2. Kinematics, Decays and Reactions
Particle and Nuclear Physics
In this section...

- Natural units
- Symmetries and conservation laws
- Relativistic kinematics
- Particle properties
- Decays
- Cross-sections
- Scattering
- Resonances
Units

The usual practice in particle and nuclear physics is to use **Natural Units**.

- **Energies** are measured in units of eV:
  - **Nuclear**  keV (10^3 eV),  MeV (10^6 eV)
  - **Particle**  GeV (10^9 eV),  TeV (10^{12} eV)

- **Masses** are quoted in units of MeV/c^2 or GeV/c^2 (using E = mc^2)
  - e.g. electron mass \( m_e = 9.11 \times 10^{-31} \text{ kg} = (9.11 \times 10^{-31})(3 \times 10^8)^2 \text{ J/c}^2 \)
    \[ = 8.20 \times 10^{-14}/1.602 \times 10^{-19} \text{ eV/c}^2 = 5.11 \times 10^5 \text{ eV/c}^2 = 0.511 \text{ MeV/c}^2 \]

- **Atomic/nuclear masses** are often quoted in unified (or atomic) mass units
  - 1 unified mass unit (u) = (mass of a \(^{12}\text{C}\) atom) / 12
    \[ 1 \text{ u} = 1 \text{ g/N}_A = 1.66 \times 10^{-27} \text{ kg} = 931.5 \text{ MeV/c}^2 \]

- **Cross-sections** are usually quoted in barns: 1b = 10^{-28} m^2.
Choose energy as the basic unit of measurement... ...and simplify by choosing $\hbar = c = 1$

<table>
<thead>
<tr>
<th>Unit</th>
<th>Natural Unit</th>
<th>SI Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>GeV</td>
<td>GeV</td>
</tr>
<tr>
<td>Momentum</td>
<td>GeV/c</td>
<td>GeV</td>
</tr>
<tr>
<td>Mass</td>
<td>GeV/c²</td>
<td>GeV</td>
</tr>
<tr>
<td>Time</td>
<td>$(\text{GeV}/\hbar)^{-1}$</td>
<td>GeV⁻¹</td>
</tr>
<tr>
<td>Length</td>
<td>$(\text{GeV}/\hbar c)^{-1}$</td>
<td>GeV⁻¹</td>
</tr>
<tr>
<td>Cross-section</td>
<td>$(\text{GeV}/\hbar c)^{-2}$</td>
<td>GeV⁻²</td>
</tr>
</tbody>
</table>

Reintroduce “missing” factors of $\hbar$ and $c$ to convert back to SI units.

- $\hbar c = 0.197 \text{ GeV fm} = 1$ Energy $\leftrightarrow$ Length
- $\hbar = 6.6 \times 10^{-25} \text{ GeV s} = 1$ Energy $\leftrightarrow$ Time
- $c = 3.0 \times 10^8 \text{ ms}^{-1} = 1$ Length $\leftrightarrow$ Time
### Units Examples

1. **cross-section**  
   \[ \sigma = 2 \times 10^{-6} \text{ GeV}^{-2} \]  
   Need to change units of energy to length. Use \( \hbar c = 0.197 \text{ GeV} \text{fm} = 1. \)
   \[ \begin{align*}
   \text{GeV}^{-1} &= 0.197 \text{ fm} \\
   \text{GeV}^{-1} &= 0.197 \times 10^{-15} \text{ m} \\
   \text{GeV}^{-2} &= 3.89 \times 10^{-32} \text{ m}^2
   \end{align*} \]
   \[ \begin{align*}
   \sigma &= 2 \times 10^{-6} \times (3.89 \times 10^{-32} \text{ m}^2) \\
   &= 7.76 \times 10^{-38} \text{ m}^2
   \end{align*} \]
   And using \( 1 \text{ b} = 10^{-28} \text{ m}^2, \sigma = 0.776 \text{ nb} \)

2. **lifetime**  
   \[ \tau = 1/\Gamma = 0.5 \text{ GeV}^{-1} \]  
   Need to change units of energy\(^{-1}\) to time. Use \( \hbar = 6.6 \times 10^{-25} \text{ GeV} \text{s} = 1. \)
   \[ \begin{align*}
   \text{GeV}^{-1} &= 6.6 \times 10^{-25} \text{ s} \\
   \tau &= 0.5 \times (6.6 \times 10^{-25} \text{ s}) = 3.3 \times 10^{-25} \text{ s}
   \end{align*} \]
   Also, can have Natural Units involving electric charge: \( \epsilon_0 = \mu_0 = \hbar = c = 1 \)

3. **Fine structure constant** (dimensionless)  
   \[ \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \sim \frac{1}{137} \]  
   becomes \( \alpha = \frac{e^2}{4\pi} \sim \frac{1}{137} \) i.e. \( e \sim 0.30(\text{n.u.}) \)
Symmetries and conservation laws

The most elegant and powerful idea in physics
Noether’s theorem:
每 differentiable symmetry of the action of a 
physical system has a corresponding conservation law.

