QCD Phenomenology at High Energy

Bryan Webber

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Lecture 2: $e^+e^-$, NLO & Parton Branching

- $e^+e^-$
  - Annihilation cross section
  - Shape distributions
  - Resummation and matching
  - Jet fractions
- NLO QCD Calculations
  - Phase space slicing
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- Parton Branching
  - Kinematics
  - Splitting functions
  - Phase space
  - 4-jet angular distribution
$e^+e^- \rightarrow \mu^+\mu^-$ is a fundamental electroweak processes. Same type of process, $e^+e^- \rightarrow q\bar{q}$, will produce hadrons. Cross sections are roughly proportional.

Since formation of hadrons is non-perturbative, how can PT give hadronic cross section? This can be understood by visualizing event in space-time:

- $e^+$ and $e^-$ collide to form $\gamma$ or $Z^0$ with virtual mass $Q = \sqrt{s}$. This fluctuates into $q\bar{q}$, $q\bar{q}g, \ldots$, occupy space-time volume $\sim 1/Q$. At large $Q$, rate for this short-distance process given by PT.
Subsequently, at much later time $\sim 1/\Lambda$, produced quarks and gluons form hadrons. This modifies outgoing state, but occurs too late to change original probability for event to happen.

Well below $Z^0$, process $e^+ e^- \rightarrow f \bar{f}$ is purely electromagnetic, with lowest-order (Born) cross section (neglecting quark masses)

$$\sigma_0 = \frac{4\pi\alpha^2}{3s} Q_f^2$$

Thus ($3 = N =$ number of possible $q\bar{q}$ colours)

$$R \equiv \frac{\sigma(e^+ e^- \rightarrow \text{hadrons})}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-)} = \frac{\sum_q \sigma(e^+ e^- \rightarrow q\bar{q})}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-)} = 3 \sum_q Q_q^2.$$ 

On $Z^0$ pole, $\sqrt{s} = M_Z$, neglecting $\gamma/Z$ interference

$$\sigma_0 = \frac{4\pi\alpha^2\kappa^2}{3\Gamma_Z^2} (a_e^2 + v_e^2) (a_f^2 + v_f^2)$$
where \( \kappa = \sqrt{2} G_F M_Z^2 / 4\pi \alpha = 1 / \sin^2(2\theta_W) \simeq 1.5 \). Hence

\[
R_Z = \frac{\Gamma(Z \rightarrow \text{hadrons})}{\Gamma(Z \rightarrow \mu^+\mu^-)} = \frac{\sum_q \Gamma(Z \rightarrow q\bar{q})}{\Gamma(Z \rightarrow \mu^+\mu^-)} = \frac{3 \sum_q (a_q^2 + v_q^2)}{a^2_\mu + v^2_\mu}
\]

- Measured cross section is about 5% higher than \( \sigma_0 \), due to QCD corrections. For massless quarks, corrections to \( R \) and \( R_Z \) are equal. To \( \mathcal{O}(\alpha_S) \) we have:

- Real emission diagrams (b):
  - Write 3-body phase-space integration as
  \[
d\Phi_3 = [...] d\alpha d\beta d\gamma dx_1 dx_2 ,
\]
  \( \alpha, \beta, \gamma \) are Euler angles of 3-parton plane,
\[ x_1 = 2p_1 \cdot q/q^2 = 2E_q/\sqrt{s}, \]
\[ x_2 = 2p_2 \cdot q/q^2 = 2E_q/\sqrt{s}. \]

- Applying Feynman rules and integrating over Euler angles:

\[ \sigma^{q\bar{q}g} = 3\sigma_0 C_F \frac{\alpha_s}{2\pi} \int dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}. \]

Integration region: \( 0 \leq x_1, x_2, x_3 \leq 1 \) where
\[ x_3 = 2k \cdot q/q^2 = 2E_g/\sqrt{s} = 2 - x_1 - x_2. \]

- Integral divergent at \( x_{1,2} = 1 \):

\[
1 - x_1 = \frac{1}{2}x_2x_3(1 - \cos \theta_{qg})
\]
\[
1 - x_2 = \frac{1}{2}x_1x_3(1 - \cos \theta_{\bar{q}g})
\]

Divergences: **collinear** when \( \theta_{qg} \to 0 \) or \( \theta_{\bar{q}g} \to 0 \); **soft** when \( E_g \to 0 \), i.e. \( x_3 \to 0 \). Singularities are not physical – simply indicate breakdown of PT when energies and/or invariant masses approach QCD scale \( \Lambda \).

