QCD Phenomenology at High Energy

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Lecture 3: DIS and Evolution Equations

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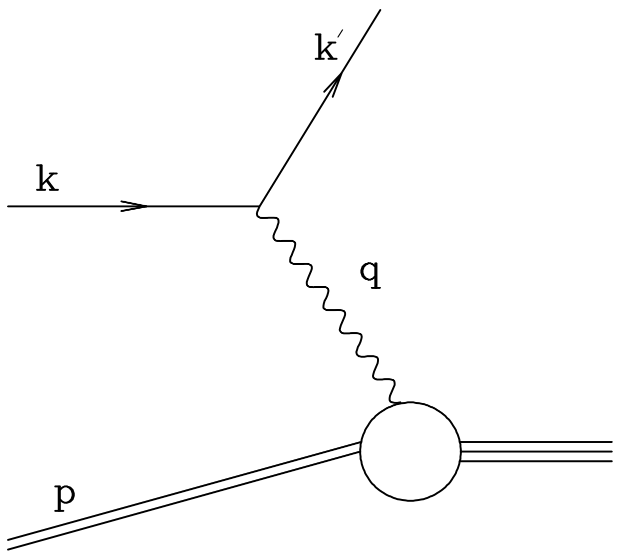
Deep Inelastic Scattering

- Consider lepton-proton scattering via exchange of virtual photon:



- Standard variables are:

$$x = \frac{-q^2}{2p \cdot q} = \frac{Q^2}{2M(E - E')}$$

$$y = \frac{q \cdot p}{k \cdot p} = 1 - \frac{E'}{E}$$

where $Q^2 = -q^2 > 0$, $M^2 = p^2$ and energies refer to target rest frame.

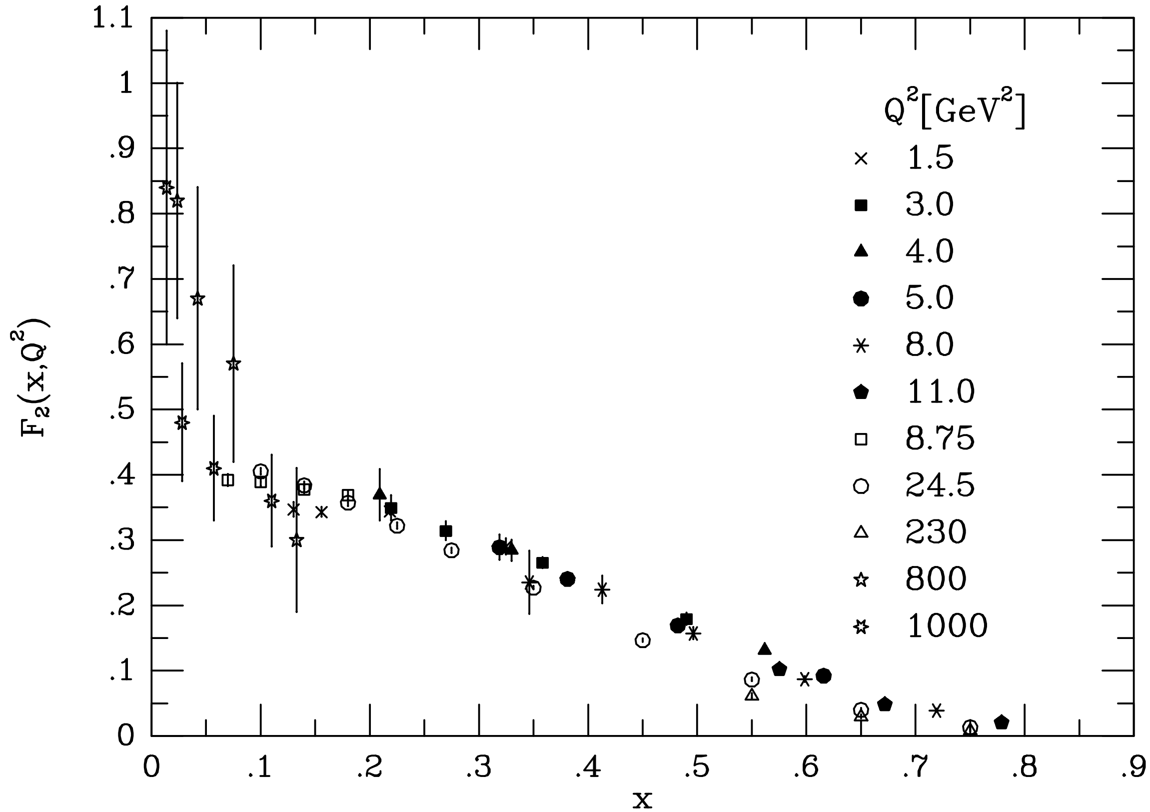
- Elastic scattering has $(p + q)^2 = M^2$, i.e. $x = 1$. Hence deep inelastic scattering (DIS) means $Q^2 \gg M^2$ and $x < 1$.

- Structure functions $F_i(x, Q^2)$ parametrise target structure as 'seen' by virtual photon. Defined in terms of cross section

$$\frac{d^2\sigma}{dxdy} = \frac{8\pi\alpha^2 ME}{Q^4}\left[\left(\frac{1+(1-y)^2}{2}\right)2xF_1 \right.$$
$$\left. +(1-y)(F_2 - 2xF_1) - (M/2E)xyF_2\right] .$$

- Bjorken limit is Q^2, $p \cdot q \rightarrow \infty$ with x fixed. In this limit structure functions obey approximate Bjorken scaling law, i.e. depend only on dimensionless variable x:

$$F_i(x, Q^2) \longrightarrow F_i(x).$$



- Figure shows F_2 structure function for proton target. Although Q^2 varies by two orders of magnitude, in first approximation data lie on universal curve.
- Bjorken scaling implies that virtual photon is scattered by *pointlike constituents* (partons) — otherwise structure functions would depend on ratio Q/Q_0, with $1/Q_0$ a length scale characterizing size of constituents.

- Parton model of DIS is formulated in a frame where target proton is moving very fast — *infinite momentum frame.*
  - Suppose that, in this frame, photon scatters from pointlike quark with fraction ξ of proton's momentum. Since $(\xi p+q)^2 = m_q^2 \ll Q^2$, we must have $\xi = Q^2/2p\cdot q = x$.
  - In terms of Mandelstam variables $\hat{s}, \hat{t}, \hat{u}$, spin-averaged matrix element squared for massless $eq \to eq$ scattering (related by crossing to $e^+e^- \to q\bar{q}$) is

$$\overline{\sum}|\mathcal{M}|^2 = 2e_q^2 e^4 \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$$

    where $\overline{\sum}$ denotes average (sum) over initial (final) colours and spins.
  - In terms of DIS variables, $\hat{t} = -Q^2$, $\hat{u} = \hat{s}(y-1)$ and $\hat{s} = Q^2/xy$. Differential cross section is then

$$\frac{d^2\hat{\sigma}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}[1+(1-y)^2]\frac{1}{2}e_q^2\delta(x-\xi).$$

  - From structure function definition (neglecting M)

$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left\{[1+(1-y)^2]F_1 + \frac{(1-y)}{x}(F_2 - 2xF_1)\right\}.$$

  - Hence structure functions for scattering from parton with momentum fraction ξ is

$$\hat{F}_2 = x e_q^2 \delta(x-\xi) = 2x\hat{F}_1 \ .$$

- Suppose probability that quark q carries momentum fraction between ξ and $\xi + d\xi$ is $q(\xi)\, d\xi$. Then

$$\begin{aligned} F_2(x) &= \sum_q \int_0^1 d\xi\ q(\xi)\ x e_q^2 \delta(x-\xi) \\ &= \sum_q e_q^2 x q(x) = 2xF_1(x)\ . \end{aligned}$$

- Relationship $F_2 = 2xF_1$ (Callan-Gross relation) follows from spin-$\frac{1}{2}$ property of quarks ($F_1 = 0$ for spin-0).

