8. Quark Model of Hadrons
Particle and Nuclear Physics
In this section...

- Hadron wavefunctions and parity
- Light mesons
- Light baryons
- Charmonium
- Bottomonium
Evidence for quarks
- The magnetic moments of proton and neutron are not $\mu_N = e\hbar/2m_p$ and 0 respectively ⇒ not point-like
- Electron-proton scattering at high $q^2$ deviates from Rutherford scattering ⇒ proton has substructure
- Hadron jets are observed in $e^+e^-$ and $pp$ collisions
- Symmetries (patterns) in masses and properties of hadron states, “quarky” periodic table ⇒ sub-structure
- Steps in $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$
- Observation of $c\bar{c}$ and $b\bar{b}$ bound states
- and much, much more...

Here, we will first consider the wave-functions for hadrons formed from light quarks ($u$, $d$, $s$) and deduce some of their static properties (mass and magnetic moments).
Then we will go on to discuss the heavy quarks ($c$, $b$).
We will cover the $t$ quark later…
Hadron Wavefunctions

Quarks are always confined in hadrons (i.e. colourless states)

Mesons
Spin 0, 1, ...

Baryons
Spin 1/2, 3/2, ...

Treat quarks as identical fermions with states labelled with spatial, spin, flavour and colour.

\[ \psi = \psi_{\text{space}} \psi_{\text{flavour}} \psi_{\text{spin}} \psi_{\text{colour}} \]

All hadrons are colour singlets, i.e. net colour zero

Mesons
\[ \psi_{\text{colour}}^{qq} = \frac{1}{\sqrt{3}} (r\bar{r} + g\bar{g} + b\bar{b}) \]

Baryons
\[ \psi_{\text{colour}}^{qqq} = \frac{1}{\sqrt{6}} (rgb + gbr + brg - grb - rbg - bgr) \]
The Parity operator, $\hat{P}$, performs spatial inversion
\[ \hat{P}|\psi(\vec{r}, t)\rangle = |\psi(-\vec{r}, t)\rangle \]

The eigenvalue of $\hat{P}$ is called Parity
\[ \hat{P}|\psi\rangle = P|\psi\rangle, \quad P = \pm 1 \]

Most particles are eigenstates of Parity and in this case $P$ represents intrinsic Parity of a particle/antiparticle.

Parity is a useful concept. If the Hamiltonian for an interaction commutes with $\hat{P}$
\[ [\hat{P}, \hat{H}] = 0 \]

then Parity is conserved in the interaction:

\textbf{Parity conserved} in the strong and EM interactions, but \textbf{not} in the weak interaction.
Parity

- Composite system of two particles with orbital angular momentum $\ell$:

$$P = P_1 P_2 (-1)^\ell$$

where $P_{1,2}$ are the intrinsic parities of particles 1, 2.

**Quantum Field Theory** tells us that

- Fermions and antifermions: opposite parity
- Bosons and antibosons: same parity

**Choose:**

- Quarks and leptons: $P_{q/\ell} = +1$
- Antiquarks and antileptons: $P_{\bar{q},\bar{\ell}} = -1$

**Gauge Bosons:** $(\gamma, g, W, Z)$ are vector fields which transform as

$$J^P = 1^-$$

$$P_\gamma = -1$$
Light Mesons

Mesons are bound $q\bar{q}$ states.
Consider ground state mesons consisting of light quarks ($u, d, s$).

\[ m_u \sim 0.3 \text{ GeV}, \quad m_d \sim 0.3 \text{ GeV}, \quad m_s \sim 0.5 \text{ GeV} \]

- **Ground State ($\ell = 0$):** Meson “spin” (total angular momentum) is given by the $q\bar{q}$ spin state.
  
  Two possible $q\bar{q}$ total spin states: $S = 0, 1$
  
  $S = 0$: pseudoscalar mesons
  
  $S = 1$: vector mesons

- **Meson Parity:** ($q$ and $\bar{q}$ have opposite parity)

  \[ P = P_qP_{\bar{q}}(-1)^\ell = (+1)(-1)(-1)^\ell = -1 \quad \text{(for $\ell = 0$)} \]

- **Flavour States:** $u\bar{d}, u\bar{s}, d\bar{u}, d\bar{s}, s\bar{u}, s\bar{d}$ and $u\bar{u}, d\bar{d}, s\bar{s}$ mixtures

  Expect: Nine $J^P = 0^-$ mesons: Pseudoscalar nonet
  
  Nine $J^P = 1^-$ mesons: Vector nonet
Basic quark multiplet – plot the quantum numbers of (anti)quarks:

**Quarks**

\[ J^P = \frac{1^+}{2} \]

**Antiquarks**

\[ J^P = \frac{1^-}{2} \]

**Mesons**

Spin \( J = 0 \) or \( 1 \)

The ideas of strangeness and isospin are historical quantum numbers assigned to different states. Essentially they count quark flavours (this was all before the formulation of the Quark Model).

