Monte Carlo Methods in Particle Physics

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See also

ESW: “QCD and Collider Physics”, C.U.P. 1996

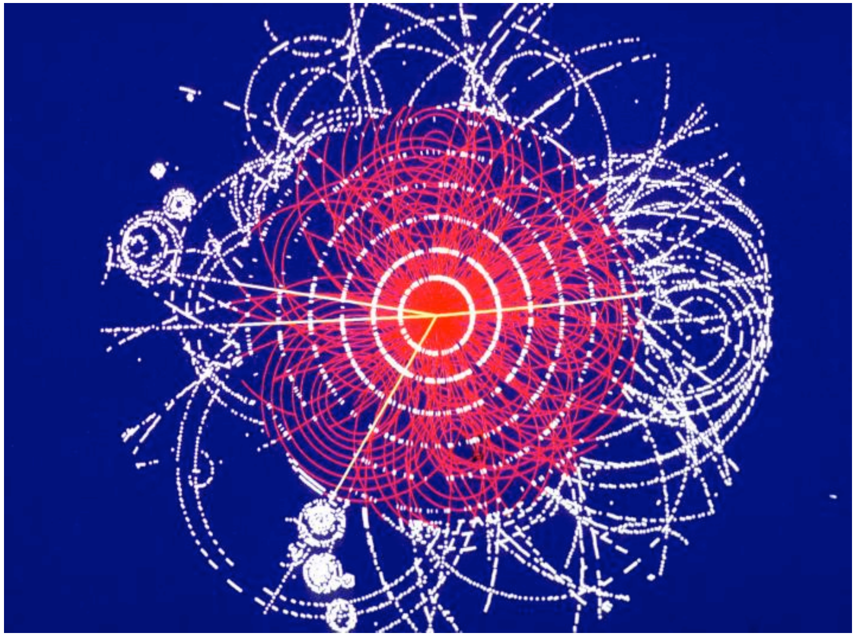
http://www.hep.phy.cam.ac.uk/theory/webber/QCDupdates.html

http://www.hep.phy.cam.ac.uk/theory/webber/QCD_03/

Thanks to Mike Seymour, Torbjörn Sjöstrand, Frank Krauss, Peter Richardson,...

Monte Carlo Event Generation

- Basic Principles
- Event Generation
- Parton Showers
- Hadronization
- Underlying Event
- Event Generator Survey
- Matching to Fixed Order
- Beyond Standard Model



Lecture1: Basics

- The Monte Carlo concept
- Event generation
- Examples: particle production and decay
- Structure of an LHC event
- Monte Carlo implementation of NLO QCD

Integrals as Averages

- Basis of all Monte Carlo methods: $$I = \int_{x_1}^{x_2} f(x)\,dx = (x_2 - x_1)\,\langle f(x) \rangle$$
- Draw N values from a uniform distribution: $$I \approx I_N \equiv (x_2 - x_1)\frac{1}{N}\sum_{i=1}^{N} f(x_i)$$
- Sum invariant under reordering: randomize
- Central limit theorem: $$I \approx I_N \pm \sqrt{V_N/N}$$

$$V = (x_2 - x_1)\int_{x_1}^{x_2}\left[f(x)\right]^2 dx - \left[\int_{x_1}^{x_2} f(x)\,dx\right]^2$$

