In this section...

- Natural units
- Symmetries and conservation laws
- Relativistic kinematics
- Particle properties
- Decays
- Cross-sections
- Scattering
- Resonances
The usual practice in particle and nuclear physics is to use Natural Units.

- **Energies** are measured in units of eV:
  - **Nuclear** keV ($10^3$ eV), MeV ($10^6$ eV)
  - **Particle** GeV ($10^9$ eV), TeV ($10^{12}$ eV)

- **Masses** are quoted in units of MeV/$c^2$ or GeV/$c^2$ (using $E = mc^2$)
  e.g. electron mass $m_e = 9.11 \times 10^{-31}$ kg = $(9.11 \times 10^{-31})(3 \times 10^8)^2$ J/$c^2$
  = $8.20 \times 10^{-14}/1.602 \times 19^{-19}$ eV/$c^2 = 5.11 \times 10^5$ eV/$c^2 = 0.511$ MeV/$c^2$

- **Atomic/nuclear masses** are often quoted in unified (or atomic) mass units
  1 unified mass unit (u) = (mass of a $^{12}\text{C}$ atom) / 12
  $1\text{u} = 1\text{ g}/N_A = 1.66 \times 10^{-27}$ kg = 931.5 MeV/$c^2$

- **Cross-sections** are usually quoted in barns: $1\text{b} = 10^{-28}$ m$^2$. 
# Units

**Natural Units**

Choose energy as the basic unit of measurement... ...and simplify by choosing $\hbar = c = 1$

<table>
<thead>
<tr>
<th>Physical Quantity</th>
<th>Unit in Natural Units</th>
<th>Unit in SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>GeV</td>
<td>GeV</td>
</tr>
<tr>
<td>Momentum</td>
<td>GeV/c</td>
<td>GeV</td>
</tr>
<tr>
<td>Mass</td>
<td>GeV/c$^2$</td>
<td>GeV</td>
</tr>
<tr>
<td>Time</td>
<td>$(\text{GeV}/\hbar)^{-1}$</td>
<td>GeV$^{-1}$</td>
</tr>
<tr>
<td>Length</td>
<td>$(\text{GeV}/\hbar c)^{-1}$</td>
<td>GeV$^{-1}$</td>
</tr>
<tr>
<td>Cross-section</td>
<td>$(\text{GeV}/\hbar c)^{-2}$</td>
<td>GeV$^{-2}$</td>
</tr>
</tbody>
</table>

Reintroduce “missing” factors of $\hbar$ and $c$ to convert back to SI units.

- $\hbar c = 0.197 \text{ GeV fm} = 1$ Energy $\leftrightarrow$ Length
- $\hbar = 6.6 \times 10^{-25} \text{ GeV s} = 1$ Energy $\leftrightarrow$ Time
- $c = 3.0 \times 10^8 \text{ ms}^{-1} = 1$ Length $\leftrightarrow$ Time
1. cross-section $\sigma = 2 \times 10^{-6} \text{ GeV}^{-2}$
   Need to change units of energy to length. Use $\hbar c = 0.197 \text{ GeVfm} = 1$.
   
   $\sigma = 2 \times 10^{-6} \times (3.89 \times 10^{-32} \text{ m}^2)$
   $= 7.76 \times 10^{-38} \text{ m}^2$
   And using $1 \text{ b} = 10^{-28} \text{ m}^2$, $\sigma = 0.776 \text{ nb}$

2. lifetime $\tau = 1/\Gamma = 0.5 \text{ GeV}^{-1}$
   Need to change units of energy$^{-1}$ to time. Use $\hbar = 6.6 \times 10^{-25} \text{ GeV s} = 1$.
   
   $\tau = 0.5 \times (6.6 \times 10^{-25} \text{ s}) = 3.3 \times 10^{-25} \text{ s}$

   Also, can have Natural Units involving electric charge: $\epsilon_0 = \mu_0 = \hbar = c = 1$

3. Fine structure constant (dimensionless)
   $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \sim \frac{1}{137}$
   becomes
   $\alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$
   i.e. $e \sim 0.30(\text{n.u.})$
Symmetries and conservation laws

The most elegant and powerful idea in physics
Noether’s theorem:
*every differentiable symmetry of the action of a physical system has a corresponding conservation law.*