\[
\text{Isospin} = \frac{1}{2}(n_u - n_d - n_{\bar{u}} + n_{\bar{d}})
\]

\[
\text{Strangeness} = n_{\bar{s}} - n_s
\]
Light Mesons

Pseudoscalar nonet

\[ J^P = 0^- \]

\[ \pi^0, \eta, \eta' \] are combinations of \( u\bar{u}, d\bar{d}, s\bar{s} \)

\[
\begin{align*}
\pi^0 & \quad K^0 \quad d\bar{s} \\
\eta & \quad \eta' \\
K^- & \quad s\bar{u} \\
\end{align*}
\]

Masses / MeV

\[ \pi(140), \ K(495) \]

\[ \eta(550), \ \eta'(960) \]

Vector nonet

\[ J^P = 1^- \]

\[ \rho^0, \phi, \omega^0 \] are combinations of \( u\bar{u}, d\bar{d}, s\bar{s} \)

\[
\begin{align*}
\rho^0 & \quad K^* \quad d\bar{s} \\
\omega^0 & \quad \phi \\
K^{*-} & \quad s\bar{u} \\
\end{align*}
\]

Masses / MeV

\[ \rho(770), \ K^*(890) \]

\[ \omega(780), \ \phi(1020) \]
The states $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$ all have zero flavour quantum numbers and can mix.

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$  \hspace{1cm}  $$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

$$\eta = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$  \hspace{1cm}  $$\omega^0 = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$

$$\eta^' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$  \hspace{1cm}  $$\phi = s\bar{s}$$

Mixing coefficients determined experimentally from meson masses and decays.

**Example:** Leptonic decays of vector mesons

$$M(\rho^0 \rightarrow e^+e^-) \sim \frac{e}{q^2} \left[ \frac{1}{\sqrt{2}}(Q_{u}e - Q_{d}e) \right]$$

$$\Gamma(\rho^0 \rightarrow e^+e^-) \propto \left[ \frac{1}{\sqrt{2}} \left( \frac{2}{3} - \left( -\frac{1}{3} \right) \right) \right]^2 = \frac{1}{2}$$

$$\Gamma(\omega^0 \rightarrow e^+e^-) \propto \left[ \frac{1}{\sqrt{2}} \left( \frac{2}{3} + \left( -\frac{1}{3} \right) \right) \right]^2 = \frac{1}{18}$$

$$\Gamma(\phi \rightarrow e^+e^-) \propto \left[ \frac{1}{3} \right]^2 = \frac{1}{9}$$

**Predict:** $\Gamma_{\rho^0} : \Gamma_{\omega^0} : \Gamma_{\phi} = 9 : 1 : 2$  \hspace{1cm}  **Experiment:** $(8.8 \pm 2.6) : 1 : (1.7 \pm 0.4)$
Meson Masses

Meson masses are only partly from constituent quark masses:

\[ m(K) > m(\pi) \Rightarrow \text{suggests } m_s > m_u, m_d \]

495 MeV   140 MeV

Not the whole story...

\[ m(\rho) > m(\pi) \Rightarrow \text{although both are } u\bar{d} \]

770 MeV   140 MeV

Only difference is the orientation of the quark spins (↑↑ vs ↑↓)

⇒ spin-spin interaction
**QED:** Hyperfine splitting in H$_2$ ($\ell = 0$)

Energy shift due to electron spin in magnetic field of proton

$$\Delta E = \vec{\mu}.\vec{B} = \frac{2}{3}\vec{\mu}_e.\vec{\mu}_p|\psi(0)|^2$$

and using $\vec{\mu} = \frac{e}{2m}\vec{S}$

$$\Delta E \propto \frac{\vec{S}_e \cdot \vec{S}_p}{m_em_p}$$

**QCD:** Colour Magnetic Interaction

Fundamental form of the interaction between a quark and a gluon is identical to that between an electron and a photon. Consequently, also have a colour magnetic interaction

$$\Delta E \propto \frac{\vec{S}_1 \cdot \vec{S}_2}{m_1m_2}$$
Meson Masses

Meson Mass Formula ($\ell = 0$)

\[ M_{q\bar{q}} = m_1 + m_2 + A \frac{\vec{S}_1 \cdot \vec{S}_2}{m_1 m_2} \]

where $A$ is a constant

For a state of spin

\[ \vec{S} = \vec{S}_1 + \vec{S}_2 \]

\[ \vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1 \cdot \vec{S}_2 \]

\[ \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2} \left( \vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2 \right) \]

\[ \vec{S}_1^2 = \vec{S}_2^2 = \vec{S}_1(\vec{S}_1 + 1) = \frac{1}{2} \left( \frac{1}{2} + 1 \right) = \frac{3}{4} \]

giving

\[ \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2} \vec{S}^2 - \frac{3}{4} \]

For $J^P = 0^-$ mesons:

\[ \vec{S}^2 = 0 \quad \Rightarrow \quad \vec{S}_1 \cdot \vec{S}_2 = -\frac{3}{4} \]

For $J^P = 1^-$ mesons:

\[ \vec{S}^2 = S(S+1) = 2 \quad \Rightarrow \quad \vec{S}_1 \cdot \vec{S}_2 = +\frac{1}{4} \]

Giving the ($\ell = 0$) Meson Mass formulae:

\[ M_{q\bar{q}} = m_1 + m_2 - \frac{3A}{4m_1 m_2} \quad (J^P = 0^-) \]

\[ M_{q\bar{q}} = m_1 + m_2 + \frac{A}{4m_1 m_2} \quad (J^P = 1^-) \]