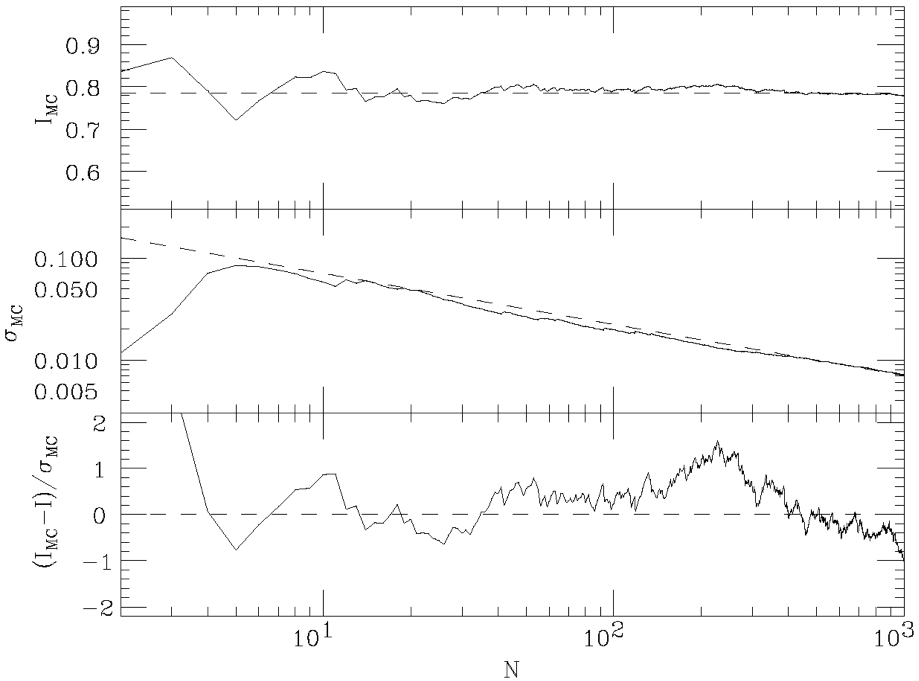
Convergence

- Monte Carlo integrals governed by Central Limit Theorem: error $\propto 1/\sqrt{N}$

  c.f. trapezium rule $\propto 1/N^2$

  Simpson's rule $\propto 1/N^4$

but only if derivatives exist and are finite: $\sqrt{1-x^2} \sim 1/N^{3/2}$

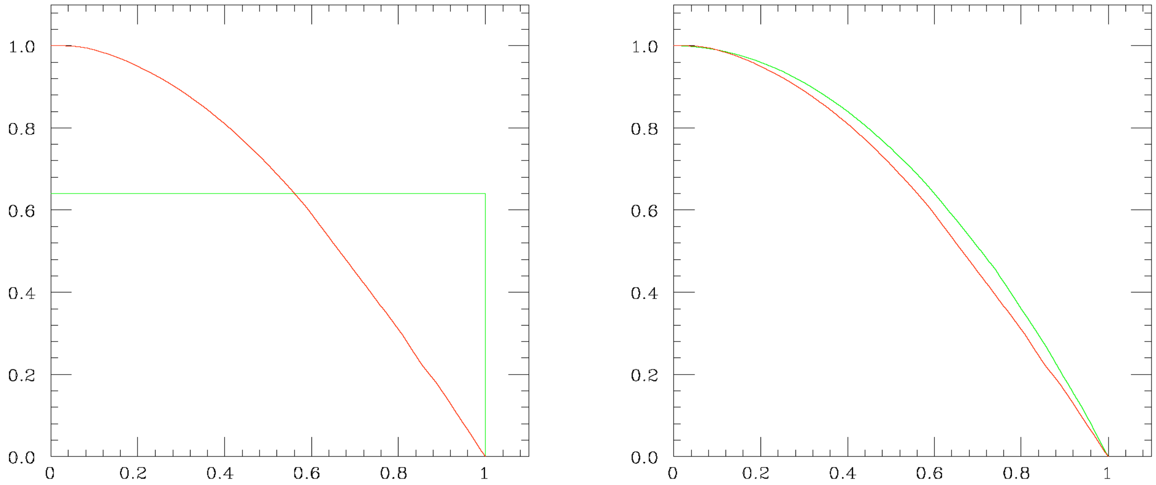


Importance Sampling

Convergence improved by putting more samples in region where function is largest.

Corresponds to a Jacobian transformation.

Hit-and-miss: accept points with probability = ratio (if < 1)



$$
\begin{aligned}
I &= \int_0^1 dx\, \cos\tfrac{\pi}{2}x \\
&= 0.637 \pm 0.308/\sqrt{N}
\end{aligned}
$$