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Conserved current</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time, ( t )</td>
<td>Energy, ( E )</td>
</tr>
<tr>
<td>Translational, ( x )</td>
<td>Linear momentum, ( p )</td>
</tr>
<tr>
<td>Rotational, ( \theta )</td>
<td>Angular momentum, ( L )</td>
</tr>
<tr>
<td>Probability</td>
<td>Total probability always 1</td>
</tr>
<tr>
<td>Lorentz invariance</td>
<td>Charge Parity Time (CPT)</td>
</tr>
<tr>
<td>Gauge</td>
<td>charge (e.g. electric, colour, weak)</td>
</tr>
</tbody>
</table>

Lorentz invariance: laws of physics stay the same for all frames moving with a uniform velocity.
Gauge invariance: observable quantities unchanged (charge, \( E \), \( \nu \)) when a field is transformed.
**Nuclear reactions**

Low energy, typically K.E. $\mathcal{O}(10 \text{ MeV}) \ll$ nucleon rest energies.

$\Rightarrow$ non-relativistic formulae ok

Exception: always treat $\beta$-decay relativistically

$(m_e \sim 0.5 \text{ MeV} < 1.3 \text{ MeV} \sim m_n - m_p)$

**Particle physics**

High energy, typically K.E. $\mathcal{O}(100 \text{ GeV}) \gg$ rest mass energies.

$\Rightarrow$ relativistic formulae usually essential.
Recall the energy $E$ and momentum $p$ of a particle with mass $m$
\[ E = \gamma m, \quad |\vec{p}| = \gamma \beta m \]
\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c} = \frac{v}{c} \]
or
\[ \gamma = \frac{E}{m}, \quad \beta = \frac{|\vec{p}|}{E} \]
and these are related by $E^2 = \vec{p}^2 + m^2$.

Interesting cases:
- when a particle is at rest, $\vec{p} = 0$, $E = m$,
- when a particle is massless, $m = 0$, $E = |\vec{p}|$,
- when a particle is ultra-relativistic $E \gg m$, $E \sim |\vec{p}|$.

Kinetic energy (K.E., or $T$) is the extra energy due to motion
\[ T = E - m = (\gamma - 1)m \]
in the non-relativistic limit $\beta \ll 1$, $T = \frac{1}{2}mv^2$. 
Relativistic Kinematics  

Four-Vectors

The kinematics of a particle can be expressed as a four-vector, e.g.

\[ p_\mu = (E, -\vec{p}), \quad p^\mu = (E, \vec{p}) \quad \text{and} \quad x_\mu = (t, -\vec{x}), \quad x^\mu = (t, \vec{x}) \]

multiply by a metric tensor to raise/lower indices

\[ p_\mu = g_{\mu\nu} p^\nu, \quad p^\mu = g^{\mu\nu} p_\nu \]

Scalar product of two four-vectors \( A^\mu = (A^0, \vec{A}) \), \( B^\mu = (B^0, \vec{B}) \) is invariant:

\[ A^\mu B_\mu = A.B = A^0 B^0 - \vec{A}.\vec{B} \]

or

\[ p^\mu p_\mu = p^\mu g_{\mu\nu} p_\nu = \sum_{\mu=0,3} \sum_{\nu=0,3} p^\mu g_{\mu\nu} p_\nu = g_{00} p_0^2 + g_{11} p_1^2 + g_{22} p_2^2 + g_{33} p_3^2 \]

\[ = E^2 - |\vec{p}|^2 = m^2 \quad \text{invariant mass} \]

\((t, \vec{x})\) and \((E, \vec{p})\) transform between frames of reference, but

\[ d^2 = t^2 - \vec{x}^2 \quad \text{Invariant interval is constant} \]

\[ m^2 = E^2 - |\vec{p}|^2 \quad \text{Invariant mass is constant} \]
A common technique to identify particles is to form the **invariant mass** from their decay products.

Remember, for a single particle $m^2 = E^2 - \vec{p}^2$.

For a system of particles, where $X \rightarrow 1 + 2 + 3...n$:

$$M_X^2 = \left( (E_1, \vec{p}_1) + (E_2, \vec{p}_2) + ... \right)^2 = \left( \sum_{i=1}^{n} E_i \right)^2 - \left( \sum_{i=1}^{n} \vec{p}_i \right)^2$$

In the specific (and common) case of a two-body decay, $X \rightarrow 1 + 2$, this reduces to

$$M_X^2 = m_1^2 + m_2^2 + 2 (E_1 E_2 - |\vec{p}_1||\vec{p}_2| \cos \theta)$$

n.b. sometimes invariant mass $M$ is called “centre-of-mass energy” $E_{CM}$, or $\sqrt{s}$
Relativistic Kinematics  Decay Example

Consider a charged pion decaying at rest in the lab frame $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$

Find the momenta of the decay products
How do we study particles and forces?