- Proton consists of three valence quarks (uud), which carry its electric charge and baryon number, and infinite sea of light $q\bar{q}$ pairs. Probed at scale Q, sea contains all quark flavours with $m_q \ll Q$. Thus at $Q \sim 1$ GeV expect

$$F_2^{em}(x) \simeq \frac{4}{9}x[u(x)+\bar{u}(x)] + \frac{1}{9}x[d(x)+\bar{d}(x)+s(x)+\bar{s}(x)]$$

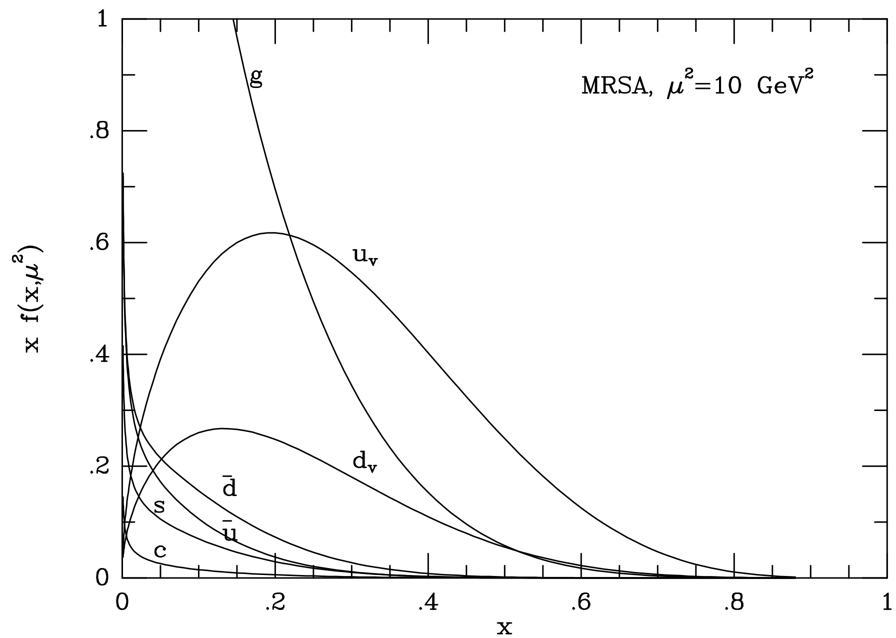
where

$$\begin{aligned} u(x) &= u_V(x) + \bar{u}(x) \\ d(x) &= d_V(x) + \bar{d}(x) \\ s(x) &= \bar{s}(x) \end{aligned}$$

with sum rules

$$\int_0^1 dx\ u_V(x) = 2\ ,\quad \int_0^1 dx\ d_V(x) = 1\ .$$

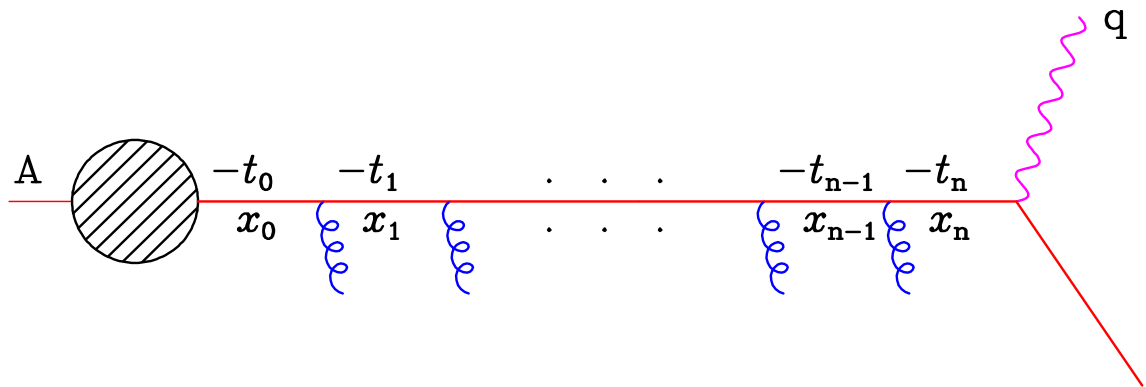
- Experimentally one finds $\sum_q \int_0^1 dx\ x[q(x) + \bar{q}(x)] \simeq 0.5$.. Thus quarks only carry about 50% of proton's momentum. Rest is carried by *gluons*. Although not directly measured in DIS, gluons participate in other hard scattering processes such as large-p_T jet and prompt photon production.



- Figure shows typical set of parton distributions extracted from fits to DIS data, at $Q^2 = 10$ GeV^2.

Scaling Violation and DGLAP Equation

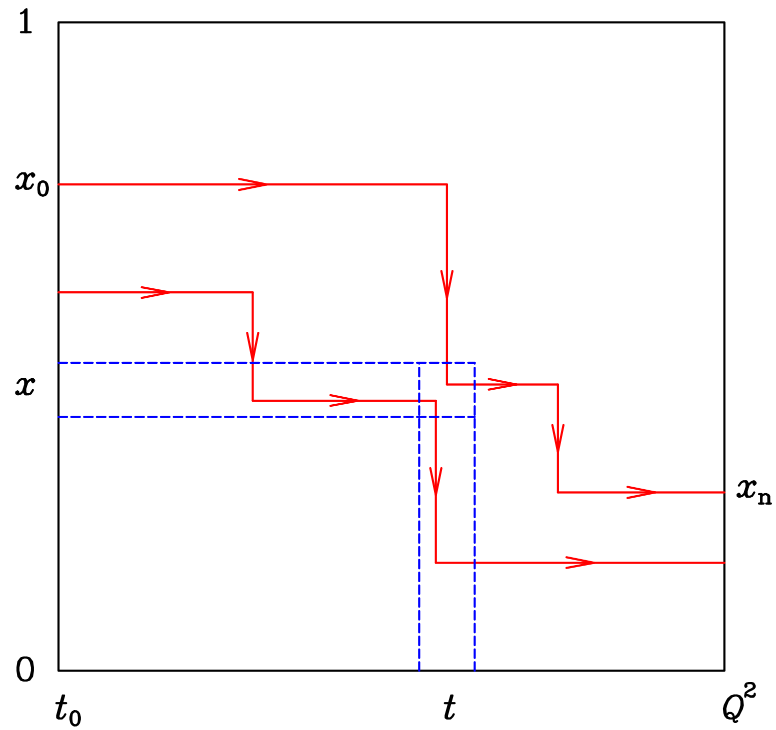
- Bjorken scaling is not exact. This is due to enhancement of higher-order contributions from small-angle parton branching, discussed earlier.



