{1/2}} + \text{const.}$

  - **β-decay** $\beta^+$, $\beta^-$, electron capture; energetics, stability
    Fermi theory – 4-fermion interaction plus 3-body phase space.
    \[
    \Gamma = \frac{G_F^2 |M_{\text{nuclear}}|^2}{2\pi^3} \int_0^{E_0} (E_0 - E_e)^2 p_e^2 dp_e
    \]
    Electron energy spectrum; Kurie plot.
    Comparative half-lives.
    Selection rules; Fermi, Gamow-Teller; allowed, forbidden.

  - **γ-decay** Dipole, quadrupole; electric, magnetic transitions.
    Selection rules.

Up next...
Section 16: Fission and Fusion