- Collinear and/or soft regions do not in fact make important contribution to \( R \). To see this, make integrals finite using dimensional regularization, \( D = 4 - 2\epsilon \) with \( \epsilon < 0 \). Then

\[
\sigma^{q\bar{q}g} = 2\sigma_0 \frac{\alpha_s}{\pi} H(\epsilon) \int dx_1 dx_2 \frac{(1 - \epsilon)(x_1^2 + x_2^2) + \epsilon(1 - x_3)}{(1 - x_3)^{\epsilon - 1}[(1 - x_1)(1 - x_2)]^{1+\epsilon}}
\]

where

\[
H(\epsilon) = \frac{3(1 - \epsilon)(4\pi)^{2\epsilon}}{(3 - 2\epsilon)\Gamma(2 - 2\epsilon)} = 1 + \mathcal{O}(\epsilon).
\]
Hence
\[
\sigma^{q\bar{q}g} = 2\sigma_0 \frac{\alpha_s}{\pi} H(\epsilon) \left[ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + O(\epsilon) \right].
\]

- Soft and collinear singularities are regulated, appearing instead as poles at \( D = 4 \).
- **Virtual gluon** contributions (a): using dimensional regularization again

\[
\sigma^{q\bar{q}} = 3\sigma_0 \left\{ 1 + \frac{2\alpha_s}{3\pi} H(\epsilon) \left[ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + O(\epsilon) \right] \right\}.
\]

- Adding real and virtual contributions, poles cancel and result is finite as \( \epsilon \to 0 \):

\[
R = 3 \sum_{q} Q_q^2 \left\{ 1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right\}.
\]

Thus \( R \) is an **infrared safe** quantity.

- Coupling \( \alpha_s \) evaluated at renormalization scale \( \mu \). UV divergences in \( R \) cancel to \( O(\alpha_s) \), so coefficient of \( \alpha_s \) independent of \( \mu \). At \( O(\alpha_s^2) \) and higher, UV divergences make coefficients renormalization scheme dependent:

\[
R = 3 K_{QCD} \sum_{q} Q_q^2,
\]

\[
K_{QCD} = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \sum_{n \geq 2} C_n \left( \frac{s}{\mu^2} \right) \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^n
\]
In $\overline{\text{MS}}$ scheme with scale $\mu = \sqrt{s}$,

$$C_2(1) = \frac{365}{24} - 11\zeta(3) - [11 - 8\zeta(3)]\frac{N_f}{12}$$

$$\simeq 1.986 - 0.115N_f$$

Coefficient $C_3$ is also known.

Scale dependence of $C_2$, $C_3$ ... fixed by requirement that, order-by-order, series should be independent of $\mu$. For example

$$C_2\left(\frac{s}{\mu^2}\right) = C_2(1) - \frac{\beta_0}{4}\log\frac{s}{\mu^2}$$

where $\beta_0 = 4\pi b = 11 - 2N_f/3$.

Scale and scheme dependence only cancels completely when series is computed to all orders. Scale change at $O(\alpha_S^n)$ induces changes at $O(\alpha_S^{n+1})$. The more terms are added, the more stable is prediction with respect to changes in $\mu$. 

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Residual scale dependence is an important source of uncertainty in QCD predictions. One can vary scale over some ‘physically reasonable’ range, e.g. $\sqrt{s}/2 < \mu < 2\sqrt{s}$, to try to quantify this uncertainty, but there is no real substitute for a full higher-order calculation.
\textbf{\(e^+e^-\) Shape Distributions}

- **Shape variables** measure some aspect of shape of hadronic final state, e.g. whether it is pencil-like, planar, spherical etc.

- For \(d\sigma/dX\) to be calculable in PT, shape variable \(X\) should be infrared safe, i.e. insensitive to emission of soft or collinear particles. In particular, \(X\) must be invariant under \(p_i \rightarrow p_j + p_k\) whenever \(p_j\) and \(p_k\) are parallel or one of them goes to zero.