- Incoming quark from target hadron, initially with low virtual mass-squared $-t_0$ and carrying a fraction x_0 of hadron's momentum, moves to more virtual masses and lower momentum fractions by successive small-angle emissions, and is finally struck by photon of virtual mass-squared $q^2 = -Q^2$.

- Cross section will depend on Q^2 and on momentum fraction distribution of partons seen by virtual photon at this scale, $D(x, Q^2)$.

- To derive evolution equation for Q^2-dependence of $D(x, Q^2)$, first introduce pictorial representation of evolution, also useful for Monte Carlo simulation.



- Represent sequence of branchings by path in (t, x)-space. Each branching is a step downwards in x, at a value of t equal to (minus) the virtual mass-squared after the branching.
- At $t = t_0$, paths have distribution of starting points $D(x_0, t_0)$ characteristic of target hadron at that scale. Then distribution $D(x, t)$ of partons at scale t is just the x-distribution of paths at that scale.
- Consider change in the parton distribution $D(x, t)$ when t is increased to $t + \delta t$. This is number of paths arriving in element $(\delta t, \delta x)$ minus number leaving that element, divided by δx.

- Number arriving is branching probability times parton density integrated over all higher momenta $x' = x/z$,

$$\begin{aligned} \delta D_{\text{in}}(x,t) &= \frac{\delta t}{t} \int_x^1 dx'\, dz \frac{\alpha_S}{2\pi} \hat{P}(z) D(x',t)\, \delta(x - zx') \\ &= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t) \end{aligned}$$

- For the number leaving element, must integrate over lower momenta $x' = zx$:

$$\begin{aligned} \delta D_{\text{out}}(x,t) &= \frac{\delta t}{t} D(x,t) \int_0^x dx'\, dz \frac{\alpha_S}{2\pi} \hat{P}(z)\, \delta(x' - zx) \\ &= \frac{\delta t}{t} D(x,t) \int_0^1 dz \frac{\alpha_S}{2\pi} \hat{P}(z) \end{aligned}$$

- Change in population of element is

$$\begin{aligned} \delta D(x,t) &= \delta D_{\text{in}} - \delta D_{\text{out}} \\ &= \frac{\delta t}{t} \int_0^1 dz\, \frac{\alpha_S}{2\pi} \hat{P}(z) \left[\frac{1}{z} D(x/z,t) - D(x,t) \right] . \end{aligned}$$

- Introduce plus-prescription with definition

$$\int_0^1 dz\, f(z)\, g(z)_+ \;=\; \int_0^1 dz\, [f(z) - f(1)]\, g(z)\ .$$

Using this we can define regularized splitting function

$$P(z) = \hat{P}(z)_+\ ,$$

and obtain Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation:

$$t\frac{\partial}{\partial t}D(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P(z)D(x/z,t)\ .$$

Beware! Note that

$$\begin{aligned}\int_x^1 dz\, f(z)g(z)_+ &= \int_0^1 dz\, \Theta(z-x)f(z)g(z)_+ \\ &= \int_x^1 dz\, [f(z)-f(1)]g(z) - f(1)\int_0^x dz\, g(z)\end{aligned}$$

- Here $D(x,t)$ represents parton momentum fraction distribution inside incoming hadron probed at scale t. In timelike branching, it represents instead hadron momentum fraction distribution produced by an outgoing parton. Boundary conditions and direction of evolution are different, but evolution equation remains the same.

Quark and Gluon Distributions

- For several different types of partons, must take into account different processes by which parton of type i can enter or leave the element $(\delta t, \delta x)$. This leads to coupled DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}D_i(x,t) = \sum_j \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P_{ij}(z)D_j(x/z,t) \equiv \frac{\alpha_S}{2\pi}P_{ij}\otimes D_j$$

- Quark $(i = q)$ can enter element via either $q \longrightarrow qg$ or $g \longrightarrow q\bar{q}$, but can only leave via $q \longrightarrow qg$. Thus plus-prescription applies only to $q \longrightarrow qg$ part, giving

$$\begin{aligned} P_{qq}(z) &= \hat{P}_{qq}(z)_+ = C_F\left(\frac{1+z^2}{1-z}\right)_+ \\ P_{qg}(z) &= \hat{P}_{qg}(z) = T_R\left[z^2 + (1-z)^2\right] \end{aligned}$$

- Gluon can arrive either from $g \longrightarrow gg$ (2 contributions) or from $q \longrightarrow qg$ (or $\bar{q} \longrightarrow \bar{q}g$). Thus number arriving is

$$
\begin{aligned}
\delta D_{g,\text{in}} &= \frac{\delta t}{t}\int_0^1 dz \frac{\alpha_S}{2\pi}\left\{ \hat{P}_{gg}(z)\left[\frac{D_g(x/z,t)}{z} + \frac{D_g(x/(1-z),t)}{1-z}\right]\right. \\
&\quad \left. + \frac{\hat{P}_{qg}(z)}{1-z}\left[D_q\left(\frac{x}{1-z},t\right) + D_{\bar{q}}\left(\frac{x}{1-z},t\right)\right]\right\} \\
&= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}\left\{2\hat{P}_{gg}(z) D_g\left(\frac{x}{z},t\right) + \hat{P}_{qg}(1-z)\left[D_q\left(\frac{x}{z},t\right) + D_{\bar{q}}\left(\frac{x}{z},t\right)\right]\right\} ,
\end{aligned}
$$

- Gluon can leave by splitting into either gg or $q\bar{q}$, so that

$$
\delta D_{g,\text{out}} = \frac{\delta t}{t} D_g(x,t) \int_0^1 dz \frac{\alpha_S}{2\pi}\left[\hat{P}_{gg}(z) + N_f \hat{P}_{qg}(z)\, dz\right] .
$$

- After some manipulation we find

$$
\begin{aligned}
P_{gg}(z) &= 2C_A\left[\left(\frac{z}{1-z} + \tfrac{1}{2}z(1-z)\right)_+ + \frac{1-z}{z}\right. \\
&\quad \left. + \tfrac{1}{2}z(1-z)\right] - \frac{2}{3}N_f T_R\,\delta(1-z) ,
\end{aligned}
$$

$$P_{gq}(z) \quad = \quad P_{g\bar{q}}(z) = \hat{P}_{qq}(1-z) = C_F \frac{1+(1-z)^2}{z} \; .$$

- Using definition of the plus-prescription, can check that

$$\left(\frac{z}{1-z} + \tfrac{1}{2} z(1-z) \right)_+ \quad = \quad \frac{z}{(1-z)_+} + \tfrac{1}{2} z(1-z) + \frac{11}{12}\delta(1-z)$$

$$\left(\frac{1+z^2}{1-z} \right)_+ \quad = \quad \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \; ,$$

so P_{qq} and P_{gg} can be written in more common forms

$$P_{qq}(z) \quad = \quad C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right]$$

$$P_{gg}(z) \quad = \quad 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \frac{1}{6}(11C_A - 4N_f T_R)\,\delta(1-z) \; .$$

Solution by Moments

- Given $D_i(x,t)$ at some scale $t = t_0$, factorized structure of DGLAP equation means we can compute its form at any other scale.
- One strategy for doing this is to take moments (Mellin transforms) with respect to x:

$$\tilde{D}_i(N,t) = \int_0^1 dx\, x^{N-1}\, D_i(x,t)\ .$$

Inverse Mellin transform is

$$D_i(x,t) = \frac{1}{2\pi i}\int_C dN\, x^{-N}\, \tilde{D}_i(N,t)\ ,$$

where contour C is parallel to imaginary axis to right of all singularities of integrand.