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Conserved current</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time, $t$</td>
<td>Energy, $E$</td>
</tr>
<tr>
<td>Translational, $x$</td>
<td>Linear momentum, $p$</td>
</tr>
<tr>
<td>Rotational, $\theta$</td>
<td>Angular momentum, $L$</td>
</tr>
<tr>
<td>Probability</td>
<td>Total probability always 1</td>
</tr>
<tr>
<td>Lorentz invariance</td>
<td>Charge Parity Time (CPT)</td>
</tr>
<tr>
<td>Gauge</td>
<td>charge (e.g. electric, colour, weak)</td>
</tr>
</tbody>
</table>

Lorentz invariance: laws of physics stay the same for all frames moving with a uniform velocity.
Gauge invariance: observable quantities unchanged (charge, $E$, $\nu$) when a field is transformed.
Nuclear reactions
Low energy, typically K.E. $O(10 \text{ MeV}) \ll$ nucleon rest energies.
  $\Rightarrow$ non-relativistic formulae ok
Exception: always treat $\beta$-decay relativistically
  $(m_e \sim 0.5 \text{ MeV} < 1.3 \text{ MeV} \sim m_n - m_p)$

Particle physics
High energy, typically K.E. $O(100 \text{ GeV}) \gg$ rest mass energies.
  $\Rightarrow$ relativistic formulae usually essential.
Recall the energy $E$ and momentum $p$ of a particle with mass $m$

$$E = \gamma m, \quad |\vec{p}| = \gamma \beta m$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c} = \frac{v}{c}$$

or

$$\gamma = \frac{E}{m}, \quad \beta = \frac{|\vec{p}|}{E}$$

and these are related by $E^2 = \vec{p}^2 + m^2$

Interesting cases

- when a particle is at rest, $\vec{p} = 0, \ E = m$,
- when a particle is massless, $m = 0, \ E = |\vec{p}|$,
- when a particle is ultra-relativistic $E \gg m, \ E \sim |\vec{p}|$.

Kinetic energy (K.E., or $T$) is the extra energy due to motion

$$T = E - m = (\gamma - 1)m$$

in the non-relativistic limit $\beta \ll 1, \ T = \frac{1}{2}mv^2$
Relativistic Kinematics \textit{Four-Vectors}

The kinematics of a particle can be expressed as a four-vector, e.g.

\[ p^\mu = (E, -\vec{p}) \quad \text{and} \quad x^\mu = (t, -\vec{x}) \]

multiply by a metric tensor to raise/lower indices

\[ p_\mu = g_{\mu\nu} p^\nu, \quad p^\mu = g^{\mu\nu} p_\nu \]

Scalar product of two four-vectors \( A^\mu = (A^0, \vec{A}) \), \( B^\mu = (B^0, \vec{B}) \) is invariant:

\[ A^\mu B_\mu = A \cdot B = A^0 B^0 - \vec{A} \cdot \vec{B} \]

or

\[ p^{\mu} p_\mu = \sum_{\mu=0,3} \sum_{\nu=0,3} p^{\mu} g_{\mu\nu} p^\nu = g_{00} p_0^2 + g_{11} p_1^2 + g_{22} p_2^2 + g_{33} p_3^2 \]

\[ = E^2 - |\vec{p}|^2 = m^2 \quad \text{invariant mass} \]

\((t, \vec{x})\) and \((E, \vec{p})\) transform between frames of reference, but

\[ d^2 = t^2 - \vec{x}^2 \quad \text{Invariant interval is constant} \]

\[ m^2 = E^2 - \vec{p}^2 \quad \text{Invariant mass is constant} \]
A common technique to identify particles is to form the invariant mass from their decay products.

Remember, for a single particle \( m^2 = E^2 - \vec{p}^2 \).

