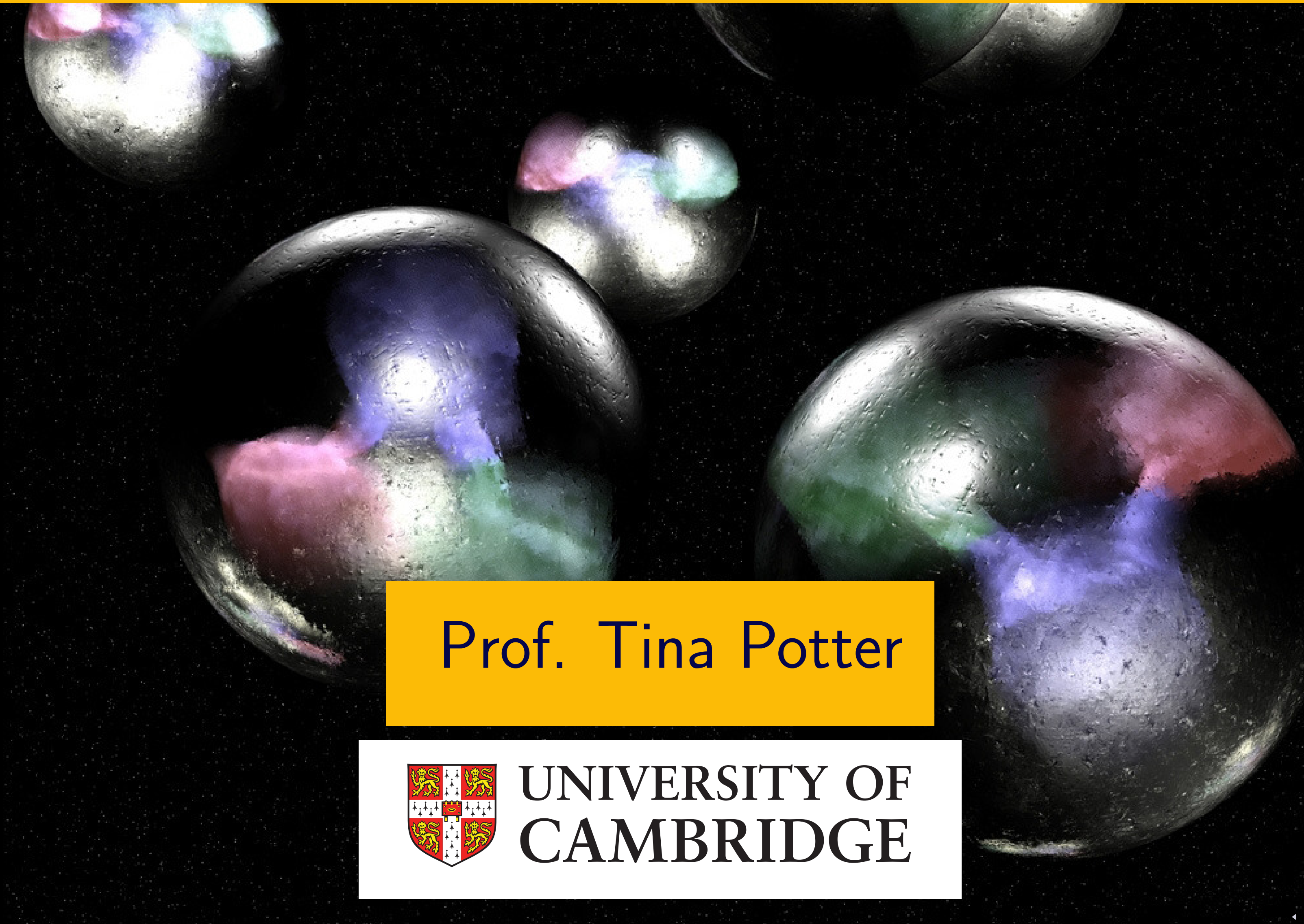
8. Quark Model of Hadrons

Particle and Nuclear Physics



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In this section...

- Hadron wavefunctions and parity
- Light mesons
- Light baryons
- Charmonium
- Bottomonium

The Quark Model of Hadrons

Evidence for quarks

- The magnetic moments of proton and neutron are not $\mu_N = e\hbar/2m_p$ and 0 respectively $\Rightarrow$ not point-like
- Electron-proton scattering at high q^2 deviates from Rutherford scattering $\Rightarrow$ proton has substructure
- Hadron jets are observed in e^+e^- and pp collisions
- Symmetries (patterns) in masses and properties of hadron states, "quarky" periodic table $\Rightarrow$ sub-structure
- Steps in $R = \sigma(e^+e^- \rightarrow \mathrm{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$
- Observation of $c\bar{c}$ and $b\bar{b}$ bound states
- and much, much more...

Here, we will first consider the wave-functions for hadrons formed from light quarks (u, d, s) and deduce some of their static properties (mass and magnetic moments).

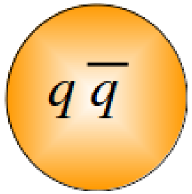
Then we will go on to discuss the heavy quarks (c, b).

We will cover the t quark later...

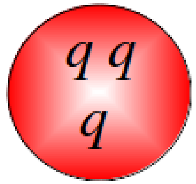
Hadron Wavefunctions

Quarks are always confined in hadrons (i.e. colourless states)

Mesons
Spin 0, 1, ...



Baryons
Spin 1/2, 3/2, ...



Treat quarks as identical fermions with states labelled with spatial, spin, flavour and colour.

$$\psi = \psi_{\text{space}}\psi_{\text{flavour}}\psi_{\text{spin}}\psi_{\text{colour}}$$

All hadrons are colour singlets, i.e. net colour zero

Mesons $$\psi^{q\bar{q}}_{\text{colour}} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

Baryons $$\psi^{qqq}_{\text{colour}} = \frac{1}{\sqrt{6}}(rgb + gbr + brg - grb - rbg - bgr)$$

Parity

- The Parity operator, $\hat{P}$, performs spatial inversion

$$\hat{P}|\psi(\vec{r},t)\rangle = |\psi(-\vec{r},t)\rangle$$

- The eigenvalue of $\hat{P}$ is called Parity

$$\hat{P}|\psi\rangle = P|\psi\rangle, \qquad P = \pm 1$$

- Most particles are eigenstates of Parity and in this case P represents intrinsic Parity of a particle/antiparticle.
- Parity is a useful concept. If the Hamiltonian for an interaction commutes with $\hat{P}$

$$\left[\hat{P},\hat{H}\right] = 0$$

then Parity is conserved in the interaction:

Parity conserved in the **strong** and **EM** interactions, but **not** in the **weak** interaction.

Parity

- Composite system of two particles with orbital angular momentum L:

$$P = P_1 P_2 (-1)^L$$

where $P_{1,2}$ are the intrinsic parities of particles 1, 2.

Quantum Field Theory tells us that

Fermions and antifermions: opposite parity

Bosons and antibosons: same parity

Choose:

Quarks and leptons: $P_{q/\ell} = +1$

Antiquarks and antileptons: $P_{\bar{q},\bar{\ell}} = -1$

Gauge Bosons: (γ, g, W, Z) are vector fields which transform as

$$J^P = 1^-$$

$$P_\gamma = -1$$

Light Mesons

Mesons are bound $q\bar{q}$ states.
Consider ground state mesons consisting of light quarks (u, d, s).

$$m_u \sim 0.3\ \mathrm{GeV},\ \ m_d \sim 0.3\ \mathrm{GeV},\ \ m_s \sim 0.5\ \mathrm{GeV}$$

- **Ground State ($L = 0$):** Meson "spin" (total angular momentum) is given by the $q\bar{q}$ spin state.

  Two possible $q\bar{q}$ total spin states: $S = 0, 1$

  $S = 0$: pseudoscalar mesons

  $S = 1$: vector mesons

- **Meson Parity:** (q and $\bar{q}$ have opposite parity)

$$P = P_q P_{\bar{q}}(-1)^L = (+1)(-1)(-1)^L = -1 \quad (\mathrm{for}\ L = 0)$$

- **Flavour States:** $u\bar{d}, u\bar{s}, d\bar{u}, d\bar{s}, s\bar{u}, s\bar{d}$ and $u\bar{u}, d\bar{d}\ s\bar{s}$ mixtures

  Expect: Nine $J^P = 0^-$ mesons: Pseudoscalar nonet

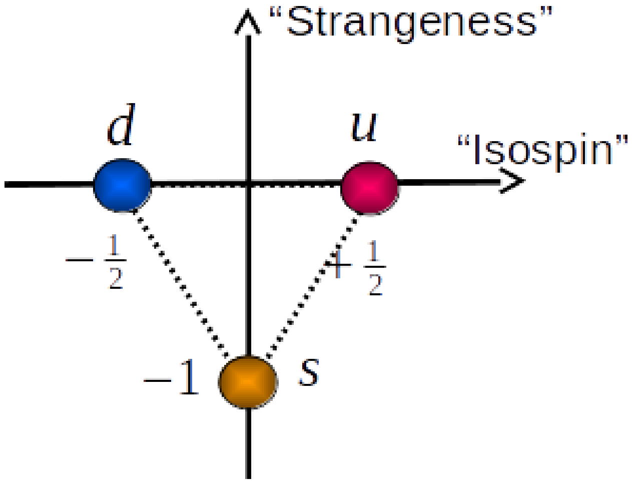
  Nine $J^P = 1^-$ mesons: Vector nonet

uds Multiplets

Basic quark multiplet – plot the quantum numbers of (anti)quarks:

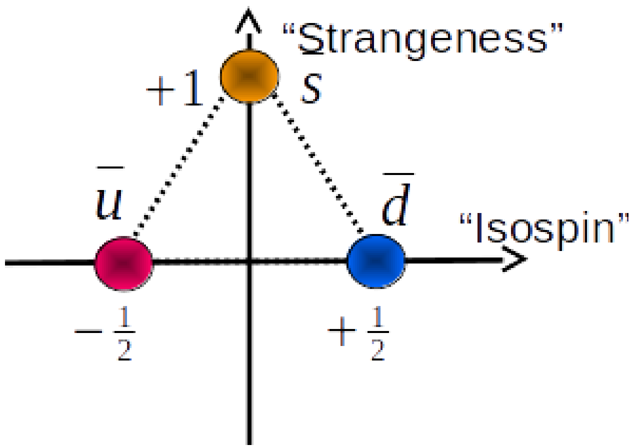
Quarks

$J^P = \frac{1}{2}^+$



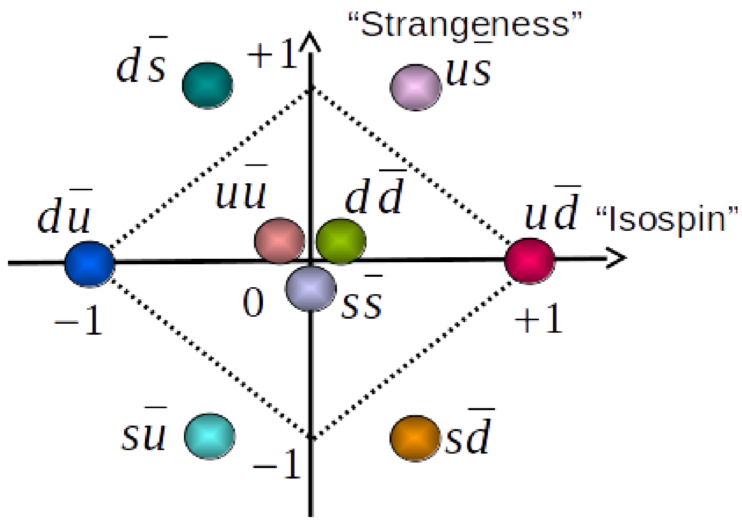
Antiquarks

$J^P = \frac{1}{2}^-$



Mesons

Spin $J = 0$ or 1



The ideas of strangeness and isospin are historical quantum numbers assigned to different states.