So $J^P = 0^-$ mesons are lighter than $J^P = 1^-$ mesons
Excellent fit obtained to masses of the different flavour pairs \((u\bar{d}, u\bar{s}, d\bar{u}, d\bar{s}, s\bar{u}, s\bar{d})\) with 

\[
\begin{align*}
m_u &= 0.305 \text{ GeV}, & m_d &= 0.308 \text{ GeV}, & m_s &= 0.487 \text{ GeV}, & A &= 0.06 \text{ GeV}^3 
\end{align*}
\]

\(\eta\) and \(\eta'\) are mixtures of states, e.g.

\[
\eta = \frac{1}{\sqrt{6}} (u\bar{u} + d\bar{d} - 2s\bar{s}) \quad M_\eta = \frac{1}{6} \left( 2m_u - \frac{3A}{4m_u^2} \right) + \frac{1}{6} \left( 2m_d - \frac{3A}{4m_d^2} \right) + \frac{4}{6} \left( 2m_s - \frac{3A}{4m_s^2} \right)
\]
Baryons made from 3 indistinguishable quarks (flavour can be treated as another quantum number in the wave-function)

\[ \psi_{\text{baryon}} = \psi_{\text{space}} \psi_{\text{flavour}} \psi_{\text{spin}} \psi_{\text{colour}} \]

\( \psi_{\text{baryon}} \) must be anti-symmetric under interchange of any 2 quarks

**Example:** \( \Omega^- (sss) \) wavefunction \((\ell = 0, J = 3/2)\)

\[ \psi_{\text{spin}} \psi_{\text{flavour}} = s \uparrow \, s \uparrow \, s \uparrow \] is symmetric \(\Rightarrow\) require antisymmetric \( \psi_{\text{colour}} \)

**Ground State** \((\ell = 0)\)

We will only consider the baryon ground states, which have zero orbital angular momentum

\[ \psi_{\text{space}} \] symmetric

\( \rightarrow \) All hadrons are colour singlets

\[ \psi_{\text{colour}} = \frac{1}{\sqrt{6}} (rgb + gbr + brg - grb - rbg - bgr) \] antisymmetric

Therefore, \( \psi_{\text{spin}} \psi_{\text{flavour}} \) must be symmetric
Baryon spin wavefunctions ($\psi_{\text{spin}}$)

Combine 3 spin 1/2 quarks: Total spin $J = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} = \frac{1}{2}$ or $\frac{3}{2}$

Consider $J = 3/2$

Trivial to write down the spin wave-function for the $|\frac{3}{2}, \frac{3}{2}\rangle$ state: $|\frac{3}{2}, \frac{3}{2}\rangle = \uparrow\uparrow\uparrow$

Generate other states using the ladder operator $\hat{J}_-$

$$\hat{J}_- |\frac{3}{2}, \frac{3}{2}\rangle = (\hat{J}_- \uparrow) \uparrow\uparrow + \uparrow (\hat{J}_- \uparrow) \uparrow + \uparrow\uparrow (\hat{J}_- \uparrow)$$

$$\sqrt{\frac{35}{22} - \frac{31}{22}} |\frac{3}{2}, \frac{1}{2}\rangle = \downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow$$

$$|\frac{3}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{3}} (\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow)$$

Giving the $J = 3/2$ states: →

All symmetric under interchange of any two spins

$|\frac{3}{2}, \frac{3}{2}\rangle = \uparrow\uparrow\uparrow$

$|\frac{3}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{3}} (\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow)$

$|\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{3}} (\uparrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow)$

$|\frac{3}{2}, \frac{1}{2}\rangle = \downarrow\downarrow\downarrow$
Baryon spin wavefunctions ($\psi_{\text{spin}}$)

Consider $J = 1/2$

First consider the case where the first 2 quarks are in a $|0, 0\rangle$ state:

$$|0, 0\rangle_{(12)} = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle_{(12)} = |0, 0\rangle_{(12)} \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \frac{1}{\sqrt{2}}(\uparrow\uparrow\uparrow - \downarrow\uparrow\uparrow)$$

$$\left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{(12)} = |0, 0\rangle_{(12)} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$

Antisymmetric under exchange 1 $\leftrightarrow$ 2.

Three-quark $J = 1/2$ states can also be formed from the state with the first two quarks in a symmetric spin wavefunction.

Can construct a three-particle state $\left| \frac{1}{2}, \frac{1}{2} \right\rangle_{(123)}$ from

$$|1, 0\rangle_{(12)} \left| \frac{1}{2}, \frac{1}{2} \right\rangle_{(3)} \quad \text{and} \quad |1, 1\rangle_{(12)} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{(3)}$$
Baryon spin wavefunctions ($\psi_{\text{spin}}$)