For a system of particles, where \( X \to 1 + 2 + 3...n \):

\[
M_X^2 = ((E_1, \vec{p}_1) + (E_2, \vec{p}_2) + ...)^2 = \left( \sum_{i=1}^{n} E_i \right)^2 - \left( \sum_{i=1}^{n} \vec{p}_i \right)^2
\]

In the specific (and common) case of a two-body decay, \( X \to 1 + 2 \), this reduces to

\[
M_X^2 = m_1^2 + m_2^2 + 2 (E_1 E_2 - |\vec{p}_1| |\vec{p}_2| \cos \theta)
\]

n.b. sometimes invariant mass \( M \) is called “centre-of-mass energy” \( E_{CM} \), or \( \sqrt{s} \)
Consider a charged pion decaying at rest in the lab frame: $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$

Find the momenta of the decay products.
How do we study particles and forces?

- **Static Properties**
  What particles/states exist?
  Mass, spin and parity \( (J^P) \), magnetic moments, bound states

- **Particle Decays**
  Most particles and nuclei are unstable.
  Allowed/forbidden decays → Conservation Laws.

- **Particle Scattering**
  Direct production of new massive particles in matter-antimatter annihilation.
  Study of particle interaction cross-sections.
  Use high-energies to study forces at short distances.

<table>
<thead>
<tr>
<th>Force</th>
<th>Typical Lifetime [s]</th>
<th>Typical cross-section [mb]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong</td>
<td>( 10^{-23} )</td>
<td>10</td>
</tr>
<tr>
<td>Electromagnetic</td>
<td>( 10^{-20} )</td>
<td>( 10^{-2} )</td>
</tr>
<tr>
<td>Weak</td>
<td>( 10^{-8} )</td>
<td>( 10^{-13} )</td>
</tr>
</tbody>
</table>
Particle Decays

Reminder

Most particles are transient states – only a few live forever ($e^-$, $p$, $\nu$, $\gamma$...).

- **Number** of particles remaining at time $t$

  $$N(t) = N(0)p(t) = N(0)e^{-\lambda t}$$

  where $N(0)$ is the number at time $t = 0$.

- **Rate of decays**

  $$\frac{dN}{dt} = -\lambda N(0)e^{-\lambda t} = -\lambda N(t)$$

  Assuming the nuclei only decay. More complicated if they are also being created.

- **Activity**

  $$A(t) = \left| \frac{dN}{dt} \right| = \lambda N(t)$$

- It’s rather common in nuclear physics to use the **half-life** (i.e. the time over which 50% of the particles decay). In particle physics, we usually quote the mean life. They are simply related:

  $$N(\tau_{1/2}) = \frac{N(0)}{2} = N(0)e^{-\lambda \tau_{1/2}} \quad \Rightarrow \quad \tau_{1/2} = \frac{\ln 2}{\lambda} = 0.693\tau$$
Decay Chains frequently occur in nuclear physics

\[ N_1 \xrightarrow{\lambda_1} N_2 \xrightarrow{\lambda_2} N_3 \rightarrow \ldots \]

*Parent*  *Daughter*  *Granddaughter*

e.g. \( ^{235}\text{U} \rightarrow ^{231}\text{Th} \rightarrow ^{231}\text{Pa} \)

\[ \tau_{1/2}(^{235}\text{U}) = 7.1 \times 10^8 \text{ years} \]
\[ \tau_{1/2}(^{231}\text{Th}) = 26 \text{ hours} \]

Activity (i.e. rate of decay) of the daughter is \( \lambda_2 N_2(t) \).

Rate of change of population of the daughter

\[ \frac{dN_2(t)}{dt} = \lambda_1 N_1(t) - \lambda_2 N_2(t) \]

Units of Radioactivity are defined as the number of decays per unit time.

Becquerel (Bq) = 1 decay per second

Curie (Ci) = \( 3.7 \times 10^{10} \) decays per second.
Particle Decays

A decay is the transition from one quantum state (initial state) to another (final or daughter).

The transition rate is given by Fermi’s Golden Rule:

$$\Gamma(i \rightarrow f) = \lambda = 2\pi |M_{fi}|^2 \rho(E_f) \quad \hbar = 1$$

where
- $\lambda$ is the number of transitions per unit time
- $M_{fi}$ is the transition matrix element
- $\rho(E_f)$ is the density of final states.

$\Rightarrow \lambda \, dt$ is the (constant) probability a particle will decay in time $dt$. 
Particle Decays

**Single Particle Decay**

Let \( p(t) \) be the probability that a particle still exists at time \( t \), given that it was known to exist at \( t = 0 \).