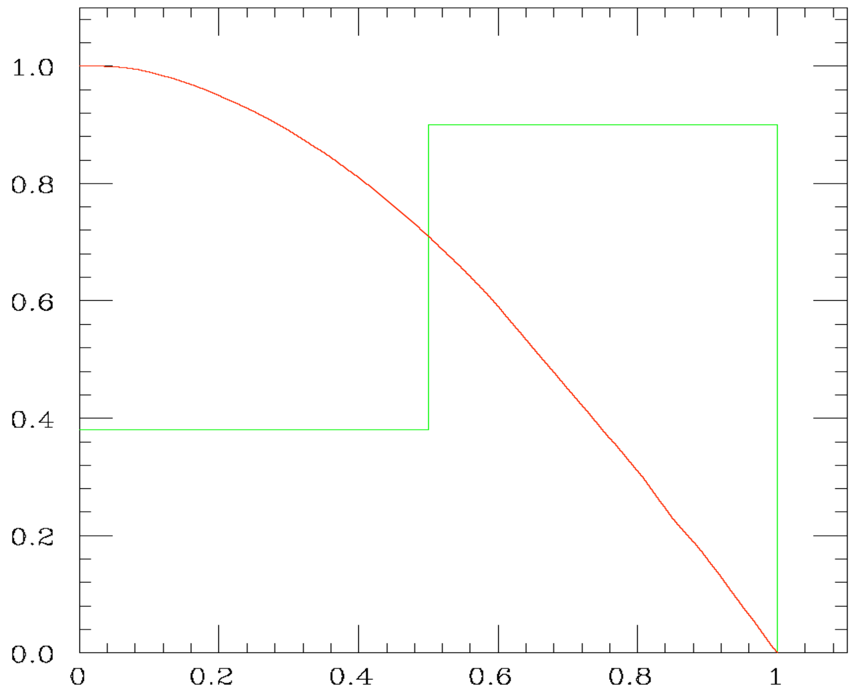
$$
\begin{aligned}
I &= \int_0^1 dx(1-x^2)\,\frac{\cos\frac{\pi}{2}x}{1-x^2} \\
&= \int d\rho\,\frac{\cos\frac{\pi}{2}x}{1-x^2}[x(\rho)] \\
&= 0.637 \pm 0.032/\sqrt{N}
\end{aligned}
$$

Stratified Sampling

Divide up integration region piecemeal and optimize to minimize total error.

Can be done automatically (eg VEGAS).

Never as good as Jacobian transformations.



N.B. Puts more points where rapidly varying, not necessarily where larger!

$$I = 0.637 \pm 0.147/\sqrt{N}$$

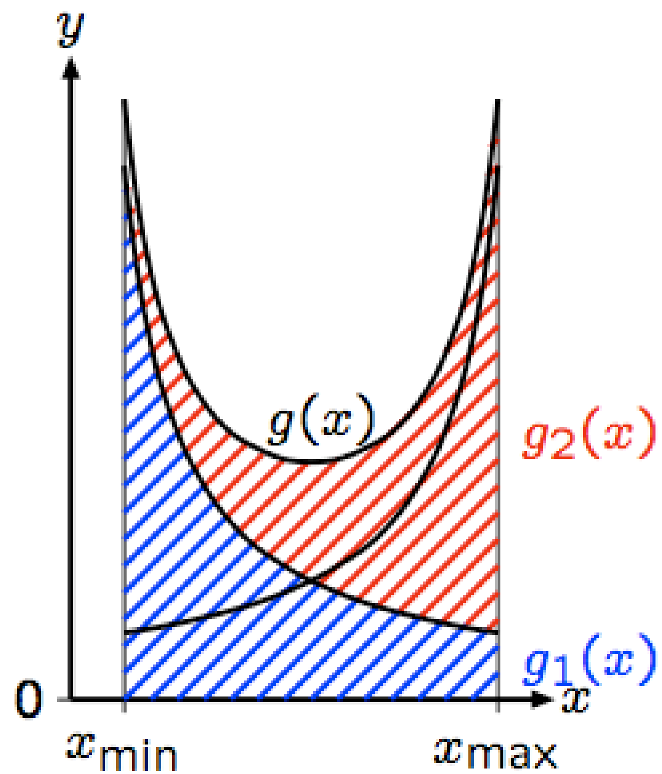
Multichannel Sampling

If $f(x) \leq g(x) = \sum_i g_i(x)$,
where all g_i "nice" (but $g(x)$ not)

1) select i with relative probability

$$A_i = \int_{x_{\rm min}}^{x_{\rm max}} g_i(x')\,{\rm d}x'$$

2) select x according to $g_i(x)$
3) select $y = R\,g(x) = R\sum_i g_i(x)$
4) while $y > f(x)$ cycle to 1)



Multi-dimensional Integration

- Formalism extends trivially to many dimensions
- Particle physics: very many dimensions,
  e.g. phase space = 3 dimensions per particles,
  LHC event ~ 250 hadrons.
- Monte Carlo error remains $\propto 1/\sqrt{N}$
- Trapezium rule $\propto 1/N^{2/d}$
- Simpson's rule $\propto 1/N^{4/d}$

Summary

Disadvantages of Monte Carlo:

- Slow convergence in few dimensions.

