APPENDIX E: LOCAL GAUGE INVARIANCE IN QED

Consider a non-relativistic charged particle in an electromagnetic field:

$$\underline{F} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$$

where $\underline{E}$ and $\underline{B}$ can be written in terms of the vector and scalar potentials, $\underline{A}$ and ϕ:

$$\underline{B} = \underline{\nabla} \times \underline{A} \ \text{ and } \ \underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}.$$

The classical Hamiltonian,

$$\underline{H} = \frac{1}{2m}\left(\underline{p} - q\underline{A}\right)^2 + q\phi,$$

can be used along with Schrödinger's equation to obtain

$$H\psi = \left[\frac{1}{2m}\left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)^2 + q\phi\right]\psi(\underline{x}, t) = \mathrm{i}\frac{\partial \psi}{\partial t}(\underline{x}, t). \tag{1}$$

where we have substituted $\underline{p} \rightarrow -\mathrm{i}\underline{\nabla}$. We now need to show that Schrödinger's equation is invariant under the local guage transformation

$$\psi \rightarrow \psi' = e^{\mathrm{i}q\alpha(\underline{x},t)}\psi$$
$$\underline{A} \rightarrow \underline{A}' = \underline{A} + \underline{\nabla}\alpha$$
$$\phi \rightarrow \phi' = \phi - \frac{\partial \alpha}{\partial t}$$

Substituting for ψ', $\underline{A}'$ and ϕ' in equation (1):

$$\left[\frac{1}{2m}\left(-\mathrm{i}\underline{\nabla} - q(\underline{A} + \underline{\nabla}\alpha)\right)^2 + q(\phi - \frac{\partial \alpha}{\partial t})\right] e^{\mathrm{i}q\alpha}\psi = \mathrm{i}\frac{\partial}{\partial t}(e^{\mathrm{i}q\alpha}\psi)$$
$$\left[\frac{1}{2m}\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)^2 + q\phi - q\frac{\partial \alpha}{\partial t}\right] e^{\mathrm{i}q\alpha}\psi = \mathrm{i}\left(e^{\mathrm{i}q\alpha}\frac{\partial \psi}{\partial t} + \mathrm{i}q\psi\frac{\partial \alpha}{\partial t}e^{\mathrm{i}q\alpha}\right).$$

The last terms on either side of the above equation cancel.
Now consider the $\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)^2$ term. In order to show local gauge invariance, we need to show that

$$\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)^2 e^{\mathrm{i}q\alpha}\psi = \left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)^2 e^{\mathrm{i}q\alpha}\psi$$

or, equivalently,

$$\left(-\mathrm{i}\underline{\nabla} - q\underline{A}'\right)^2 \psi' = \left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)^2 e^{\mathrm{i}q\alpha}\psi.$$

Now,

$$\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)^2 e^{\mathrm{i}q\alpha}\psi = \left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right) \cdot \left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right) e^{\mathrm{i}q\alpha}\psi$$

and

$$\underline{\nabla}\left(e^{\mathrm{i}q\alpha}\psi\right) = e^{\mathrm{i}q\alpha}\left(\underline{\nabla} + \mathrm{i}q\underline{\nabla}\alpha\right)\psi.$$

Therefore,

$$\begin{aligned}\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right) e^{\mathrm{i}q\alpha}\psi &= e^{\mathrm{i}q\alpha}\left(-\mathrm{i}\underline{\nabla} + q\underline{\nabla}\alpha - q\underline{A} - q\underline{\nabla}\alpha\right)\psi \\ &= e^{\mathrm{i}q\alpha}\left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)\psi\end{aligned}$$

and

$$\begin{aligned}\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)^2\psi' &= \left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right) e^{\mathrm{i}q\alpha}\left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)\psi \\ &= e^{\mathrm{i}q\alpha}\left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)^2\psi.\end{aligned}$$

Hence,

$$\underline{\left(-\mathrm{i}\underline{\nabla} - q\underline{A}'\right)^2\psi' = e^{\mathrm{i}q\alpha}\left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)^2\psi}$$

and Schrödinger's equation is invariant under a local gauge transformation.