- After Mellin transformation, convolution in DGLAP equation becomes simply a product:

$$t\frac{\partial}{\partial t}\tilde{D}_i(x,t) = \sum_j \gamma_{ij}(N,\alpha_S)\tilde{D}_j(N,t)$$

where moments of splitting functions give PT expansion of anomalous dimensions γ_{ij}:

$$\begin{aligned}\gamma_{ij}(N,\alpha_S) &= \sum_{n=0}^{\infty}\gamma_{ij}^{(n)}(N)\left(\frac{\alpha_S}{2\pi}\right)^{n+1}\\ \gamma_{ij}^{(0)}(N) &= \tilde{P}_{ij}(N) = \int_0^1 dz\, z^{N-1}\, P_{ij}(z)\end{aligned}$$

- From above expressions for $P_{ij}(z)$ we find

$$\begin{aligned}
\gamma_{qq}^{(0)}(N) &= C_F\left[-\frac{1}{2}+\frac{1}{N(N+1)}-2\sum_{k=2}^{N}\frac{1}{k}\right] \\
\gamma_{qg}^{(0)}(N) &= T_R\left[\frac{(2+N+N^2)}{N(N+1)(N+2)}\right] \\
\gamma_{gq}^{(0)}(N) &= C_F\left[\frac{(2+N+N^2)}{N(N^2-1)}\right] \\
\gamma_{gg}^{(0)}(N) &= 2C_A\left[-\frac{1}{12}+\frac{1}{N(N-1)}+\frac{1}{(N+1)(N+2)}\right. \\
&\quad \left.-\sum_{k=2}^{N}\frac{1}{k}\right]-\frac{2}{3}N_f T_R\ .
\end{aligned}$$

- Consider combination of parton distributions which is flavour non-singlet, e.g. $D_V = D_{q_i} - D_{\bar{q}_i}$ or $D_{q_i} - D_{q_j}$. Then mixing with the flavour-singlet gluons drops out and solution for fixed α_S is

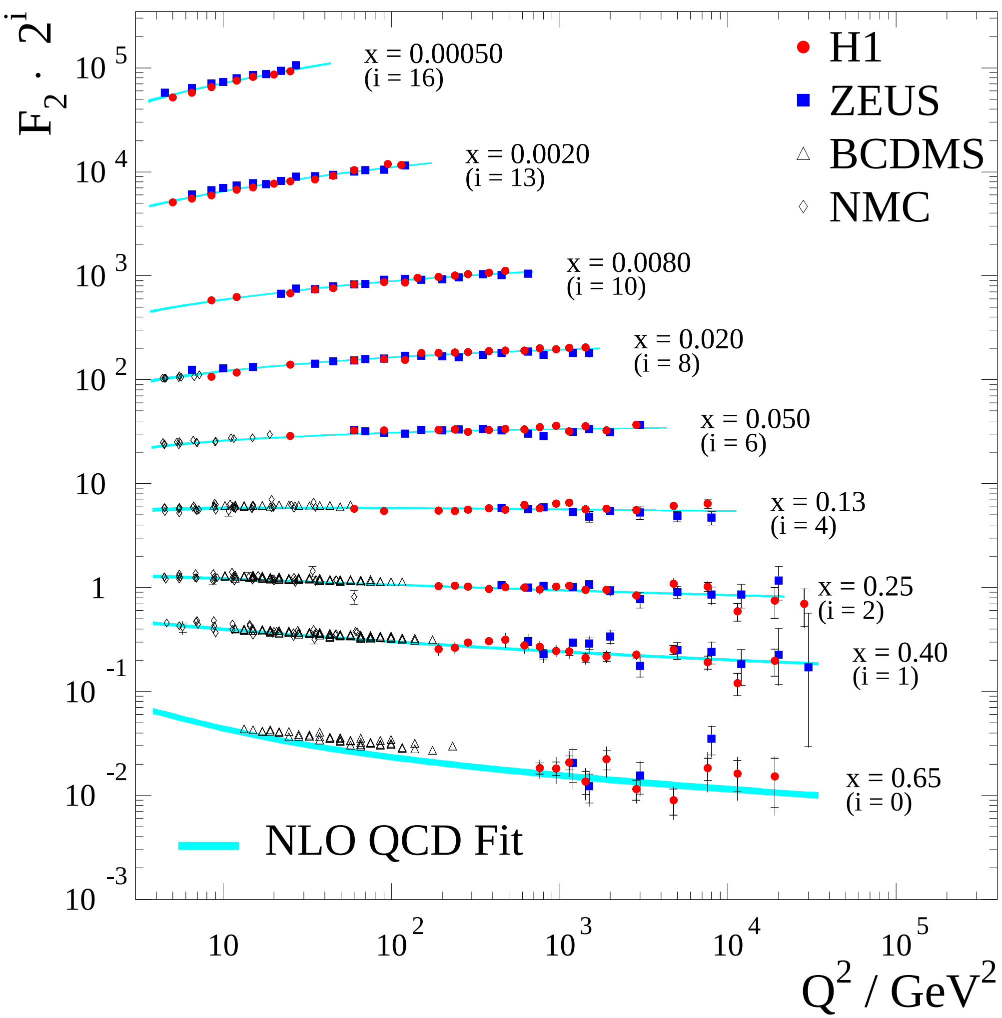
$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{t}{t_0}\right)^{\gamma_{qq}(N,\alpha_S)} ,$$

- We see that dimensionless function D_V, instead of being scale-independent function of x as expected from dimensional analysis, has scaling violation: its moments vary like powers of scale t (hence the name anomalous dimensions).
- For running coupling $\alpha_S(t)$, scaling violation is power-behaved in $\ln t$ rather than t. Using leading-order formula $\alpha_S(t) = 1/b\ln(t/\Lambda^2)$, we find

$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{\alpha_S(t_0)}{\alpha_S(t)}\right)^{d_{qq}(N)}$$

where $d_{qq}(N) = \gamma_{qq}^{(0)}(N)/2\pi b$.
- Now $d_{qq}(1) = 0$ and $d_{qq}(N) < 0$ for $N \geq 2$. Thus as t increases $\tilde{D}_V(N,t)$ is constant for $N = 1$ (*valence sum rule*) and decreases at larger N.