Advantages of Monte Carlo:

- Fast convergence in many dimensions.
- Arbitrarily complex integration regions (finite discontinuities not a problem).
- Few points needed to get first estimate (“feasibility limit”).
- Every additional point improves accuracy (“growth rate”).
- Easy error estimate.

Phase Space

$$\sigma = \frac{1}{2s}\int |\mathcal{M}|^2 \, d\Pi_n(\sqrt{s})$$

$$\Gamma = \frac{1}{2M}\int |\mathcal{M}|^2 \, d\Pi_n(M)$$

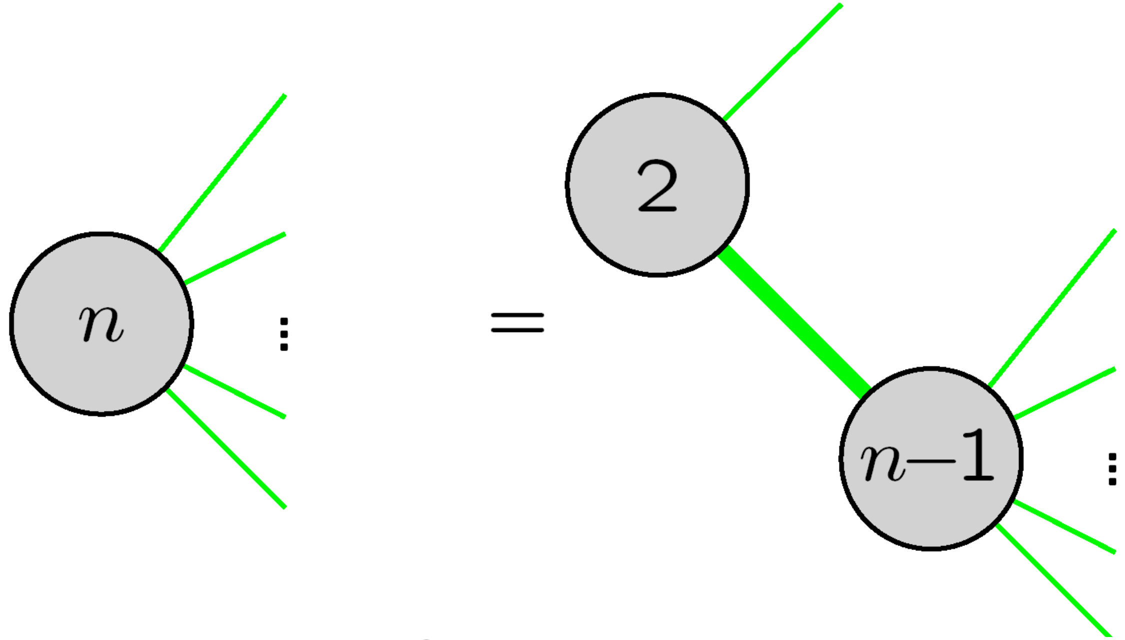
Phase space:

$$d\Pi_n(M) = \left[\prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3(2E_i)}\right](2\pi)^4\delta^{(4)}\left(p_0 - \sum_{i=1}^{n} p_i\right)$$

Two-body easy:

$$d\Pi_2(M) = \frac{1}{8\pi}\,\frac{2p}{M}\,\frac{d\Omega}{4\pi}$$

Other cases by recursive subdivision:



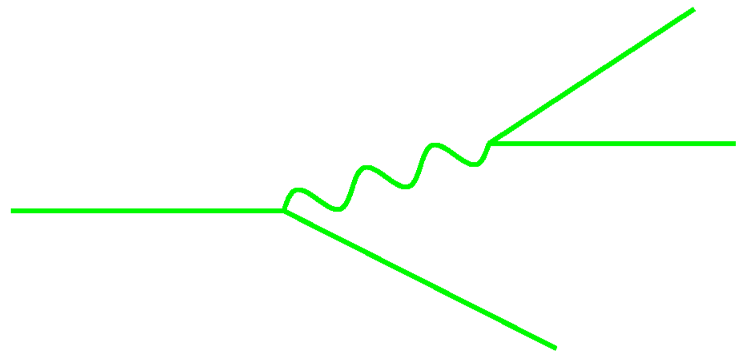
$$d\Pi_n(M) = \frac{1}{2\pi}\int_0^{(M-m)^2} dm_x^2\, d\Pi_2(M)\, d\Pi_{n-1}(m_x)$$

Or by 'democratic' algorithms: RAMBO, MAMBO
Can be better, but matrix elements rarely flat.