- **Static Properties**
  What particles/states exist?
  Mass, spin and parity ($J^P$), magnetic moments, bound states

- **Particle Decays**
  Most particles and nuclei are unstable.
  Allowed/forbidden decays $\rightarrow$ Conservation Laws.

- **Particle Scattering**
  Direct production of new massive particles in matter-antimatter annihilation.
  Study of particle interaction cross-sections.
  Use high-energies to study forces at short distances.

<table>
<thead>
<tr>
<th>Force</th>
<th>Typical Lifetime [s]</th>
<th>Typical cross-section [mb]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong</td>
<td>$10^{-23}$</td>
<td>10</td>
</tr>
<tr>
<td>Electromagnetic</td>
<td>$10^{-20}$</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>Weak</td>
<td>$10^{-8}$</td>
<td>$10^{-13}$</td>
</tr>
</tbody>
</table>
**Particle Decays**  

Reminder

Most particles are transient states – only a few live forever ($e^-$, $p$, $\nu$, $\gamma$...).

- **Number** of particles remaining at time $t$
  
  $$N(t) = N(0)p(t) = N(0)e^{-\lambda t}$$

  where $N(0)$ is the number at time $t = 0$.

- **Rate of decays**
  
  $$\frac{dN}{dt} = -\lambda N(0)e^{-\lambda t} = -\lambda N(t)$$

  Assuming the nuclei only decay. More complicated if they are also being created.

- **Activity**
  
  $$A(t) = \left| \frac{dN}{dt} \right| = \lambda N(t)$$

- Its rather common in nuclear physics to use the **half-life** (i.e. the time over which 50% of the particles decay). In particle physics, we usually quote the mean life. They are simply related:

  $$N(\tau_{1/2}) = \frac{N(0)}{2} = N(0)e^{-\lambda \tau_{1/2}} \quad \Rightarrow \quad \tau_{1/2} = \frac{\ln 2}{\lambda} = 0.693\tau$$
Particle Decays  Multiple Particle Decay

**Decay Chains** frequently occur in nuclear physics

\[
N_1 \xrightarrow{\lambda_1} N_2 \xrightarrow{\lambda_2} N_3 \rightarrow \ldots
\]

*Parent*  *Daughter*  *Granddaughter*

**e.g.** \( ^{235}\text{U} \rightarrow ^{231}\text{Th} \rightarrow ^{231}\text{Pa} \)

\[
\tau_{1/2}(^{235}\text{U}) = 7.1 \times 10^8 \text{ years}
\]

\[
\tau_{1/2}(^{231}\text{Th}) = 26 \text{ hours}
\]

**Activity** (i.e. rate of decay) of the daughter is \( \lambda_2 N_2(t) \).

Rate of change of population of the daughter

\[
\frac{dN_2(t)}{dt} = \lambda_1 N_1(t) - \lambda_2 N_2(t)
\]

**Units of Radioactivity** are defined as the number of decays per unit time.

- **Becquerel (Bq)** = 1 decay per second
- **Curie (Ci)** = \( 3.7 \times 10^{10} \) decays per second.
A decay is the transition from one quantum state (initial state) to another (final or daughter).

The transition rate is given by Fermi’s Golden Rule:

\[ \Gamma(i \rightarrow f) = \lambda = 2\pi |M_{fi}|^2 \rho(E_f) \]

where \( \lambda \) is the number of transitions per unit time,
\( M_{fi} \) is the transition matrix element,
\( \rho(E_f) \) is the density of final states.

\[ \Rightarrow \lambda \, dt \] is the (constant) probability a particle will decay in time \( dt \).
Let $p(t)$ be the probability that a particle still exists at time $t$, given that it was known to exist at $t = 0$.