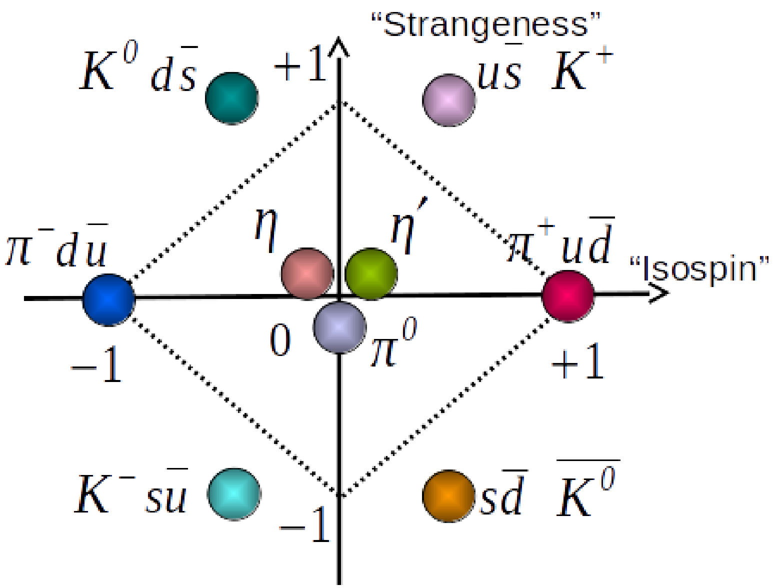
Essentially they count quark flavours (this was all before the formulation of the Quark Model).

$$\text{Isospin} = \tfrac{1}{2}(n_u - n_d - n_{\bar{u}} + n_{\bar{d}})$$
$$\text{Strangeness} = n_{\bar{s}} - n_s$$

Light Mesons

Pseudoscalar nonet

$J^P = 0^-$



π^0, η, η' are combinations of $u\bar{u}$, $d\bar{d}$, $s\bar{s}$

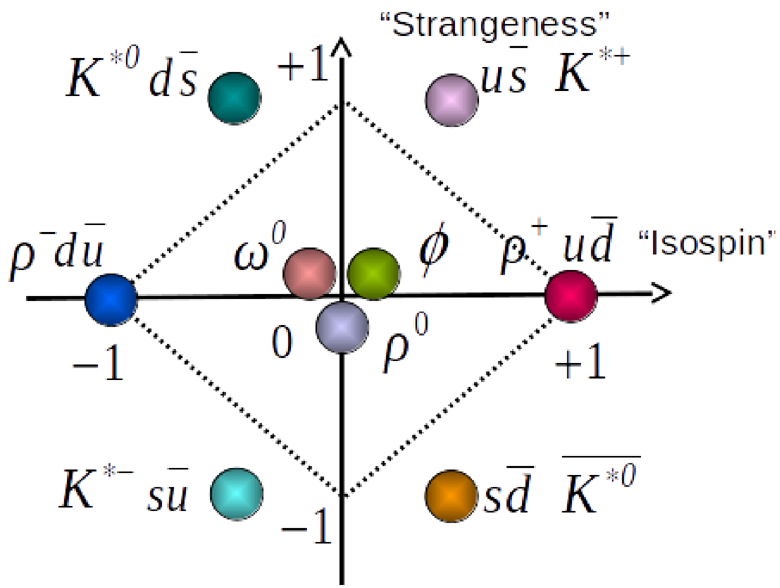
Masses / MeV

$\pi(140)$, $K(495)$

$\eta(550)$, $\eta'(960)$

Vector nonet

$J^P = 1^-$



ρ^0, ϕ, ω^0 are combinations of $u\bar{u}$, $d\bar{d}$, $s\bar{s}$

Masses/ MeV

$\rho(770)$, $K^*(890)$

$\omega(780)$, $\phi(1020)$

$u\bar{u}$, $d\bar{d}$, $s\bar{s}$ States

The states $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$ all have zero flavour quantum numbers and can mix

$$J^P = 0^-$$

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\eta = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

$$J^P = 1^-$$

$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\omega^0 = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$$\phi = s\bar{s}$$

Mixing coefficients determined experimentally from meson masses and decays.

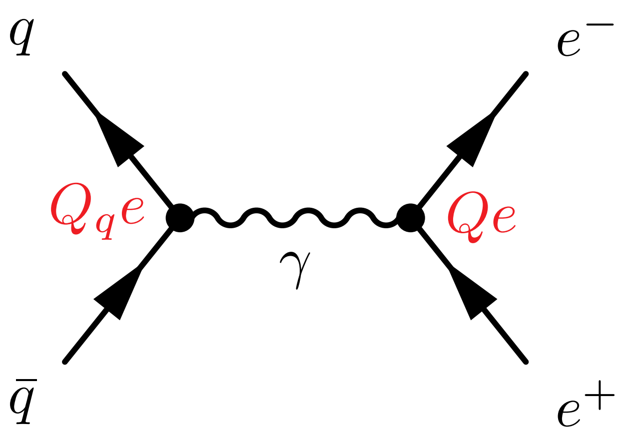
Example: Leptonic decays of vector mesons

$$M(\rho^0 \to e^+e^-) \sim \frac{e}{q^2}\left[\frac{1}{\sqrt{2}}(Q_u e - Q_d e)\right]$$

$$\Gamma(\rho^0 \to e^+e^-) \propto \left[\frac{1}{\sqrt{2}}\left(\frac{2}{3} - \left(-\frac{1}{3}\right)\right)\right]^2 = \frac{1}{2}$$

$$\Gamma(\omega^0 \to e^+e^-) \propto \left[\frac{1}{\sqrt{2}}\left(\frac{2}{3} + \left(-\frac{1}{3}\right)\right)\right]^2 = \frac{1}{18}$$

$$\Gamma(\phi \to e^+e^-) \propto \left[\frac{1}{3}\right]^2 = \frac{1}{9}$$



$$M \sim Q_q\alpha \qquad \Gamma \sim Q_q^2\alpha^2$$

Predict: $\Gamma_{\rho^0} : \Gamma_{\omega^0} : \Gamma_\phi = 9 : 1 : 2$ **Experiment:** $(8.8 \pm 2.6) : 1 : (1.7 \pm 0.4)$

Meson Masses

Meson masses are only partly from constituent quark masses:

$$\underset{495\ \mathrm{MeV}}{m(K)} > \underset{140\ \mathrm{MeV}}{m(\pi)} \quad \Rightarrow \text{suggests } m_s > m_u, m_d$$

Not the whole story...

$$\underset{770\ \mathrm{MeV}}{m(\rho)} > \underset{140\ \mathrm{MeV}}{m(\pi)} \quad \Rightarrow \text{although both are } u\bar{d}$$

Only difference is the orientation of the quark spins ($\uparrow\uparrow$ vs $\uparrow\downarrow$)

$\Rightarrow$ **spin-spin interaction**

Meson Masses *Spin-spin Interaction*

QED: Hyperfine splitting in H_2 $(L = 0)$

Energy shift due to electron spin in magnetic field of proton

$$\Delta E = \vec{\mu}.\vec{B} = \frac{2}{3}\vec{\mu}_e.\vec{\mu}_p|\psi(0)|^2$$

and using $\vec{\mu} = \frac{e}{2m}\vec{S}$

$$\Delta E \propto \alpha \frac{\vec{S}_e}{m_e}\frac{\vec{S}_p}{m_p}$$

QCD: Colour Magnetic Interaction

Fundamental form of the interaction between a quark and a gluon is identical to that between an electron and a photon. Consequently, also have a colour magnetic interaction

$$\Delta E \propto \alpha_s \frac{\vec{S}_1}{m_1}\frac{\vec{S}_2}{m_2}$$

Meson Masses *Meson Mass Formula ($L = 0$)*

$$M_{q\bar{q}} = m_1 + m_2 + A\frac{\vec{S}_1}{m_1}\frac{\vec{S}_2}{m_2}$$ where A is a constant

For a state of spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ $\quad \vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1.\vec{S}_2$

$$\vec{S}_1.\vec{S}_2 = \frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right) \qquad \vec{S}_1^2 = \vec{S}_2^2 = \vec{S}_1(\vec{S}_1 + 1) = \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4}$$

giving $\vec{S}_1.\vec{S}_2 = \frac{1}{2}\vec{S}^2 - \frac{3}{4}$

For $J^P = 0^-$ mesons: $\vec{S}^2 = 0$ $\quad \Rightarrow \vec{S}_1.\vec{S}_2 = -3/4$

For $J^P = 1^-$ mesons: $\vec{S}^2 = S(S+1) = 2$ $\quad \Rightarrow \vec{S}_1.\vec{S}_2 = +1/4$

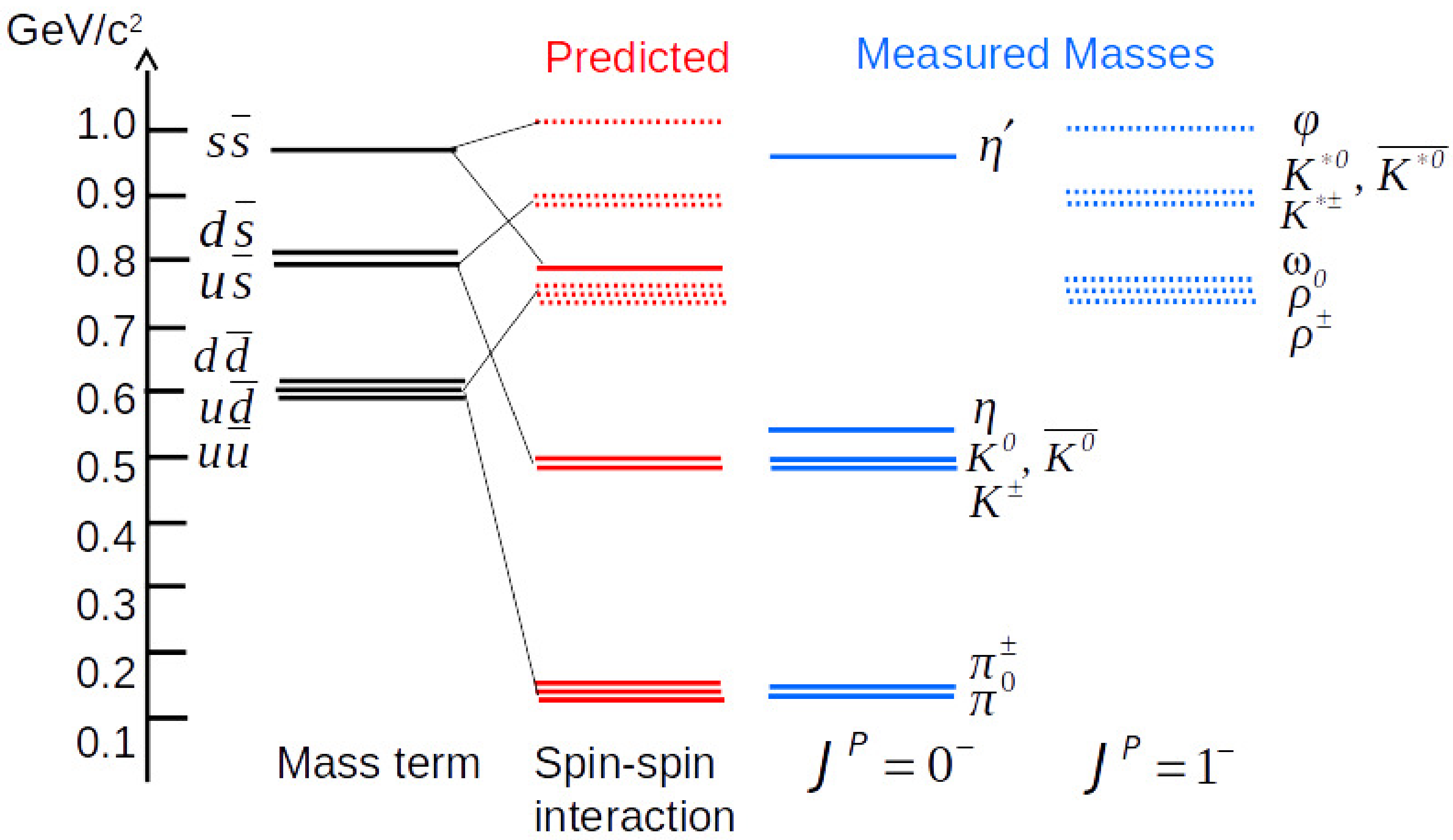
Giving the ($L = 0$) Meson Mass formulae:

$$M_{q\bar{q}} = m_1 + m_2 - \frac{3A}{4m_1 m_2} \quad (J^P = 0^-)$$

$$M_{q\bar{q}} = m_1 + m_2 + \frac{A}{4m_1 m_2} \quad (J^P = 1^-)$$

So $J^P = 0^-$ mesons are lighter than $J^P = 1^-$ mesons

Meson Masses



Excellent fit obtained to masses of the different flavour pairs $(u\bar{d},\ u\bar{s},\ d\bar{u},\ d\bar{s},\ s\bar{u},\ s\bar{d})$ with

$$m_u = 0.305 \text{ GeV}, \quad m_d = 0.308 \text{ GeV}, \quad m_s = 0.487 \text{ GeV}, \quad A = 0.06 \text{ GeV}^3$$

η and η' are mixtures of states, e.g.

$$\eta = \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right) \qquad M_\eta = \frac{1}{6}\left(2m_u - \frac{3A}{4m_u^2}\right) + \frac{1}{6}\left(2m_d - \frac{3A}{4m_d^2}\right) + \frac{4}{6}\left(2m_s - \frac{3A}{4m_s^2}\right)$$

Baryons

Baryons made from 3 indistinguishable quarks (flavour can be treated as another quantum number in the wave-function)

$$\psi_{\text{baryon}} = \psi_{\text{space}}\,\psi_{\text{flavour}}\,\psi_{\text{spin}}\,\psi_{\text{colour}}$$

ψ_{baryon} must be anti-symmetric under interchange of any 2 quarks

Example: $\Omega^-(sss)$ wavefunction $(L=0, J=3/2)$

$\psi_{\text{spin}}\,\psi_{\text{flavour}} = s\uparrow s\uparrow s\uparrow$ is symmetric $\Rightarrow$ require antisymmetric ψ_{colour}

Ground State $(L=0)$

We will only consider the baryon ground states, which have zero orbital angular momentum