Particle Decays

Simplest example

e.g. top quark decay:



$$|\mathcal{M}|^2 = \frac{1}{2}\left(\frac{8\pi\alpha}{\sin^2\theta_w}\right)^2 \frac{p_t\cdot p_\ell\, p_b\cdot p_\nu}{(m_W^2 - M_W^2)^2 + \Gamma_W^2 M_W^2}$$

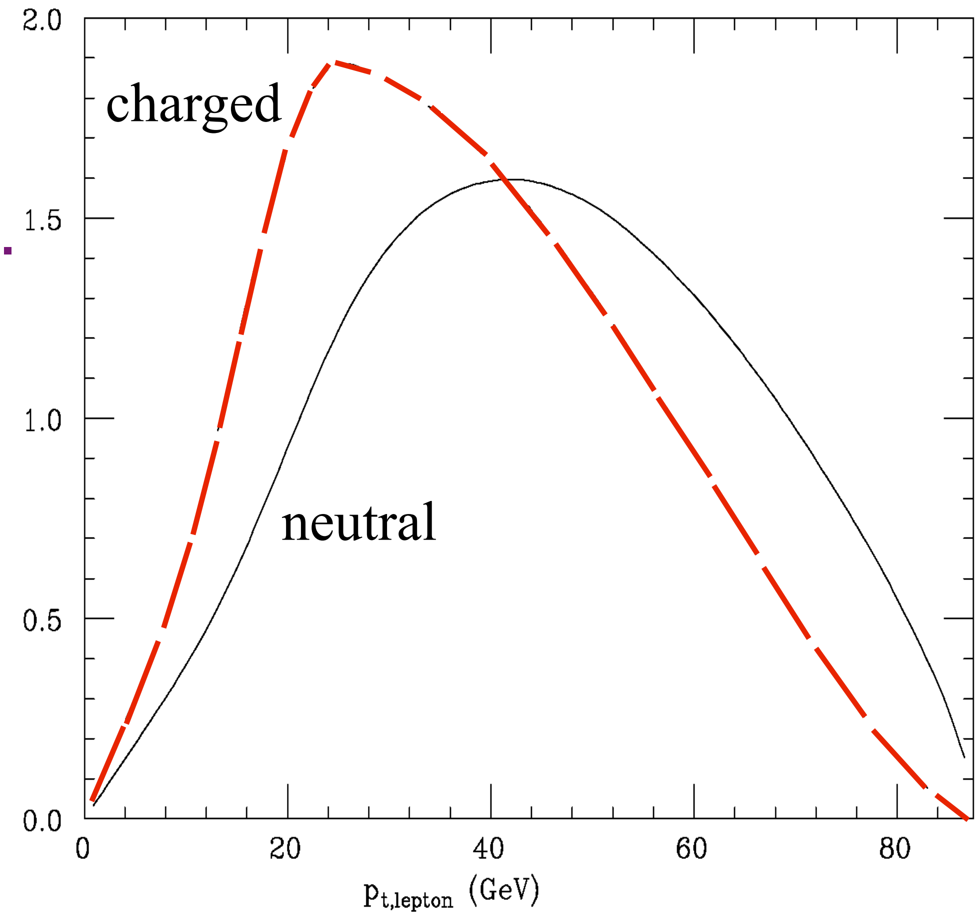
$$\Gamma = \frac{1}{2M}\frac{1}{128\pi^3}\int |\mathcal{M}|^2 dm_W^2 \left(1-\frac{m_W^2}{M^2}\right)\frac{d\Omega}{4\pi}\frac{d\Omega_W}{4\pi}$$

Breit-Wigner peak of W very strong: must be removed by Jacobian factor

Associated Distributions

- Big advantage of Monte Carlo integration:
- simply histogram any associated quantities.
- Almost any other technique requires new integration for each observable.
- Can apply arbitrary cuts/smearing.