Probability for particle decay in the next time interval $\, dt$ is
$$p(t)\lambda\, dt$$
Probability that particle survives the next is
$$p(t + \, dt) = p(t)(1 - \lambda\, dt)$$

$$p(t)(1 - \lambda\, dt) = p(t + \, dt) = p(t) + \frac{dp}{dt}\, dt$$

$$\frac{dp}{dt} = -p(t)\lambda$$

$$\int_1^{p} \frac{dp}{p} = -\int_0^{t} \lambda\, dt$$

$$\Rightarrow p(t) = e^{-\lambda t} \quad \text{Exponential Decay Law}$$

Probability that a particle lives until time $t$ and then decays in time $\, dt$ is
$$p(t)\lambda\, dt = \lambda e^{-\lambda t}\, dt$$
Particle Decays

Single Particle Decay

- The average lifetime of the particle

\[ \tau = \langle t \rangle = \int_0^\infty t \lambda e^{-\lambda t} \, dt = \left[ -te^{-\lambda t} \right]_0^\infty + \int_0^\infty e^{-\lambda t} \, dt = \left[ -\frac{1}{\lambda} e^{-\lambda t} \right]_0^\infty = \frac{1}{\lambda} \]

- Finite lifetime \( \Rightarrow \) uncertain energy \( \Delta E \), (c.f. Resonances, Breit-Wigner)
- Decaying states do not correspond to a single energy – they have a width \( \Delta E \)

\[ \Delta E \tau \sim \hbar \Rightarrow \Delta E \sim \frac{\hbar}{\tau} = \hbar \lambda \] \( \hbar = 1 \) (n.u.)

- The width, \( \Delta E \), of a particle state is therefore
  - Inversely proportional to the lifetime \( \tau \)
  - Proportional to the decay rate \( \lambda \) (or equal in natural units)
Decay of Resonances

QM description of decaying states
Consider a state formed at $t = 0$ with energy $E_0$ and mean lifetime $\tau$

$$\psi(t) = \psi(0)e^{-iE_0 t}e^{-t/2\tau}$$

$$|\psi(t)|^2 = |\psi(0)|^2 e^{-t/\tau}$$

i.e. the probability density decays exponentially (as required).

The frequencies (i.e. energies, since $E = \omega$ if $\hbar = 1$) present in the wavefunction are given by the Fourier transform of $\psi(t)$, i.e.

$$f(\omega) = f(E) = \int_{0}^{\infty} \psi(t)e^{iEt} dt = \int_{0}^{\infty} \psi(0)e^{-t(iE_0 + \frac{1}{2\tau})}e^{iEt} dt$$

$$= \int_{0}^{\infty} \psi(0)e^{-t(i(E_0 - E) + \frac{1}{2\tau})} dt = \frac{i\psi(0)}{(E_0 - E) - \frac{i}{2\tau}}$$

Probability of finding state with energy $E = f(E) \ast f(E)$ is

$$P(E) = |f(E)|^2 = \frac{|\psi(0)|^2}{(E_0 - E)^2 + \frac{1}{4\tau^2}}$$
Probability for producing the decaying state has this energy dependence, i.e. resonant when $E = E_0$

$$P(E) \propto \frac{1}{(E_0 - E)^2 + 1/4\tau^2}$$

Consider full-width at half-maximum $\Gamma$

$$P(E = E_0) \propto 4\tau^2$$

$$P(E = E_0 \pm \frac{\Gamma}{2}) \propto \frac{1}{(E_0 - E_0 \pm \frac{\Gamma}{2})^2 + 1/4\tau^2} = \frac{1}{\frac{\Gamma^2}{4} + \frac{1}{4\tau^2}}$$

$$P(E = E_0 \pm \frac{\Gamma}{2}) = \frac{1}{2}P(E = E_0), \quad \Rightarrow \quad \frac{1}{\frac{\Gamma^2}{4} + \frac{1}{4\tau^2}} = 2\tau^2$$

Total width (using natural units) $\Gamma = \frac{1}{\tau} = \lambda$
Partial Decay Widths

Particles can often decay with more than one decay mode e.g. $Z \rightarrow e^+ e^-$, or $\mu^+ \mu^-$, or $q \bar{q}$ etc, each with its own transition rate, i.e. from initial state $i$ to final state $f$:

$$\lambda_f = 2\pi |M_{fi}|^2 \rho(E_f)$$

The total decay rate is given by

$$\lambda = \sum_f \lambda_f$$

This determines the average lifetime

$$\tau = \frac{1}{\lambda}$$

The total width of a particle state is defined by the partial widths

$$\Gamma_f = \lambda_f$$

The proportion of decays to a particular decay mode is called the branching fraction or branching ratio

$$B_f = \frac{\Gamma_f}{\Gamma}, \quad \sum_f B_f = 1$$
Reactions and Cross-sections

The strength of a particular reaction between two particles is specified by the interaction cross-section.

