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Lent/Easter Terms 2024

Part II Particle and Nuclear Physics
Examples Sheet

==================== **Part 1, Chapters 1-4** ====================

1. Chapter 1: Classification of Particles
   Explain the meaning of the terms *fermion, quark, lepton, hadron, nucleus,* and *boson* as used in the classification of particles.

2. Chapter 2: Natural Units

   (a) Explain what is meant by *natural* units and the *Heaviside-Lorentz* system.

   (b) Calculate a value for the proton mass in natural units, assuming $m_p = 1.67 \times 10^{-27}$ kg.

   (c) The muon decay rate is given by Sargent's Rule

   $$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3},$$

   where m_μ is the muon mass (106 MeV) and G_F is the Fermi constant (1.166×10^{-5} GeV^{-2}). Given $\tau = 1/\Gamma$, calculate the muon lifetime in seconds.

   (d) The cross section for $e^+e^- \to \mu^+\mu^-$ is measured as 0.5 pb. What is this cross-section in natural units?

   [Note that $c = 3.0 \times 10^8\,\mathrm{ms^{-1}}$; $\hbar = 6.6 \times 10^{-25}$ GeVs; $\hbar c = 197$ MeVfm; 1 barn $= 10^{-28}\,\mathrm{m^2}$]

3. Chapter 2: Relativistic Kinematics
   Consider the electromagnetic decay of the rho meson, $\rho \to \pi^0\gamma$. Calculate the energies of the photon and pion in the ρ^0 rest frame. The π^0 goes on to rapidly decay to two photons. What range of energies will these two photons take in the ρ rest frame, assuming the ρ decays as above? Draw a rough sketch of the energy distribution of all the photons that might be detected from a ρ decay.

   Discuss how the observation of a range of energies for the electron from neutron decay ($n \to pe^-\bar{\nu}_e$) led to the prediction of the existance of the neutrino.

   [The masses of the ρ and π^0 are 770 MeV and 135 MeV, respectively.]

4. Chapter 2: Radioactive Decay
   The decay chain ^{211}Bi$\to^{207}$Tl$\to^{207}$Pb is observed for an initially pure sample of 5×10^{11} Bq of ^{211}Bi. The half life of ^{211}Bi is 2.14 minutes and that of ^{207}Tl is 4.88 minutes; ^{207}Pb is stable. Write down the rate equations for this system, and show that the number of Tl atoms present at time t is given by

   $$N_{\mathrm{Tl}}(t) = X\left(e^{-\lambda_{\mathrm{Bi}}t} - e^{-\lambda_{\mathrm{Tl}}t}\right),$$

   where the λ values represent the corresponding decay rates and X is a constant. What is the maximum ^{207}Tl activity and at what time does it occur?

5. Chapter 2: Cross sections
Define the terms *total cross-section* and *differential cross-section* for scattering processes.

A beam of neutrons with an intensity 10^5 particles per second traverses a thin foil of ^{235}U with a density of $200\,\mathrm{kgm^{-3}}$ and thickness of 0.5 mm. There are three possible outcomes for the neutron-uranium interaction:

   i. elastic scattering of the neutron, with a cross-section 0.1 b;
   ii. neutron capture followed by the emission of a γ-ray, with a cross-section 70 b;
   iii. neutron capture followed by fission, with a cross-section 200 b;

Determine

   (a) the intensity of the neutron beam transmitted by the foil;
   (b) the rate of fission reactions occurring in the foil induced by the incident beam;
   (c) the rate of γ-rays induced by the incident beam;
   (d) the flux of neutrons elastically scattered out of the beam at a point 10 m from the foil, assuming that the neutrons are scattered isotropically.

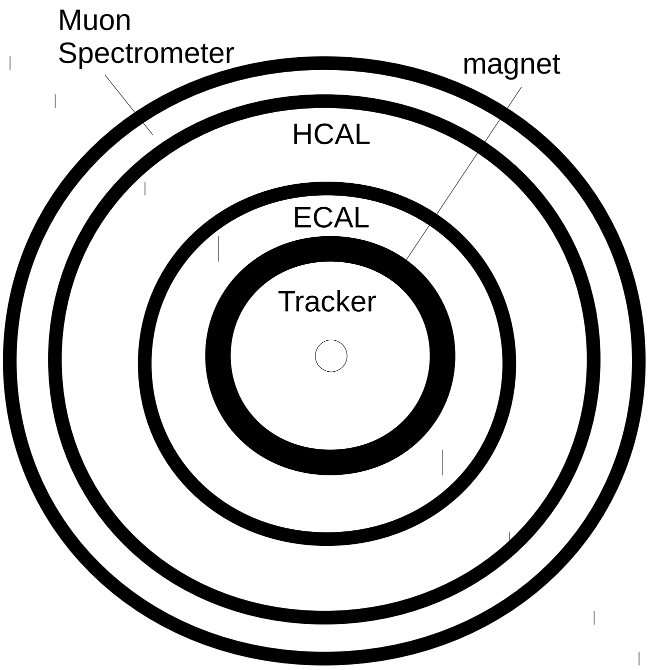
6. Chapter 2: Breit-Wigner Formula
The Breit-Wigner formula for a reaction cross-section is given by

$$\sigma(E) = \frac{\pi g}{p_i^2} \frac{\Gamma_i \Gamma_f}{(E - E_0)^2 + \Gamma^2/4}.$$

Explain the meaning of the symbols in this equation, and outline its derivation.

The maximum value of the cross section for radiative capture of neutrons in ^{113}Cd (i.e. the process $n+^{113}\mathrm{Cd} \rightarrow\ ^{114}\mathrm{Cd}+\gamma$) is 20.6 kb and is reached at a neutron energy of 178 meV, where the elastic width Γ_n is 0.6 meV and the radiative width Γ_γ is 112.4 meV. The spin of ^{113}Cd in its ground state is $J = \frac{1}{2}$. Calculate the elastic cross-section at resonance and find the spin of the compound nucleus formed.
[The mass of the neutron is 939.6 MeV. *]*

7. Chapter 3: Detector Signatures
For each e^+e^- process below, sketch the signature in a typical cylindrical detector e.g.



