APPENDIX C: Scattering from a Yukawa potential

Consider relativistc elastic scattering from a Yukawa potential

$$V(\vec{r}) = \frac{g\,\mathrm{e}^{-mr}}{r}$$

The matrix element is given by

$$|M_{if}|^2 = \left| \int \mathrm{e}^{-\mathrm{i}\vec{q}.\vec{r}}\, V(\vec{r})\, \mathrm{d}^3\vec{r} \right|^2$$

In order to perform the integral, choose the z axis to lie along $\vec{r}$. Then $\vec{q}.\vec{r} = -qr\cos\theta$ and

$$\begin{aligned}
\int \mathrm{e}^{-\mathrm{i}\vec{q}.\vec{r}}\, V(\vec{r})\, \mathrm{d}^3\vec{r} &= \int_0^\infty \int_0^{2\pi} \int_0^\pi V(r)\, \mathrm{e}^{iqr\cos\theta}\, r^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}r \\
&= \int_0^\infty \int_{-1}^{+1} 2\pi V(r)\, \mathrm{e}^{iqr\cos\theta}\, r^2\, \mathrm{d}(\cos\theta)\, \mathrm{d}r \\
&= \int_0^\infty 2\pi V(r) \left(\frac{\mathrm{e}^{iqr} - \mathrm{e}^{-iqr}}{iqr} \right) r^2\, \mathrm{d}r \\
&= \int_0^\infty 2\pi g \frac{\mathrm{e}^{-mr}}{r} \left(\frac{\mathrm{e}^{iqr} - \mathrm{e}^{-iqr}}{iqr} \right) r^2\, \mathrm{d}r \\
&= \int_0^\infty 2\pi g\, \mathrm{e}^{-mr} \left(\frac{\mathrm{e}^{iqr} - \mathrm{e}^{-iqr}}{iq} \right) \mathrm{d}r \\
&= \int_0^\infty \frac{2\pi g}{iq} \left(\mathrm{e}^{-r(m-iq)} - \mathrm{e}^{-r(m+iq)} \right) \mathrm{d}r \\
&= \frac{2\pi g}{iq} \left(\frac{1}{m - \mathrm{i}q} - \frac{1}{m + \mathrm{i}q} \right) = \frac{2\pi g}{iq} \frac{2iq}{m^2 + q^2} \\
&= \frac{4\pi g}{m^2 + q^2}
\end{aligned}$$

The matrix element is then

$$|M_{if}|^2 = \frac{16\pi^2 g^2}{(m^2 + q^2)^2}$$

The Yukawa potential is a general potential, and can be extended to other potentials, e.g. for the Coulomb potential

$$V(\vec{r}) = -\frac{Z\alpha}{r}$$

using $g = Z\alpha$ and $m = 0$, the matrix element for Rutherford Scattering is

$$|M_{if}|^2 = \frac{16\pi^2 Z^2 \alpha^2}{q^4}$$