Probability for particle decay in the next time interval \( dt \) is \( = p(t)\lambda dt \)

Probability that particle survives the next is \( = p(t + dt) = p(t)(1 - \lambda dt) \)

\[
p(t)(1 - \lambda dt) = p(t + dt) = p(t) + \frac{dp}{dt} dt
\]

\[
\frac{dp}{dt} = -p(t)\lambda
\]

\[
\int_1^p \frac{dp}{p} = - \int_0^t \lambda dt
\]

\[
\Rightarrow p(t) = e^{-\lambda t} \quad \text{Exponential Decay Law}
\]

Probability that a particle lives until time \( t \) and then decays in time \( dt \) is

\[
p(t)\lambda dt = \lambda e^{-\lambda t} dt
\]
The average lifetime of the particle

\[ \tau = \langle t \rangle = \int_0^\infty t \lambda e^{-\lambda t} \, dt = \left[ -te^{-\lambda t} \right]_0^\infty + \int_0^\infty e^{-\lambda t} \, dt = \left[ -\frac{1}{\lambda} e^{-\lambda t} \right]_0^\infty = \frac{1}{\lambda} \]

\[ \tau = \frac{1}{\lambda} \quad \text{and} \quad p(t) = e^{-t/\tau} \]

Finite lifetime ⇒ uncertain energy \( \Delta E \), (c.f. Resonances, Breit-Wigner)
Decaying states do not correspond to a single energy – they have a width \( \Delta E \)

\[ \Delta E \tau \sim \hbar \quad \Rightarrow \quad \Delta E \sim \frac{\hbar}{\tau} = \hbar \lambda \quad \text{with} \quad \hbar = 1 \text{ (n.u.)} \]

The width, \( \Delta E \), of a particle state is therefore
- Inversely proportional to the lifetime \( \tau \)
- Proportional to the decay rate \( \lambda \) (or equal in natural units)
QM description of decaying states
Consider a state formed at $t = 0$ with energy $E_0$ and mean lifetime $\tau$

$$\psi(t) = \psi(0)e^{-iE_0t}e^{-t/2\tau}$$

$$|\psi(t)|^2 = |\psi(0)|^2 e^{-t/\tau}$$

i.e. the probability density decays exponentially (as required).

The frequencies (i.e. energies, since $E = \omega$ if $\hbar = 1$) present in the wavefunction are given by the Fourier transform of $\psi(t)$, i.e.

$$f(\omega) = f(E) = \int_0^\infty \psi(t)e^{iEt} \, dt = \int_0^\infty \psi(0)e^{-t(iE_0 + \frac{1}{2\tau})}e^{iEt} \, dt$$

$$= \int_0^\infty \psi(0)e^{-t(i(E_0-E) + \frac{1}{2\tau})} \, dt = \frac{i\psi(0)}{(E_0 - E) - \frac{i}{2\tau}}$$

Probability of finding state with energy $E = f(E) \ast f(E)$ is

$$P(E) = |f(E)|^2 = \frac{|\psi(0)|^2}{(E_0 - E)^2 + \frac{1}{4\tau^2}}$$
Probability for producing the decaying state has this energy dependence, i.e. resonant when $E = E_0$

$$P(E) \propto \frac{1}{(E_0 - E)^2 + 1/4\tau^2}$$

Consider full-width at half-maximum $\Gamma$

$$P(E = E_0) \propto 4\tau^2$$

$$P(E = E_0 \pm \frac{1}{2}\Gamma) \propto \frac{1}{(E_0 - E_0 \mp \frac{1}{2}\Gamma)^2 + 1/4\tau^2} = \frac{1}{\frac{\Gamma^2}{4} + \frac{1}{4\tau^2}}$$

$$P(E = E_0 \pm \frac{1}{2}\Gamma) = \frac{1}{2}P(E = E_0), \quad \Rightarrow \quad \frac{1}{\frac{\Gamma^2}{4} + \frac{1}{4\tau^2}} = 2\tau^2$$

Total width (using natural units) $\Gamma = \frac{1}{\tau} = \lambda$
Partial Decay Widths

Particles can often decay with more than one decay mode e.g. $Z \rightarrow e^+ e^-$, or $\mu^+ \mu^-$, or $q\bar{q}$ etc, each with its own transition rate, i.e. from initial state $i$ to final state $f$: $\lambda_f = 2\pi |M_{fi}|^2 \rho(E_f)$

The total decay rate is given by $\lambda = \sum_f \lambda_f$

This determines the average lifetime $\tau = \frac{1}{\lambda}$

The total width of a particle state is defined by the partial widths $\Gamma = \lambda = \sum_f \lambda_f$

$\Gamma_f = \lambda_f$

The proportion of decays to a particular decay mode is called the branching fraction or branching ratio $B_f = \frac{\Gamma_f}{\Gamma}, \quad \sum_f B_f = 1$
The strength of a particular reaction between two particles is specified by the interaction cross-section.