ψ_{space} symmetric

$\rightarrow$ All hadrons are colour singlets

$$\psi_{\text{colour}} = \frac{1}{\sqrt{6}}(rgb + gbr + brg - grb - rbg - bgr)$$ antisymmetric

Therefore, $\psi_{\text{spin}}\,\psi_{\text{flavour}}$ must be symmetric

Baryon spin wavefunctions (ψ_{spin})

Combine 3 spin 1/2 quarks: Total spin $J = \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \text{ or } \frac{3}{2}$

Consider $J = 3/2$

Trivial to write down the spin wave-function for the $\left|\frac{3}{2}, \frac{3}{2}\right\rangle$ state: $\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \uparrow\uparrow\uparrow$

Generate other states using the ladder operator $\hat{J}_-$

$$\hat{J}_- \left|\frac{3}{2}, \frac{3}{2}\right\rangle = (\hat{J}_- \uparrow)\uparrow\uparrow + \uparrow(\hat{J}_- \uparrow)\uparrow + \uparrow\uparrow(\hat{J}_- \uparrow)$$

$$\sqrt{\frac{3}{2}\frac{5}{2} - \frac{3}{2}\frac{1}{2}} \left|\frac{3}{2}, \frac{1}{2}\right\rangle = \downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow$$

$$\hat{J}_- |j, m\rangle = \sqrt{j(j+1) - m(m-1)}\, |j, m-1\rangle$$

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow)$$

Giving the $J = 3/2$ states: $\longrightarrow$

All symmetric under interchange of any two spins

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \uparrow\uparrow\uparrow$$

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow)$$

$$\left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow)$$

$$\left|\frac{3}{2}, -\frac{3}{2}\right\rangle = \downarrow\downarrow\downarrow$$

Baryon spin wavefunctions ($\psi_{\rm spin}$)

Consider $J = 1/2$

First consider the case where the first 2 quarks are in a $|0,0\rangle$ state:

$$|0,0\rangle_{(12)} = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_{(123)} = |0,0\rangle_{(12)}\left|\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \qquad \left|\frac{1}{2},-\frac{1}{2}\right\rangle_{(123)} = |0,0\rangle_{(12)}\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$

Antisymmetric under exchange $1 \leftrightarrow 2$.

Three-quark $J = 1/2$ states can also be formed from the state with the first two quarks in a symmetric spin wavefunction.

Can construct a three-particle state $\left|\frac{1}{2},\frac{1}{2}\right\rangle_{(123)}$ from

$$|1,0\rangle_{(12)}\left|\frac{1}{2},\frac{1}{2}\right\rangle_{(3)} \qquad \text{and} \qquad |1,1\rangle_{(12)}\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{(3)}$$

Baryon spin wavefunctions ($\psi_{\rm spin}$)

Taking the linear combination

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = a\left|1,1\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + b\left|1,0\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle$$

with $a^2+b^2=1$. Act upon both sides with $\hat{J}_+$

$$\hat{J}_+\left|\frac{1}{2},\frac{1}{2}\right\rangle = a\left[\left(\hat{J}_+\left|1,1\right\rangle\right)\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|1,1\right\rangle\left(\hat{J}_+\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)\right] + b\left[\left(\hat{J}_+\left|1,0\right\rangle\right)\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|1,0\right\rangle\left(\hat{J}_+\left|\frac{1}{2},\frac{1}{2}\right\rangle\right)\right]$$

$$0 = a\left|1,1\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \sqrt{2}b\left|1,1\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle$$

$$a = -\sqrt{2}b \qquad \hat{J}_+\left|j,m\right\rangle = \sqrt{j(j+1)-m(m+1)}\left|j,m+1\right\rangle$$

which with $a^2+b^2=1$ implies $a=\sqrt{\frac{2}{3}},\ b=-\sqrt{\frac{1}{3}}$

Giving

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1,1\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|1,0\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$\left|1,1\right\rangle = \uparrow\uparrow$$

$$\left|1,0\right\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow)$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \qquad \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow - \uparrow\downarrow\downarrow)$$

Symmetric under interchange $1 \leftrightarrow 2$

Three-quark spin wavefunctions

$J = 3/2$

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle = \uparrow\uparrow\uparrow$$

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow)$$

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow)$$

$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \downarrow\downarrow\downarrow$$

Symmetric under interchange of any 2 quarks

$J = 1/2$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$

Antisymmetric under interchange of $1 \leftrightarrow 2$

$J = 1/2$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow - \uparrow\downarrow\downarrow)$$

Symmetric under interchange of $1 \leftrightarrow 2$

$\psi_{\text{spin}}\,\psi_{\text{flavour}}$ must be symmetric under interchange of any 2 quarks.

Three-quark spin wavefunctions

Consider 3 cases:

1. **Quarks all same flavour: $uuu,\ ddd,\ sss$**
   - ψ_{flavour} is symmetric under interchange of any two quarks
   - Require ψ_{spin} to be symmetric under interchange of any two quarks
   - Only satisfied by $J = 3/2$ states
   - There are no $J = 1/2$ $uuu,\ ddd,\ sss$ baryons with $L = 0$.

   Three $J = 3/2$ states: $uuu,\ ddd,\ sss$

2. **Two quarks have same flavour: $uud,\ uus,\ ddu,\ dds,\ ssu,\ ssd$**
   - For the like quarks ψ_{flavour} is symmetric
   - Require ψ_{spin} to be symmetric under interchange of like quarks $1 \leftrightarrow 2$
   - Satisfied by $J = 3/2$ and $J = 1/2$ states

   Six $J = 3/2$ states and six $J = 1/2$ states: $uud,\ uus,\ ddu,\ dds,\ ssu,\ ssd$

Three-quark spin wavefunctions

3. **All quarks have different flavours: *uds***

   Two possibilities for the (ud) part:

   - Flavour Symmetric $\frac{1}{\sqrt{2}}(ud + du)$
     - Require ψ_{spin} to be symmetric under interchange of ud
     - Satisfied by $J = 3/2$ and $J = 1/2$ states

   One $J = 3/2$ and one $J = 1/2$ state: uds

   - Flavour Antisymmetric $\frac{1}{\sqrt{2}}(ud - du)$
     - Require ψ_{spin} to be antisymmetric under interchange of ud
     - Only satisfied by $J = 1/2$ state

   One $J = 1/2$ state: uds

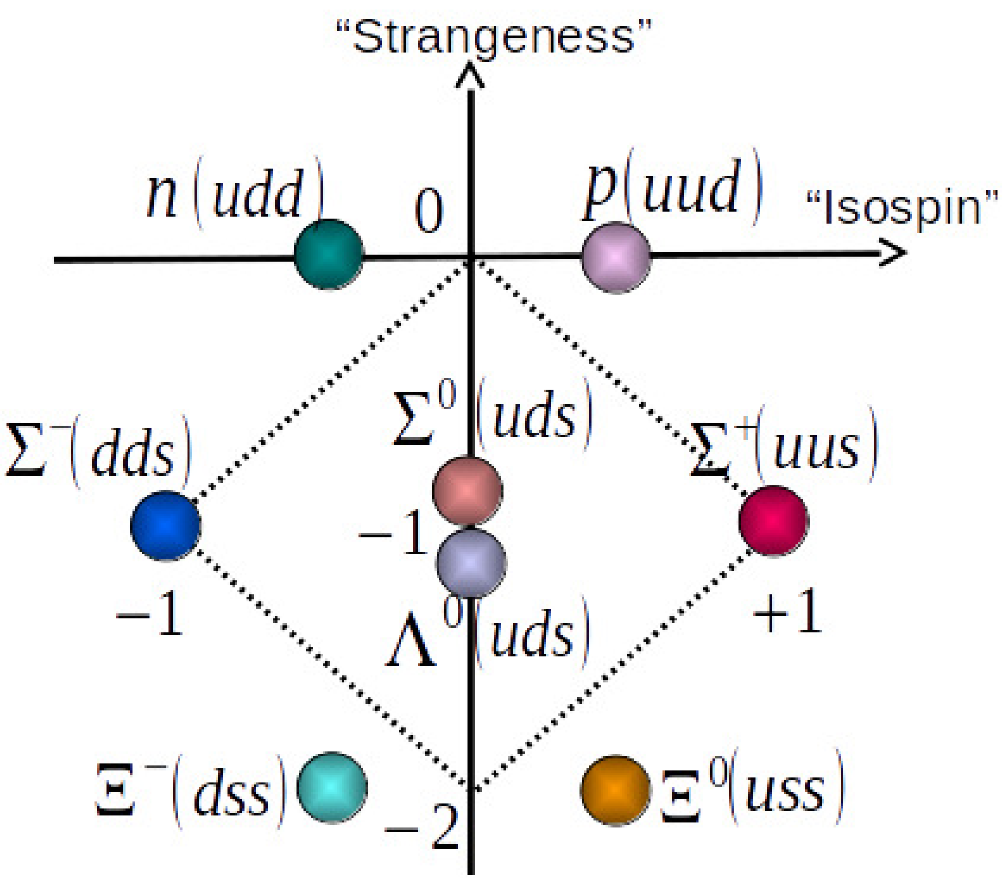
Quark Model predicts that light baryons appear in

Decuplets (10) of spin $3/2$ states

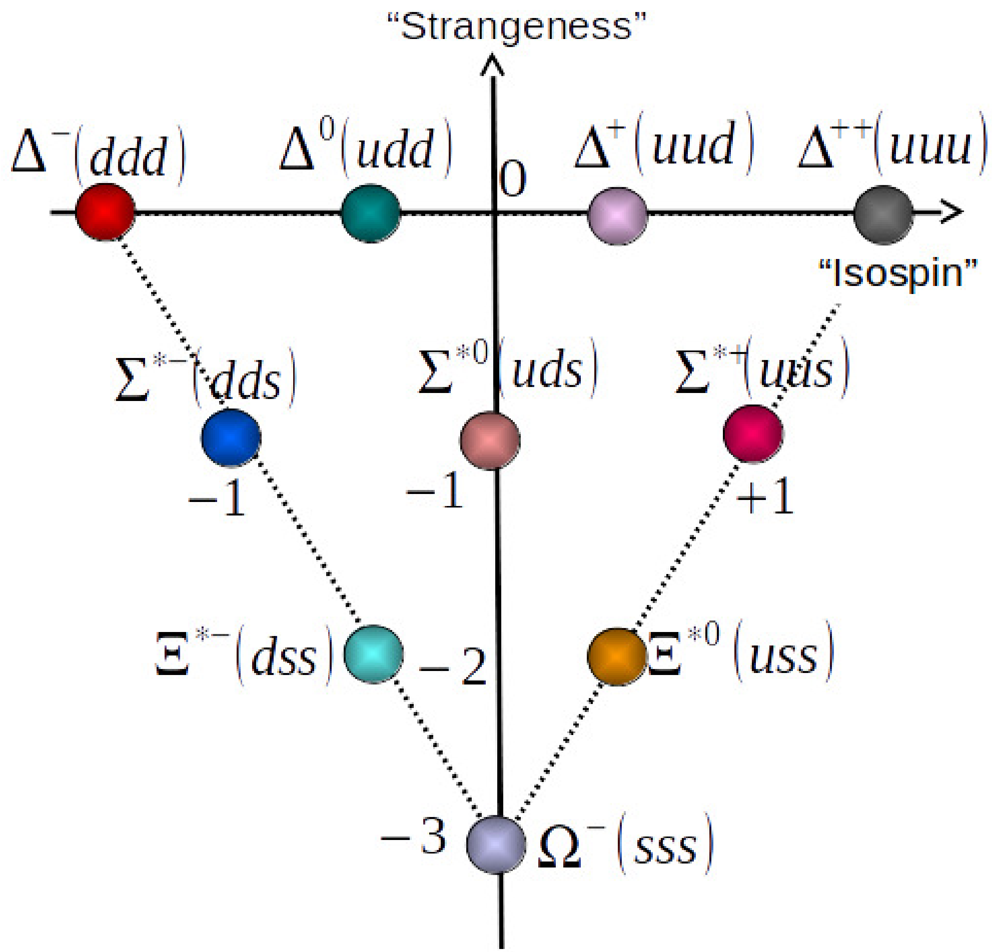
Octets (8) of spin $1/2$ states

Baryon Multiplets

Octet $J^P = \frac{1}{2}^+$



Decuplet $J^P = \frac{3}{2}^+$



Antibaryons are in separate multiplets

Example:

Antiparticle of $\Sigma^+(uus)$ is $\bar{\Sigma}^-(\bar{u}\bar{u}\bar{s})$, $J^P = \frac{1}{2}^-$ and not $\Sigma^-(dds)$, $J^P = \frac{1}{2}^+$

$$M_{qqq} = m_1 + m_2 + m_3 + A'\left(\frac{\vec{S}_1}{m_1}.\frac{\vec{S}_2}{m_2} + \frac{\vec{S}_1}{m_1}.\frac{\vec{S}_3}{m_3} + \frac{\vec{S}_2}{m_2}.\frac{\vec{S}_3}{m_3}\right)$$

where A' is a constant

Example: All quarks have the same mass, $m_1 = m_2 = m_3 = m_q$

$$M_{qqq} = 3m_q + A'\sum_{i<j}\frac{\vec{S}_i.\vec{S}_j}{m_q^2}$$

$$\vec{S}^2 = \left(\vec{S}_1 + \vec{S}_2 + \vec{S}_3\right)^2 = \vec{S}_1^2 + \vec{S}_2^2 + \vec{S}_3^2 + 2\sum_{i<j}\vec{S}_i.\vec{S}_j$$