- Since larger-N moments emphasise larger x, this means that $D_V(x,t)$ *decreases* at large x and *increases* at small x. Physically, this is due to increase in the phase space for gluon emission by quarks as t increases, leading to loss of momentum. This is clearly visible in data:



- For flavour-singlet combination, define $\Sigma = \sum_i \left(D_{q_i} + D_{\bar{q}_i}\right)$. Then we obtain

$$
\begin{aligned}
t\frac{\partial \Sigma}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{qq}\otimes\Sigma + 2N_f P_{qg}\otimes D_g\right] \\
t\frac{\partial D_g}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{gq}\otimes\Sigma + P_{gg}\otimes D_g\right]\ .
\end{aligned}
$$

- Thus flavour-singlet quark distribution Σ mixes with gluon distribution D_g: evolution equation for moments has matrix form

$$
t\frac{\partial}{\partial t}\begin{pmatrix}\tilde{\Sigma}\\ \tilde{D}_g\end{pmatrix} = \begin{pmatrix}\gamma_{qq} & 2N_f\gamma_{qg}\\ \gamma_{gq} & \gamma_{gg}\end{pmatrix}\begin{pmatrix}\tilde{\Sigma}\\ \tilde{D}_g\end{pmatrix}
$$

- Singlet anomalous dimension matrix has two real eigenvalues $\gamma_\pm$ given by

$$
\gamma_\pm = \tfrac{1}{2}[\gamma_{gg} + \gamma_{qq} \pm \sqrt{(\gamma_{gg}-\gamma_{qq})^2 + 8N_f\gamma_{gq}\gamma_{qg}}]\ .
$$

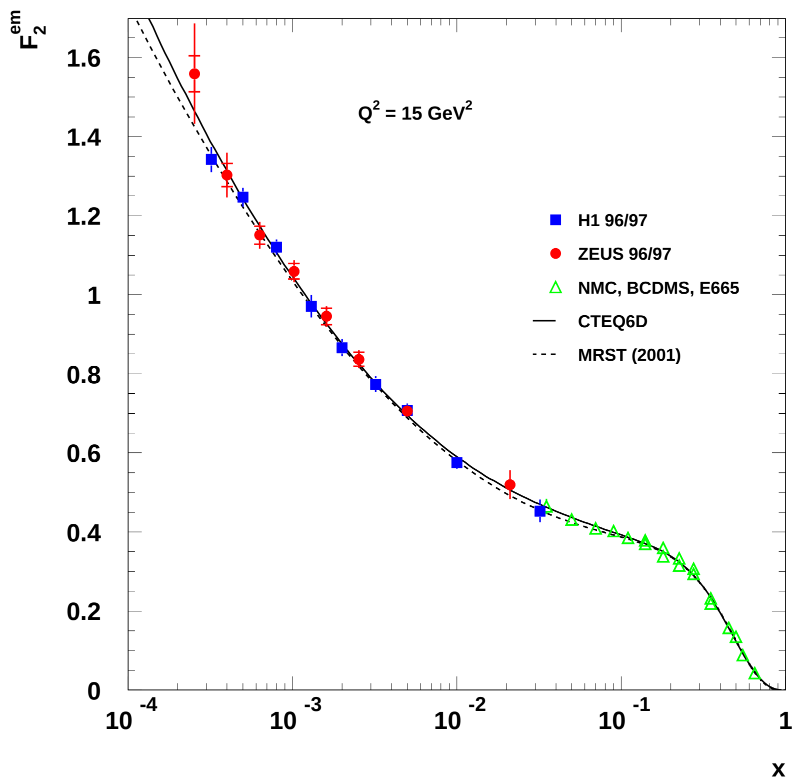
- Expressing $\tilde{\Sigma}$ and $\tilde{D}_g$ as linear combinations of eigenvectors $\tilde{\Sigma}_+$ and $\tilde{\Sigma}_-$, we find they evolve as superpositions of terms of above form with $\gamma_\pm$ in place of γ_{qq}.

Small x

- At small x, corresponding to $N \longrightarrow 1$,

$$\gamma_+ \longrightarrow \gamma_{gg} \longrightarrow \infty \ , \qquad \gamma_- \longrightarrow \gamma_{qq} \longrightarrow 0 \ ,$$

Therefore we expect structure functions to grow rapidly at small x, which is as observed:



- Higher-order corrections also become large in this region:

$$\begin{aligned}
\gamma_{qq}^{(1)}(N) &\rightarrow \frac{40C_F N_f T_R}{9(N-1)} \\
\gamma_{qg}^{(1)}(N) &\rightarrow \frac{40C_A T_R}{9(N-1)} \\
\gamma_{gq}^{(1)}(N) &\rightarrow \frac{9C_F C_A - 40C_F N_f T_R}{9(N-1)} \\
\gamma_{gg}^{(1)}(N) &\rightarrow \frac{(12C_F - 46C_A) N_f T_R}{9(N-1)} \,.
\end{aligned}$$

- Thus we find

$$\begin{aligned}
\gamma_+ &\rightarrow \frac{2C_A}{N-1}\frac{\alpha_S}{2\pi}\left[1+\frac{(26C_F-23C_A)N_f}{18C_A}\frac{\alpha_S}{2\pi}+\ldots\right] \\
&= \frac{2C_A}{N-1}\frac{\alpha_S}{2\pi}\left[1-0.64N_f\frac{\alpha_S}{2\pi}+\ldots\right]
\end{aligned}$$

where neglected terms are either non-singular at $N=1$ or higher-order in α_S. Thus NLO correction is relatively small.

- In general one finds (BFKL) that for $N \longrightarrow 1$

$$\gamma_+ \longrightarrow \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{\gamma^{(n,m)}}{(N-1)^m} \left(\frac{\alpha_S}{2\pi}\right)^n$$

Each inverse power of $(N-1)$ corresponds to a $\log x$ enhancement at small x. However, it happens that $\gamma^{(2,2)}$ and $\gamma^{(3,3)}$ are zero. This is the main reason why substantial deviations from NLO QCD are not yet seen in DIS at small x.