(a) $e^+e^- \to \mu^+\mu^-$

(b) $e^+e^- \to e^+e^-\mu^+\mu^-$

(c) $e^+e^- \to e^+e^-\gamma$

(d) $e^+e^- \to q\bar{q}g$

(e) $e^+e^- \to \tau^+\tau^-$, where the taus decay as $\tau^+ \to \mu^+\nu_\mu\bar{\nu}_\tau$ and $\tau^- \to \pi^-\pi^0\nu_\tau$

(f) $e^+e^- \to \pi^+n\bar{p}\pi^0K^+K^-$

Calculate the average distance travelled by a tau produced in a $e^+e^- \to \tau^+\tau^-$ collision with $E_{CM} = 10$ GeV. Explain why we don't observe taus directly in typical cylindrical detectors. *[The tau lifetime is 2.9×10^{-13} s and the tau mass is 1.777 GeV]*

8. Chapter 3: Detector Resolution
In an experiment, the momentum resolution in the tracker for a 10 GeV particle is 10%, while the energy resolution in the electromagnetic calorimeter is 0.16%.

(a) Calculate the momentum and energy resolutions for a 0.5 GeV electron and a 100 GeV electron. Which sub-detector would give the more reliable estimate in each case?

(b) Would a 10 GeV muon or a 100 GeV muon be measured more accurately in a typical cylindrical detector? Explain your reasoning.

9. Chapter 3: Collider Kinematics
   Calculate the centre-of-mass energy for

   i. a fixed target experiment using a 50 GeV electron beam on a proton target,

   ii. a collider experiment using a 50 GeV electron beam and a 50 GeV positron beam.

   Comment on whether a Z boson or a Higgs boson may be produced from the collisions in each case.

   What would be the length of a linear collider used to produce the electron-positron beams in ii), assuming RF cavities capable of providing 16 GeV per km?

   If proton beams are collided instead, why might the centre-of-mass energy be lower than the value calculated in ii)?

10. Chapter 4: Virtual Particles
    Show that the process $\gamma \to e^+e^-$ is kinematically forbidden in a vacuum, but is possible in matter.

    One such possible interaction of a photon in matter is $e^-\gamma \to e^-e^+e^-$. What is the minimum photon energy for this process to occur? How does this change for the photon striking a far more massive object M, $M\gamma \to Me^+e^-$? You may assume $m_M \gg m_e$.

==================== **Part 2, Chapters 5-8** ====================

11. Chapter 5: Feynman Diagrams
    Define the terms *scattering amplitude, decay rate*, and *scattering rate*.
    Draw all lowest order Feynman diagrams and write down the form of the scattering amplitudes $\mathcal{M}$ for the following processes:

    (a) $\mu^- \to e^- \nu_\mu \bar{\nu}_e$
    (b) $p \to n e^+ \nu_e$
    (c) $\pi^0 \to \gamma\gamma$
    (d) $\rho \to \pi^+ \pi^-$
    (e) $\pi^0 \to \pi^- e^+ \nu_e$
    (f) $e^- \gamma \to e^- e^+ e^-$
    (g) $e^- \gamma \to \nu \bar{\nu} e^-$
    (h) $e^+ e^- \to e^+ e^-$
    (i) $e^- e^- \to e^- e^-$
    (j) $\nu_e e^- \to \nu_e e^-$
    (k) $\nu_e e^+ \to \nu_e e^+$
    (l) $\nu_e \mu^- \to \nu_e \mu^-$
    (m) $\nu_\tau p \to \tau^+ n$

    (n) $e^+ e^- \to \mu^+ \mu^-$ at $E_{CM} = 10$ GeV vs $E_{CM} = 90$ GeV.

    Are any of these processes forbidden? Where multiple diagrams are possible, note which would dominate (if any).

12. Chapter 6: Electromagnetic Decays
    The neutral pion, π^0, is a $J^P = 0^-$, $(u\bar{u} - d\bar{d})/\sqrt{2}$ state and decays via the four possibilities listed below.

Process	Branching fraction
$\pi^0 \to \gamma\gamma$	0.9882
$\pi^0 \to \gamma e^+ e^-$	0.0117
$\pi^0 \to e^+ e^- e^+ e^-$	3×10^{-5}
$\pi^0 \to e^+ e^-$	6×10^{-8}

    Draw a Feynman diagram for each of the pion decays and use Fermi's Golden Rule $\Gamma = 2\pi|\mathcal{M}|^2 \rho$ to *roughly* explain the relative branching fractions. For this question, you may assume the coupling of the photon to a charged particle is $Q\sqrt{\alpha} = Q/\sqrt{137}$ for simplicity.

13. Chapter 6: Drell Yan Production
    Draw a typical Feynman diagram for Drell Yan production at a hadron collider. Find the ratio of the Drell Yan production rate for $\pi^- p : \pi^+ \pi^- : p\bar{p} : pp$.

14. Chapter 6: Quark Charge and Colour
    Estimate $R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$ at $E_{CM} = 6$ GeV.
    How would R change if

    i. the bottom quark mass was 1 GeV?
    ii. the electric charge for up-type quarks was $+\frac{3}{4}$ and the down-type quarks was $-\frac{1}{2}$?

    In each case, what number of colours would give the best agreement if the measured value of R was $3\frac{1}{3}$ at this E_{CM}?

15. Chapter 7: Quark/Gluon Production

   (a) Draw the lowest order Feynman diagrams for $u\bar{u} \to gg$. What would you expect for $R = \frac{\sigma(u\bar{u} \to gg)}{\sigma(d\bar{d} \to gg)}$ in a $p\bar{p}$ collider?

   (b) Draw the lowest order Feynman diagrams for $gq \to qg$ and $gq \to q\gamma$ and sketch their signatures in a typical detector. How would the energies of the final state particles be determined? What would the ratio of u to d events be for each case in a $p\bar{p}$ collider? What happens to the quarks in the proton and antiproton that do not directly participate in the scattering?