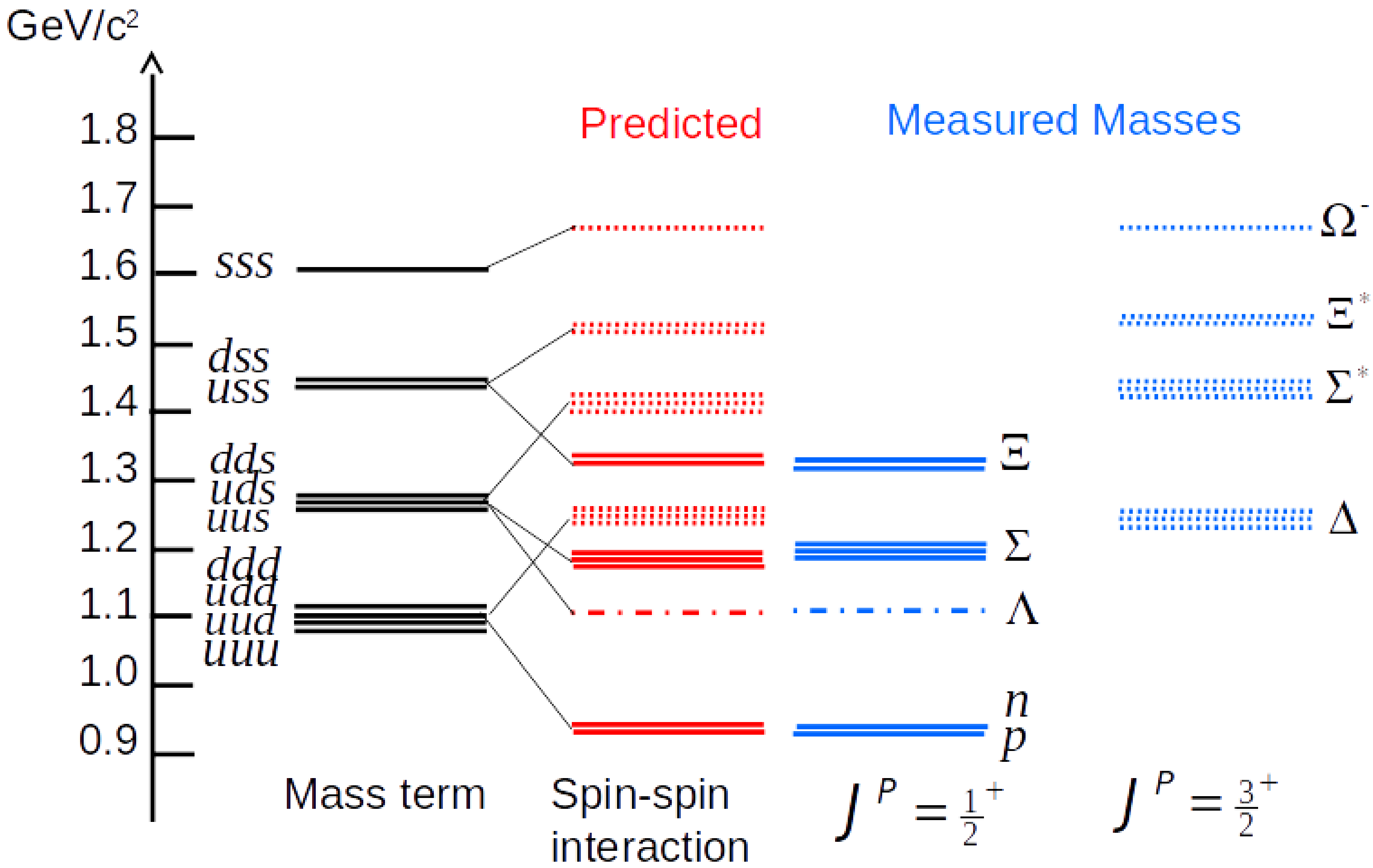
$$2\sum_{i<j}\vec{S}_i.\vec{S}_j = S(S+1) - 3\frac{1}{2}(\frac{1}{2}+1) = S(S+1) - \frac{9}{4}$$

$$\sum_{i<j}\vec{S}_i.\vec{S}_j = -\frac{3}{4}\ \left(J = \frac{1}{2}\right) \qquad \sum_{i<j}\vec{S}_i.\vec{S}_j = +\frac{3}{4}\ \left(J = \frac{3}{2}\right)$$

e.g. proton (uud) compared with Δ (uud) – same quark content

$$M_p = 3m_u - \frac{3A'}{4m_u^2}, \qquad M_\Delta = 3m_u + \frac{3A'}{4m_u^2}$$

Baryon Masses



Excellent agreement using

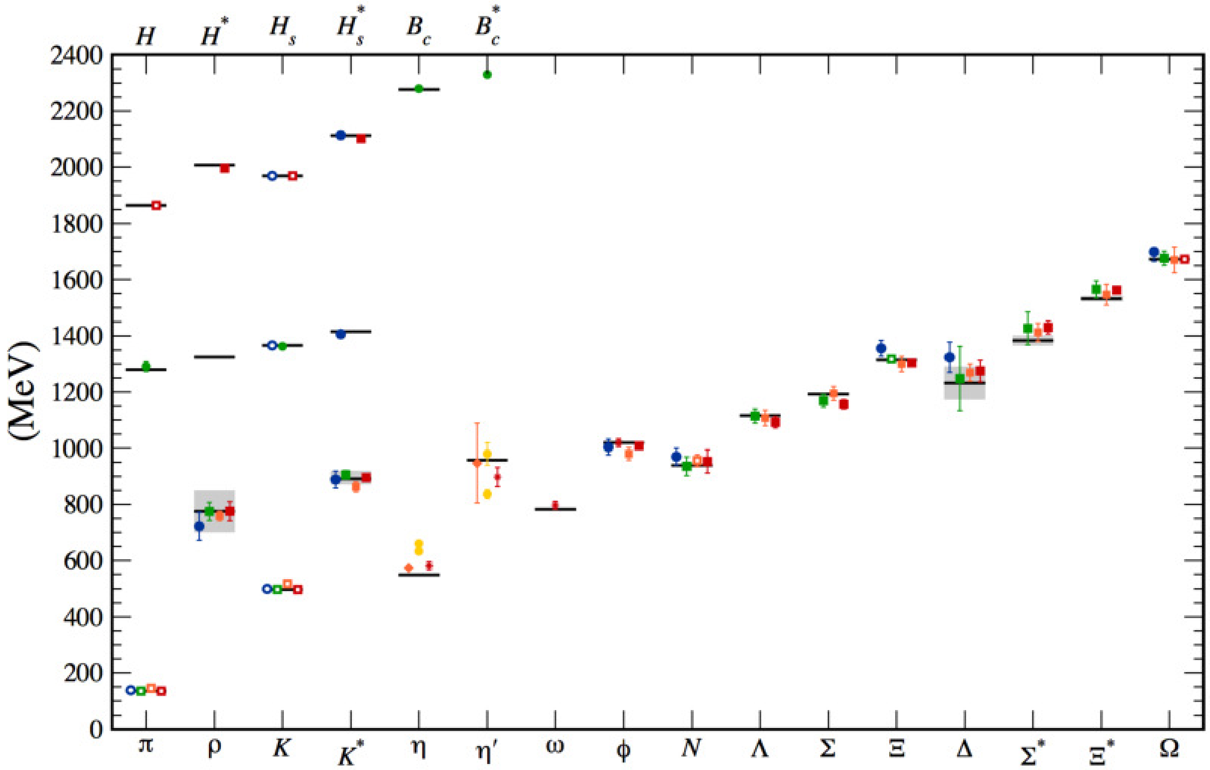
Colour factor of 2

$m_u = 0.362\ \mathrm{GeV},\ m_d = 0.366\ \mathrm{GeV},\ m_s = 0.537\ \mathrm{GeV},\ A' = 0.026\ \mathrm{GeV}^3 \sim A/2$

Constituent quark mass depends on hadron wave-function and includes cloud of gluons and qq pairs $\Rightarrow$ slightly different values for mesons and baryons.

Hadron masses in QCD

- Calculation of hadron masses in QCD is a hard problem – can't use perturbation theory.
- Need to solve field equations exactly – only feasible on a discrete lattice of space-time points.
- Needs specialised supercomputing (Pflops) + clever techniques.
- Current state of the art (after 40 years of work)...



Baryon Magnetic Moments

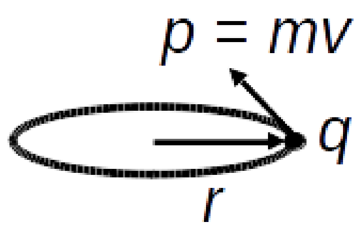
Magnetic dipole moments arise from

- the orbital motion of charged quarks
- the intrinsic spin-related magnetic moments of the quarks.

Orbital Motion

Classically, current loop

$$\mu = IA = \frac{qv}{2\pi r}\pi r^2 = \frac{qpr}{2m} = \frac{q}{2m}L_z$$



Quantum mechanically, get the same result

$$\hat{\mu} = g_L \frac{q}{2m}\hat{L}_z$$

g_L is the "g-factor"
$g_L = 1$ charged particles
$g_L = 0$ neutral particles

Intrinsic Spin

The magnetic moment operator due to the intrinsic spin of a particle is

$$\hat{\mu} = g_s \frac{q}{2m}\hat{S}_z$$

g_s is the "spin g-factor"
$g_s = 2$ for Dirac spin $1/2$ point-like particles.

Baryon Magnetic Moments

The magnetic dipole moment is the maximum measurable component of the magnetic dipole moment operator

$$\mu_L = \left\langle \psi_{\rm space} \left| g_L \frac{q}{2m} \hat{L}_z \right| \psi_{\rm space} \right\rangle \qquad \mu_s = \left\langle \psi_{\rm spin} \left| g_s \frac{q}{2m} \hat{S}_z \right| \psi_{\rm spin} \right\rangle$$

For an electron

$$\mu_L = -g_L \frac{e}{2m_e} \hbar L \qquad \mu_s = -g_s \frac{e}{2m_e} \frac{\hbar}{2}$$

$$= \; -\mu_B L \qquad = \; -\mu_B$$

where $\mu_B = e\hbar/2m_e$ is the Bohr Magneton

Observed difference from $g_s = 2$ is due to higher order corrections in QED

$$\mu_s = -\mu_B \left[1 + \frac{\alpha}{2\pi} + O(\alpha^2) + \ldots \right] \qquad \alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$$

If the proton and neutron were point-like particles,

$$\mu_L = g_L \frac{e}{2m_p} \hbar L \qquad\qquad \mu_s = g_s \frac{e}{2m_p} \frac{\hbar}{2} = \frac{1}{2} g_s \mu_N$$

where $\mu_N = e\hbar/2m_p$ is the Nuclear Magneton

Expect: p spin 1/2, charge $+e$ $\quad \mu_s = \mu_N$

n spin 1/2, charge 0 $\quad \mu_s = 0$

Observe: $p \quad \mu_s = +2.793\mu_N \quad \rightarrow \quad g_s = +5.586$

$n \quad \mu_s = -1.913\mu_N \quad \rightarrow \quad g_s = -3.826$

Observation shows that p and n are not point-like $\Rightarrow$ evidence for quarks.

$\Rightarrow$ use quark model to estimate baryon magnetic moments.

Baryon Magnetic Moments *in the Quark Model*

Assume that bound quarks within baryons behave as Dirac point-like spin 1/2 particles with fractional charge q_q.

Then quarks will have magnetic dipole moment operator and magnitude:

$$\vec{\mu}_q = \frac{q_q}{m_q}\hat{S}_z \qquad \mu_q = \left\langle \psi^q_{\text{spin}} \left| \frac{q_q}{m_q}\hat{S}_z \right| \psi^q_{\text{spin}} \right\rangle = \frac{q_q \hbar}{2m_q}$$

where m_q is the quark mass.

Therefore $$\mu_u = \frac{2}{3}\frac{e\hbar}{2m_u}, \qquad \mu_d = -\frac{1}{3}\frac{e\hbar}{2m_d}, \qquad \mu_s = -\frac{1}{3}\frac{e\hbar}{2m_s}$$

For quarks bound within an $L = 0$ baryon, the baryon magnetic moment is the expectation value of the sum of the individual quark magnetic moment operators:

$$\hat{\mu}_{\text{baryon}} = \frac{q_1}{m_1}\hat{S}_{1z} + \frac{q_2}{m_2}\hat{S}_{2z} + \frac{q_3}{m_3}\hat{S}_{3z}; \qquad \mu_{\text{baryon}} = \left\langle \psi^B_{\text{spin}} \left| \hat{\mu}_B \right| \psi^B_{\text{spin}} \right\rangle$$

where ψ^B_{spin} is the baryon spin wavefunction.

