APPENDIX H: GAMOW FACTOR IN ALPHA DECAY

The probability for an α particle to tunnel through the Coulomb barrier can be written as

$$P = \prod_i \exp\{-2G\}$$

where G is the Gamow Factor,

$$G = \int_R^{R'} \frac{\left[2m\left(V(r) - E_0\right)\right]^{1/2}}{\hbar}\, dr.$$

R is the radius of the nucleus of mass number Z, R' is the radius at which the α particle escapes, m is the mass of the α particle, $V(r) = 2\left(Z-2\right)e^2/4\pi\epsilon_0 r \equiv B/r$ is the Coulomb potential, and E_0 is the energy release in the decay.
The α particle escapes the nucleus when $r = R'$. Hence, the potential $V(R') = E_0$ and $R' = B/E_0$. Therefore,

$$\begin{aligned} G &= \left(\frac{2m}{\hbar^2}\right)^{1/2} \int_R^{R'} \left[\frac{B}{r} - E_0\right]^{1/2} dr \\ &= \left(\frac{2mB}{\hbar^2}\right)^{1/2} \int_R^{R'} \left[\frac{1}{r} - \frac{1}{R'}\right]^{1/2} dr \end{aligned}$$

In order to perform the integration, let $r = R'\cos^2\theta$ and $dr = -2R'\cos\theta\sin\theta\; d\theta$. Then

$$\begin{aligned} \int \left[\frac{1}{r} - \frac{1}{R'}\right]^{1/2} dr &= \int \left[\frac{1}{R'\cos^2\theta} - \frac{1}{R'}\right]^{1/2} \left(-2R'\cos\theta\sin\theta\right) d\theta \\ &= \int -2R'^{1/2}\sin^2\theta\; d\theta \\ &= R'^{1/2}\left[\sin 2\theta - \theta\right] \end{aligned}$$

Now, using $\cos\theta = \left(r/R'\right)^{1/2}$, $\sin\theta = \left(1 - r/R'\right)^{1/2}$ and $\sin 2\theta = 2\sin\theta\cos\theta$, then

$$\begin{aligned} R'^{1/2}\left[\sin 2\theta - \theta\right]_R^{R'} &= R'^{1/2}\left[2\left(1 - r/R'\right)^{1/2}\left(r/R'\right)^{1/2} - \cos^{-1}\left(r/R'\right)^{1/2}\right]_R^{R'} \\ &= R'^{1/2}\left[\cos^{-1}\left(R/R'\right)^{1/2} - 2\{\left(1 - R/R'\right)\left(R/R'\right)\}^{1/2}\right] \end{aligned}$$

Hence, the Gamow factor

$$G = \left(\frac{2mB}{\hbar^2}\right)^{1/2} R'^{1/2}\left[\cos^{-1}\left(R/R'\right)^{1/2} - 2\{\left(1 - R/R'\right)\left(R/R'\right)\}^{1/2}\right]$$

or

$$\underline{G = \left(\frac{2m}{E_0}\right)^{1/2} \frac{B}{\hbar}\left[\cos^{-1}\left(R/R'\right)^{1/2} - 2\{\left(1 - R/R'\right)\left(R/R'\right)\}^{1/2}\right]}$$

For most practical cases, $R \ll R'$, so the term in square brackets is $\approx \pi/2$ and G becomes

$$\underline{G \approx \left(\frac{2m}{E_0}\right)^{1/2} \frac{B}{\hbar}\frac{\pi}{2}}$$