Cross-section $\sigma$ – the effective target area presented to the incoming particle for it to cause the reaction.

Units: $\sigma$ 1 barn (b) = $10^{-28}m^2$  Area

$\sigma$ is defined as the reaction rate per target particle $\Gamma$, per unit incident flux $\Phi$

$$\Gamma = \Phi \sigma$$

where the flux $\Phi$ is the number of beam particles passing through unit area per second.

$\Gamma$ is given by Fermi’s Golden Rule (previously used $\lambda$).
Scattering with a beam

Consider a beam of particles incident upon a target:

Beam of \( N \) particles per unit time in an area \( A \)

Target of \( n \) nuclei per unit volume
Target thickness \( dx \) is small

Number of target particles in area \( A \), \( N_T = nA \, dx \)
Effective area for absorption = \( \sigma N_T = \sigma nA \, dx \)
Incident flux \( \Phi = \frac{N}{A} \)
Number of particles scattered per unit time
\[
= -dN = \Phi \sigma N_T = \frac{N}{A} \sigma nA \, dx
\]

\( \sigma = \frac{-dN}{nN} \, dx \)
Attenuation of a beam

Beam attenuation in a target of thickness $L$:

- Thick target $\sigma n L \gg 1$:

  \[
  \int_{N_0}^{N} -\frac{dN}{N} = \int_{0}^{L} \sigma n \, dx
  \]

  \[
  N = N_0 e^{-\sigma n L}
  \]

  This is exact. i.e. the beam attenuates exponentially.

- Thin target $\sigma n L \ll 1$, $e^{-\sigma n L} \sim 1 - \sigma n L$

  \[
  N = N_0(1 - \sigma n L)
  \]

  Useful approximation for thin targets.

  Or, the number scattered = $N_0 - N = N_0\sigma n L$

Mean free path between interactions = $1/\sigma n$ often referred to as “interaction length”. 
The angular distribution of the scattered particles is not necessarily uniform

** n.b. \( d\Omega \) can be considered in position space, or momentum space **

Number of particles scattered per unit time into \( d\Omega \) is \( dN_{d\Omega} = d\sigma \Phi N_T \)

** Differential cross-section **

units: area/steradian

\[
\frac{d\sigma}{d\Omega} = \frac{dN_{d\Omega}}{(\Phi \times N_T \times d\Omega)}
\]

The differential cross-section is the number of particles scattered per unit time and solid angle, divided by the incident flux and by the number of target nuclei, \( N_T \), defined by the beam area.

Most experiments do not cover \( 4\pi \) solid angle, and in general we measure \( d\sigma/d\Omega \).

Angular distributions provide more information than the total cross-section about the mechanism of the interaction, e.g. angular momentum.
Different types of interaction can occur between particles

\[
e^+e^- \rightarrow \gamma, \text{ or } e^+e^- \rightarrow Z...
\]

\[
\sigma_{\text{tot}} = \sum_i \sigma_i
\]

where the \(\sigma_i\) are called partial cross-sections for different final states.

**Types of interaction**

- **Elastic scattering:** \(a + b \rightarrow a + b\)
  
  only the momenta of \(a\) and \(b\) change

- **Inelastic scattering:** \(a + b \rightarrow c + d\)
  
  final state is not the same as initial state
Scattering in QM

Consider a beam of particles scattering from a fixed potential \( V(r) \):

\[
\vec{q} = \vec{p}_f - \vec{p}_i
\]

“momentum transfer”

NOTE: using natural units \( \vec{p} = \hbar \vec{k} \rightarrow \vec{p} = \vec{k} \) etc

The scattering rate is characterised by the interaction cross-section

\[
\sigma = \frac{\Gamma}{\Phi} = \frac{\text{Number of particles scattered per unit time}}{\text{Incident flux}}
\]

How can we calculate the cross-section?