- Examples are **Thrust** and **C-parameter**:

\[
T = \max \frac{\sum_i |p_i \cdot n|}{\sum_i |p_i|} \\
C = 3 \frac{\sum_{i,j} |p_i| |p_j| \sin^2 \theta_{ij}}{2 (\sum_i |p_i|)^2}
\]

After maximization, unit vector \(n\) defines *thrust axis*.

- In Born approximation final state is \(q\bar{q}\) and \(1 - T = C = 0\). Non-zero contribution at \(\mathcal{O}(\alpha_S)\) comes from \(e^+e^- \rightarrow q\bar{q}g\). Recall distribution of \(x_i = 2E_i/\sqrt{s}\):

\[
\frac{1}{\sigma} \frac{d^2\sigma}{dx_1 dx_2} = C_F \frac{\alpha_S}{2\pi} \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}.
\]

Distribution of shape variable \(X\) is obtained by integrating over \(x_1\) and \(x_2\) with constraint \(\delta(X - f_X(x_1, x_2, x_3 = 2 - x_1 - x_2))\), i.e. along contour of constant \(X\) in \((x_1, x_2)\)-plane.
For thrust, \( f_T = \max\{x_1, x_2, x_3\} \) and we find

\[
\frac{1}{\sigma} \frac{d\sigma}{dT} = C_F \frac{\alpha_S}{2\pi} \left[ \frac{2(3T^2 - 3T + 2)}{T(1-T)} \log \left( \frac{2T-1}{1-T} \right) \right.
\]

\[
\left. - \frac{3(3T-2)(2-T)}{(1-T)} \right].
\]

This diverges as \( T \to 1 \), due to soft and collinear gluon singularities. Virtual gluon contribution is negative and proportional to \( \delta(1-T) \), such that correct total cross section is obtained after integrating over \( \frac{2}{3} \leq T \leq 1 \), the physical region for two- and three-parton final states.

Corrections up to \( \mathcal{O}(\alpha_S^3) \) are known. Comparisons with data provide test of QCD matrix elements, through shape of distribution, and measurement of \( \alpha_S \), from overall rate. Care must be taken near \( T = 1 \) where (a) hadronization effects become large, and (b) large higher-order terms of the form \( \alpha_S^n \log^{2n-1}(1-T)/(1-T) \) appear in \( \mathcal{O}(\alpha_S^n) \).
Figure shows thrust distribution measured at LEP1 (DELPHI data) compared with LO theory for vector gluon (solid) or scalar gluon (dashed).

To describe event shape distributions over a wider range, we must include higher-order corrections and resum leading and next-to-leading logarithms of \((1 - T')\) to all orders (NNLRA).
Resummation and Matching

For resummation, it is convenient to introduce the event shape fraction

\[ f(\tau) = \int_{1-\tau}^{1} dT \frac{1}{\sigma} \frac{d\sigma}{dT}. \]

This quantity satisfies exponentiation, by which we mean that

\[ f(\tau) = C(\alpha_s) \exp G(\alpha_s, L) + D(\alpha_s, \tau) \]

where \( L = \ln(1/\tau) \), \( C(\alpha_s) \) is a power series in \( \alpha_s \),

\[ G(\alpha_s, L) = \sum_{n=1}^{\infty} \sum_{m=1}^{n+1} G_{nm} \left( \frac{\alpha_s}{2\pi} \right)^n L^m \]

\[ \equiv L g_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \cdots \]

and the remainder \( D(\alpha_s, \tau) \) vanishes as \( \tau \to 0 \). (We suppress dependence on renormalization scale \( \mu \) for the moment.)

Whereas the event fraction itself has up to two factors of \( L \) for each power of \( \alpha_s \), its logarithm has only one extra factor of \( L \) for each \( \alpha_s \). The double logs come purely from the expansion of the exponential function.
The function $g_1(u = \alpha_S L)$ that resums leading logs is

$$g_1(u) = -\frac{C_F}{\pi b^2 u} [(1 - 2bu) \ln(1 - 2bu) - 2(1 - bu) \ln(1 - bu)] .$$

where $b$, the first $\beta$-function coefficient, is $(33 - 2N_f)/12\pi$.

