A mathematically more elegant regularization procedure is to use dimensional regularization, with the number of space-time dimensions now $n > 4$. Recall that this was encountered in the previous chapter as a method for performing ultraviolet regularization in the context of renormalization. Here the method is being extended to *real* gluon emission in addition to loop diagrams. Going to $n \neq 4$ dimensions affects both the phase space and the traces of the Dirac matrices in the $q\bar{q}g$ cross-section calculation. As a result, Eq. (3.11) becomes

$$\sigma^{q\bar{q}g}(\epsilon) \;=\; \sigma_0\, 3\sum_q Q_q^2\; H(\epsilon) \int \frac{dx_1 dx_2}{P(x_1,x_2)}\,\frac{2\alpha_s}{3\pi}\left[\frac{(1-\epsilon)(x_1^2+x_2^2)+2\epsilon(1-x_3)}{(1-x_1)(1-x_2)} - 2\epsilon\right] \tag{3.12}$$

with $x_3 = 2 - x_1 - x_2$, $\epsilon = (4-n)/2$, and $P(x_1,x_2) = [(1-x_1)(1-x_2)(1-x_3)]^\epsilon$

$$H(\epsilon) = \frac{3(1-\epsilon)}{(3-2\epsilon)\Gamma(2-2\epsilon)}(4\pi)^{2\epsilon} = 1 + O(\epsilon)\;, \tag{3.13}$$

where $\Gamma$ is the Euler gamma function given in Eq. (2.57). With the three-body phase space integrals recast in $n$ dimensions, the soft and collinear singularities are regulated, appearing instead as *poles* at $n = 4$. Performing the integrations in Eq. (3.12) gives

$$\sigma^{q\bar{q}g}(\epsilon) \;=\; \sigma_0\; 3\sum_q Q_q^2\,\frac{C_F\alpha_s}{2\pi}\; H(\epsilon)\;\left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + O(\epsilon)\right]\;. \tag{3.14}$$

The virtual gluon contribution can be calculated in a similar fashion, with dimensional regularization again used to control the infra-red divergences in the loops. The result is

$$\sigma^{q\bar{q}(g)}(\epsilon) \;=\; \sigma_0\; 3\sum_q Q_q^2\,\frac{C_F\alpha_s}{2\pi}\; H(\epsilon)\;\left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + O(\epsilon)\right]\;. \tag{3.15}$$

When the two contributions Eqs. (3.14) and (3.15) are added together, the poles exactly cancel and the result is *finite* in the limit $\epsilon \to 0$:

$$R \;=\; 3\sum_q Q_q^2\;\left\{1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right\}. \tag{3.16}$$