e.g. lepton momentum in top decays:



Cross Sections

Additional integrations over incoming parton densities:

$$\begin{aligned}\sigma(s) &= \int_0^1 dx_1 f_1(x_1) \int_0^1 dx_2 f_2(x_2)\, \hat{\sigma}(x_1 x_2 s) \\ &= \int_0^1 \frac{d\tau}{\tau}\hat{\sigma}(\tau s) \int_\tau^1 \frac{dx}{x}\, x f_1(x)\, \tfrac{\tau}{x} f_2(\tfrac{\tau}{x})\end{aligned}$$

$\hat{\sigma}(\hat{s})$ can have strong peaks, eg Z Breit-Wigner: need Jacobian factors.

Hard to make process-independent.

Leading Order Monte Carlo Calculations

Now have everything we need to make leading order cross section calculations and distributions

Can be largely automated…

- MADGRAPH
- GRACE
- COMPHEP
- AMEGIC++
- ALPGEN

But…

- Fixed parton/jet multiplicity
- No control of large logs
- Parton level

→ Need hadron level event generators

Event Generators

Up to here, only considered Monte Carlo as a numerical integration method.

If function being integrated is a probability density (positive definite), trivial to convert it to a simulation of physical process = an event generator.

Simple example: $\sigma = \int_0^1 \frac{d\sigma}{dx}\, dx$

Naive approach: 'events' x with 'weights' $d\sigma/dx$

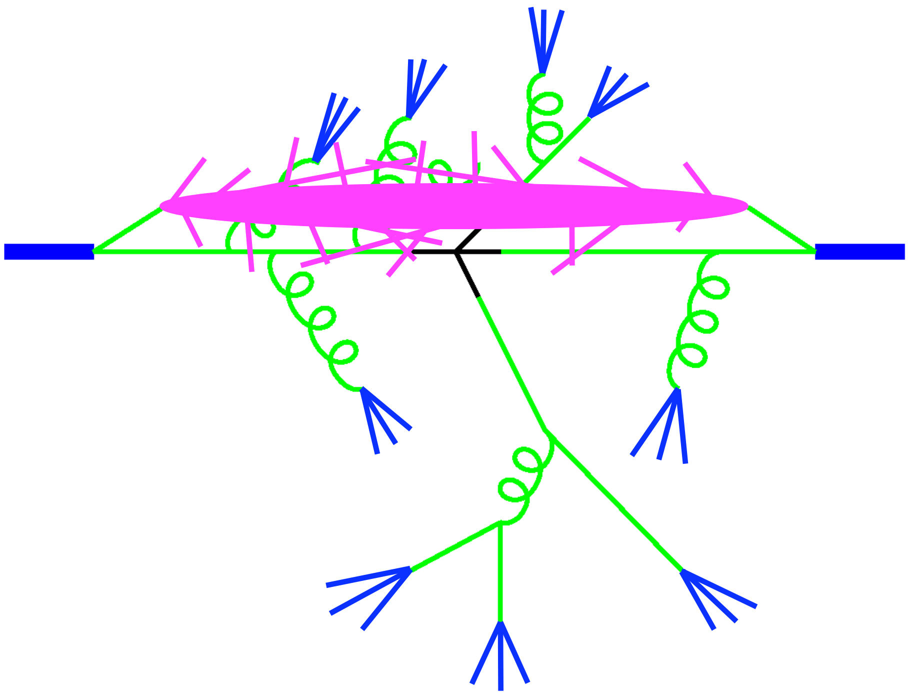
Can generate unweighted events by keeping them with probability $(d\sigma/dx)/(d\sigma/dx)_{\mathrm{max}}$: give them all weight σ_{tot}

Happen with same frequency as in nature.

Efficiency: $\frac{(d\sigma/dx)_{\mathrm{avge}}}{(d\sigma/dx)_{\mathrm{max}}}$ = fraction of generated events kept.

Structure of LHC Events

1. Hard process
2. Parton shower
3. Hadronization
4. Underlying event



We'll return to this later...

Monte Carlo Calculations of NLO QCD

Two separate divergent integrals:

$$\sigma_{NLO} = \int_{m+1} d\sigma^R + \int_m d\sigma^V$$

Must combine before numerical integration.