Use Fermi’s Golden Rule to get the transition rate

\[
\Gamma = 2\pi |M_{fi}|^2 \rho(E_f)
\]

where \( M_{fi} \) is the matrix element and \( \rho(E_f) \) is the density of final states.
Scattering in QM

1\textsuperscript{st} order Perturbation Theory using plane wave solutions of form

\[ \psi = N e^{-i(Et - \mathbf{p} \cdot \mathbf{r})} \]

Require:

1. Wave-function normalisation
2. Matrix element in perturbation theory \( M_{fi} \)
3. Expression for incident flux \( \Phi \)
4. Expression for density of states \( \rho(E_f) \)

Normalisation

Normalise wave-functions to one particle in a box of side \( L \):

\[ |\psi|^2 = N^2 = 1/L^3 \]

\[ N = (1/L)^{3/2} \]
Scattering in QM

2 Matrix Element

This contains the interesting physics of the interaction:

\[ M_{fi} = \langle \psi_f | \hat{H} | \psi_i \rangle = \int \psi_f^* \hat{H} \psi_i \, d^3 \vec{r} = \int Ne^{-i\vec{p}_i \cdot \vec{r}} V(\vec{r}) Ne^{i\vec{p}_f \cdot \vec{r}} \, d^3 \vec{r} \]

\[ M_{fi} = \frac{1}{L^3} \int e^{-i\vec{q} \cdot \vec{r}} V(\vec{r}) \, d^3 \vec{r} \]

where \( \vec{q} = \vec{p}_f - \vec{p}_i \)

3 Incident Flux

Consider a “target” of area \( A \) and a beam of particles travelling at velocity \( v_i \) towards the target. Any incident particle within a volume \( v_i A \) will cross the target area every second.

\[ \Phi = \frac{v_i A}{A} n = v_in \]

where \( n \) is the number density of incident particles = 1 per \( L^3 \)

Flux = number of incident particles crossing unit area per second

\[ \Phi = v_i / L^3 \]
Density of States \textit{also known as “phase space”}

For a box of side $L$, states are given by the periodic boundary conditions:

$$\mathbf{p} = (p_x, p_y, p_z) = \frac{2\pi}{L} (n_x, n_y, n_z)$$

Each state occupies a volume $(2\pi/L)^3$ in $p$ space (neglecting spin).

Number of states between $p$ and $p + \, dp$ in solid angle $d\Omega$

$$dN = \left(\frac{L}{2\pi}\right)^3 d^3\mathbf{p} = \left(\frac{L}{2\pi}\right)^3 p^2 \, dp \, d\Omega$$

$$\therefore \rho(p) = \frac{dN}{dp} = \left(\frac{L}{2\pi}\right)^3 p^2 \, d\Omega$$

Density of states in energy $E^2 = p^2 + m^2 \Rightarrow 2E \, dE = 2p \, dp \Rightarrow \frac{dE}{dp} = \frac{p}{E}$

$$\rho(E) = \frac{dN}{dE} = \frac{dN}{dp} \frac{dp}{dE} = \left(\frac{L}{2\pi}\right)^3 p^2 \frac{E}{p} \, d\Omega$$

For relativistic scattering ($E \sim p$) \hspace{1cm} $\rho(E) = \left(\frac{L}{2\pi}\right)^3 E^2 \, d\Omega$
Putting all the parts together:

\[
\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2 v_i} \left| \int e^{-i\vec{q} \cdot \vec{r}} V(\vec{r}) \, d^3 \vec{r} \right|^2 p_f E_f \, d\Omega
\]

For relativistic scattering, \( v_i = c = 1 \) and \( p \sim E \)

**Born approximation for the differential cross-section**

\[
\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2} \left| \int e^{-i\vec{q} \cdot \vec{r}} V(\vec{r}) \, d^3 \vec{r} \right|^2
\]

n.b. may have seen the non-relativistic version, using \( m^2 \) instead of \( E^2 \)
Consider relativistic elastic scattering in a Coulomb potential

\[ V(\vec{r}) = -\frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{Z\alpha}{r} \]

Special case of Yukawa potential \( V = ge^{-mr}/r \) with \( g = Z\alpha \) and \( m = 0 \) (see Appendix C)

\[ |M_{if}|^2 = \frac{16\pi^2 Z^2\alpha^2}{q^4} \]

\[ |\vec{q}|^2 = |\vec{p}_f|^2 + |\vec{p}_i|^2 - 2\vec{p}_i \cdot \vec{p}_f \]

For elastic scattering, \( |\vec{p}_i| = |\vec{p}_f| = |\vec{p}| \)

\[ = 2|\vec{p}|^2(1 - \cos \theta) = 4E^2 \sin^2 \frac{\theta}{2} \]

\[ \frac{d\sigma}{d\Omega} = \frac{4E^2 Z^2\alpha^2}{q^4} = \frac{4E^2 Z^2\alpha^2}{16E^4 \sin^4 \frac{\theta}{2}} = \frac{Z^2\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \]
Some particle interactions take place via an intermediate resonant state which then decays:

\[ a + b \rightarrow Z^* \rightarrow c + d \]

Two-stage picture: (Bohr Model)

**Formation**  
\[ a + b \rightarrow Z^* \]

Occurs when the collision energy \( E_{CM} \) is the natural frequency (i.e., mass) of a resonant state.