Taking the linear combination

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle = a |1, 1\rangle \left| \frac{1}{2}, -\frac{1}{2}\right\rangle + b |1, 0\rangle \left| \frac{1}{2}, \frac{1}{2}\right\rangle$$

with $a^2 + b^2 = 1$. Act upon both sides with $\hat{J}_+$

$$\hat{J}_+ \left| \frac{1}{2}, \frac{1}{2} \right\rangle = a \left[ \left( \hat{J}_+ |1, 1\rangle \right) \left| \frac{1}{2}, -\frac{1}{2}\right\rangle + |1, 1\rangle \left( \hat{J}_+ \left| \frac{1}{2}, -\frac{1}{2}\right\rangle \right) \right] + b \left[ \left( \hat{J}_+ |1, 0\rangle \right) \left| \frac{1}{2}, \frac{1}{2}\right\rangle + |1, 0\rangle \left( \hat{J}_+ \left| \frac{1}{2}, \frac{1}{2}\right\rangle \right) \right]$$

$$0 = a |1, 1\rangle \left| \frac{1}{2}, \frac{1}{2}\right\rangle + \sqrt{2}b |1, 1\rangle \left| \frac{1}{2}, \frac{1}{2}\right\rangle$$

$$a = -\sqrt{2}b$$

$$\hat{J}_+ |j, m\rangle = \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle$$

which with $a^2 + b^2 = 1$ implies $a = \sqrt{\frac{2}{3}}, b = -\sqrt{\frac{1}{3}}$

Giving

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}} |1, 1\rangle \left| \frac{1}{2}, -\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}} |1, 0\rangle \left| \frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$|1, 1\rangle = \uparrow\uparrow$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}} (\uparrow\downarrow + \downarrow\uparrow)$$

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle = \frac{1}{\sqrt{6}} (2 \uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\downarrow)$$

$$\left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{1}{\sqrt{6}} (2 \downarrow\uparrow\downarrow - \downarrow\downarrow\uparrow - \uparrow\downarrow\uparrow)$$

Symmetric under interchange $1 \leftrightarrow 2$
Three-quark spin wavefunctions

\[ |\frac{3}{2}, \frac{3}{2}\rangle = \uparrow\uparrow\uparrow \]

\[ |\frac{3}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{3}}(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow) \]

\[ |\frac{3}{2}, -\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow + \uparrow\uparrow\downarrow) \]

Symmetric under interchange of any 2 quarks

\[ |\frac{3}{2}, -\frac{3}{2}\rangle = \downarrow\downarrow\downarrow \]

\[ |\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \]

Antisymmetric under interchange of 1 \(\leftrightarrow\) 2

\[ |\frac{1}{2}, -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \]

Symmetric under interchange of 1 \(\leftrightarrow\) 2

\[ |\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \]

\[ |\frac{1}{2}, -\frac{1}{2}\rangle = \frac{1}{\sqrt{6}}(2\downarrow\uparrow\downarrow - \downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow) \]

\[ \psi_{\text{spin}} \psi_{\text{flavour}} \] must be symmetric under interchange of any 2 quarks.
Three-quark spin wavefunctions

Consider 3 cases:

1. **Quarks all same flavour:** \( uuu, ddd, sss \)
   - \( \psi_{\text{flavour}} \) is symmetric under interchange of any two quarks
   - Require \( \psi_{\text{spin}} \) to be symmetric under interchange of any two quarks
   - Only satisfied by \( J = \frac{3}{2} \) states
   - There are no \( J = \frac{1}{2} \) \( uuu, ddd, sss \) baryons with \( \ell = 0 \).
   - Three \( J = \frac{3}{2} \) states: \( uuu, ddd, sss \)

2. **Two quarks have same flavour:** \( uud, uus, ddu, dds, ssu, ssd \)
   - For the like quarks \( \psi_{\text{flavour}} \) is symmetric
   - Require \( \psi_{\text{spin}} \) to be symmetric under interchange of like quarks \( 1 \leftrightarrow 2 \)
   - Satisfied by \( J = \frac{3}{2} \) and \( J = \frac{1}{2} \) states
   - Six \( J = \frac{3}{2} \) states and six \( J = \frac{1}{2} \) states: \( uud, uus, ddu, dds, ssu, ssd \)
All quarks have different flavours: $uds$

Two possibilities for the $(ud)$ part:

- **Flavour Symmetric** $\frac{1}{\sqrt{2}}(ud + du)$
  - Require $\psi_{\text{spin}}$ to be symmetric under interchange of $ud$
  - Satisfied by $J = 3/2$ and $J = 1/2$ states

  One $J = 3/2$ and one $J = 1/2$ state: $uds$

- **Flavour Antisymmetric** $\frac{1}{\sqrt{2}}(ud - du)$
  - Require $\psi_{\text{spin}}$ to be antisymmetric under interchange of $ud$
  - Only satisfied by $J = 1/2$ state

  One $J = 1/2$ state: $uds$

**Quark Model predicts that light baryons appear in**

- Decuplets (10) of spin 3/2 states
- Octets (8) of spin 1/2 states
Antibaryons are in separate multiplets

Example:
Antiparticle of $\Sigma^+(uus)$ is $\bar{\Sigma}^-(\bar{u}\bar{u}s)$, $J^P = \frac{1}{2}^-$ and not $\Sigma^-(dds)$, $J^P = \frac{1}{2}^+$
Baryon Masses