Jet definition could be arbitrarily complicated.

$$d\sigma^R = d\Pi_{m+1} \, |\mathcal{M}_{m+1}|^2 \, F^J_{m+1}(p_1, \ldots, p_{m+1})$$

How to combine without knowing F^J?

Two solutions:

phase space slicing and subtraction method.

Illustrate with simple one-dim. example:

$$|\mathcal{M}_{m+1}|^2 \equiv \frac{1}{x}\mathcal{M}(x)$$

x = gluon energy or two-parton invariant mass.

Divergences regularized by $d = 4 - 2\epsilon$ dimensions.

$$|\mathcal{M}_m^{\text{one-loop}}|^2 \equiv \frac{1}{\epsilon}\mathcal{V}$$

Cross section in d dimensions is:

$$\sigma = \int_0^1 \frac{dx}{x^{1+\epsilon}}\,\mathcal{M}(x)\;F_1^J(x) + \frac{1}{\epsilon}\mathcal{V}\;F_0^J$$

Infrared safety: $F_1^J(0) = F_0^J$

KLN cancellation theorem: $\mathcal{M}(0) = \mathcal{V}$

Phase space slicing

Introduce arbitrary cutoff $\delta \ll 1$:

$$
\begin{aligned}
\sigma &= \int_0^\delta \frac{dx}{x^{1+\epsilon}}\,\mathcal{M}(x)\;F_1^J(x) + \int_\delta^1 \frac{dx}{x^{1+\epsilon}}\,\mathcal{M}(x)\;F_1^J(x) + \frac{1}{\epsilon}\mathcal{V}\;F_0^J \\
&\approx \int_0^\delta \frac{dx}{x^{1+\epsilon}}\,\mathcal{V}\;F_0^J + \int_\delta^1 \frac{dx}{x}\,\mathcal{M}(x)\;F_1^J(x) + \frac{1}{\epsilon}\mathcal{V}\;F_0^J \\
&= \int_\delta^1 \frac{dx}{x}\,\mathcal{M}(x)\;F_1^J(x) + \log(\delta)\mathcal{V}\;F_0^J
\end{aligned}
$$

Two separate finite integrals ➡ Monte Carlo.

Becomes exact for $\delta \to 0$ but numerical errors blow up

➡ compromise (trial and error).

Systematized by Giele-Glover-Kosower.

JETRAD, DYRAD, EERAD, . . .

Subtraction method

Exact identity:

$$
\begin{aligned}
\sigma &= \int_0^1 \frac{dx}{x^{1+\epsilon}}\,\mathcal{M}(x)\;F_1^J(x) - \int_0^1 \frac{dx}{x^{1+\epsilon}}\,\mathcal{V}\;F_0^J \\
&\qquad\qquad + \int_0^1 \frac{dx}{x^{1+\epsilon}}\,\mathcal{V}\;F_0^J + \frac{1}{\epsilon}\mathcal{V}\;F_0^J \\
&= \int_0^1 \frac{dx}{x}\left(\mathcal{M}(x)\;F_1^J(x) - \mathcal{V}\;F_0^J\right) + \mathcal{O}(1)\;\mathcal{V}\;F_0^J .
\end{aligned}
$$

→ Two separate finite integrals again.

Subtraction method

Exact identity:

$$\sigma = \int_0^1 \frac{dx}{x}\left(\mathcal{M}(x)\ F_1^J(x) - \mathcal{V}\ F_0^J\right) + \mathcal{O}(1)\ \mathcal{V}\ F_0^J .$$

Two separate finite integrals again.

Much harder: subtracted cross section must be valid everywhere in phase space.

Systematized in

S. Catani and M.H. Seymour, Nucl. Phys. B485 (1997) 291.

S. Catani, S. Dittmaier, M.H. Seymour and Z. Trocsanyi, Nucl. Phys. B627 (2002) 189.

→ any observable in any process

→ analytical integrals done once-and-for-all

EVENT2, DISENT, NLOJET++, MCFM, …

Summary

- Monte Carlo is a very convenient numerical integration method.
- Well-suited to particle physics: difficult integrands, many dimensions.
- Integrand positive definite → event generator.
- Fully exclusive → treat particles exactly like in data.

→ need to understand/model hadronic final state.

N.B. NLO QCD programs are not event generators:

Not positive definite.

But full numerical treatment of arbitrary observables.

We'll discuss later how to combine/match NLO and EGs