**Decay**  
\[ Z^* \rightarrow c + d \]

The decay of the resonance \( Z^* \) is independent of the mode of formation and depends only on the properties of the \( Z^* \). May be multiple decay modes.
The **resonance cross-section** is given by

\[ \sigma = \frac{\Gamma}{\Phi} \quad \text{with} \quad \Gamma = 2\pi |M_{fi}|^2 \rho(E_f) \]

**same as Born Approx.**

incident flux \( \Phi = \frac{v_i}{L^3} \)

density of states \( \rho(p) = \frac{dN}{dp} = \left(\frac{L}{2\pi}\right)^3 p^2 \, d\Omega \)

\[ \rightarrow \rho(E) = \frac{dN}{dE} = \left(\frac{L}{2\pi}\right)^3 p^2 \frac{E}{v} \, d\Omega \]

\[ = \left(\frac{L}{2\pi}\right)^3 p^2 \frac{1}{v} \, d\Omega \]

The matrix element \( M_{fi} \) is given by 2\textsuperscript{nd} order Perturbation Theory

\[ M_{fi} = \sum_Z \frac{M_{iZ} M_{Zf}}{E - E_Z} \]

n.b. 2\textsuperscript{nd} order effects are large since

\( E - E_Z \) is small \( \rightarrow \) large perturbation

where the sum runs over all intermediate states.

Near resonance, effectively only one state \( Z \) contributes.
Consider one intermediate state described by
\[ \psi(t) = \psi(0)e^{iE_0 t}e^{-t/2\tau} = \psi(0)e^{-i(E_0 - i\Gamma/2)t} \]
this describes a states with energy \( E_0 - i\Gamma/2 \)

\[ |M_{fi}|^2 = \frac{|M_{iZ}|^2 |M_{Zf}|^2}{(E - E_0)^2 + \frac{\Gamma^2}{4}} \]

Rate of decay of Z:
\[ \Gamma_{Z\rightarrow f} = 2\pi |M_{Zf}|^2 \rho(E_f) = 2\pi |M_{Zf}|^2 \frac{4\pi p_f^2}{(2\pi)^3 v_f} = |M_{Zf}|^2 \frac{p_f^2}{\pi v_f} \]

Rate of formation of Z:
\[ \Gamma_{i\rightarrow Z} = 2\pi |M_{iZ}|^2 \rho(E_i) = 2\pi |M_{iZ}|^2 \frac{4\pi p_i^2}{(2\pi)^3 v_i} = |M_{iZ}|^2 \frac{p_i^2}{\pi v_i} \]

nb. \( |M_{Zi}|^2 = |M_{iZ}|^2 \).

Hence \( M_{iZ} \) and \( M_{Zf} \) can be expressed in terms of partial widths.
Resonance Cross-Section

Putting everything together:

\[
\frac{d\sigma}{d\Omega} = \frac{p_f^2}{(2\pi)^2 \nu_i \nu_f} |M_{fi}|^2
\]

\[
\Rightarrow \sigma = 4\pi \frac{p_f^2}{(2\pi)^2 \nu_i \nu_f} \frac{\Gamma_{Z\rightarrow i}}{p_f^2} \frac{\Gamma_{Z\rightarrow f}}{p_i^2} \frac{(E - E_0)^2 + \frac{\Gamma^2}{4}}{p_i^2}
\]

We need to include one more piece of information to account for spin...
Resonance Cross-Section

Breit-Wigner Cross-Section

\[ \sigma = \frac{\pi \frac{g}{p_i^2} \cdot \frac{\Gamma_{Z\rightarrow i}\Gamma_{Z\rightarrow f}}{(E - E_0)^2 + \frac{\Gamma^2}{4}}}{ } \]

The \( g \) factor takes into account the spin:

\[ a + b \rightarrow Z^* \rightarrow c + d, \quad g = \frac{2J_Z + 1}{(2J_a + 1)(2J_b + 1)} \]

and is the ratio of the number of spin states for the resonant state to the total number of spin states for the \( a+b \) system,

i.e. the probability that \( a+b \) collide in the correct spin state to form \( Z^* \).