16. Chapter 7: The Strong Coupling Constant
Sketch the strong coupling constant α_s as a function of the energy scale. Why does this suggest that quarks cannot exist as free particles? Outline two methods for measuring α_s.

17. Chapter 8: Hadron States
In the lectures we showed the three light quarks (u, d, s) form eight $J^P = \frac{1}{2}^+$ states and ten $J^P = \frac{3}{2}^+$ states. If quarks were spin-0 particles, what baryon states could be formed? Assume all other quark properties remain the same.

18. Chapter 8: Hadron Masses

   (a) What are the quark model mass predictions for the following mesons: K^+, η, ω, η'? Do they agree with the measured masses? If not, can you suggest why this may be?
   [Assume $m_u = m_d = 310$ MeV, $m_s = 483$ MeV and the spin-spin interaction coefficient $A = 0.0615$ GeV^3.]

   (b) What are the quark model mass predictions for the following baryons: Δ, Ξ? Do they agree with the measured masses?
   [Assume in this case $m_u = m_d = 360$ MeV, $m_s = 540$ MeV and the spin-spin interaction coefficient $A = 0.026$ GeV^3.]

   (c) The baryons Λ^0 and Σ^0 have the same quark composition (uds) and both are members of the same $J^P = \frac{1}{2}^+$ baryon octet. Explain why their masses are different (1.116 GeV and 1.193 GeV respectively), and suggest why their lifetimes are very different (2.6×10^{-10} s and 7×10^{-20} s respectively).

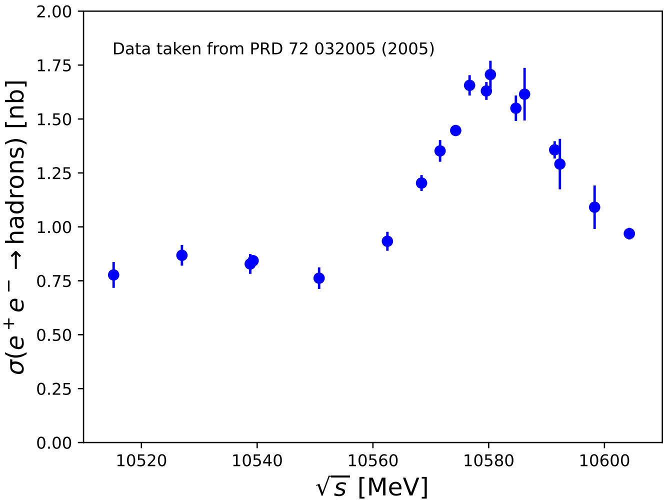
19. Chapter 8: Spin and Parity
When π^- mesons are stopped in deuterium they form "pionic atoms" (π^-d) which usually undergo transitions to an atomic s-state ($\ell = 0$), whereupon the capture reaction $\pi^-\text{d} \to n\,n$ occurs and destroys them. (The fact that capture normally occurs in an s-state is established from studies of the X-rays emitted in the transitions before capture). Given that the deuteron has spin-parity $J^P = 1^+$ and the pion has $J = 0$, show that these observations imply that the pion has negative intrinsic parity.

20. Chapter 8: Magnetic moments
The proton has quark content uud and magnetic moment $2.8\,\mu_N$, while the Σ^+ baryon has quark content uus and magnetic moment $2.4\,\mu_N$. Use this information to estimate the magnetic moment of the Σ_b^+ baryon (uub).
[Assume in this case $m_u = m_d = 0.3$ GeV, $m_s = 0.5$ GeV and $m_b = 5$ GeV.]

21. Chapter 8: The Upsilon Resonance
The BABAR experiment made measurements of $\sigma(e^+e^- \to \text{hadrons})$ for $\sqrt{s}$ in the region of the $\Upsilon(4S)$ $(b\bar{b})$ resonance and some of the results are shown in the figure. The Υ resonance is described by the Breit-Wigner cross section

$$\sigma(i \to f) = \frac{g\pi}{p^2} \frac{\Gamma_i \Gamma_f}{(E - E_0)^2 + \Gamma^2/4}$$

and is known to decay to hadrons close to 100% of the time.



(a) What spin-parity states may be produced in e^+e^- collisions?

(b) Estimate the mass and total width of the $\Upsilon(4S)$ meson.

(c) The detector used to make these cross-section measurements was not fully efficient. Estimate the efficiency by comparing the measured cross section to the theoretical prediction. You may assume the efficiency of the detector was independent of $\sqrt{s}$ and the branching ratio for $\Upsilon(4S) \to e^+e^-$ is 2.5×10^{-5}.

(d) Draw Feynman diagrams for the decays $\Upsilon \to e^+e^-$ and $\Upsilon \to$ hadrons for the $\Upsilon(3S)$ and $\Upsilon(4S)$ resonances. Explain why the $\Upsilon(3S)$ resonance has values of Γ_{ee} similar to that of $\Upsilon(4S)$, but has a total width which is smaller by at least two orders of magnitude.
[B^+ $(\bar{b}u)$ and $B^0(\bar{b}d)$ have masses of about 5280 MeV. *The Υ(3S) resonance has a mass of* 10355 MeV. *]*

22. Chapter 8: Mixed Flavour States
The partial width for the leptonic decay of the ρ^0 meson, $\rho^0 \to e^+e^-$, is 7 keV. Estimate the partial width for the leptonic decay of the ω^0 meson, $\omega^0 \to e^+e^-$.
The partial width for the pionic decay of the ρ^0 meson, $\rho^0 \to \pi^0\gamma$, is 77 keV. Estimate the partial width for the pionic decay of the ω^0 meson, $\omega^0 \to \pi^0\gamma$.
[The π^0 and ρ^0 are both $(u\bar{u} - d\bar{d})/\sqrt{(2)}$ states, while ω^0 is a $(u\bar{u} + d\bar{d})/\sqrt{(2)}$ state. The ρ^0 and ω^0 are both $J^P = 1^-$ states, while the π^0 is 0^-.]