Parton Showers

- DGLAP equations are convenient for evolution of parton distributions. To study structure of final states, a slightly different form is useful. Consider again simplified treatment with only one type of parton branching. Introduce the Sudakov form factor:

$$\Delta(t) \equiv \exp\left[-\int_{t_0}^{t}\frac{dt'}{t'}\int dz\,\frac{\alpha_S}{2\pi}\hat{P}(z)\right] \;,$$

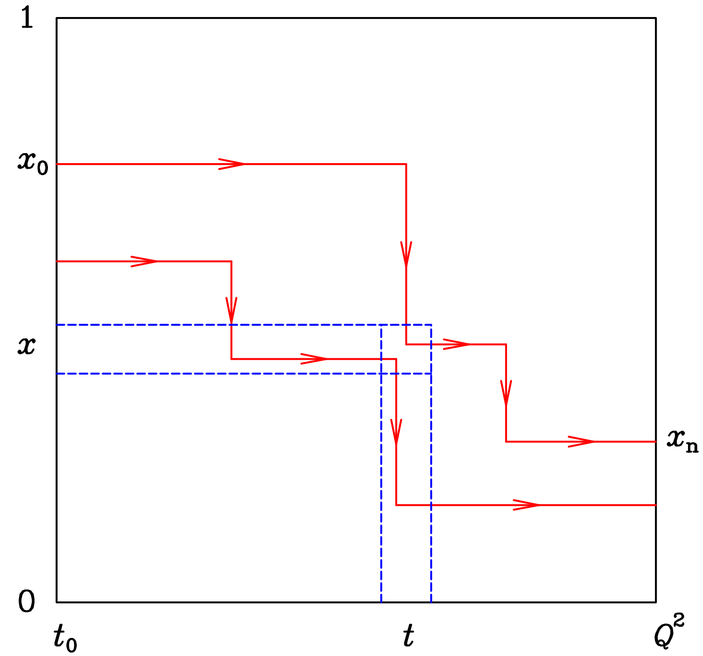
  Then

$$\begin{aligned} t\frac{\partial}{\partial t}D(x,t) &= \int\frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)D(x/z,t) + \frac{D(x,t)}{\Delta(t)}t\frac{\partial}{\partial t}\Delta(t) \;, \\ t\frac{\partial}{\partial t}\left(\frac{D}{\Delta}\right) &= \frac{1}{\Delta}\int\frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)D(x/z,t) \;. \end{aligned}$$

- This is similar to DGLAP, except D is replaced by D/Δ and regularized splitting function P replaced by unregularized $\hat{P}$. Integrating,

$$\begin{aligned} D(x,t) &= \Delta(t)D(x,t_0) \\ &\quad + \int_{t_0}^{t}\frac{dt'}{t'}\frac{\Delta(t)}{\Delta(t')}\int\frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)D(x/z,t') \;. \end{aligned}$$

- This has simple interpretation. First term is contribution from paths that do not branch between scales t_0 and t. Thus Sudakov form factor $\Delta(t)$ is probability of evolving from t_0 to t without branching. Second term is contribution from paths which have their last branching at scale t'. Factor of $\Delta(t)/\Delta(t')$ is probability of evolving from t' to t without branching.



- Generalization to several species of partons straightforward. Species i has Sudakov form factor

$$\Delta_i(t) \equiv \exp\left[-\sum_j \int_{t_0}^{t} \frac{dt'}{t'} \int dz \, \frac{\alpha_S}{2\pi} \hat{P}_{ji}(z)\right] \; ,$$

which is probability of it evolving from t_0 to t without branching. Then

$$t\frac{\partial}{\partial t}\left(\frac{D_i}{\Delta_i}\right) = \frac{1}{\Delta_i}\sum_j \int \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}_{ij}(z) D_j(x/z,t) \; .$$

Infrared Cutoff

- In DGLAP equation, infrared singularities of splitting functions at $z = 1$ are regularized by plus-prescription. However, in above form we must introduce an explicit infrared cutoff, $z < 1 - \epsilon(t)$. Branchings with z above this range are unresolvable: emitted parton is too soft to detect. Sudakov form factor with this cutoff is probability of evolving from t_0 to t without any resolvable branching.
- Sudakov form factor sums enhanced virtual (parton loop) as well as real (parton emission) contributions. No-branching probability is the sum of virtual and unresolvable real contributions: both are divergent but their sum is finite.
- Infrared cutoff $\epsilon(t)$ depends on what we classify as resolvable emission. For timelike branching, natural resolution limit is given by cutoff on parton virtual mass-squared, $t > t_0$. When parton energies are much larger than virtual masses, transverse momentum in $a \to bc$ is

$$p_T^2 = z(1-z)p_a^2 - (1-z)p_b^2 - zp_c^2 > 0 \ .$$

Hence for $p_a^2 = t$ and $p_b^2,\ p_c^2 > t_0$ we require

$$z(1-z) > t_0/t \ ,$$

that is,

$$z,\ 1-z > \epsilon(t) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-4t_0/t} \simeq t_0/t \ .$$

- Quark Sudakov form factor is then

$$\Delta_q(t) \simeq \exp\left[-\int_{2t_0}^{t}\frac{dt'}{t'}\int_{t_0/t'}^{1-t_0/t'} dz\frac{\alpha_S}{2\pi}\hat{P}_{qq}(z)\right] \; .$$

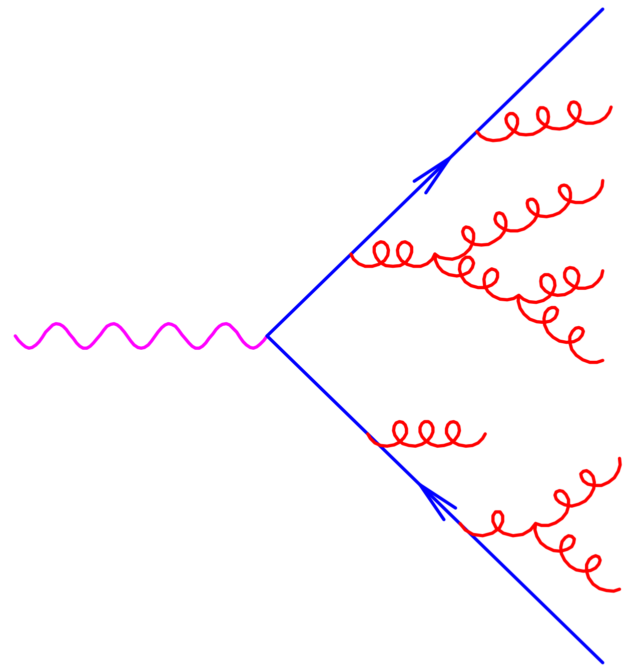
- Careful treatment of running coupling suggests its argument should be $p_T^2 \sim z(1-z)t'$. Then at large t

$$\Delta_q(t) \sim \left(\frac{\alpha_S(t)}{\alpha_S(t_0)}\right)^{p\ln t} \; ,$$

($p =$ a constant), which tends to zero faster than any negative power of t.

- Infrared cutoff discussed here follows from kinematics. We shall see later that QCD dynamics effectively reduces phase space for parton branching, leading to a more restrictive effective cutoff.

- Each emitted (timelike) parton can itself branch. In that case t evolves downwards towards cutoff value t_0, rather than upwards towards hard process scale Q^2. Due to successive branching, a parton cascade or shower develops. Each outgoing line is source of new cascade, until all outgoing lines have stopped branching. At this stage, which depends on cutoff scale t_0, outgoing partons have to be converted into hadrons via a hadronization model.



- Figure shows (schematically) a typical parton shower in $Z^0 \longrightarrow$ hadrons: for a resolution scale $t_0 \sim 1$ GeV^2, about 7 gluons are emitted.