Baryon Magnetic Moments *in the Quark Model*

Example: Magnetic moment of a proton

Baryon Magnetic Moments *in the Quark Model*

Repeat for the other $L = 0$ baryons. Predict $\dfrac{\mu_n}{\mu_p} = -\dfrac{2}{3}$

compared to the experimentally measured value of -0.685

Baryon	**μ_B in Quark Model**	**Predicted [μ_N]**	**Observed [μ_N]**
p (uud)	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_d$	$+2.79$	$+2.793$
n (ddu)	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_u$	-1.86	-1.913
Λ (uds)	μ_s	-0.61	-0.614 ± 0.005
Σ^+ (uus)	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_s$	$+2.68$	$+2.46 \pm 0.01$
Ξ^0 (ssu)	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_u$	-1.44	-1.25 ± 0.014
Ξ^- (ssd)	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_d$	-0.51	-0.65 ± 0.01
Ω^- (sss)	$3\mu_s$	-1.84	-2.02 ± 0.05

Reasonable agreement with data using

$$m_u = m_d = 0.336\ \mathrm{GeV},\ \ m_s \sim 0.509\ \mathrm{GeV}$$

Hadron Decays

- Hadrons are eigenstates of the strong force.
- Hadrons will decay via the strong interaction to lighter mass states if energetically feasible (i.e. mass of parent > mass of daughters).
- And, angular momentum and parity must be conserved in strong decays.

Examples:

$$\rho^0 \rightarrow \pi^+\pi^- \qquad\qquad \Delta^{++} \rightarrow p\pi^+$$

$$\underset{769}{m(\rho^0)} > \underset{140}{m(\pi^+)} + \underset{140\ \mathrm{MeV}}{m(\pi^-)} \qquad\qquad \underset{1231}{m(\Delta^{++})} > \underset{938}{m(p)} + \underset{140\ \mathrm{MeV}}{m(\pi^+)}$$

Hadron Decays

Also need to check for identical particles in the final state.

Examples:

$$\omega^0 \to \pi^0\pi^0$$

$$\underset{782}{m(\omega^0)} > \underset{135}{m(\pi^0)} + \underset{135\ \mathrm{MeV}}{m(\pi^0)}$$

$$\omega^0 \to \pi^+\pi^-\pi^0$$

$$\underset{782}{m(\omega^0)} > \underset{140}{m(\pi^+)} + \underset{140}{m(\pi^-)} + \underset{135\ \mathrm{MeV}}{m(\pi^0)}$$

Hadron Decays

Hadrons can also decay via the electromagnetic interaction.

Examples:

$$\rho^0 \to \pi^0 \gamma \qquad\qquad \Sigma^0 \to \Lambda^0 \gamma$$

$$\underset{769}{m(\rho^0)} > \underset{135\ \mathrm{MeV}}{m(\pi^0)} + m(\gamma) \qquad\qquad \underset{1193}{m(\Sigma^0)} > \underset{1116\ \mathrm{MeV}}{m(\Lambda^0)} + m(\gamma)$$

The lightest mass states ($p,\ K^\pm,\ K^0,\ \bar{K}^0,\ \Lambda,\ n$) require a change of quark flavour in the decay and therefore decay via the weak interaction (see later).

Summary of light (*uds*) hadrons

- Baryons and mesons are composite particles (complicated).
- However, the naive Quark Model can be used to make predictions for masses/magnetic moments.
- The predictions give reasonably consistent values for the constituent quark masses:

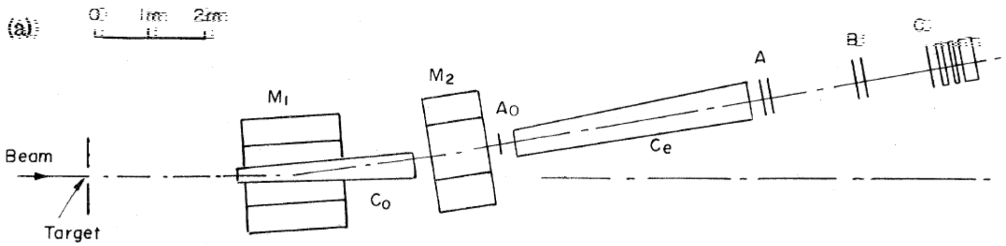
	$m_{u/d}$	m_s
Meson Masses	307 MeV	487 MeV
Baryon Masses	364 MeV	537 MeV
Baryon Magnetic Moments	336 MeV	509 MeV

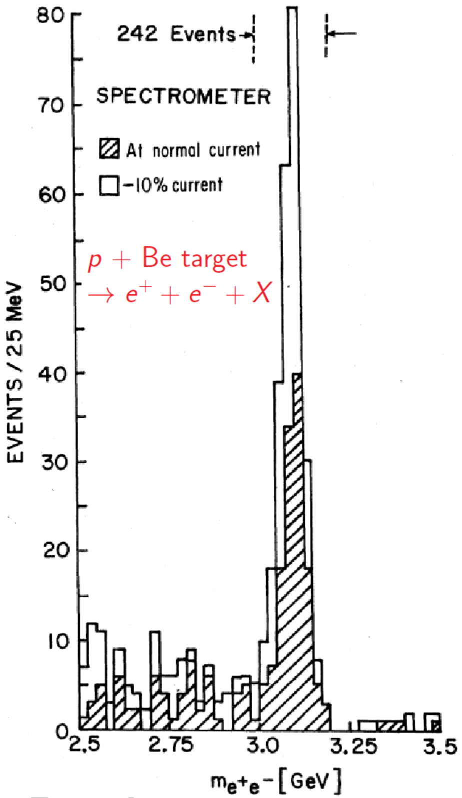
$$m_u \sim m_d \sim 335 \text{ MeV}, \quad m_s \sim 510 \text{ MeV}$$

- Hadrons will decay via the strong interaction to lighter mass states if energetically feasible.
- Hadrons can also decay via the EM interaction.
- The lightest mass states require a change of quark flavour to decay and therefore decay via the weak interaction (see later).

Heavy hadrons *The November Revolution*

Brookhaven National Laboratory
Led by Samuel Ting

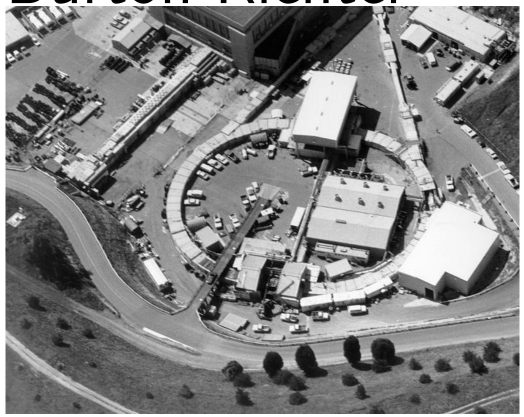




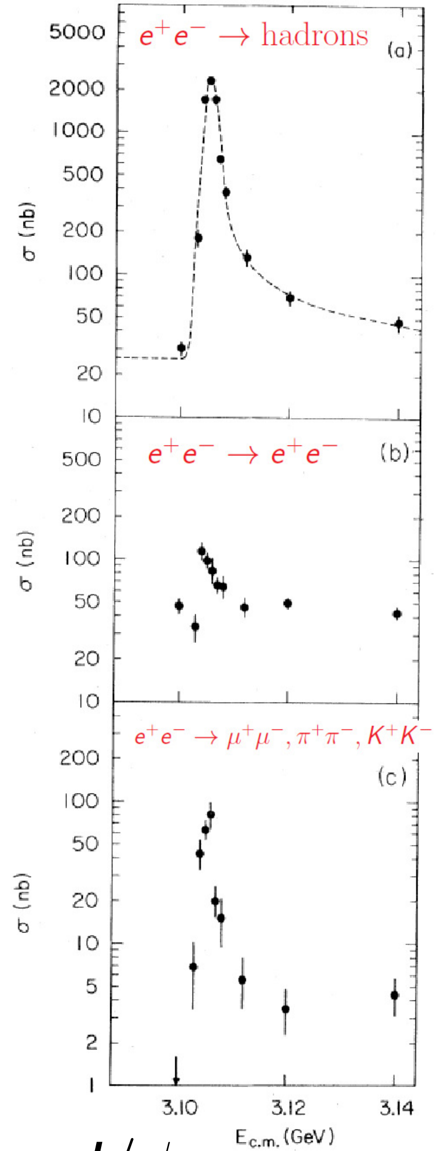
J particle:
PRL 33 (1974) 1404

Stanford Linear Accelerator Center, SPEAR

Led by Burton Richter



ψ particle:
PRL 33 (1974) 1406



Both experiments announced discovery on 11 November 1974 $\Rightarrow J/\psi$
1976 Nobel Prize awarded to Ting and Richter.

Heavy hadrons *Charmonium*

1974: Discovery of a narrow resonance in e^+e^- collisions at $\sqrt{s} \sim 3.1\ \mathrm{GeV}$

$$J/\psi(3097)$$

Observed width $\sim 3\ \mathrm{MeV}$, all due to experimental resolution.
Actual Total Width, $\Gamma_{J/\psi} \sim 97\ \mathrm{keV}$

Branching fractions:

$$B(J/\psi \to \text{ hadrons}) \sim 88\%$$

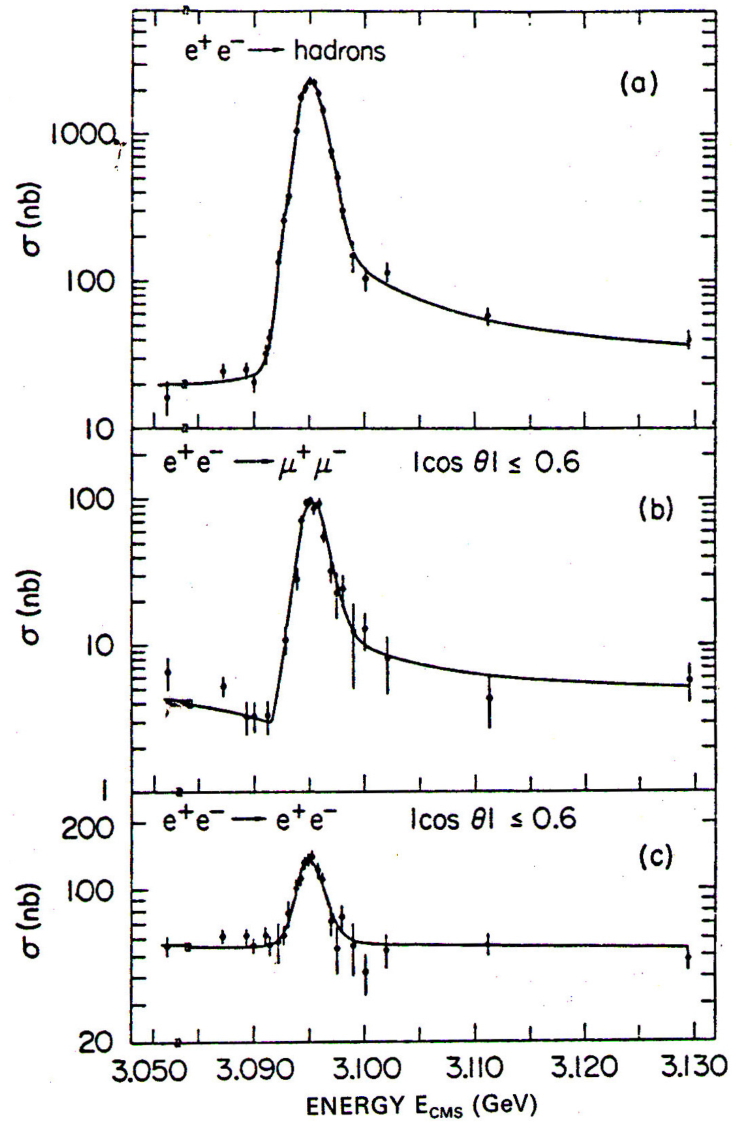
$$B(J/\psi \to \mu^+\mu^-) \sim (J/\psi \to e^+e^-) \sim 6\%$$

Partial widths:

$$\Gamma_{J/\psi \to \text{ hadrons}} \sim 87\ \mathrm{keV}$$

$$\Gamma_{J/\psi \to \mu^+\mu^-} \sim \Gamma_{J/\psi \to e^+e^-} \sim 5\ \mathrm{keV}$$

Mark II Experiment, SLAC, 1978



Resonance seen in

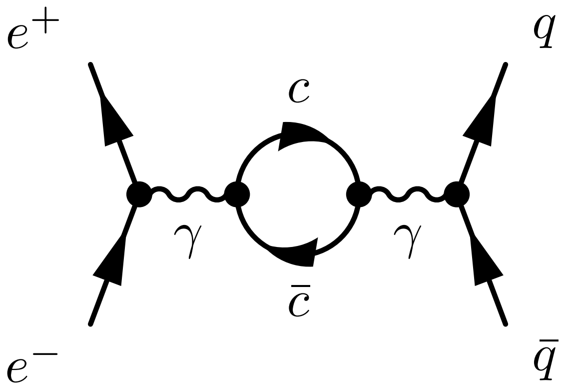
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