Cross-section \( \sigma \) – the effective target area presented to the incoming particle for it to cause the reaction.

Units: \( \sigma \quad 1 \text{ barn (b)} = 10^{-28} \text{m}^2 \quad \text{Area} \)

\( \sigma \) is defined as the reaction rate per target particle \( \Gamma \), per unit incident flux \( \Phi \)

\[ \Gamma = \Phi \sigma \]

where the flux \( \Phi \) is the number of beam particles passing through unit area per second.

\( \Gamma \) is given by Fermi’s Golden Rule (previously used \( \lambda \)).
Scattering with a beam

Consider a beam of particles incident upon a target:

Beam of \( N \) particles per unit time in an area \( A \)

Target of \( n \) nuclei per unit volume

Target thickness \( dx \) is small

Number of target particles in area \( A \), \( N_T = nA \, dx \)
Effective area for absorption = \( \sigma N_T = \sigma nA \, dx \)
Incident flux \( \Phi = N/A \)
Number of particles scattered per unit time
\[
= -dN = \Phi \sigma N_T = \frac{N}{A} \sigma nA \, dx
\]

\[
\sigma = \frac{-dN}{nN \, dx}
\]
Attenuation of a beam

Beam attenuation in a target of thickness $L$:

- **Thick target** $\sigma nL \gg 1$:

  $$\int_{N_0}^N -\frac{dN}{N} = \int_0^L \sigma n\,dx$$

  $$N = N_0 e^{-\sigma nL}$$

  This is exact.

  i.e. the beam attenuates *exponentially*.

- **Thin target** $\sigma nL \ll 1$, $e^{-\sigma nL} \sim 1 - \sigma nL$

  $$N = N_0(1 - \sigma nL)$$

  Useful approximation for thin targets.

  Or, the number scattered $= N_0 - N = N_0\sigma nL$

Mean free path between interactions $= 1/n\sigma$

often referred to as “interaction length”.
Differential Cross-section

The angular distribution of the scattered particles is not necessarily uniform

** n.b. $d\Omega$ can be considered in position space, or momentum space **

Number of particles scattered per unit time into $d\Omega$ is $dN_{d\Omega} = d\sigma \Phi N_T$

**Differential cross-section**
units: area/steradian

$$\frac{d\sigma}{d\Omega} = \frac{dN_{d\Omega}}{(\Phi \times N_T \times d\Omega)}$$

The differential cross-section is the number of particles scattered per unit time and solid angle, divided by the incident flux and by the number of target nuclei, $N_T$, defined by the beam area.

Most experiments do not cover $4\pi$ solid angle, and in general we measure $d\sigma / d\Omega$.

Angular distributions provide more information than the total cross-section about the mechanism of the interaction, e.g. angular momentum.
Different types of interaction can occur between particles
e.g. $e^+ e^- \rightarrow \gamma$, or $e^+ e^- \rightarrow Z$...

$$\sigma_{tot} = \sum_i \sigma_i$$

where the $\sigma_i$ are called partial cross-sections for different final states.

**Types of interaction**

- **Elastic scattering**: $a + b \rightarrow a + b$
  
  only the momenta of $a$ and $b$ change

- **Inelastic scattering**: $a + b \rightarrow c + d$
  
  final state is not the same as initial state
Scattering in QM

Consider a beam of particles scattering from a fixed potential $V(r)$:

\[ \vec{q} = \vec{p}_f - \vec{p}_i \]

“momentum transfer”

NOTE: using natural units $\vec{p} = \hbar \vec{k}$ \rightarrow \vec{p} = \vec{k}$ etc

The scattering rate is characterised by the interaction cross-section

\[ \sigma = \frac{\Gamma}{\Phi} = \frac{\text{Number of particles scattered per unit time}}{\text{Incident flux}} \]

How can we calculate the cross-section?