Soft Gluon Coherence

- Parton branching formalism discussed so far takes account of collinear enhancements to all orders in PT. There are also soft enhancements: When external line with momentum p and mass m (not necessarily small) emits gluon with momentum q, propagator factor is

$$\frac{1}{(p \pm q)^2 - m^2} = \frac{\pm 1}{2p \cdot q} = \frac{\pm 1}{2\omega E(1 - v\cos\theta)}$$

where ω is emitted gluon energy, E and v are energy and velocity of parton emitting it, and θ is angle of emission. This diverges as $\omega \to 0$, for any velocity and emission angle.

- Including numerator, soft gluon emission gives a colour factor times universal, spin-independent factor in amplitude

$$F_{\mathrm{soft}} = \frac{p \cdot \epsilon}{p \cdot q}$$

where ϵ is polarization of emitted gluon. For example, emission from quark gives numerator factor $N \cdot \epsilon$, where

$$\begin{aligned} N^\mu &= (\not p + \not q + m)\gamma^\mu u(p) \underset{\omega \to 0}{\longrightarrow} (\gamma^\nu \gamma^\mu p_\nu + \gamma^\mu m) u(p) \\ &= (2p^\mu - \gamma^\mu \not p + \gamma^\mu m) u(p) = 2p^\mu u(p) \;. \end{aligned}$$

(using Dirac equation for on-mass-shell spinor $u(p)$).

- Universal factor F_{soft} coincides with classical eikonal formula for radiation from current p^μ, valid in long-wavelength limit.

- No soft enhancement of radiation from off-mass-shell internal lines, since associated denominator factor $(p+q)^2 - m^2 \to p^2 - m^2 \neq 0$ as $\omega \to 0$.
- Enhancement factor in amplitude for each external line implies cross section enhancement is sum over all pairs of external lines $\{i, j\}$:

$$d\sigma_{n+1} = d\sigma_n \frac{d\omega}{\omega} \frac{d\Omega}{2\pi} \frac{\alpha_S}{2\pi} \sum_{i,j} C_{ij} W_{ij}$$

where $d\Omega$ is element of solid angle for emitted gluon, C_{ij} is a colour factor, and radiation function W_{ij} is given by

$$W_{ij} = \frac{\omega^2 p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q} = \frac{1 - v_i v_j \cos\theta_{ij}}{(1 - v_i \cos\theta_{iq})(1 - v_j \cos\theta_{jq})} \, .$$

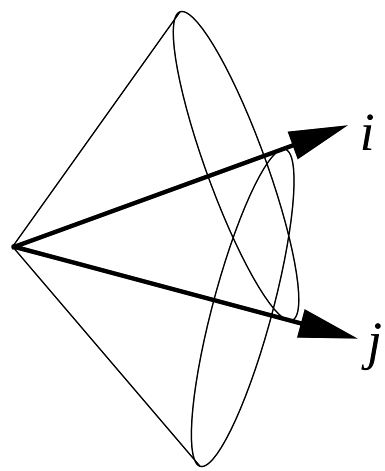
Colour-weighted sum of radiation functions $C_{ij} W_{ij}$ is antenna pattern of hard process.

- Radiation function can be separated into two parts containing collinear singularities along lines i and j. Consider for simplicity massless particles, $v_{i,j} = 1$. Then $W_{ij} = W_{ij}^i + W_{ij}^j$ where

$$W_{ij}^i = \frac{1}{2}\left(W_{ij} + \frac{1}{1-\cos\theta_{iq}} - \frac{1}{1-\cos\theta_{jq}}\right) .$$

- This function has remarkable property of angular ordering. Write angular integration in polar coordinates w.r.t. direction of i, $d\Omega = d\cos\theta_{iq}\, d\phi_{iq}$. Performing azimuthal integration, we find

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} W_{ij}^i = \frac{1}{1-\cos\theta_{iq}} \quad \text{if } \theta_{iq} < \theta_{ij}, \text{ otherwise } 0.$$



Thus, after azimuthal averaging, contribution from W_{ij}^i is confined to cone, centred on direction of i, extending in angle to direction of j. Similarly, W_{ij}^j, averaged over ϕ_{jq}, is confined to cone centred on line j extending to direction of i.

Angular Ordering

- To prove angular ordering property, write

$$1 - \cos\theta_{jq} = a - b\cos\phi_{iq}$$

where

$$a = 1 - \cos\theta_{ij}\cos\theta_{iq}\ , \qquad b = \sin\theta_{ij}\sin\theta_{iq}\ .$$

Defining $z = \exp(i\phi_{iq})$, we have

$$I_{ij}^{i} \equiv \int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{1-\cos\theta_{jq}} = \frac{1}{i\pi b}\oint \frac{dz}{(z-z_+)(z-z_-)}$$

where z-integration contour the unit circle and

$$z_\pm = \frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}\ .$$

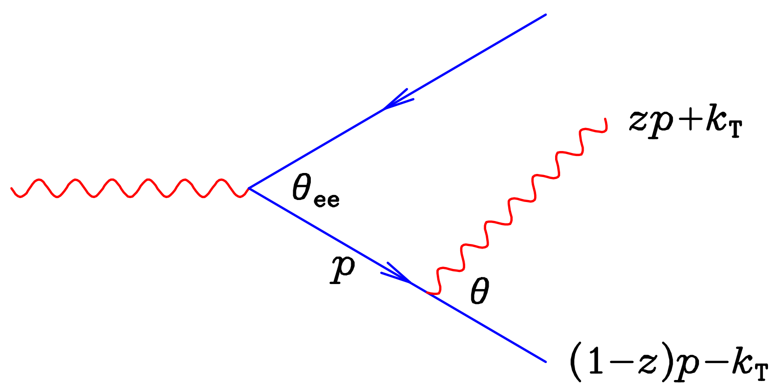
Now only pole at $z = z_-$ can lie inside unit circle, so

$$I_{ij}^{i} = \sqrt{\frac{1}{a^2 - b^2}} = \frac{1}{|\cos\theta_{iq} - \cos\theta_{ij}|}\ .$$

Hence

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} W_{ij}^i = \frac{1}{2(1-\cos\theta_{iq})}[1 + (\cos\theta_{iq} - \cos\theta_{ij}) I_{ij}^i]$$
$$= \frac{1}{1-\cos\theta_{iq}} \quad \text{if } \theta_{iq} < \theta_{ij}, \text{ otherwise } 0.$$

- Angular ordering is coherence effect common to all gauge theories. In QED it causes Chudakov effect – suppression of soft bremsstrahlung from e^+e^- pairs, which has simple explanation in old-fashioned (time-ordered) perturbation theory.