==================== Part 3, Chapters 9-12 ====================

23. Chapter 9: W boson and the Number of Neutrino Species
The number of neutrino species can be estimated using the total width of the W boson. Using the Standard Model prediction of the partial width for $\mathrm{W}^- \to \mathrm{e}^-\bar{\nu}_\mathrm{e}$ decays,

$$\Gamma(\mathrm{W}^- \to \mathrm{e}^-\bar{\nu}_\mathrm{e}) = \frac{\mathrm{G_F}}{\sqrt{2}}\frac{\mathrm{M_W^3}}{6\pi},$$

the mass of the W boson, $M_\mathrm{W} = 80.385 \pm 0.015$ GeV and the total width, $\Gamma_\mathrm{W} = 2.085 \pm 0.042$ GeV, estimate the number of light neutrino species. Make clear your assumptions.
[$G_\mathrm{F} = 1.2 \times 10^{-5}$ GeV^{-2}. *]*

24. Chapter 9: Helicity in the Weak Interaction

    (a) Draw the lowest order Feynman diagrams for $\pi^- \to \mu^-\bar{\nu}_\mu$ and $\pi^- \to e^-\bar{\nu}_e$. Using arguments of lepton universality and density of states only, how would you expect the rates of these two decays to compare?

    (b) Calculate the velocity with which the electron and muon are emitted in the pion rest frame. Note if the velocities you calculate are relativistic or non-relativistic.
    [Assume $m_e = 0.511$ MeV, $m_\mu = 106$ MeV, *and* $m_\pi = 140$ MeV.*]*

    (c) The probability for a W-boson to couple to the $\pm$ helicity state of a lepton is equal to $\frac{1}{2}(1 \mp \frac{v}{c})$. What is the consequence for the $\pi^- \to \mu^-\bar{\nu}_\mu$ and $\pi^- \to e^-\bar{\nu}_e$ decay rates?

25. Chapter 9: Weak Force and Conservation
Consider each of the groups of processes given below. In each group, with the aid of Feynman diagrams using the Standard Model vertices, determine which processes are allowed and which are forbidden. By considering the strength of the forces involved, rank the processes in each group in order of expected rate.

    (a) $D_s^+ \to K^+\pi^0$, $D_s^+ \to K^+K^0$, $D_s^+ \to \pi^+\phi$

    (b) $B^0 \to D^-\pi^+$, $B^0 \to \pi^+\pi^-$, $B^0 \to J/\psi K^0$

    (c) $\pi^- \to \mu^-\bar{\nu}_\mu$, $K^- \to \mu^-\bar{\nu}_\mu$, $B^- \to \mu^-\bar{\nu}_\mu$

    (d) $\rho^0 \to \nu\bar{\nu}$, $\pi^0 \to \nu\bar{\nu}$, $\pi^0 \to \mu^+\mu^-$

26. Chapter 10: Z coupling

    (a) In the GWS theory, the couplings of the Z boson to fermions is described by $g_{L,R} \propto (I_3)_{L,R} - Q\sin^2\theta_W$, where L/R denotes a left/right-handed fermion, I_3 denotes weak isospin, Q is the electric charge, and the weak mixing angle is $\theta_W = 29^o$. Assuming $\Gamma(Z \to f\bar{f}) \propto g_L^2 + g_R^2$, predict the branching ratios for the Z boson to decay to hadrons, neutrinos, and $\tau^+\tau^-$.

    (b) In the OPAL experiment at LEP, the cross section for $e^+e^- \to \tau^+\tau^-$ was measured at various centre-of-mass energies. Some of the results are shown below.

$E_{\rm cm}$/GeV	$\sigma(e^+e^- \to \tau^+\tau^-)$/nb
88.481	0.2769 ± 0.0235
89.442	0.4892 ± 0.0091
90.223	0.8331 ± 0.0368
91.283	1.4988 ± 0.0213
91.969	1.1892 ± 0.0235
92.971	0.7089 ± 0.0105
93.717	0.4989 ± 0.0276

Plot these data and make estimates of the Z boson mass, m_Z, the total width of the Z boson, Γ_Z, and the partial decay width to $\tau^+\tau^-$, $\Gamma_{\tau\tau}$. Compare the branching fraction for $Z \to \tau^+\tau^-$ with your theoretical prediction and comment.

Why is the measured resonance curve asymmetric? Indicate what other effects need to be taken into account when accurately determining m_Z, $\Gamma_{\rm Z}$ and $\Gamma_{\tau\tau}$.

27. Chapter 10: Number of Generations from LEP
Consider a fourth lepton generation exists with $m_L- = 40$ GeV and $m_\nu \sim 0$. Draw Feynman diagrams for possible production mechanisms of this fourth degeneration at a electron-positron collider such as LEP. What might you expect for the possible decays of the charged lepton? Draw Feynman diagrams for the L^- decays and predict the branching ratios for each final state.

Use Sargent's rule for the partial decay rate for $X \to \nu_X e^- \bar{\nu}_e$

$$\Gamma_{X\to e} = \frac{G_F^2 m_X^5}{192\pi^3},$$

to calculate the L^- partial decay rate to electrons and the expected lifetime of L^- in seconds.

Outline how the precision electroweak measurements at LEP ruled out such a fourth generation.

28. Chapter 11: A Higgs Boson Factory
A Higgs factory is being considered to study the properties of the spin-0 Higgs boson with mass 125 GeV. The Higgs boson could be produced through the resonant reaction $\mu^+\mu^- \to H$, with a cross-section described by the Breit-Wigner formula. The partial decay width of the Higgs boson (H) to fermions is proportional to m_f^2.

(a) Explain why a $\mu^+\mu^-$ collider is being considered for a Higgs factory rather than e^+e^- or pp collisions.

(b) Find the ratio of $\Gamma(H \to \tau^+\tau^-)$: $\Gamma(H \to c\bar{c})$: $\Gamma(H \to b\bar{b})$. You may assume $m_\tau = 1.77$ GeV, $m_c = 1.5$ GeV, and $m_b = 4.5$ GeV.