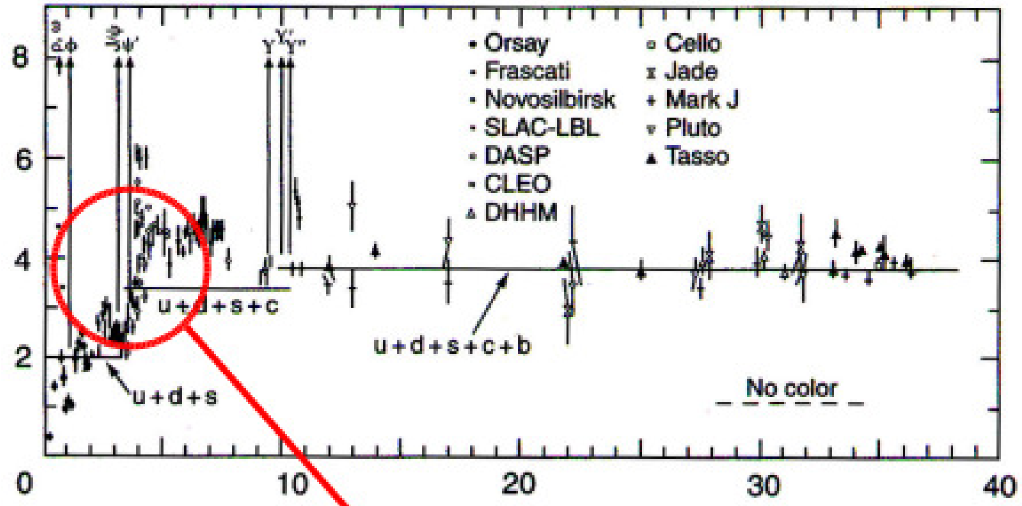
Zoom into the charmonium ($c\bar{c}$) region

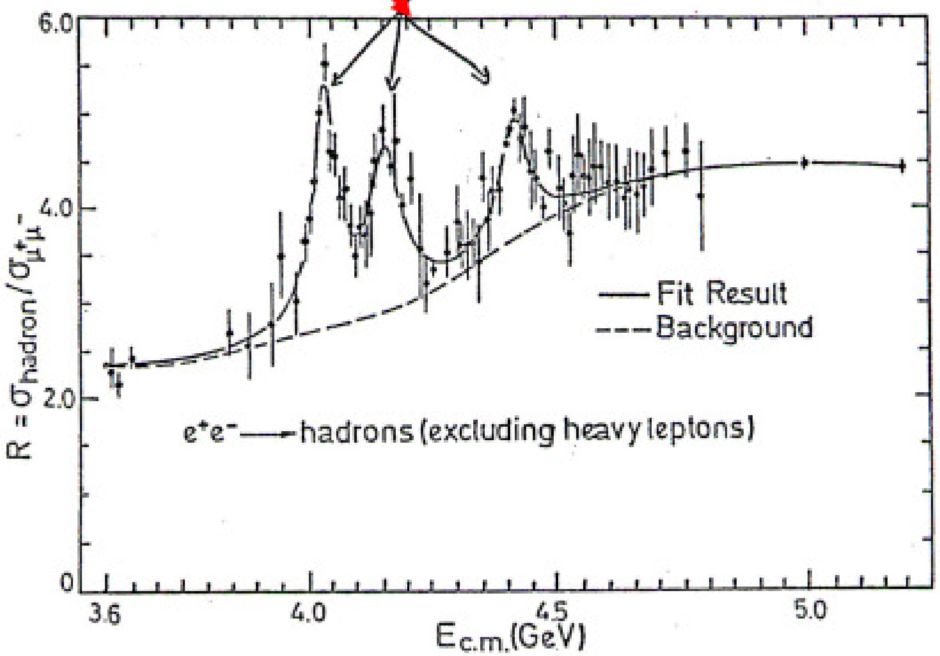
$$\sqrt{s} \sim 2m_c$$

mass of charm quark, $m_c \sim 1.5$ GeV

Resonances due to formation of bound unstable $c\bar{c}$ states. The lowest energy of these is the narrow J/ψ state.

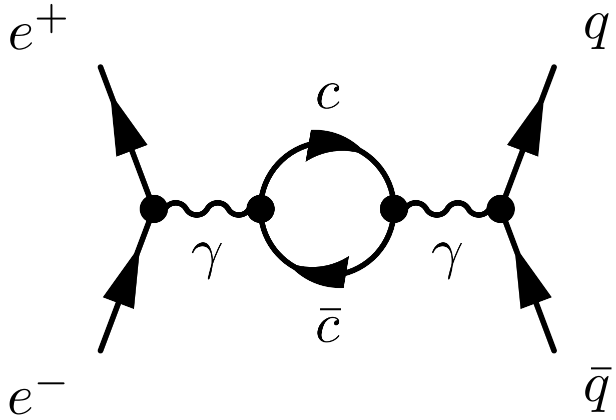






Charmonium

$c\bar{c}$ bound states produced directly in e^+e^- collisions must have the same spin and parity as the photon



$$J^P = 1^-$$

However, expect that a whole spectrum of bound $c\bar{c}$ states should exist (analogous to e^+e^- bound states, positronium)

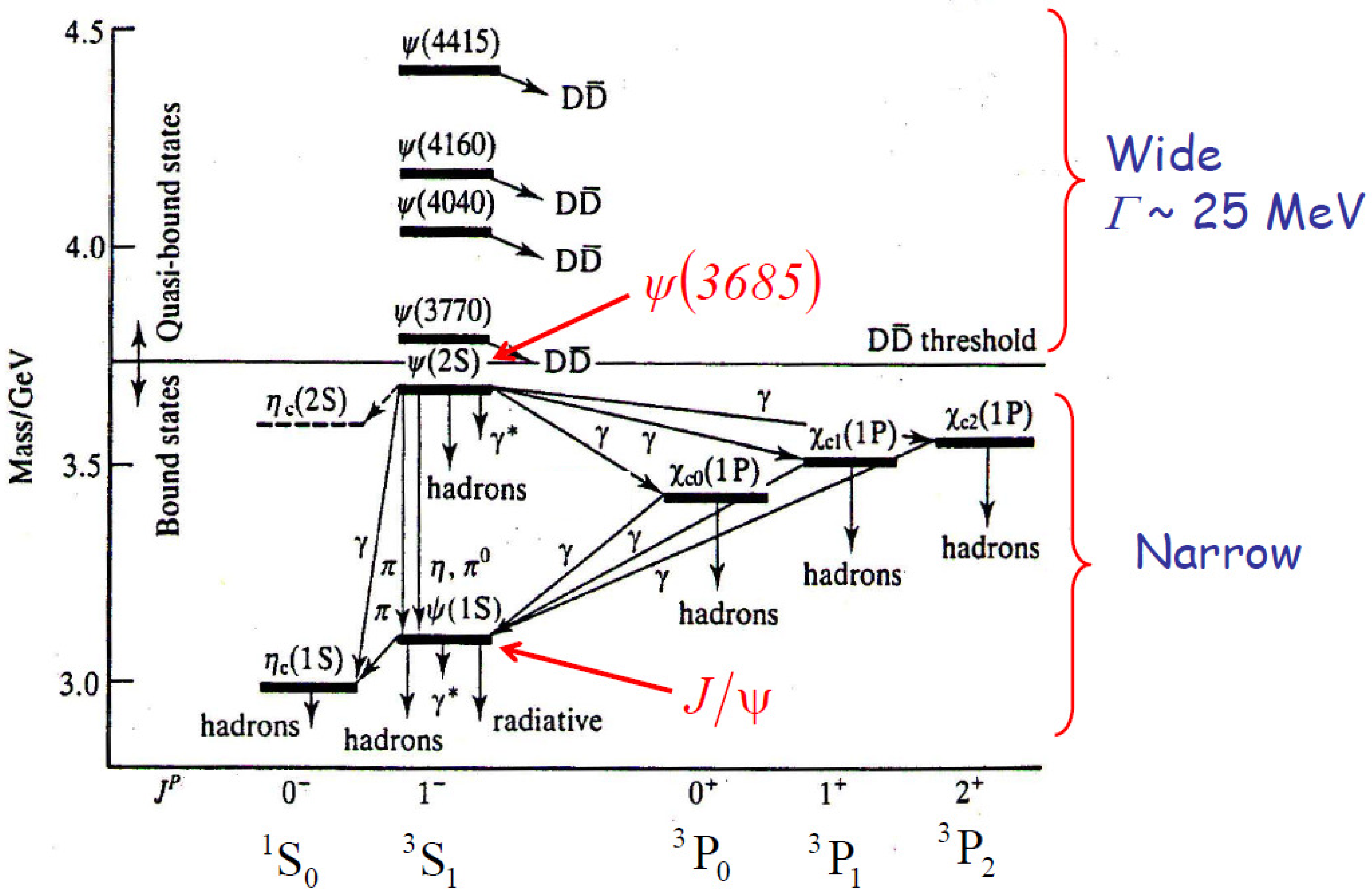
$$n = 1 \quad L = 0 \qquad S = 0, 1 \qquad {}^1S_0, {}^3S_1$$

$$n = 2 \quad L = 0, 1 \qquad S = 0, 1 \qquad {}^1S_0, {}^3S_1, {}^1P_1, {}^3P_{0,1,2}$$

$$\ldots \text{etc}$$

$$\text{Parity} = (-1)(-1)^L \qquad {}^{2S+1}L_J$$

The Charmonium System

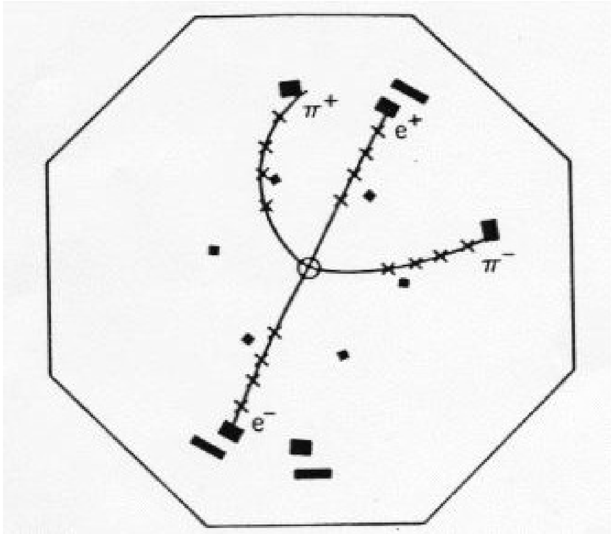


The Charmonium System

All $c\bar{c}$ bound states can be observed via their decay:

Example: Hadronic decay

$$\psi(3685) \to J/\psi\ \pi^+\pi^-$$



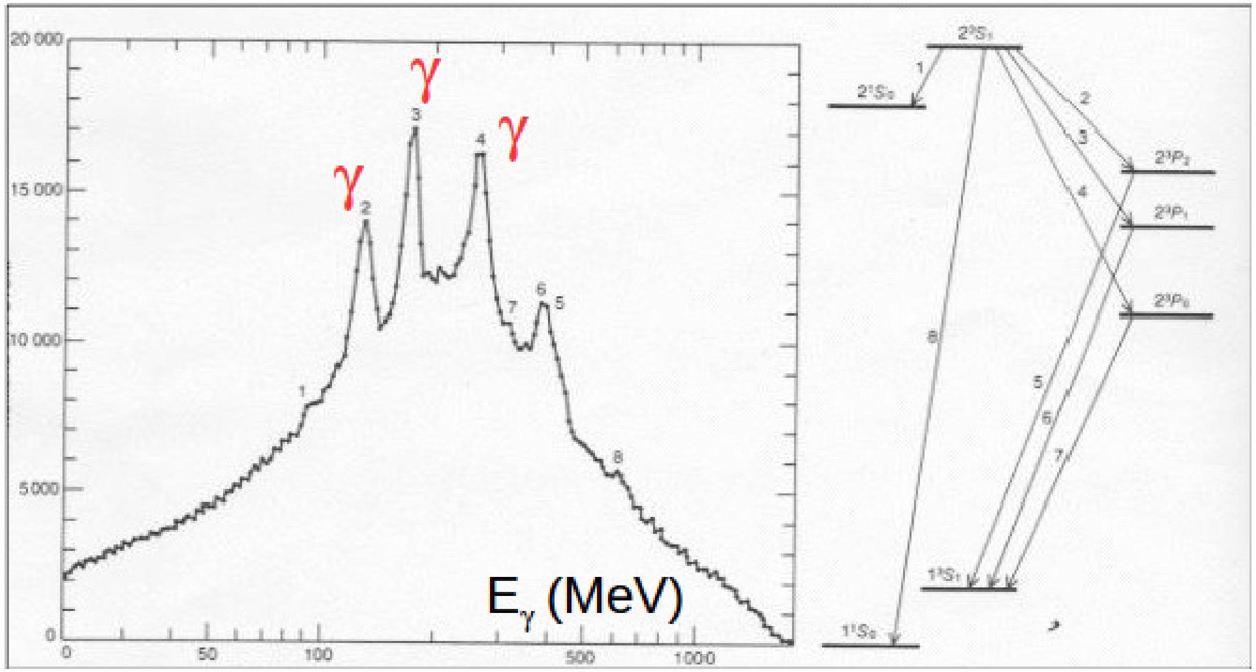
Example: Photonic decay

$$\psi(3685) \to \chi + \gamma$$

$$\chi \to J/\psi + \gamma$$

Peaks in γ spectrum

Charmonium Spectroscopy



The Charmonium System

Knowing the $c\bar{c}$ energy levels provides a probe of the QCD potential.