Baryon Mass Formula ($\ell = 0$)

\[ M_{qqq} = m_1 + m_2 + m_3 + A' \left( \frac{\vec{S}_1 \cdot \vec{S}_2}{m_1 m_2} + \frac{\vec{S}_1 \cdot \vec{S}_3}{m_1 m_3} + \frac{\vec{S}_2 \cdot \vec{S}_3}{m_2 m_3} \right) \]

where $A'$ is a constant

Example: All quarks have the same mass, $m_1 = m_2 = m_3 = m_q$

\[ M_{qqq} = 3m_q + A' \sum_{i<j} \frac{\vec{S}_i \cdot \vec{S}_j}{m_q^2} \]

\[ \vec{S}^2 = \left( \vec{S}_1 + \vec{S}_2 + \vec{S}_3 \right)^2 = \vec{S}_1^2 + \vec{S}_2^2 + \vec{S}_3^2 + 2 \sum_{i<j} \vec{S}_i \cdot \vec{S}_j \]

\[ 2 \sum_{i<j} \vec{S}_i \cdot \vec{S}_j = S(S+1) - 3 \frac{1}{2} \left( 1 + 1 \right) = S(S+1) - \frac{9}{4} \]

\[ \sum_{i<j} \vec{S}_i \cdot \vec{S}_j = -\frac{3}{4} \left( J = \frac{1}{2} \right) \quad \sum_{i<j} \vec{S}_i \cdot \vec{S}_j = +\frac{3}{4} \left( J = \frac{3}{2} \right) \]

e.g. proton ($uud$) compared with $\Delta$ ($uud$) – same quark content

\[ M_p = 3m_u - \frac{3A'}{4m_u^2}, \quad M_\Delta = 3m_u + \frac{3A'}{4m_u^2} \]
Excellent agreement using

\[ m_u = 0.362 \text{ GeV}, \quad m_d = 0.366 \text{ GeV}, \quad m_s = 0.537 \text{ GeV}, \quad A' = 0.026 \text{ GeV}^3 \sim A/2 \]

Constituent quark mass depends on hadron wave-function and includes cloud of gluons and qq pairs \( \Rightarrow \) slightly different values for mesons and baryons.
Hadron masses in QCD

- Calculation of hadron masses in QCD is a hard problem – can’t use perturbation theory.
- Need to solve field equations exactly – only feasible on a discrete lattice of space-time points.
- Needs specialised supercomputing (Pflops) + clever techniques.
- Current state of the art (after 40 years of work)...
Baryon Magnetic Moments

Magnetic dipole moments arise from

- the orbital motion of charged quarks
- the intrinsic spin-related magnetic moments of the quarks.

**Orbital Motion**

Classically, current loop

\[ \mu = IA = \frac{qv}{2\pi r} \pi r^2 = \frac{qpr}{2m} = \frac{q}{2m}L_z \]

Quantum mechanically, get the same result

\[ \hat{\mu} = g_\ell \frac{q}{2m} \hat{L}_z \]

\( g_\ell \) is the "g-factor"
\( g_\ell = 1 \) charged particles
\( g_\ell = 0 \) neutral particles

**Intrinsic Spin**

The magnetic moment operator due to the intrinsic spin of a particle is

\[ \hat{\mu} = g_s \frac{q}{2m} \hat{S}_z \]

\( g_s \) is the "spin g-factor"
\( g_s = 2 \) for Dirac spin 1/2 point-like particles.
The magnetic dipole moment is the maximum measurable component of the magnetic dipole moment operator

$$\mu_\ell = \left\langle \psi_{\text{space}} \left| g_\ell \frac{q}{2m} \hat{L}_z \right| \psi_{\text{space}} \right\rangle$$

$$\mu_s = \left\langle \psi_{\text{spin}} \left| g_s \frac{q}{2m} \hat{S}_z \right| \psi_{\text{spin}} \right\rangle$$

For an electron

$$\mu_\ell = -g_\ell \frac{e}{2m_e} \hbar \ell$$

$$\mu_s = -g_s \frac{e}{2m_e} \frac{\hbar}{2}$$

$$= -\mu_B \ell$$

$$= -\mu_B$$

where $$\mu_B = e\hbar/2m_e$$ is the Bohr Magneton

Observed difference from $$g_s = 2$$ is due to higher order corrections in QED

$$\mu_s = -\mu_B \left[ 1 + \frac{\alpha}{2\pi} + O(\alpha^2) + \ldots \right]$$

$$\alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$$
If the proton and neutron were point-like particles,

\[
\mu_{\ell} = g_{\ell} \frac{e}{2m_p} \hbar \ell \\
\mu_s = g_s \frac{e}{2m_p} \frac{\hbar}{2} = \frac{1}{2} g_s \mu_N
\]

where \( \mu_N = e\hbar/2m_p \) is the Nuclear Magneton

**Expect:**
- \( p \) spin 1/2, charge +e \( \mu_s = \mu_N \)
- \( n \) spin 1/2, charge 0 \( \mu_s = 0 \)

**Observe:**
- \( p \) \( \mu_s = +2.793\mu_N \) \( \rightarrow \) \( g_s = +5.586 \)
- \( n \) \( \mu_s = -1.913\mu_N \) \( \rightarrow \) \( g_s = -3.826 \)

Observation shows that \( p \) and \( n \) are not point-like \( \Rightarrow \) evidence for quarks. 
\( \Rightarrow \) use quark model to estimate baryon magnetic moments.
Assume that bound quarks within baryons behave as Dirac point-like spin 1/2 particles with fractional charge $q_q$. Then quarks will have magnetic dipole moment operator and magnitude:

$$\vec{\mu}_q = \frac{q_q}{m_q} \hat{S}_z$$
$$\mu_q = \left\langle \psi_{\text{spin}}^q \left| \frac{q_q}{m_q} \hat{S}_z \right| \psi_{\text{spin}}^q \right\rangle = \frac{q_q \hbar}{2m_q}$$

where $m_q$ is the quark mass.