At small $u$, $g_1(u) \sim -C_F u/\pi$, giving

$$f(\tau) \sim \exp(-\alpha_S C_F L^2 / \pi)$$

in the limit $\alpha_S L \ll 1$. We see that the dominant effect of resummation is to suppress the event fraction at small $\tau$ (large $L$), leading to a turn-over instead of a divergence in the distribution at high thrust.

The NLL function $g_2(u)$ is also known. It has a dependence on the renormalization scale $\mu$,

$$g_2(u, \mu) = g_2(u, Q) - 2bu^2 \frac{dg_1}{du} \ln \left( \frac{Q}{\mu} \right) ,$$

which cancels the NLL scale dependence of $g_1(\alpha_S L)$.

To match the NLLA resummed shape fraction to the NLO fixed order prediction without double counting, simplest procedure is the so-called log matching scheme, in which one writes

$$\ln f(\tau) = K(\alpha_S) + G(\alpha_S, L) + H(\alpha_S, \tau)$$

where $K(\alpha_S)$ is a power series in $\alpha_S$ and $H(\alpha_S, \tau)$ is a remainder that vanishes as $\tau \to 0$. 

Writing the NLO prediction as

\[ f(\tau) = 1 + \frac{\alpha_S}{2\pi} A(\tau) + \left( \frac{\alpha_S}{2\pi} \right)^2 B(\tau) + O(\alpha_S^3) , \]

we have

\[ \ln f(\tau) = \frac{\alpha_S}{2\pi} A(\tau) + \left( \frac{\alpha_S}{2\pi} \right)^2 \left\{ B(\tau) - \frac{1}{2}[A(\tau)]^2 \right\} + O(\alpha_S^3) . \]

To match the predictions to NLO, we should add \( G(\alpha_S, L) \) to this expression after subtracting its first- and second-order parts, which are already included in \( A(\tau) \) and \( B(\tau) \). Hence the resummed prediction with \( K(\alpha_S) \) and \( H(\alpha_S, \tau) \) evaluated to second order is

\[
\begin{align*}
\ln f(\tau) &= L g_1(\alpha_S L) + g_2(\alpha_S L) + \frac{\alpha_S}{2\pi} \left[ A(\tau) - G_{11} L - G_{12} L^2 \right] \\
&\quad + \left( \frac{\alpha_S}{2\pi} \right)^2 \left\{ B(\tau) - \frac{1}{2}[A(\tau)]^2 - G_{22} L^2 - G_{23} L^3 \right\} ,
\end{align*}
\]

where the coefficients \( G_{nm} \) are obtained by expanding the functions \( g_1 \) and \( g_2 \) to second order.
Resulting expression (NLO+NLLA) fits the data over a much wider range than NLO alone – in fact, better than NNLO.

A Gehrmann-De Ridder et al., arXiv:0712.0327
Jet Fractions

● To define fraction \( f_n \) of \( n \)-jet final states \((n = 2, 3, \ldots)\), must specify jet algorithm.

● Most common is \( k_T \) or Durham algorithm:
  ❖ Define jet resolution \( y_{\text{cut}} \) (dimensionless).
  ❖ For each pair of final-state momenta \( p_i, p_j \) define
    \[
    y_{ij} = 2 \min\{E_i^2, E_j^2\}(1 - \cos \theta_{ij})/s
    \]
  ❖ If \( y_{IJ} = \min\{y_{ij}\} < y_{\text{cut}} \), combine \( I, J \) into one object \( K \) with \( p_K = p_I + p_J \).
  ❖ Repeat until \( y_{IJ} > y_{\text{cut}} \). Then remaining objects are jets.

● Variation of jet fractions with energy provides further evidence of running \( \alpha_S \)
  ❖ Fit is to NLO 2-jet fraction and mean number of jets, \( \langle N \rangle \).

\[\begin{array}{c|c|c|c|c}
\hline
\sqrt{s} (\text{GeV}) & 0.095 & 0.1 & 0.105 & 0.11 \ \\
\hline
\alpha_S (N) & 0.095 & 0.115 & 0.125 & \ \\
\hline
\end{array}\]

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**OPAL**

- Measured \( \alpha_S \) values
- \( \alpha_S (M_Z=0.1177) \)
- \( \alpha_{\text{exp}}=0.1087 \)
- \( \alpha_{\text{exp}}=0.0636 \)
Jet fractions now calculated to $\mathcal{O}(\alpha_s^3)$, i.e. NLO for 4 jets, NNLO 3 jets, N$^3$LO for 2 jets.