- Because QCD is a theory of a strong confining force (self-interacting gluons), it is **very** difficult to calculate the exact form of the QCD potential from first principles.
- However, it is possible to experimentally "determine" the QCD potential by finding an appropriate form which gives the observed charmonium states.
- In practise, the QCD potential

$$V_{\mathrm{QCD}} = -\frac{4}{3}\frac{\alpha_s}{r} + kr$$

with $\alpha_s = 0.2$ and $k = 1\ \mathrm{GeVfm}^{-1}$ provides a good description of the experimentally observed levels in the charmonium system.

Why is the J/ψ so narrow?

Consider the charmonium $^3S_1$ states:

1^3S_1 $\psi(3097)$ $\Gamma \sim 0.09$ MeV

2^3S_1 $\psi(3685)$ $\Gamma \sim 0.24$ MeV

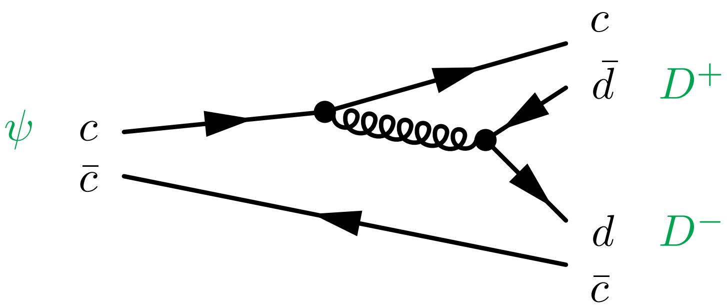
3^3S_1 $\psi(3767)$ $\Gamma \sim 25$ MeV

4^3S_1 $\psi(4040)$ $\Gamma \sim 50$ MeV

The width depends on whether the decay to lightest mesons containing c quarks, $D^-(d\bar{c})$, $D^+(c\bar{d})$, is kinematically possible or not:

$$m(D^\pm) = 1869.4 \pm 0.5 \text{ MeV}$$

$$\boldsymbol{m(\psi) > 2m(D)}$$

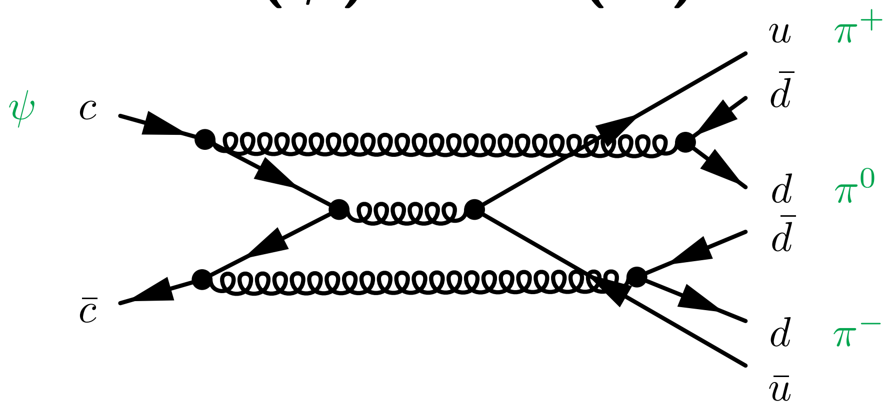


$\psi \to D^+D^-$ allowed

"ordinary" strong decay

$\Rightarrow$ large width

$$\boldsymbol{m(\psi) < 2m(D)}$$



Zweig Rule: Unconnected lines in the Feynman diagram lead to suppression of the decay amplitude

$\Rightarrow$ narrow width

Charmed Hadrons

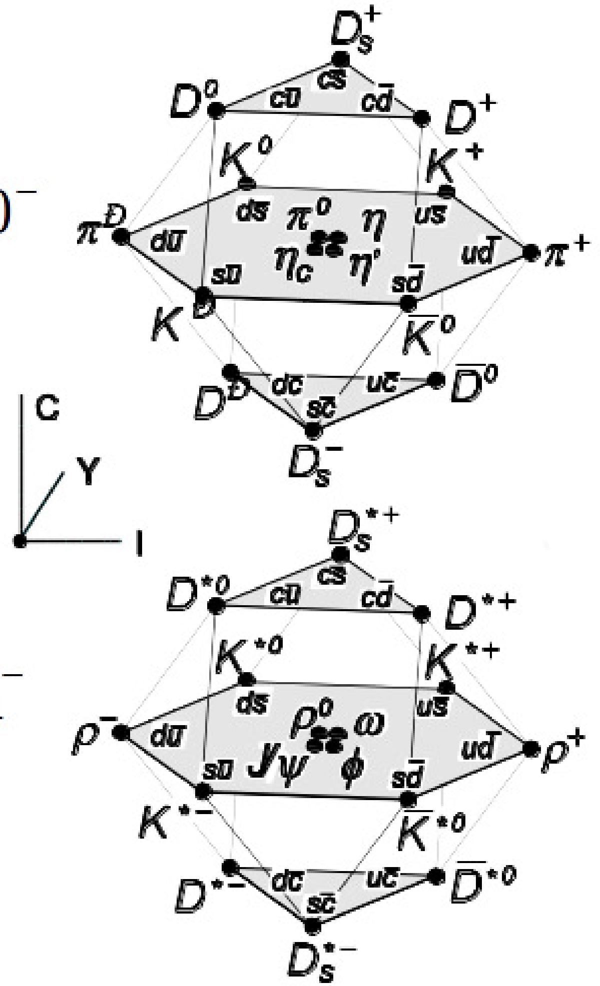
The existence of the c quark $\Rightarrow$ expect to see charmed mesons and baryons (i.e. containing a c quark).

Extend quark symmetries to 3 dimensions:

Mesons

$J^P = 0^-$

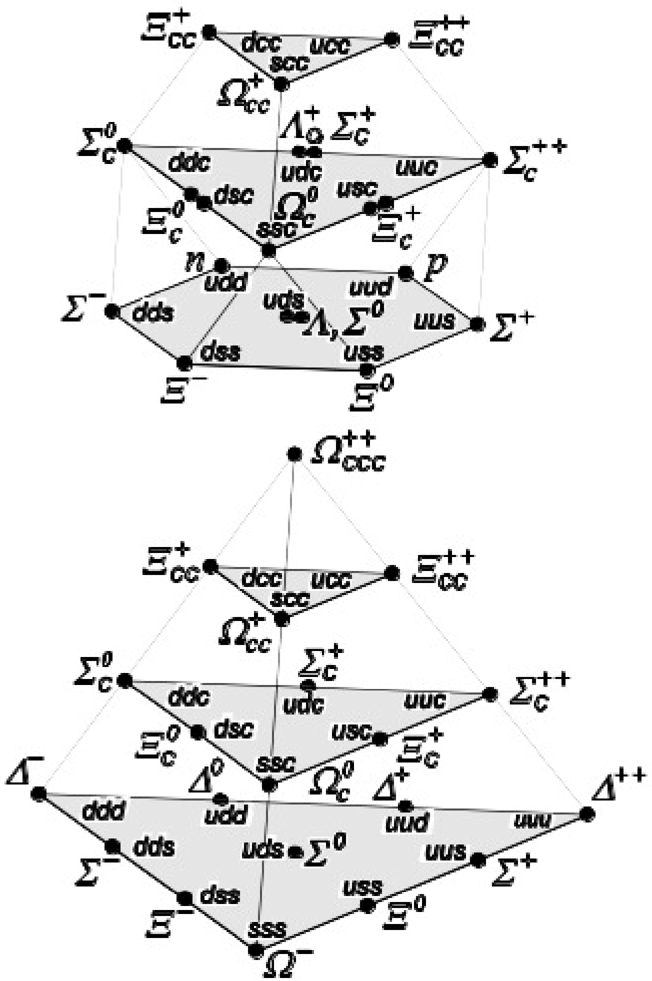
$J^P = 1^-$



Baryons

$J^P = \frac{1}{2}^+$

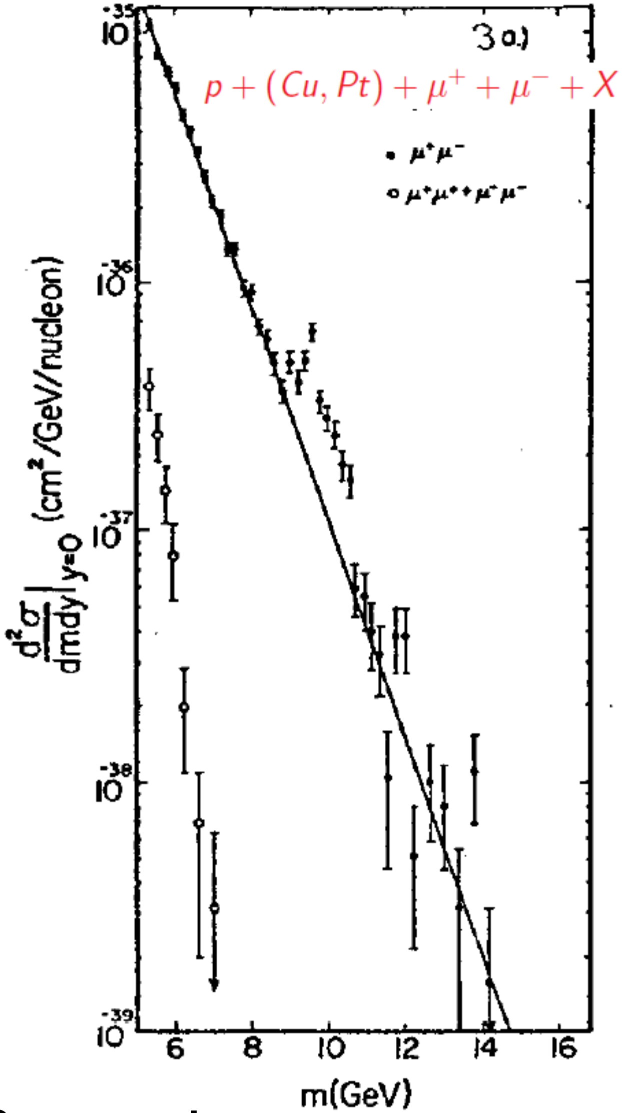
$J^P = \frac{3}{2}^+$



Heavy hadrons *the Υ $(b\bar{b})$*

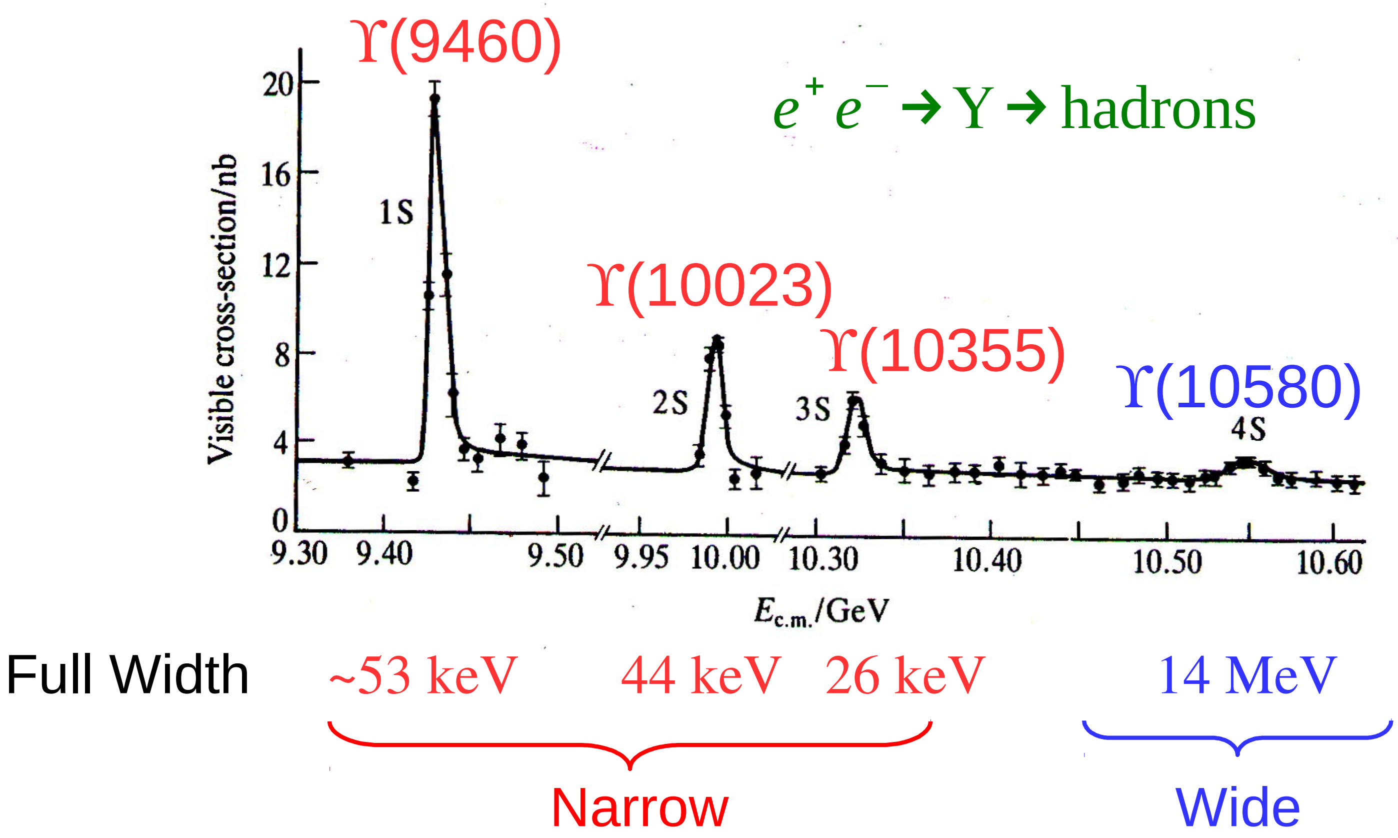
E288 collaboration, Fermilab

Led by Leon Lederman



- 1977: Discovery of the $\Upsilon(9460)$ resonance state.
- Lowest energy ${}^3S_1$ bound $b\bar{b}$ state (bottomonium).
- $\Rightarrow m_b \sim 4.7$ GeV