Therefore

$$\mu_u = \frac{2}{32m_u} e\hbar, \quad \mu_d = -\frac{1}{32m_d} e\hbar, \quad \mu_s = -\frac{1}{32m_s} e\hbar$$

For quarks bound within an $\ell = 0$ baryon, the baryon magnetic moment is the expectation value of the sum of the individual quark magnetic moment operators:

$$\hat{\mu}_{\text{baryon}} = \frac{q_1}{m_1} \hat{S}_{1z} + \frac{q_2}{m_2} \hat{S}_{2z} + \frac{q_3}{m_3} \hat{S}_{3z}$$
$$\mu_{\text{baryon}} = \left\langle \psi_{\text{spin}}^B \left| \hat{\mu}_B \right| \psi_{\text{spin}}^B \right\rangle$$

where $\psi_{\text{spin}}^B$ is the baryon spin wavefunction.
Example: Magnetic moment of a proton
Baryon Magnetic Moments \textit{in the Quark Model}

Repeat for the other $\ell = 0$ baryons. Predict $\frac{\mu_n}{\mu_p} = -\frac{2}{3}$ compared to the experimentally measured value of $-0.685$

<table>
<thead>
<tr>
<th>Baryon $\mu_B$ in Quark Model</th>
<th>Predicted $[\mu_N]$</th>
<th>Observed $[\mu_N]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$ $(uud)$ $\frac{4}{3}\mu_u - \frac{1}{3}\mu_d$</td>
<td>+2.79</td>
<td>+2.793</td>
</tr>
<tr>
<td>$n$ $(ddu)$ $\frac{4}{3}\mu_d - \frac{1}{3}\mu_u$</td>
<td>−1.86</td>
<td>−1.913</td>
</tr>
<tr>
<td>$\Lambda$ $(uds)$ $\mu_s$</td>
<td>−0.61</td>
<td>−0.614 ± 0.005</td>
</tr>
<tr>
<td>$\Sigma^+$ $(uus)$ $\frac{4}{3}\mu_u - \frac{1}{3}\mu_s$</td>
<td>+2.68</td>
<td>+2.46 ± 0.01</td>
</tr>
<tr>
<td>$\Xi^0$ $(ssu)$ $\frac{4}{3}\mu_s - \frac{1}{3}\mu_u$</td>
<td>−1.44</td>
<td>−1.25 ± 0.014</td>
</tr>
<tr>
<td>$\Xi^-$ $(ssd)$ $\frac{4}{3}\mu_s - \frac{1}{3}\mu_d$</td>
<td>−0.51</td>
<td>−0.65 ± 0.01</td>
</tr>
<tr>
<td>$\Omega^-$ $(sss)$ $3\mu_s$</td>
<td>−1.84</td>
<td>−2.02 ± 0.05</td>
</tr>
</tbody>
</table>

Reasonable agreement with data using $m_u = m_d = 0.336$ GeV, $m_s \sim 0.509$ GeV
Hadron Decays

- Hadrons are eigenstates of the strong force.
- Hadrons will decay via the strong interaction to lighter mass states if energetically feasible (i.e. mass of parent > mass of daughters).
- And, angular momentum and parity must be conserved in strong decays.

Examples:

\[ \rho^0 \rightarrow \pi^+ \pi^- \]
\[ m(\rho^0) > m(\pi^+) + m(\pi^-) \]
\[ 769 \quad 140 \quad 140 \text{ MeV} \]

\[ \Delta^{++} \rightarrow p\pi^+ \]
\[ m(\Delta^{++}) > m(p) + m(\pi^+) \]
\[ 1231 \quad 938 \quad 140 \text{ MeV} \]
Hadron Decays

Also need to check for identical particles in the final state.

Examples:

\[ \omega^0 \rightarrow \pi^0 \pi^0 \]

\[ m(\omega^0) > m(\pi^0) + m(\pi^0) \]

782 \quad 135 \quad 135 \text{ MeV}

\[ \omega^0 \rightarrow \pi^+ \pi^- \pi^0 \]

\[ m(\omega^0) > m(\pi^+) + m(\pi^-) + m(\pi^0) \]

782 \quad 140 \quad 140 \quad 135 \text{ MeV}
Hadron Decays

Hadrons can also decay via the electromagnetic interaction. Examples:

\[ \rho^0 \rightarrow \pi^0 \gamma \]
\[ m(\rho^0) > m(\pi^0) + m(\gamma) \]

\[ \Sigma^0 \rightarrow \Lambda^0 \gamma \]
\[ m(\Sigma^0) > m(\Lambda^0) + m(\gamma) \]