- Resummation of $\log y_{\text{cut}}$ would improve the fit at small $y_{\text{cut}}$.

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A Gehrmann-De Ridder et al., arXiv:0802.0813
NLO QCD Calculations

Consider $m$-jet cross section $\sigma^J$, defined according to some (infrared-safe) jet definition. In NLO, two separate divergent integrals:

$$\sigma_{NLO}^J = \int_{m+1} d\sigma^J_R + \int_{m} d\sigma^J_V$$

Must combine before numerical integration.

❖ Jet definition could be arbitrarily complicated:

$$d\sigma^J_R = d\Phi_{m+1}|\mathcal{M}_{m+1}|^2 F^J_{m+1}(p_1, \ldots, p_{m+1})$$

How to combine without knowing $F^J$?

❖ Two solutions: phase space slicing and subtraction method.
Illustrate with simple one-variable example

\[ |\mathcal{M}_{m+1}|^2 = \frac{1}{x} \mathcal{M}(x) \]

\( x \) could be gluon energy or two-parton invariant mass fraction \((0 < x < 1)\).

- IR divergences regularized by \( D = 4 - 2\epsilon \) dimensions \((\epsilon < 0)\).

\[ |\mathcal{M}_{m}^{\text{one-loop}}|^2 = \frac{1}{\epsilon} \mathcal{V} \]

- Cross section in \( D \) dimensions is

\[ \sigma^J = \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x) F_1^J(x) + \frac{1}{\epsilon} \mathcal{V} F_0^J \]

- Infrared safety: \( F_1^J(0) = F_0^J \)

- KLN cancellation theorem: \( \mathcal{M}(0) = \mathcal{V} \)
Phase Space Slicing

- Introduce arbitrary cutoff $\delta \ll 1$:

$$
\sigma^J = \int_0^\delta \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x) F_1^J(x) + \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x) F_1^J(x) + \frac{1}{\epsilon} V F_0^J
$$

$$
\approx \int_0^\delta \frac{dx}{x^{1+\epsilon}} V F_0^J + \int_0^1 \frac{dx}{x} \mathcal{M}(x) F_1^J(x) + \frac{1}{\epsilon} V F_0^J
$$

$$
= \int_0^1 \frac{dx}{x} \mathcal{M}(x) F_1^J(x) + \log(\delta) V F_0^J
$$

- Two separate finite integrals: becomes exact for $\delta \to 0$ but huge cancellations
  $\Rightarrow$ numerical errors blow up $\Rightarrow$ compromise (trial and error).
- Systematized by Giele-Glover-Kosower: JETRAD, DYRAD, EERAD, . . .
Subtraction Method

- **Exact identity:**

\[
\sigma^J = \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x) F_1^J(x) - \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{V} F_0^J + \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{V} F_0^J + \frac{1}{\epsilon} \mathcal{V} F_0^J
\]

\[
= \int_0^1 \frac{dx}{x} \left( \mathcal{M}(x) F_1^J(x) - \mathcal{V} F_0^J \right) + \mathcal{O}(1) \mathcal{V} F_0^J
\]

- Two separate finite integrals again.
- Much harder: subtracted cross section must be valid and calculable everywhere in phase space.
- Systematized by Catani-Seymour-Dittmaier-Nagy-Trocsanyi: EVENT2, DISENT, MCFM, NLOJET++, . . .
Parton Branching

- Leading soft and collinear enhanced terms in QCD matrix elements (and corresponding virtual corrections) can be identified and summed to all orders. Consider splitting of outgoing parton $a$ into $b + c$.