Use Fermi’s Golden Rule to get the transition rate

\[ \Gamma = 2\pi |M_{fi}|^2 \rho(E_f) \]

where $M_{fi}$ is the matrix element and $\rho(E_f)$ is the density of final states.
Scattering in QM

1st order Perturbation Theory using plane wave solutions of form

$$\psi = N e^{-i(Et - \vec{p} \cdot \vec{r})}$$

Require:
1. Wave-function normalisation
2. Matrix element in perturbation theory $M_{fi}$
3. Expression for incident flux $\Phi$
4. Expression for density of states $\rho(E_f)$

Normalisation

Normalise wave-functions to one particle in a box of side $L$:

$$|\psi|^2 = N^2 = 1/L^3$$

$$N = (1/L)^{3/2}$$
Scattering in QM

2 Matrix Element
This contains the interesting physics of the interaction:

\[ M_{fi} = \langle \psi_f | \hat{H} | \psi_i \rangle = \int \psi_f^* \hat{H} \psi_i \, d^3 \vec{r} = \int Ne^{-i \vec{p}_f \cdot \vec{r}} V(\vec{r}) Ne^{i \vec{p}_i \cdot \vec{r}} \, d^3 \vec{r} \]

\[ M_{fi} = \frac{1}{L^3} \int e^{-i \vec{q} \cdot \vec{r}} V(\vec{r}) \, d^3 \vec{r} \]

where \( \vec{q} = \vec{p}_f - \vec{p}_i \)

3 Incident Flux
Consider a “target” of area \( A \) and a beam of particles travelling at velocity \( v_i \) towards the target. Any incident particle within a volume \( v_i A \) will cross the target area every second.

\[ \Phi = \frac{v_i A}{A} n = v_i n \]

where \( n \) is the number density of incident particles = 1 per \( L^3 \)

Flux = number of incident particles crossing unit area per second

\[ \Phi = \frac{v_i}{L^3} \]
Density of States  

also known as “phase space”

For a box of side \(L\), states are given by the periodic boundary conditions:

\[
\vec{p} = (p_x, p_y, p_z) = \frac{2\pi}{L} (n_x, n_y, n_z)
\]

Each state occupies a volume \((2\pi/L)^3\) in \(p\) space (neglecting spin).

Number of states between \(p\) and \(p + dp\) in solid angle \(d\Omega\)

\[
dN = \left(\frac{L}{2\pi}\right)^3 d^3\vec{p} = \left(\frac{L}{2\pi}\right)^3 p^2 dp d\Omega
\]

\[
\therefore \rho(p) = \frac{dN}{dp} = \left(\frac{L}{2\pi}\right)^3 p^2 d\Omega
\]

Density of states in energy \(E^2 = p^2 + m^2\) \(\Rightarrow 2E \, dE = 2p \, dp\) \(\Rightarrow \frac{dE}{dp} = \frac{p}{E}\)

\[
\rho(E) = \frac{dN}{dE} = \frac{dN}{dp} \frac{dp}{dE} = \left(\frac{L}{2\pi}\right)^3 p^2 \frac{E}{p} d\Omega
\]

For relativistic scattering \((E \sim p)\)

\[
\rho(E) = \left(\frac{L}{2\pi}\right)^3 E^2 d\Omega
\]
Scattering in QM

Putting all the parts together:

$$d\sigma = \frac{1}{\Phi} 2\pi |M_f|^2 \rho(E_f) = \frac{L^3}{v_i} 2\pi \left| \frac{1}{L^3} \int e^{-i\vec{q} \cdot \vec{r}} V(\vec{r}) \, d^3 \vec{r} \right|^2 \left( \frac{L}{2\pi} \right)^3 pf E_f \, d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2 v_i} \left| \int e^{-i\vec{q} \cdot \vec{r}} V(\vec{r}) \, d^3 \vec{r} \right|^2 pf E_f$$

For relativistic scattering, $v_i = c = 1$ and $p \sim E$

**Born approximation for the differential cross-section**

$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2} \left| \int e^{-i\vec{q} \cdot \vec{r}} V(\vec{r}) \, d^3 \vec{r} \right|^2$$

n.b. may have seen the non-relativistic version, using $m^2$ instead of $E^2$
Rutherford Scattering

Consider relativistic elastic scattering in a Coulomb potential

\[ V(\vec{r}) = -\frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{Z\alpha}{r} \]

Special case of Yukawa potential \( V = g e^{-mr}/r \)

with \( g = Z\alpha \) and \( m = 0 \) (see Appendix C)

\[ |M_{if}|^2 = \frac{16\pi^2 Z^2\alpha^2}{q^4} \]

\[ |\vec{q}|^2 = |\vec{p}_f|^2 + |\vec{p}_i|^2 - 2\vec{p}_i \cdot \vec{p}_f \]

elastic scattering, \( |\vec{p}_i| = |\vec{p}_f| = |\vec{p}| \)

\[ = 2|\vec{p}|^2(1 - \cos \theta) = 4E^2 \sin^2 \frac{\theta}{2} \]

\[ \frac{d\sigma}{d\Omega} = \frac{4E^2 Z^2\alpha^2}{q^4} = \frac{4E^2 Z^2\alpha^2}{16E^4 \sin^4 \frac{\theta}{2}} = \frac{Z^2\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \]
Cross-section for Resonant Scattering