The lightest mass states (\( p, K^\pm, K^0, \bar{K}^0, \Lambda, n \)) require a change of quark flavour in the decay and therefore decay via the weak interaction (see later).
Summary of light \((uds)\) hadrons

- Baryons and mesons are composite particles (complicated).
- However, the naive Quark Model can be used to make predictions for masses/magnetic moments.
- The predictions give reasonably consistent values for the constituent quark masses:

<table>
<thead>
<tr>
<th></th>
<th>(m_{u/d})</th>
<th>(m_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Meson Masses</td>
<td>307 MeV</td>
<td>487 MeV</td>
</tr>
<tr>
<td>Baryon Masses</td>
<td>364 MeV</td>
<td>537 MeV</td>
</tr>
<tr>
<td>Baryon Magnetic Moments</td>
<td>336 MeV</td>
<td>509 MeV</td>
</tr>
</tbody>
</table>

\[ m_u \sim m_d \sim 335 \text{ MeV}, \quad m_s \sim 510 \text{ MeV} \]

- Hadrons will decay via the **strong** interaction to lighter mass states if energetically feasible.
- Hadrons can also decay via the **EM** interaction.
- The lightest mass states require a change of quark flavour to decay and therefore decay via the **weak** interaction (see later).
Heavy hadrons

The November Revolution

Brookhaven National Laboratory
Led by Samuel Ting

Stanford Linear Accelerator Center, SPEAR
Led by Burton Richter

Both experiments announced discovery on 11 November 1974 $\Rightarrow J/\psi$

1976 Nobel Prize awarded to Ting and Richter.
1974: Discovery of a narrow resonance in $e^+e^-$ collisions at $\sqrt{s} \sim 3.1$ GeV

$J/\psi(3097)$

Observed width $\sim 3$ MeV, all due to experimental resolution.
Actual Total Width, $\Gamma_{J/\psi} \sim 97$ keV

Branching fractions:

$B(J/\psi \rightarrow \text{hadrons}) \sim 88\%$

$B(J/\psi \rightarrow \mu^+\mu^-) \sim (J/\psi \rightarrow e^+e^-) \sim 6\%$

Partial widths:

$\Gamma_{J/\psi \rightarrow \text{hadrons}} \sim 87$ keV

$\Gamma_{J/\psi \rightarrow \mu^+\mu^-} \sim \Gamma_{J/\psi \rightarrow e^+e^-} \sim 5$ keV
Resonance seen in
\[ R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \]

Zoom into the charmonium \((c\bar{c})\) region
\[ \sqrt{s} \sim 2m_c \]

mass of charm quark, \(m_c \sim 1.5\) GeV

Resonances due to formation of bound unstable \(c\bar{c}\) states. The lowest energy of these is the narrow \(J/\psi\) state.
Charmonium

c\bar{c} bound states produced directly in $e^+e^-$ collisions must have the same spin and parity as the photon

![Diagram of quark interaction](image)

$$J^P = 1^-$$

However, expect that a whole spectrum of bound $c\bar{c}$ states should exist (analogous to $e^+e^-$ bound states, positronium)

- $n = 1$, $L = 0$, $S = 0, 1$, $^1S_0, ^3S_1$
- $n = 2$, $L = 0, 1$, $S = 0, 1$, $^1S_0, ^3S_1, ^1P_1, ^3P_0, ^1P_2$

... etc

$$\text{Parity} = (-1)(-1)^L \quad 2S+1L_J$$
The Charmonium System

Quark Model of Hadrons

- Hadrons are composed of quarks.
- Bound states and quasi-bound states are shown.
- Mass/GeV is indicated on the y-axis.
- J^P notation is used for quantum numbers.

Wide
\[ \Gamma \approx 25 \text{ MeV} \]

Narrow

\[ J/\psi \]
All $c\bar{c}$ bound states can be observed via their decay:

**Example:** Hadronic decay

$$\psi(3685) \rightarrow J/\psi \, \pi^+\pi^-$$

**Example:** Photonic decay

$$\psi(3685) \rightarrow \chi + \gamma$$

$$\chi \rightarrow J/\psi + \gamma$$

Peaks in $\gamma$ spectrum

Charmonium Spectroscopy
Knowing the $c\bar{c}$ energy levels provides a probe of the QCD potential.

- Because QCD is a theory of a strong confining force (self-interacting gluons), it is very difficult to calculate the exact form of the QCD potential from first principles.
- However, it is possible to experimentally “determine” the QCD potential by finding an appropriate form which gives the observed charmonium states.
- In practice, the QCD potential

$$V_{\text{QCD}} = -\frac{4\alpha_s}{3} \frac{1}{r} + kr$$

with $\alpha_s = 0.2$ and $k = 1$ GeVfm$^{-1}$ provides a good description of the experimentally observed levels in the charmonium system.
Why is the $J/\psi$ so narrow?