(c) The sum of the branching ratios for Higgs to $\tau^+\tau^-$, $c\bar{c}$, and $b\bar{b}$ is 67%. Calculate the cross section for $\mu^+\mu^- \to H \to b\bar{b}$ at the peak of the resonance. Express your answer in natural units and in barns.

(d) Explain why the branching ratio for $H \to W^+W^-$ is non-zero, despite the fact that $m_H < 2m_W$.

29. Chapter 12: Neutrino Oscillations
A beam of neutrinos can interact with nucleons in a stationary target, either undergoing elastic scattering or producing a charged lepton.

(a) Draw the lowest order Feynman diagrams for the elastic and inelastic scattering of neutrinos with nucleons.

(b) Calculate the minimum energy of the ν which would permit e^- production. How would this threshold energy change for μ^- or τ^- production?

(c) Show that if there are two neutrino mass eigenstates ν_2 and ν_3 with masses m_2 and m_3 and energies E_2 and E_3, mixed so that

$$\begin{aligned}\nu_\mu &= \nu_2 \cos\theta + \nu_3 \sin\theta \\ \nu_\tau &= -\nu_2 \sin\theta + \nu_3 \cos\theta\end{aligned}$$

then the number of muon neutrinos observed at a distance L from the muon source is

$$|\nu_\mu(L)|^2 \approx |\nu_\mu(L=0)|^2 \times \left[1 - \sin^2(2\theta)\,\sin^2\left\{A\left(\frac{(m_2^2 - m_3^2)L}{p}\right)\right\}\right]$$

where A is a constant.

(d) In 2005, the MINOS experiment studied neutrino oscillations by pointing a beam of $1-5$ GeV muon neutrinos from Fermilab to the MINOS far detector 730 km away. The experiment aimed to make a precise measurement of $m_3^2 - m_2^2$. Sketch the expected energy spectrum of muon neutrinos at the MINOS detector if $\sin^2(2\theta) = 0.90$ and $m_3^2 - m_2^2 = 2.5 \times 10^{-3}$ eV^2. Assume that the energy spectrum of neutrinos produced by the beam at Fermilab was of uniform intensity in the range $1-5$ GeV and zero elsewhere (i.e. a top-hat function).

(e) If muon neutrinos oscillate into tau neutrinos, will any τ leptons (produced by charged current interactions) be observed in the MINOS far detector? How would your answer change for the future DUNE experiment, which will use a similar ν_μ beam produced at Fermilab aimed at a detector 1300 km away at the Sanford Underground Research Facility?

30. Chapter 12: Grand Unified Theories
Grand Unified Theories predict that protons can decay through the annihilation of two valence quarks to create an antilepton and antiquark via the exchange of a very heavy intermediate boson: $p \to l^+\pi^0$ or $p \to \nu\pi^+$. The non-observation of proton decay can be used to set stringent limits on GUTs.

(a) Assume two new massive bosons exist in nature, $X^{-4/3}$ and $Y^{-1/3}$. Sketch the possible Feynman diagrams for the decay of a proton, indicating the final-state particles. Explain why these Feynman diagrams are non-Standard-Model diagrams.

(b) The Super Kamiokande experiment was designed to search for proton decay as well as neutrino interactions. How might Super K detect the $p \to e^+\pi^0$ decays and distinguish them from neutrino interactions? *Hint: you may want to recall q.12 if you find you are stuck.*

(c) Show the energy of the pion is $E_\pi = (m_p^2 + m_\pi^2 - m_e^2)/2m_p$ in the laboratory frame and the invariant mass of the photons is $m_{\gamma\gamma}^2 = 2E_1E_2(1 - \cos\theta)$ in any frame of reference.

(d) Finally, show the maximum opening angle between the two photons is $\theta_{max} = 2\sin^{-1}(m_\pi/E_\pi)$ in the laboratory frame.

=================== **Part 4, Chapters 13-15** ===================

31. Chapter 13: The Semi-Empirical Mass Formula
The Semi-Empirical mass formula (SEMF) for *nuclear* masses may be written in the form

$$M(A,Z) = Zm_p + (A-Z)m_n - a_{\mathrm{V}}A + a_{\mathrm{S}}A^{\frac{2}{3}} + a_{\mathrm{C}}\frac{Z^2}{A^{\frac{1}{3}}} + a_{\mathrm{A}}\frac{(A-2Z)^2}{A} + \delta(A,Z),$$

where m_p and m_n are the masses of the proton and neutron respectively.
[m_p = 938.3 MeV, m_n = 939.6 MeV, m_e = 0.511 MeV, a_V = 15.8 MeV, a_S = 18.0 MeV, a_A = 23.5 MeV, nuclear radius $R = R_0 A^{1/3}$ with $R_0 = 1.2\,\mathrm{fm}$]

(a) Explain the physical significance and functional form of the various terms. Which terms are important for nuclear fission and fusion and why?

(b) Show that the Coulomb term constant a_C can be written as

$$a_{\mathrm{C}} = \frac{3e^2}{20\pi\epsilon_0 R_0},$$

assuming the nucleus can be treated as a sphere of uniform charge density. Calculate the value of a_C.

(c) Show that the value of Z of the most stable isobar of mass number A is

$$Z = \frac{m_n - m_p + 4a_{\mathrm{A}}}{2a_{\mathrm{C}}A^{-\frac{1}{3}} + 8a_{\mathrm{A}}/A}.$$

Use this to predict the Z value for the most stable nuclei with $A = 118$ and $A = 201$, and compare with nuclear data, which you can find on the web (e.g. `https://www.nndc.bnl.gov/nudat3/`). Predict the most stable super-heavy nucleus with mass number 302.

(d) On the typical scale of a nucleus, gravitational effects can be safely ignored. However, a neutron star may be considered as nucleus consisting entirely of neutrons and here we can no longer ignore gravity. By treating a nucleus as a sphere of uniform mass density, show the effect of gravitational forces on the binding energy may be accounted for by adding a term $-a_{\mathrm{G}}A^{\frac{5}{3}}$ to the SEMF (neglecting the proton-neutron mass difference). Calculate a value for a_G and estimate the *lightest* mass for a neutron star.