Some particle interactions take place via an intermediate resonant state which then decays

\[ a + b \rightarrow Z^* \rightarrow c + d \]

Two-stage picture: (Bohr Model)

Formation \[ a + b \rightarrow Z^* \]

Occurs when the collision energy \( E_{CM} \sim \) the natural frequency (i.e. mass) of a resonant state.

Decay \[ Z^* \rightarrow c + d \]

The decay of the resonance \( Z^* \) is independent of the mode of formation and depends only on the properties of the \( Z^* \).

May be multiple decay modes.
Resonance Cross-Section

The resonance cross-section is given by

$$\sigma = \frac{\Gamma}{\Phi} \quad \text{with} \quad \Gamma = 2\pi |M_{fi}|^2 \rho(E_f)$$

$$d\sigma = \frac{1}{\Phi} 2\pi |M_{fi}|^2 \rho(E_f) \quad **$$

$$= \frac{L^3}{v_i} 2\pi |M_{fi}|^2 \frac{p_f^2 L^3}{v_f (2\pi)^3} d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{p_f^2}{(2\pi)^2 v_i v_f} |M_{fi}|^2$$

factors of L cancel

as before, $M \propto 1/L^3$

The matrix element $M_{fi}$ is given by 2\textsuperscript{nd} order Perturbation Theory

$$M_{fi} = \sum_Z \frac{M_{iZ} M_{Zf}}{E - E_Z}$$

n.b. 2\textsuperscript{nd} order effects are large since
$E - E_Z$ is small $\rightarrow$ large perturbation

where the sum runs over all intermediate states.

Near resonance, effectively only one state $Z$ contributes.
Consider one intermediate state described by

$$\psi(t) = \psi(0) e^{-iE_0 t} e^{-t/2\tau} = \psi(0) e^{-i\left(E_0 - i\Gamma/2\right)t}$$

this describes a states with energy = $E_0 - i\Gamma/2$

$$|M_{fi}|^2 = \frac{|M_{iZ}|^2 |M_{Zf}|^2}{(E - E_0)^2 + \Gamma^2/4}$$

Rate of decay of $Z$:

$$\Gamma_{Z\rightarrow f} = 2\pi |M_{Zf}|^2 \rho(E_f) = 2\pi |M_{Zf}|^2 \frac{4\pi p_f^2}{(2\pi)^3 \nu_f} = |M_{Zf}|^2 \frac{p_f^2}{\pi \nu_f}$$

Rate of formation of $Z$:

$$\Gamma_{i\rightarrow Z} = 2\pi |M_{iZ}|^2 \rho(E_i) = 2\pi |M_{iZ}|^2 \frac{4\pi p_i^2}{(2\pi)^3 \nu_i} = |M_{iZ}|^2 \frac{p_i^2}{\pi \nu_i}$$

nb. $|M_{Zi}|^2 = |M_{iZ}|^2$.

Hence $M_{iZ}$ and $M_{Zf}$ can be expressed in terms of partial widths.
Resonance Cross-Section

Putting everything together:

\[
\frac{d\sigma}{d\Omega} = \frac{p_f^2}{(2\pi)^2 \nu_i \nu_f} |M_{fi}|^2
\]

\[
\Rightarrow \sigma = \frac{4\pi p_f^2}{(2\pi)^2 \nu_i \nu_f} \frac{\pi \nu_f \pi \nu_i}{p_f^2} \frac{\Gamma_{Z\rightarrow i} \Gamma_{Z\rightarrow f}}{p_i^2 (E - E_0)^2 + \frac{\Gamma^2}{4}} = \frac{\pi}{p_i^2 (E - E_0)^2 + \frac{\Gamma^2}{4}} \frac{\Gamma_{Z\rightarrow i} \Gamma_{Z\rightarrow f}}{p_f^2}
\]

We need to include one more piece of information to account for spin...
Resonance Cross-Section

Breit-Wigner Cross-Section

$$\sigma = \frac{\pi g}{p_i^2} \cdot \frac{\Gamma_{Z \rightarrow i} \Gamma_{Z \rightarrow f}}{(E - E_0)^2 + \Gamma^2 / 4}$$

The $g$ factor takes into account the spin

$$a + b \rightarrow Z^* \rightarrow c + d, \quad g = \frac{2J_Z + 1}{(2J_a + 1)(2J_b + 1)}$$

and is the ratio of the number of spin states for the resonant state to the total number of spin states for the $a+b$ system,

i.e. the probability that $a+b$ collide in the correct spin state to form $Z^*$.