- Can assume $p_b^2, p_c^2 \ll p_a^2 \equiv t$. Opening angle is $\theta = \theta_a + \theta_b$, energy fraction is
  \[ z = \frac{E_b}{E_a} = 1 - \frac{E_c}{E_a} . \]

- For small angles
  \[ t = 2E_b E_c (1 - \cos \theta) = z(1 - z)E_a^2 \theta^2 , \]
  \[ \theta = \frac{1}{E_a} \sqrt{\frac{t}{z(1 - z)}} = \frac{\theta_b}{1 - z} = \frac{\theta_c}{z} . \]
Consider first $g \to gg$ branching:

- Amplitude has triple-gluon vertex factor

$$gf^{ABC} \epsilon^\alpha_a \epsilon^\beta_b \epsilon^\gamma_c [g_{\alpha \beta} (p_a - p_b)\gamma + g_{\beta \gamma} (p_b - p_c)\alpha + g_{\gamma \alpha} (p_c - p_a)\beta]$$

$\epsilon_i^\mu$ is polarization vector for gluon $i$. All momenta defined as outgoing here, so $p_a = -p_b - p_c$. Using this and $\epsilon_i \cdot p_i = 0$, vertex factor becomes

$$-2gf^{ABC} [(\epsilon_a \cdot \epsilon_b)(\epsilon_c \cdot p_b) - (\epsilon_b \cdot \epsilon_c)(\epsilon_a \cdot p_b) - (\epsilon_c \cdot \epsilon_a)(\epsilon_b \cdot p_c)] .$$

- Resolve polarization vectors into $\epsilon_i^{in}$ in plane of branching and $\epsilon_i^{out}$ normal to plane, so that

$$\epsilon_i^{in} \cdot \epsilon_j^{in} = \epsilon_i^{out} \cdot \epsilon_j^{out} = -1$$

$$\epsilon_i^{in} \cdot \epsilon_j^{out} = \epsilon_i^{out} \cdot p_j = 0 .$$
For small $\theta$, neglecting terms of order $\theta^2$, we have

$$\epsilon_a^{\text{in}} \cdot p_b = -E_b \theta = -z (1 - z) E_a \theta$$
$$\epsilon_b^{\text{in}} \cdot p_c = +E_c \theta = (1 - z) E_a \theta$$
$$\epsilon_c^{\text{in}} \cdot p_b = -E_b \theta = -z E_a \theta .$$

Vertex factor proportional to $\theta$, together with propagator factor of $1/t \propto 1/\theta^2$, gives $1/\theta$ collinear singularity in amplitude.

$(n + 1)$-parton matrix element squared (in small-angle region) is given in terms of that for $n$ partons:

$$|M_{n+1}|^2 \sim \frac{4g^2}{t} C_A F(z; \epsilon_a, \epsilon_b, \epsilon_c) |M_n|^2$$
where colour factor $C_A = 3$ comes from $f^{ABC} f^{ABC}$ and functions $F$ are given below

<table>
<thead>
<tr>
<th>$\epsilon_a$</th>
<th>$\epsilon_b$</th>
<th>$\epsilon_c$</th>
<th>$F(z; \epsilon_a, \epsilon_b, \epsilon_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>in</td>
<td>in</td>
<td>in</td>
<td>$(1 - z)/z + z/(1 - z) + z(1 - z)$</td>
</tr>
<tr>
<td>in</td>
<td>out</td>
<td>out</td>
<td>$z(1 - z)$</td>
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<tr>
<td>out</td>
<td>in</td>
<td>out</td>
<td>$(1 - z)/z$</td>
</tr>
<tr>
<td>out</td>
<td>out</td>
<td>in</td>
<td>$z/(1 - z)$</td>
</tr>
</tbody>
</table>

❖ Sum/averaging over polarizations gives

$$C_A \langle F \rangle \equiv \hat{P}_{gg}(z) = C_A \left[ \frac{1 - z}{z} + \frac{z}{1 - z} + z(1 - z) \right].$$

This is (unregularized) gluon splitting function.

❖ Enhancements at $z \to 0$ ($b$ soft) and $z \to 1$ ($c$ soft) due to soft gluon polarized in plane of branching.

❖ Correlation between polarization and plane of branching (angle $\phi$):

$$F_{\phi} \propto \sum_{\epsilon_{b,c}} | \cos \phi \mathcal{M}(\epsilon_{a}^{in}, \epsilon_{b}, \epsilon_{c}) + \sin \phi \mathcal{M}(\epsilon_{a}^{out}, \epsilon_{b}, \epsilon_{c})|^2$$

$$= \frac{1 - z}{z} + \frac{z}{1 - z} + z(1 - z) + z(1 - z) \cos 2\phi.$$

Hence branching in plane of gluon polarization preferred.
Consider next $g \rightarrow q\bar{q}$ branching:

