Appendix D: Interaction via Particle Exchange

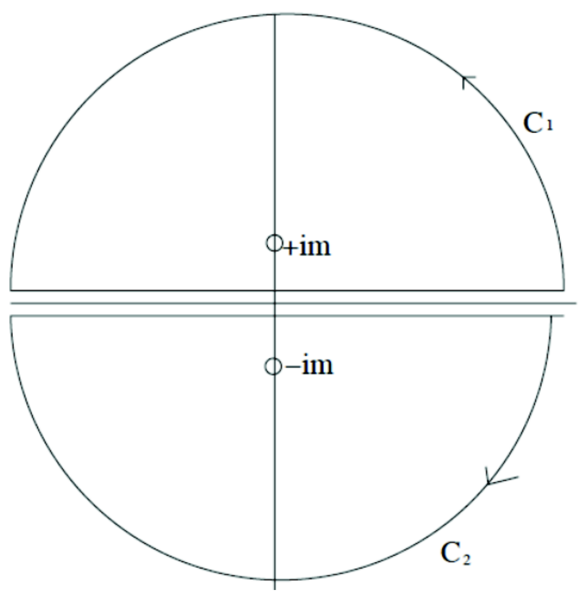
We need to evaluate the following integral in order to determine the energy shift when in state i when a particle of mass m is exchanged between particle 1 and particle 2,

$$\Delta E_i^{1\to 2} = -\frac{g^2}{2(2\pi)^2}\int_0^\infty \frac{p^2}{p^2+m^2}\frac{\mathrm{e}^{\mathrm{i}pr}-\mathrm{e}^{-\mathrm{i}pr}}{\mathrm{i}pr}dp$$

Start by rewriting

$$\Delta E_i^{1\to 2} = -\frac{1}{2}\frac{g^2}{2(2\pi)^2}\int_{-\infty}^\infty \frac{p}{p^2+m^2}\frac{\mathrm{e}^{\mathrm{i}pr}-\mathrm{e}^{-\mathrm{i}pr}}{\mathrm{i}r}dp$$

using the fact that the integrand is even in p. The integrand has poles at $p=\pm\mathrm{i}m$ (see the figure). The integrals with the $\mathrm{e}^{\mathrm{i}pr}$ and $\mathrm{e}^{-\mathrm{i}pr}$ terms are performed separately. This is because one chooses an infinite semi-circular contour to do the integration over, in such a way that on the circular piece the contribution from infinity vanishes. This happens if the integrand contains a decaying exponential in $|p|$. For $\mathrm{e}^{\mathrm{i}pr}$, this happens for $p=+\mathrm{i}|p|$ and so one closes the contour in the upper half plane (C_1 in the figure). For $\mathrm{e}^{-\mathrm{i}pr}$, we want $p=-\mathrm{i}|p|$, and so close the contour in the lower half plane (C_2 in the figure).



The whole integral is thus:

$$-\frac{g^2}{2(2\pi)^2}\left[\oint_{C_1}\frac{p}{p^2+m^2}\frac{\mathrm{e}^{\mathrm{i}pr}}{\mathrm{i}r}dp - \oint_{C_2}\frac{p}{p^2+m^2}\frac{\mathrm{e}^{-\mathrm{i}pr}}{\mathrm{i}r}dp\right].$$

The residue of the pole at $p=\mathrm{i}m$ in the first integrand is:

$$\lim_{p\to\mathrm{i}m}\frac{(p-\mathrm{i}m)}{(p-\mathrm{i}m)(p+\mathrm{i}m)}\frac{p}{\mathrm{i}r}\mathrm{e}^{\mathrm{i}pr} = \frac{1}{2\mathrm{i}r}\mathrm{e}^{-mr}$$

and the residue of the pole at $p=-\mathrm{i}m$ in the second integrand is:

$$\lim_{p\to-\mathrm{i}m}\frac{(p+\mathrm{i}m)}{(p-\mathrm{i}m)(p+\mathrm{i}m)}\frac{-p\,\mathrm{e}^{-\mathrm{i}pr}}{\mathrm{i}r} = -\frac{1}{2\mathrm{i}r}\mathrm{e}^{-mr}.$$

Cauchy's residue theorem tells us that the contour integral over an anti-clockwise contour is $2\pi\mathrm{i}$ multiplied by the sum of the residues of the poles enclosed by the contour. For a clockwise contour, there is an additional minus sign. Noting that C_1 is anti-clockwise, and C_2 is clockwise, one has:

$$\begin{aligned}\Delta E_i^{1\to 2} &= -\frac{g^2}{2(2\pi)^2} 2\pi\mathrm{i}\left[\frac{\mathrm{e}^{-mr}}{2\mathrm{i}r} + \frac{\mathrm{e}^{-mr}}{2\mathrm{i}r}\right] \\ &= \underline{-\frac{g^2}{8\pi}\frac{\mathrm{e}^{-mr}}{r}}\end{aligned}$$

as given in the notes.