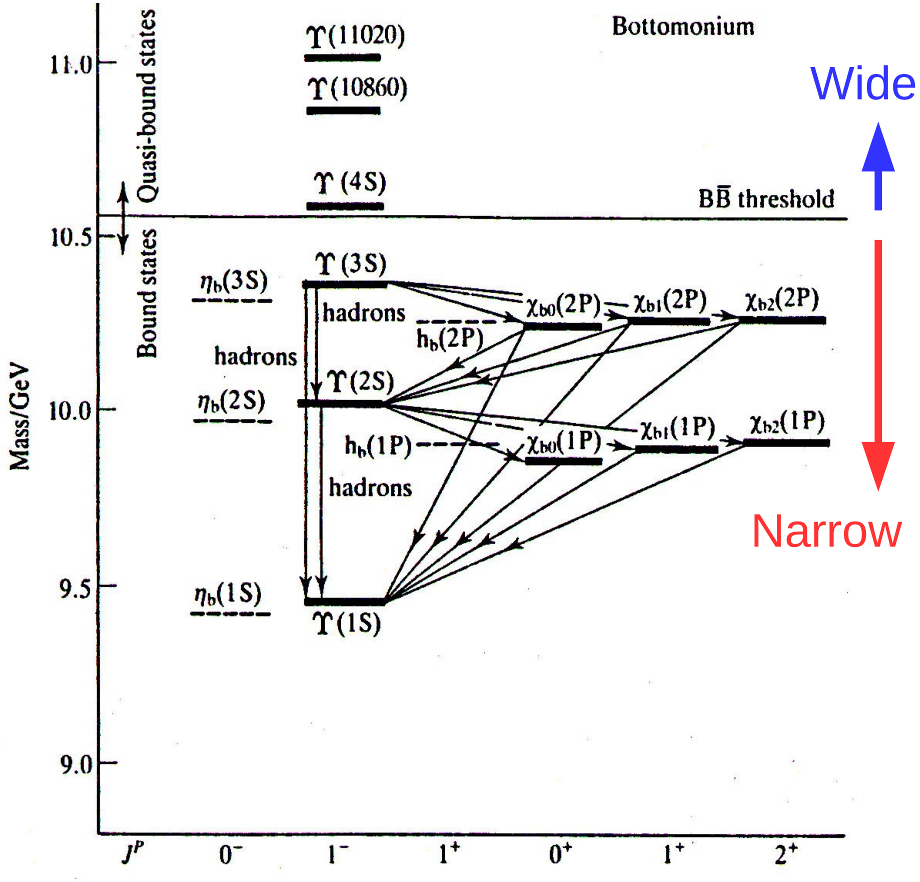
Similar properties to the ψ

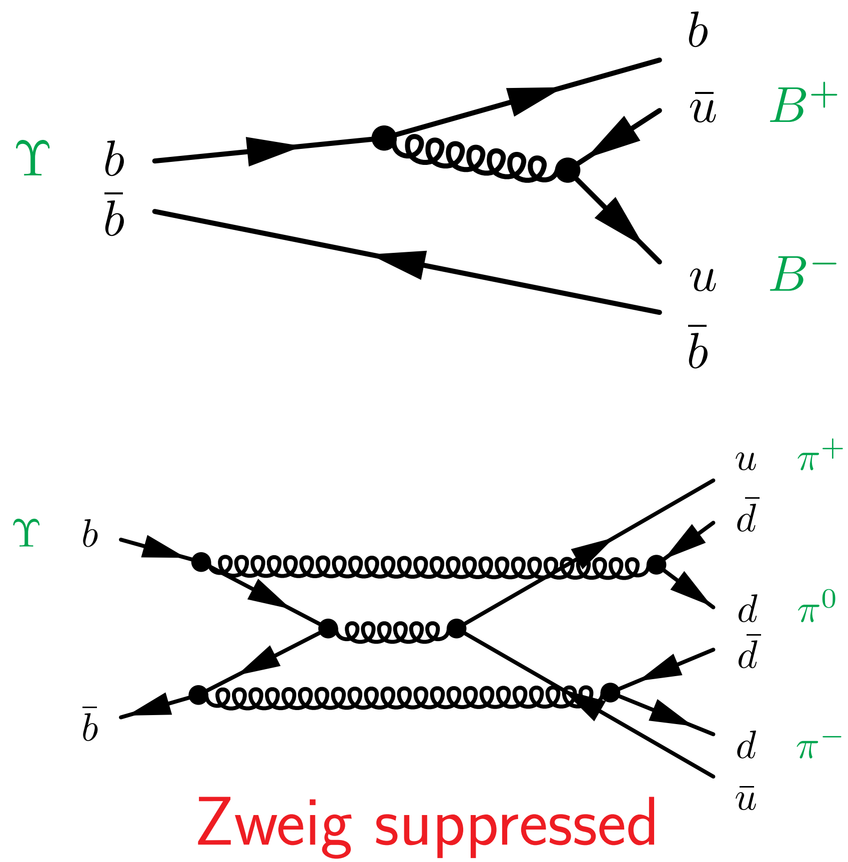


Υ particle: PRL 39 (1977) 252-255

Bottomonium

- Bottomonium is the analogue of charmonium for b quark.
- Bottomonium spectrum well described by same QCD potential as used for charmonium.
- Evidence that QCD potential does not depend on quark type.





Bottom Hadrons

Extend quark symmetries to 4 dimensions (difficult to draw!)

Examples:

Mesons $\quad (J^P = 0^-): \quad B^-(b\bar{u}); \quad B^0(\bar{b}d); \quad B_s^0(\bar{b}s); \quad B_c^-(b\bar{c})$

The B_c^- is the heaviest hadron discovered so far: $m(B_c^-) = 6.4 \pm 0.4\ \mathrm{GeV}$

$$(J^P = 1^-): \quad B^{*-}(b\bar{u}); \quad B^{*0}(\bar{b}d); \quad B_s^{*0}(\bar{b}s)$$

The mass of the B^* mesons is only 50 MeV above the B meson mass. Expect only electromagnetic decays $B^* \rightarrow B\gamma$.

Baryons $\quad \left(J^P = \frac{1}{2}^+\right): \quad \Lambda_b(bud); \quad \Sigma_b(buu); \quad \Xi_b(bus)$

Summary of heavy hadrons

- c and b quarks were first observed in bound state resonances ("onia").
- Consequences of the existence of c and b quarks are
  - Spectra of $c\bar{c}$ (charmonium) and $b\bar{b}$ (bottomonium) bound states
  - Peaks in $R = \frac{\sigma(e^+e^- \to \text{ hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$
  - Existence of mesons and baryons containing c and b quarks
- The majority of charm and bottom hadrons decay via the weak interaction (strong and electromagnetic decays are forbidden by energy conservation).

- The t quark is very heavy and decays rapidly via the weak interaction before a $t\bar{t}$ bound state (or any other hadron) can be formed.

$$\tau_t \sim 10^{-25}\,\mathrm{s} \quad t_{\mathrm{hadronisation}} \sim 10^{-22}\,\mathrm{s}$$

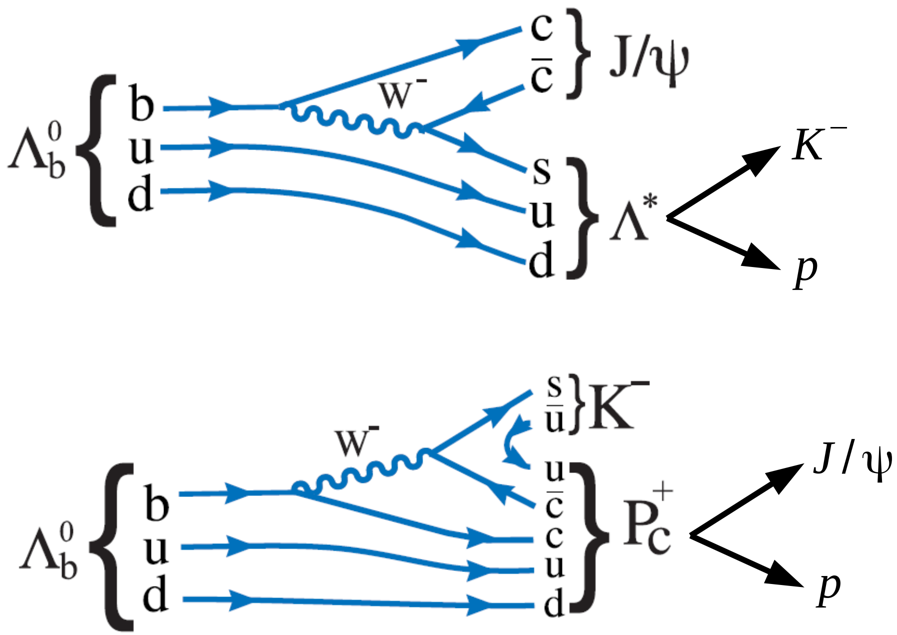
Rapid decay because $m(t) > m(W)$ so weak interaction is no longer weak.

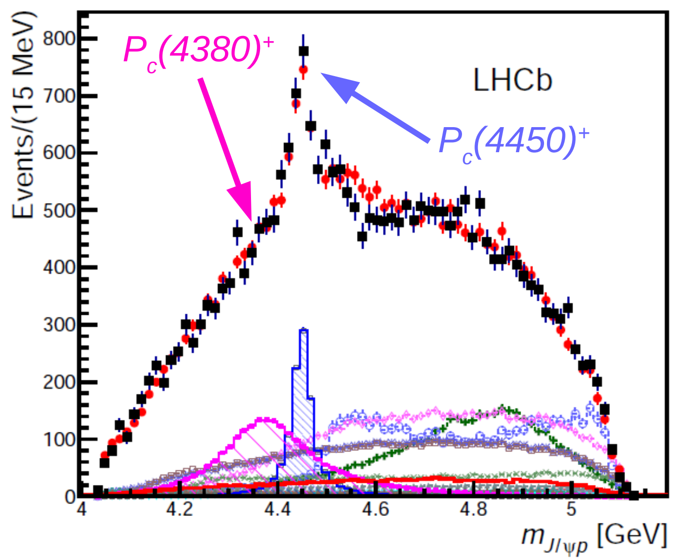
$$\begin{pmatrix} m(u) = 335 \text{ MeV} \\ m(d) = 335 \text{ MeV} \end{pmatrix} \begin{pmatrix} m(c) = 1.5 \text{ GeV} \\ m(s) = 510 \text{ MeV} \end{pmatrix} \begin{pmatrix} m(t) = 175 \text{ GeV} \\ m(b) = 4.5 \text{ GeV} \end{pmatrix}$$

Tetraquarks and Pentaquarks *(non-examinable)*

Quark Model of Hadrons is not limited to $q\bar{q}$ or qqq content.

Recent observations from *LHCb* show unquestionable discovery of pentaquark states, PRL 115, 072001 (2015).





+ others more recently.

How are these quarks bound? $qqqqq$? $qq + qqq$? $qq + qq + q$?

A few tetraquarks discovered by *Belle* and *BESIII*

e.g. $Z(4430)^-$, $c\bar{c}d\bar{u}$ discovered by *Belle* and confirmed by *LHCb*

LHCb has discovered many more!

Summary

- Evidence for hadron sub-structure – quarks
- Hadron wavefunctions and allowed states
- Hadron masses and magnetic moments
- Hadron decays (strong, EM, weak)
- Heavy hadrons: charmonium and bottomonium
- Recent tetraquark and pentaquark discoveries

Problem Sheet: q.17-22

Up next...
Section 9: The Weak Force