Useful points to remember:

- \( p_i \) is calculated in the centre-of-mass frame (\( \sigma \) is independent of frame of reference!)
- \( p_i \sim \) lab momentum if the target is heavy (often true in nuclear physics, but not in particle physics).
- \( E \) is the total energy (if two particles colliding, \( E = E_1 + E_2 \))
- \( \Gamma \) is the total decay rate
- \( \Gamma_{Z\rightarrow i} \) and \( \Gamma_{Z\rightarrow f} \) are the partial decay rates
Total cross-section

\[ \sigma_{\text{tot}} = \sum_f \sigma(i \rightarrow f) \]

Replace \( \Gamma_f \) by \( \Gamma \) in the Breit-Wigner formula

Elastic cross-section

\[ \sigma_{\text{el}} = \sigma(i \rightarrow i) \]

so, \( \Gamma_f = \Gamma_i \)

On peak of resonance \((E = E_0)\)

\[ \sigma_{\text{peak}} = \frac{4\pi g \Gamma_i \Gamma_f}{p_i^2 \Gamma^2} \]

Thus

\[ \sigma_{\text{el}} = \frac{4\pi g B_i^2}{p_i^2}, \quad \sigma_{\text{tot}} = \frac{4\pi g B_i}{p_i^2}, \quad B_i = \frac{\Gamma_i}{\Gamma} = \frac{\sigma_{\text{el}}}{\sigma_{\text{tot}}} \]

By measuring \( \sigma_{\text{tot}} \) and \( \sigma_{\text{el}} \), can cancel \( B_i \) and infer \( g \) and hence the spin of the resonant state.
Can produce the same resonance from different initial states, decaying into various final states, e.g.

\[
p + ^{29}_{\text{Ni}} \rightarrow ^{30}_{\text{Zn}} + ^{63}_{\text{Cu}} + \alpha + ^{60}_{\text{Ni}} \rightarrow ^{62}_{\text{Zn}} + n + n + 2n
\]

\[
\sigma[^{60}_{\text{Ni}}(\alpha, n)^{63}_{\text{Zn}}] \sim \sigma[^{63}_{\text{Cu}}(p, n)^{63}_{\text{Zn}}]
\]

n.b. common notation for nuclear reactions:

\[
a + A \rightarrow b + B \equiv A(a, b)B
\]

Energy of \( p \) selected to give same c.m. energy as for \( \alpha \) interaction.
The Z boson

\[ \Gamma_Z \sim 2.5 \text{ GeV} \]

\[ \tau = \frac{1}{\Gamma_Z} = 0.4 \text{ GeV}^{-1} \]

\[ = 0.4 \times \hbar \]

\[ = 2.5 \times 10^{-25} \text{ s} \]

\( (\hbar = 6.6 \times 10^{-25} \text{ GeV s}) \)
Resonances $\pi^- p$ scattering example

Resonance observed at $p_\pi \sim 0.3$ GeV, $E_{CM} \sim 1.25$ GeV

\[
\begin{align*}
\sigma_{\text{total}} &= \sigma(\pi^- p \rightarrow R \rightarrow \text{anything}) \sim 72 \text{ mb} \\
\sigma_{\text{elastic}} &= \sigma(\pi^- p \rightarrow R \rightarrow \pi^- p) \sim 28 \text{ mb}
\end{align*}
\]
Resonances  \( \pi^- p \) scattering example
Summary

- Units: MeV, GeV, barns
- Natural units: $\hbar = c = 1$
- Relativistic kinematics: most particle physics calculations require this!
- Revision of scattering theory: cross-section, Born Approximation.
- Resonant scattering
- Breit-Wigner formula (important in both nuclear and particle physics):
  \[ \sigma = \frac{\pi g}{p_i^2} \frac{\Gamma_{Z\rightarrow i} \Gamma_{Z\rightarrow f}}{(E - E_0)^2 + \frac{\Gamma^2}{4}} \]
- Measure total and elastic $\sigma$ to measure spin of resonance.

Up next...
Section 3: Colliders and Detectors