Consider the charmonium $^3S_1$ states:

- $1^3S_1 \psi(3097) \Gamma \sim 0.09 \text{ MeV}$
- $2^3S_1 \psi(3685) \Gamma \sim 0.24 \text{ MeV}$
- $3^3S_1 \psi(3777) \Gamma \sim 25 \text{ MeV}$
- $4^3S_1 \psi(4040) \Gamma \sim 50 \text{ MeV}$

The width depends on whether the decay to lightest mesons containing $c$ quarks, $D^-(d\bar{c}), \, D^+(c\bar{d})$, is kinematically possible or not:

$m(D^\pm) = 1869.4 \pm 0.5 \text{ MeV}$

$m(\psi) > 2m(D)$

$\psi \rightarrow D^+D^-$ allowed

“ordinary” strong decay

$\Rightarrow$ large width

$m(\psi) < 2m(D)$

Zweig Rule: Unconnected lines in the Feynman diagram lead to suppression of the decay amplitude

$\Rightarrow$ narrow width
Charmed Hadrons

The existence of the $c$ quark $\Rightarrow$ expect to see charmed mesons and baryons (i.e. containing a $c$ quark).

Extend quark symmetries to 3 dimensions:

Mesons

$$J^P = 0^-$$

Baryons

$$J^P = \frac{1}{2}^+$$

$$J^P = 1^-$$

$$J^P = \frac{3}{2}^+$$
Heavy hadrons \( \Upsilon \ (b\bar{b}) \)

E288 collaboration, Fermilab
Led by Leon Lederman

- 1977: Discovery of the \( \Upsilon(9460) \) resonance state.
- Lowest energy \( 3S_1 \) bound \( b\bar{b} \) state (bottomonium).

\[ \Rightarrow m_b \sim 4.7 \text{ GeV} \]

Similar properties to the \( \psi \)

\[ e^+e^- \rightarrow \Upsilon \rightarrow \text{hadrons} \]

\( \Upsilon \) particle: PRL 39 (1977) 252-255
**Bottomonium**

- Bottomonium is the analogue of charmonium for $b$ quark.
- Bottomonium spectrum well described by same QCD potential as used for charmonium.
- Evidence that QCD potential does not depend on quark type.

---

Zweig suppressed
Extend quark symmetries to 4 dimensions (difficult to draw!)

Examples:

**Mesons** \( (J^P = 0^-) \) : \( B^-(b\bar{u}); \  B^0(\bar{b}d); \  B^0_s(\bar{b} s); \  B^-_c(b\bar{c}) \)

The \( B^-_c \) is the heaviest hadron discovered so far: \( m(B^-_c) = 6.4 \pm 0.4 \ \text{GeV} \)

\( (J^P = 1^-) \) :  \( B^{*-}(b\bar{u}); \  B^{*-0}(\bar{b}d); \  B^{*-0}_s(\bar{b}s) \)

The mass of the \( B^* \) mesons is only 50 \( \text{MeV} \) above the \( B \) meson mass. Expect only electromagnetic decays \( B^* \rightarrow B\gamma \).

**Baryons** \( (J^P = \frac{1^+}{2}) \) : \( \Lambda_b(bud); \  \Sigma_b(buu); \  \Xi_b(bus) \)
Summary of heavy hadrons

- $c$ and $b$ quarks were first observed in bound state resonances ("onia").
- Consequences of the existence of $c$ and $b$ quarks are
  - Spectra of $c\bar{c}$ (charmonium) and $b\bar{b}$ (bottomonium) bound states
  - Peaks in $R = \frac{\sigma(e^+e^-\rightarrow \text{hadrons})}{\sigma(e^+e^-\rightarrow \mu^+\mu^-)}$
  - Existence of mesons and baryons containing $c$ and $b$ quarks
- The majority of charm and bottom hadrons decay via the weak interaction (strong and electromagnetic decays are forbidden by energy conservation).

The $t$ quark is very heavy and decays rapidly via the weak interaction before a $t\bar{t}$ bound state (or any other hadron) can be formed.

$$\tau_t \sim 10^{-25}\,\text{s} \quad t_{\text{hadronisation}} \sim 10^{-22}\,\text{s}$$

Rapid decay because $m(t) > m(W)$ so weak interaction is no longer weak.

$$\begin{pmatrix} m(u) = 335 \text{ MeV} \\ m(d) = 335 \text{ MeV} \end{pmatrix} \begin{pmatrix} m(c) = 1.5 \text{ GeV} \\ m(s) = 510 \text{ MeV} \end{pmatrix} \begin{pmatrix} m(t) = 175 \text{ GeV} \\ m(b) = 4.5 \text{ GeV} \end{pmatrix}$$
Tetraquarks and Pentaquarks

Quark Model of Hadrons is not limited to $q\bar{q}$ or $qqq$ content. Recent observations from LHCb show unquestionable discovery of pentaquark states, PRL 115, 072001 (2015).

How are these quarks bound? $qqqqq$? $qq + qqq$? $qq + qq + q$?

A few tetraquarks discovered by Belle and BESIII e.g. $Z(4430)^-$, $c\bar{c}d\bar{u}$ discovered by Belle and confirmed by LHCb. LHCb has discovered many more!
Summary

- Evidence for hadron sub-structure – quarks
- Hadron wavefunctions and allowed states
- Hadron masses and magnetic moments
- Hadron decays (strong, EM, weak)
- Heavy hadrons: charmonium and bottomonium
- Recent tetraquark and pentaquark discoveries

Problem Sheet: q.17-22

Up next...
Section 9: The Weak Force