32. Chapter 13: The SEMF Asymmetry and Pairing Terms
The form and magnitude of the asymmetry term can be estimated using the Fermi Gas model. This involves treating the N neutrons and Z protons as free fermions of mass m moving in a box of volume $V = \frac{4}{3}\pi R_0^3 A$. The model therefore only accounts for the kinetic energy of the nucleons, and not their potential energy.

A standard calculation, which you have done before (at least for a cubic box), gives the density of states for each species (including spin degeneracy) as

$$g(\epsilon) = BA\epsilon^{\frac{1}{2}} \quad \text{where} \quad B = \frac{4\sqrt{2}m^{\frac{3}{2}}R_0^3}{3\pi\hbar^3}\ .$$

You need not prove this unless you want to practice.

(a) Show that the Fermi energy for the neutrons is $\epsilon_{\rm F} = \left(\frac{3N}{2BA}\right)^{\frac{2}{3}}$

(b) Calculate the Fermi energy $\bar{\epsilon}_{\rm F}$ and the corresponding nucleon momentum for the symmetric case $N = Z = \frac{1}{2}A$.

(c) Show that the total kinetic energy of the nucleons is given by $\frac{3}{5}\left(\frac{3}{2BA}\right)^{\frac{2}{3}}\left(N^{\frac{5}{3}} + Z^{\frac{5}{3}}\right)$.

(d) Expand about the symmetric point $N = Z = \frac{1}{2}A$ by writing $N = \frac{1}{2}A(1+\alpha)$ and $Z = \frac{1}{2}A(1-\alpha)$ to show that the asymmetry energy has the form $a_{\rm A}\frac{(N-Z)^2}{A}$, where $a_{\rm A} = \frac{1}{3}\bar{\epsilon}_{\rm F}$.

(e) One contribution to the pairing energy can also be estimated from this model, reflecting the stepwise increase of the kinetic energy resulting from the exclusion principle. This would be expected to be approximately equal to the energy spacing of levels at the Fermi level, i.e. $1/g(\epsilon_{\rm F})$.

Show that this is, for the $N = Z = \frac{1}{2}A$ case $\frac{4\bar{\epsilon}_{\rm F}}{3A}$. Evaluate and compare with the fitted value in the SEMF for a typical value of $A \sim 100$.

33. Chapter 13: Nuclear Size
The ground state of a ^{17}F nucleus sits 2.25 MeV above the ground state of a ^{17}O nucleus. What is the maximum energy of the positron emitted in the β^+ decay of ^{17}F? Estimate the charge radius of a nucleus with 17 nucleons.
[Consider the nuclear (or atomic) mass differences in terms of the maximum positron energy, and again in terms of the change in the SEMF. $m_p = 938.272$ MeV, $m_n = 939.566$ MeV.*]*

34. Chapter 14: The Nuclear Shell Model
Outline the basis of the Nuclear Shell Model and show how it accounts for magic numbers. How can the shell model be used to predict the spins and parities of nuclear ground states?

Use the shell model to predict the spins and parities of the ground stats of the nuclides listed below and compare to the experimental values given. Comment on any discrepancies you find.

$^{3}_{2}$He	$^{9}_{4}$Be	$^{7}_{3}$Li	$^{12}_{6}$C	$^{13}_{6}$C	$^{15}_{7}$N	$^{17}_{8}$O	$^{23}_{11}$Na	$^{131}_{54}$Xe	$^{207}_{82}$Pb
$\frac{1}{2}^+$	$\frac{3}{2}^-$	$\frac{3}{2}^-$	0^+	$\frac{1}{2}^-$	$\frac{1}{2}^-$	$\frac{5}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^+$	$\frac{1}{2}^-$

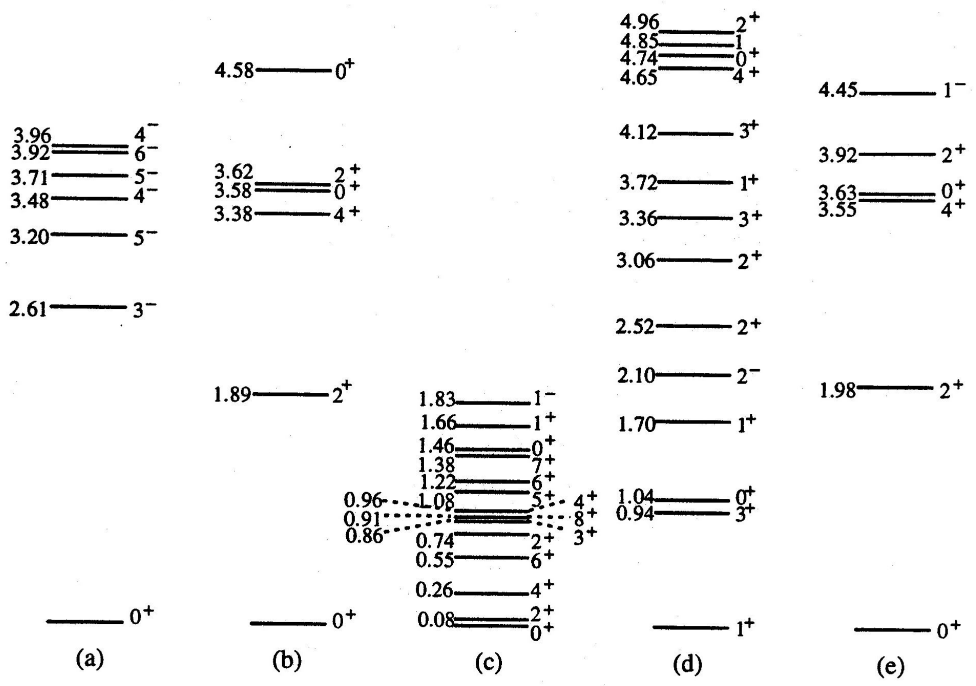
Assume the following ordering of levels:

$1s_{\frac{1}{2}}\ 1p_{\frac{3}{2}}\ 1p_{\frac{1}{2}}\ 1d_{\frac{5}{2}}\ 1d_{\frac{3}{2}}\ 2s_{\frac{1}{2}}\ 1f_{\frac{7}{2}}\ 1f_{\frac{5}{2}}\ 2p_{\frac{3}{2}}\ 2p_{\frac{1}{2}}\ 1g_{\frac{9}{2}}\ 1g_{\frac{7}{2}}\ 2d_{\frac{5}{2}}\ 2d_{\frac{3}{2}}\ 1h_{\frac{11}{2}}\ 3s_{\frac{1}{2}}\ 1h_{\frac{9}{2}}\ 2f_{\frac{7}{2}}\ 3p_{\frac{3}{2}}\ 1i_{\frac{13}{2}}\ 3p_{\frac{1}{2}}\ 2f_{\frac{5}{2}}\cdots$

35. Chapter 14: Energy Levels
The diagram below shows the low-lying energy levels for the nuclides:

$$^{18}_{10}\mathrm{Ne} \quad ^{166}_{68}\mathrm{Er} \quad ^{18}_{9}\mathrm{F} \quad ^{208}_{82}\mathrm{Pb} \quad ^{18}_{8}\mathrm{O}$$

The schemes are drawn to the same scale, with energies (in MeV) with respect to the ground state and the spin and parity (J^P) values given for each level. Identify which scheme corresponds to each nuclide and explain as fully as you can which features of the levels support your choices.



36. Chapter 14: Rotational Excitations
The spin-parity and excitation energies of the five lowest-energy states of $^{174}_{72}\mathrm{Hf}$ are

J^P	0^+	2^+	4^+	6^+	8^+
$E/$ keV	0	91	297	608	1009

Show that these states are consistent with being rotational excitations and obtain a value for the moment of inertia of the $^{174}_{72}\mathrm{Hf}$ nucleus. Compare your result to the expectation if $^{174}_{72}\mathrm{Hf}$ is assumed to be a rigid spherical rotator.
[A solid sphere of mass m and radius R has moment of inertia $U = \frac{2}{5}mR^2$.
Hint: remember $\hbar = 6.66 \times 10^{-25}$ GeV s or $\hbar = 1.05 \times 10^{-4}$ fm^2 kg s^{-1}]

37. Chapter 15: Carbon Dating

   (a) The $^{14}_{6}\mathrm{C}$ half-life is 5730 years. What is its average lifetime?

   (b) An organic artefact has been discovered in an Egyptian tomb and carbon dating shows it to have an activity of 0.13 Bq per gram. What is the age of the artefact?

   (c) Use reasonable assumptions to estimate the oldest organic artefact we may age using Carbon dating.

38. Chapter 15: Alpha Decay
An isotope of plutonium, $^{239}\mathrm{Pu}$, is an alpha-emitter with a half-life of 24,120 years. What is the initial activity of 1 kg of $^{239}\mathrm{Pu}$?

39. Chapter 15: Beta Decay
What are the conditions under which the three types of β-decay are kinematically allowed? Use these conditions to determine which of the following $A = 142$ isobars would you expect to be stable, and how would you expect the others to decay. Use Sargent's rule to estimate which of the unstable ones should have the shortest lifetimes, and which the longest. Does this match with your expectation from the classification of the β^+ and β^- decays?

Nuclide	Atomic Mass / m_u	J^P
$^{142}_{57}$La	141.9141	2^-
$^{142}_{58}$Ce	141.9092	0^+
$^{142}_{59}$Pr	141.9100	2^-
$^{142}_{60}$Nd	141.9077	0^+
$^{142}_{61}$Pm	141.9130	1^+
$^{142}_{62}$Sm	141.9152	0^+

[The mass of the electron is 0.00055 m_u.]

40. Chapter 15: Fermi Theory and Sargent's Rule
Show that the electron momentum spectrum in β-decay using Fermi theory can be written as

$$\frac{d\Gamma}{dp_e} = \frac{G_F^2}{2\pi^3}\left(E_0 - E_e\right)^2 p_e^2,$$

where G_F is the Fermi constant, E_e and p_e are the energy and momentum of the electron and E_0 is the total energy released. You may treat the electron and neutrino as massless.

Show that the average kinetic energy carried off by the electron in β decay is $E_0/2$ when the electron is highly relativistic, and $E_0/3$ when the electron is non-relativistic.

When the electron is highly relativistic, show that the total decay rate is given approximately by

$$\Gamma = \frac{G_F^2 E_0^5}{60\pi^3}$$

The E_0^5 dependence is sometimes known as Sargent's Rule.

41. Chapter 15: Gamma Decay
In an experiment, the first three excited states of $^{17}_{9}$F were studied and the following gamma transitions were observed

E1 : 2.6 MeV, 4.2 MeV, 4.7 MeV
M1 : 1.6 MeV
E2 : 0.5 MeV, 1.6 MeV

In addition, a weaker transition of energy 3.1 MeV was seen. The 0.5 MeV gamma-ray corresponds to a transition between the first excited state and the ground state. Use the nuclear shell model to predict the spin-parity of the ground state of $^{17}_{9}$F. Assuming that the first excited state is a single particle excitation of a nucleon to a *nearby (not necessarily closest)* higher energy level, suggest the likely spin-parity assignment for this excited state and discuss whether this is consistent with the observed gamma transition (and lack of any others).

Draw a possible decay scheme for $^{17}_{9}$F showing the energy levels of the first three excited states, the spin-parity assignments and the gamma-ray transitions given above. Explain your reasoning clearly. What might be the likely nature of the 3.1 MeV gamma transition?

42. Chapter 16: Induced Fission
Using the SEMF, estimate the excitation energies of the $^{236}_{92}\mathrm{U}^*$ and $^{239}_{92}\mathrm{U}^*$ nuclear states formed when $^{235}_{92}\mathrm{U}$ and $^{238}_{92}\mathrm{U}$ nuclei, respectively, capture a neutron of negligible kinetic energy. Identify the term in the SEMF which is primarily responsible for the difference in the predicted excitation energies and the ground state for both these cases.