Useful points to remember:

- $p_i$ is calculated in the centre-of-mass frame ($\sigma$ is independent of frame of reference!)
- $p_i \sim$ lab momentum if the target is heavy (often true in nuclear physics, but not in particle physics).
- $E$ is the total energy (if two particles colliding, $E = E_1 + E_2$)
- $\Gamma$ is the total decay rate
- $\Gamma_{Z \rightarrow i}$ and $\Gamma_{Z \rightarrow f}$ are the partial decay rates
Resonance Cross-Section

- Total cross-section
  \[ \sigma_{\text{tot}} = \sum_{f} \sigma(i \rightarrow f) \]

  Replace \( \Gamma_f \) by \( \Gamma \) in the Breit-Wigner formula.

- Elastic cross-section
  \[ \sigma_{\text{el}} = \sigma(i \rightarrow i) \]

  so, \( \Gamma_f = \Gamma_i \)

- On peak of resonance \((E = E_0)\)
  \[ \sigma_{\text{peak}} = \frac{4\pi g \Gamma_i \Gamma_f}{p_i^2 \Gamma^2} \]

  Thus
  \[ \sigma_{\text{el}} = \frac{4\pi g B_i^2}{p_i^2}, \quad \sigma_{\text{tot}} = \frac{4\pi g B_i}{p_i^2}, \quad B_i = \frac{\Gamma_i}{\Gamma} = \frac{\sigma_{\text{el}}}{\sigma_{\text{tot}}} \]

By measuring \( \sigma_{\text{tot}} \) and \( \sigma_{\text{el}} \), can cancel \( B_i \) and infer \( g \) and hence the spin of the resonant state.
Can produce the same resonance from different initial states, decaying into various final states, e.g.

\[
\begin{align*}
\text{p} + ^{63}\text{Cu} & \rightarrow ^{64}\text{Zn}^* \\
\alpha + ^{60}\text{Ni} & \rightarrow ^{63}\text{Zn} + \text{n} \\
& \rightarrow ^{62}\text{Cu} + \text{n} + \text{p} \\
& \rightarrow ^{62}\text{Zn} + 2\text{n}
\end{align*}
\]

\[
\sigma[^{60}\text{Ni}(\alpha, \text{n})^{63}\text{Zn}] \sim \sigma[^{63}\text{Cu}(\text{p}, \text{n})^{63}\text{Zn}]
\]

n.b. common notation for nuclear reactions:

\[a + A \rightarrow b + B \equiv A(a, b)B\]

Energy of \( p \) selected to give same c.m. energy as for \( \alpha \) interaction.
Resonances  Particle Physics Example

The $Z$ boson

$$\Gamma_Z \sim 2.5 \text{ GeV}$$

$$\tau = \frac{1}{\Gamma_Z} = 0.4 \text{ GeV}^{-1}$$

$$= 0.4 \times \hbar$$

$$= 2.5 \times 10^{-25} \text{ s}$$

($\hbar = 6.6 \times 10^{-25} \text{ GeV s}$)
Resonances \(\pi^- p\) scattering example

Resonance observed at \(p_{\pi} \sim 0.3\) GeV, \(E_{\text{CM}} \sim 1.25\) GeV

\[
\sigma_{\text{total}} = \sigma(\pi^- p \rightarrow R \rightarrow \text{anything}) \sim 72\ \text{mb}
\]

\[
\sigma_{\text{elastic}} = \sigma(\pi^- p \rightarrow R \rightarrow \pi^- p) \sim 28\ \text{mb}
\]
Resonances $\pi^- p$ scattering example
Summary

- Units: MeV, GeV, barns
- Natural units: $\hbar = c = 1$
- Relativistic kinematics: most particle physics calculations require this!
- Revision of scattering theory: cross-section, Born Approximation.
- Resonant scattering
- Breit-Wigner formula (important in both nuclear and particle physics):
  \[
  \sigma = \frac{\pi g}{p_i^2} \frac{\Gamma_{Z\to i} \Gamma_{Z\to f}}{(E - E_0)^2 + \Gamma^2/4}
  \]
- Measure total and elastic $\sigma$ to measure spin of resonance.

Up next...
Section 3: Colliders and Detectors