Appendix D: Interaction via Particle Exchange

We need to evaluate the following integral in order to determine the energy shift when in state $i$ when a particle of mass $m$ is exchanged between particle 1 and particle 2,

$$
\Delta E_{i}^{1 \rightarrow 2} = -\frac{g^2}{2(2\pi)^2} \int_{0}^{\infty} \frac{p^2}{p^2 + m^2} \left( e^{ipr} - e^{-ipr} \right) dp
$$

Start by rewriting

$$
\Delta E_{i}^{1 \rightarrow 2} = -\frac{1}{2} \frac{g^2}{2(2\pi)^2} \int_{-\infty}^{\infty} \frac{p}{p^2 + m^2} \left( e^{ipr} - e^{-ipr} \right) \frac{ir}{ir} dp
$$

using the fact that the integrand is even in $p$. The integrand has poles at $p = \pm im$ (see the figure). The integrals with the $e^{ipr}$ and $e^{-ipr}$ terms are performed separately. This is because one chooses an infinite semi-circular contour to do the integration over, in such a way that on the circular piece the contribution from infinity vanishes. This happens if the integrand contains a decaying exponential in $|p|$. For $e^{ipr}$, this happens for $p = +i|p|$ and so one closes the contour in the upper half plane ($C_1$ in the figure). For $e^{-ipr}$, we want $p = -i|p|$, and so close the contour in the lower half plane ($C_2$ in the figure).

![Diagram](image)

The whole integral is thus:

$$
-\frac{g^2}{2(2\pi)^2} \left[ \oint_{C_1} \frac{p}{p^2 + m^2} e^{ipr} \frac{dr}{ir} dp - \oint_{C_2} \frac{p}{p^2 + m^2} e^{-ipr} \frac{dr}{ir} dp \right].
$$

The residue of the pole at $p = im$ in the first integrand is:

$$
\lim_{p \to im} \frac{(p - im)}{(p - im)(p + im)} \frac{p}{ir} e^{ipr} = \frac{1}{2ir} e^{-mr}
$$

and the residue of the pole at $p = -im$ in the second integrand is:

$$
\lim_{p \to -im} \frac{(p + im)}{(p - im)(p + im)} \frac{-p}{ir} e^{-ipr} = -\frac{1}{2ir} e^{-mr}.
$$
Cauchy’s residue theorem tells us that the contour integral over an anti-clockwise contour is $2\pi i$ multiplied by the sum of the residues of the poles enclosed by the contour. For a clockwise contour, there is an additional minus sign. Noting that $C_1$ is anti-clockwise, and $C_2$ is clockwise, one has:

$$\Delta E_{1\rightarrow 2} = -\frac{g^2}{2(2\pi)^2} 2\pi i \left[ \frac{e^{-mr}}{2ir} + \frac{e^{-mr}}{2ir} \right]$$

$$= -\frac{g^2}{8\pi} \frac{e^{-mr}}{r}$$

as given in the notes.