- Vertex factor is
  \[-ig\bar{u}^b \gamma_\mu \epsilon^\mu_a v^c\]
  where $u^b$ and $v^c$ are quark and antiquark spinors.
- Spin-averaged splitting function is
  \[T_R \langle F' \rangle \equiv \hat{P}_{qg}(z) = T_R \left[ z^2 + (1 - z)^2 \right].\]

No soft ($z \rightarrow 0$ or 1) singularities since these are associated only with gluon emission.
- Vector quark-gluon coupling implies (for $m_q \simeq 0$) $q$ and $\bar{q}$ helicities always opposite (helicity conservation).
- Correlation between gluon polarization and plane of branching:
  \[F_\phi = z^2 + (1 - z)^2 - 2z(1 - z)\cos 2\phi\]
  i.e. strong preference for splitting perpendicular to polarization.
Branching $q \rightarrow qg$:

- Spin-averaged splitting function is

$$C_F \langle F \rangle \equiv \hat{P}_{qq}(z) = C_F \frac{1 + z^2}{1 - z}.$$ 

- Helicity conservation ensures that quark does not change helicity in branching.
- Gluon polarized in plane of branching preferred, polarization angular correlation being

$$F_\phi = \frac{1 + z^2}{1 - z} + \frac{2z}{1 - z} \cos 2\phi.$$
Phase Space

- Phase space factors before and after branching are related by

\[ d\Phi_{n+1} = d\Phi_n \frac{1}{4(2\pi)^3} dt \, dz \, d\phi . \]

- Hence cross sections before and after branching are related by

\[ d\sigma_{n+1} = d\sigma_n \frac{dt}{t} \frac{dz}{2\pi} \frac{d\phi}{2\pi} \alpha_S CF \]

where \( C \) and \( F \) are colour factor and polarization-dependent \( z \)-distribution introduced earlier. Integrating over azimuthal angle gives

\[ d\sigma_{n+1} = d\sigma_n \frac{dt}{t} \frac{dz}{2\pi} \alpha_S \hat{P}_{ba}(z) . \]

where \( \hat{P}_{ba}(z) \) is \( a \rightarrow b \) splitting function.
Angular correlations are illustrated by the angular distribution in $e^+e^- \rightarrow 4$ jets. Bengtsson-Zerwas angle $\chi_{BZ}$ is angle between the planes of two lowest and two highest energy jets:

$$\cos \chi_{BZ} = \frac{(\mathbf{p}_1 \times \mathbf{p}_2) \cdot (\mathbf{p}_3 \times \mathbf{p}_4)}{|\mathbf{p}_1 \times \mathbf{p}_2| \cdot |\mathbf{p}_3 \times \mathbf{p}_4|}.$$
Lowest-order diagrams for 4-jet production shown below. Two hardest jets tend to follow directions of primary $q\bar{q}$.

“Double bremsstrahlung” diagrams give negligible correlations.

$g \rightarrow q\bar{q}$ give strong anti-correlation (“Abelian” curve), because gluon tends to be polarized in plane of primary jets and prefers to split perpendicular to polarization.

$g \rightarrow gg$ occurs more often parallel to polarization. Although its correlation is much weaker than in $g \rightarrow q\bar{q}$, $g \rightarrow gg$ is dominant in QCD due to larger colour factor and soft gluon enhancements.

Thus B-Z angular distribution is flatter than in an Abelian theory.
Combining with fits to event shape distributions allows determination of the colour factors $C_A$ and $C_F$. 

![Graph showing combined result and SU(3) QCD](image_url)
Summary of Lecture 2

- $e^+e^-$ annihilation cross section — an infrared-safe quantity.
  - NNLO prediction shows good stability w.r.t. renormalization scale.

- $e^+e^-$ shape distributions and jet fractions (suitably defined) also infrared safe.
  - But require resummation of large logs, e.g. $\ln(1 - T)$.
  - Complete NNLO calculations now available.

- NLO (and beyond) calculations require special methods to deal with infrared divergences.
  - Phase space slicing method — simpler but numerical problems.
  - Subtraction method — more difficult but exact in principle.

- Parton branching approximation sums leading collinear enhanced terms.
  - Formulated in terms of $1 \rightarrow 2$ parton splitting functions.
  - Spin correlations explain qualitative features of 4-jet angular distribution.