The observed excitation energies following low energy (thermal) neutron capture by $^{235}_{92}\mathrm{U}$ and $^{238}_{92}\mathrm{U}$ are approximately 6.5 MeV and 4.8 MeV, respectively. The fission activation energies for $^{235}_{92}\mathrm{U}$ and $^{238}_{92}\mathrm{U}$ are approximately 6.2 MeV and 6.6 MeV, respectively. Explain why thermal neutrons can induce rapid fission of $^{235}_{92}\mathrm{U}$ but not of $^{238}_{92}\mathrm{U}$. Discuss the implications of the energy dependence of the cross sections for neutron induced fission for the design of nuclear reactors which use uranium as a fuel.

43. Chapter 16: Moderating Neutrons for Fission
Compute the maximum fractional energy loss which a non-relativistic neutron can undergo in a single elastic collision with

i. a $^{12}_{6}\mathrm{C}$ nucleus
ii. a $^{10}_{5}\mathrm{B}$ nucleus.

For each case, calculate the minimum number of collisions which would be required in order to bring a 2.5 MeV fission neutron down to a thermal energy of 0.025 eV. What are the advantages and disadvantages of using each material as a moderator in nuclear reactions?

44. Chapter 16: Fusion
Estimate the size of the Coulomb barrier between two $^{13}_{6}\mathrm{C}$ nuclei which needs to be overcome before they can undergo fusion, and thus estimate the temperature needed to bring about fusion in this case.

Numerical answers

1) ; 2) b) 938 MeV, c) 2.2 μs, d) 1.3×10^{-9} GeV^{-2} ; 3) $E_\pi = 397$ MeV, $E_\gamma = 373$ MeV, from π $E_\gamma = 67.5$ MeV, in lab $11.8 - 385$ MeV; 4) 4.5 min, 1.2×10^{11} Bq ; 5) a) 99308 s^{-1}, b) 513 s^{-1}, c) 179 s^{-1}, d) 2×10^{-4} s^{-1} ; 6) 2.8×10^{-13} eV^{-2}, 0; 7) 0.23 mm; 8) a) 0.5% & 0.7%, 100% & 0.05%; 9) i) 10 GeV, ii) 100 GeV, 6.25 km; 10) 2 MeV, 1 MeV; 11) ; 12) ; 13) 8:5:17:∼0; 14) $\frac{10}{3}$, i) $\frac{11}{3}$ & 3, ii) 4.9 & 2; 15) a) 4, b) gq 2, $q\gamma$ 8; 16) ; 17) ; 18) a) K^+ 485 MeV, η 559 MeV, ω 780 MeV, η' 349 MeV, b) Δ 1230 MeV, Ξ 1329 MeV; 19) -1; 20) $2.38\pm0.14\,\mu_N$; 21) a) 1^-, b) 10580 MeV& 25 MeV, c) 27%; 22) 0.8 keV, 693 keV; 23) ∼3; 24) b) $\beta_e = 1$, $\beta_\mu = 0.27$; 25) ; 26) a) 69.1%, 20.5%, 3.5%, b) m_Z 91.2 GeV, Γ_Z 2.6 GeV, $\Gamma_{\tau\tau}$ 7 MeV, BR 2.9%; 27) 3×10^{-20} s; 28) b) 1:2.2:19.4, c) 20 pb ; 29) b) 0 GeV, 0.11 GeV, 3.5 GeV; 30) ; 31) b) 0.72 MeV, c) 50, 80, 114, d) a_G 5.8×10^{-37} MeV, ∼5% solar mass; 32) b) 33.3 MeV, 250 MeV, d) a_A 11 MeV; 33) 1.7 MeV, 4.1 fm; 34) ; 35) ; 36) 2.3×10^{-24} kg fm^2, 5.19×10^{-24} kg fm^2; 37) a) 8267 yrs, b) 3215 yrs, c) ∼60,000 yrs; 38) 2.27×10^{12} Bq; 39) ; 40) ; 41) 1st exc. $\frac{1}{2}^+$; 42) 6.7 MeV& 5.2 MeV; 43) i) 28% & 55, ii) 33% & 46; 44) 1.1×10^{11} K.

Suggested Tripos Questions

Relativistic Kinematics: 2016 1(c), 2014 3, 2004 (3) C12(b) (not last part)

Breit-Wigner resonances, production and decay rates: 2020 A3, 2018 A1(a), 2005 (3) A3

Feynman Diagrams: 2016 3(a), 2009 (3) A1(b), 2008 (3) A4

QCD: 2018 B3 last part, 2014 3 last part, 2013 1(b)

Hadron physics and quark model: 2018 B3, 2016 1(b), 2010 A4

Weak interaction: 2020 2, 2018 B4, 2011 3

Electroweak unification: 2018 A1(b), 2015 3, 2013 1(a)

Neutrino Oscillation: 2009 (3) A4

Semi-Empirical Mass Formula: 2018 B2, 2010 A1(a)

Nuclear Forces & Scattering: 2016 4, 2012 1(a)

Shell Model: 2020 5(a)(b), 2017 1(b), 2016 1(a)

Nuclear excitations: 2017 3 (last part), 2015 1(a)

Nuclear decay: 2004 (3) A1

α-decay: 2017 3, 2010 A1(b), 2007 A3

β-decay: 2019 3, 2016 3(b)(c)(d), 2015 4

γ-decay: 2020 5(c), 2018 A1(b), 2014 4, 2010 A3 last part

Fission and Fusion: 2011 4

Supervisions

Supervisions might follow this pattern
Supo 1: Q1-10, Chapters 1-4, covered by week 2.5 LT
Supo 2: Q11-22, Chapters 5-8, covered by week 5.5 LT
Supo 3: Q23-30, Chapters 9-12, covered by week 8 LT
Supo 4: Q31-44, Chapters 13-16, covered by week 2 ET