- ❖ Consider emission of soft photon at angle θ from electron in pair with opening angle $\theta_{ee} < \theta$. For simplicity assume $\theta_{ee}, \theta \ll 1$.
- ❖ Transverse momentum of photon is $k_T \sim zp\theta$ and energy imbalance at $e \to e\gamma$ vertex is

$$\Delta E \sim k_T^2/zp \sim zp\theta^2 \; .$$

- Time available for emission is $\Delta t \sim 1/\Delta E$. In this time transverse separation of pair will be $\Delta b \sim \theta_{ee}\Delta t$.
- For non-negligible probability of emission, photon must resolve this transverse separation of pair, so

$$\Delta b > \lambda/\theta \sim (zp\theta)^{-1}$$

  where λ is photon wavelength.
- This implies that

$$\theta_{ee}(zp\theta^2)^{-1} > (zp\theta)^{-1} \, ,$$

  and hence $\theta_{ee} > \theta$. Thus soft photon emission is suppressed at angles larger than opening angle of pair, which is angular ordering.
- Photons at larger angles cannot resolve electron and positron charges separately – they see only total charge of pair, which is zero, implying no emission.

- More generally, if i and j come from branching of parton k, with (colour) charge $\boldsymbol{Q}_k = \boldsymbol{Q}_i + \boldsymbol{Q}_k$, then radiation outside angular-ordered cones is emitted coherently by i and j and can be treated as coming directly from (colour) charge of k.

Coherent Branching

- Angular ordering provides basis for coherent parton branching formalism, which includes leading soft gluon enhancements to all orders.
- In place of virtual mass-squared variable t in earlier treatment, use angular variable

$$\zeta = \frac{p_b \cdot p_c}{E_b\, E_c} \simeq 1 - \cos\theta$$

  as evolution variable for branching $a \to bc$, and impose angular ordering $\zeta' < \zeta$ for successive branchings. Iterative formula for n-parton emission becomes

$$d\sigma_{n+1} = d\sigma_n \frac{d\zeta}{\zeta} dz \frac{\alpha_S}{2\pi} \hat{P}_{ba}(z)\ .$$

- In place of virtual mass-squared cutoff t_0, must use angular cutoff ζ_0 for coherent branching. This is to some extent arbitrary, depending on how we classify emission as unresolvable. Simplest choice is $\zeta_0 = t_0/E^2$ for parton of energy E.
- For radiation from particle i with finite mass-squared t_0, radiation function becomes

$$\omega^2 \left(\frac{p_i \cdot p_j}{p_i \cdot q\, p_j \cdot q} - \frac{p_i^2}{(p_i \cdot q)^2} \right) \simeq \frac{1}{\zeta} \left(1 - \frac{t_0}{E^2 \zeta} \right)\ ,$$

  so angular distribution of radiation is cut off at $\zeta = t_0/E^2$. Thus t_0 can still be interpreted as minimum virtual mass-squared.

- With this cutoff, most convenient definition of evolution variable is not ζ itself but rather

$$\tilde{t} = E^2\zeta \geq t_0 \ .$$

Angular ordering condition $\zeta_b, \zeta_c < \zeta_a$ for timelike branching $a \to bc$ (a outgoing) becomes

$$\tilde{t}_b < z^2\tilde{t} \ , \quad \tilde{t}_c < (1-z)^2\tilde{t}$$

where $\tilde{t} = \tilde{t}_a$ and $z = E_b/E_a$. Thus cutoff on z becomes

$$\sqrt{t_0/\tilde{t}} < z < 1 - \sqrt{t_0/\tilde{t}} \ .$$

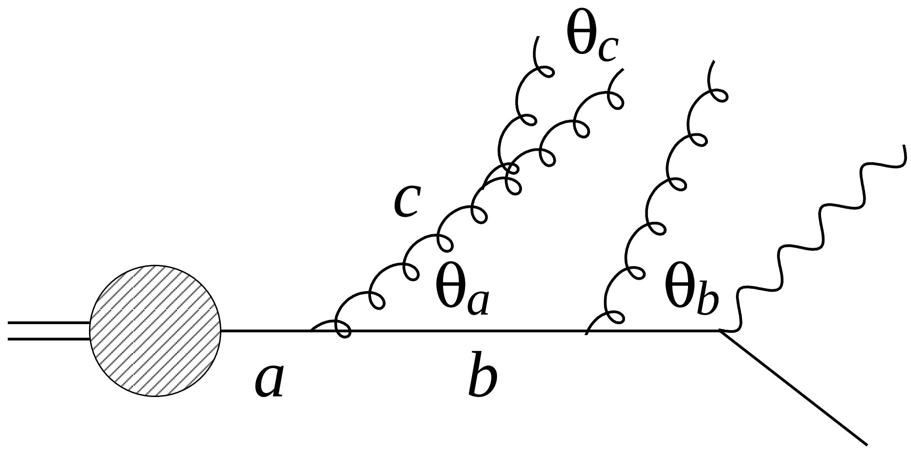
- Neglecting masses of b and c, virtual mass-squared of a and transverse momentum of branching are

$$t = z(1-z)\tilde{t} \ , \quad p_t^2 = z^2(1-z)^2\tilde{t} \ .$$

Thus for coherent branching Sudakov form factor of quark becomes

$$\tilde{\Delta}_q(\tilde{t}) = \exp\left[-\int_{4t_0}^{\tilde{t}} \frac{dt'}{t'} \int_{\sqrt{t_0/t'}}^{1-\sqrt{t_0/t'}} \frac{dz}{2\pi}\alpha_S(z^2(1-z)^2t')\hat{P}_{qq}(z)\right]$$

At large $\tilde{t}$ this falls more slowly than form factor without coherence, due to the suppression of soft gluon emission by angular ordering.



- Note that for spacelike branching $a \to bc$ (a incoming, b spacelike), angular ordering condition is

$$\theta_b > \theta_a > \theta_c \ .$$

However, kinematics implies $E_b\theta_b > E_a\theta_a$ and in this case $E_b < E_a$, so angular ordering does not impose an extra constraint on branching. Therefore gluon emission is not suppressed by coherence in spacelike branching.

  - This permits the rapid rise of structure functions at small x.
  - We shall see that the production of low-momentum hadrons in *jet fragmentation*, controlled by timelike branching at small x, is quite different – strongly suppressed by QCD coherence.

Summary of Lecture 3

- Deep inelastic lepton scattering (DIS) reveals parton structure of hadrons.
  - Pointlike constituents $\Rightarrow$ Bjorken scaling.
  - Sum rules reveal properties of partons.
  - Gluons inferred from missing momentum.
- Logarithmic violation of Bjorken scaling follows from QCD.
  - Leading contribution due to multiple small-angle parton branching..
- Parton distributions evolve according to DGLAP equations.
  - These involve convolutions $\Rightarrow$ solve by taking moments (x^{N-1})
  - Divergences as $N \longrightarrow 1$ lead to rapid increase in parton distributions at small x.
- Emitted partons can also branch, leading to parton showers.
  - Showers determine broad structure of final state.
  - Sudakov form factor gives probability of evolution without resolvable branching.
  - Can follow parton showers until evolution scale becomes too low for perturbation theory $\Rightarrow$ infrared cutoff. Then need hadronization model.
- Soft gluon emission also gives enhanced higher-order contributions.
  - Must sum emission from different partons coherently.
  - Main effect of coherence is angular ordering $\Rightarrow$ use angular evolution variable.
  - Soft gluon emission suppressed.
  - Not a major effect in DIS (initial-state showers).