QCD and Collider Phenomenology

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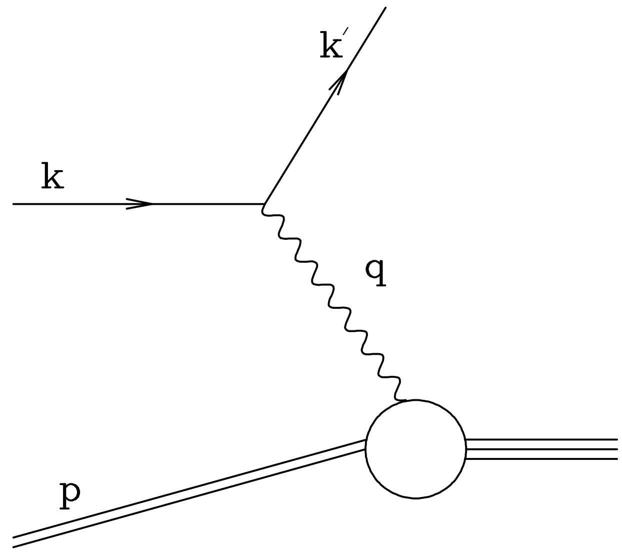
IPMU/KEK Lectures 2009

Lecture 1: DIS and Parton Showering

- Deep Inelastic Scattering
  - Parton model
  - Asymptotic freedom
  - Scaling violation and DGLAP equation
  - Quark and gluon distributions
  - Solution by moments
  - Small x
- Parton Showers
  - Sudakov form factor
  - Infrared cutoff
  - Polarization effects

Deep Inelastic Scattering

- Consider lepton-proton scattering via exchange of virtual photon:



- Standard variables are:

$$x = \frac{-q^2}{2p \cdot q} = \frac{Q^2}{2M(E - E')}$$

$$y = \frac{q \cdot p}{k \cdot p} = 1 - \frac{E'}{E}$$

where $Q^2 = -q^2 > 0$, $M^2 = p^2$ and energies refer to target rest frame.

- Elastic scattering has $(p+q)^2 = M^2$, i.e. $x = 1$. Hence deep inelastic scattering (DIS) means $Q^2 \gg M^2$ and $x < 1$.

- Structure functions $F_i(x, Q^2)$ parametrise target structure as 'seen' by virtual photon. Defined in terms of cross section

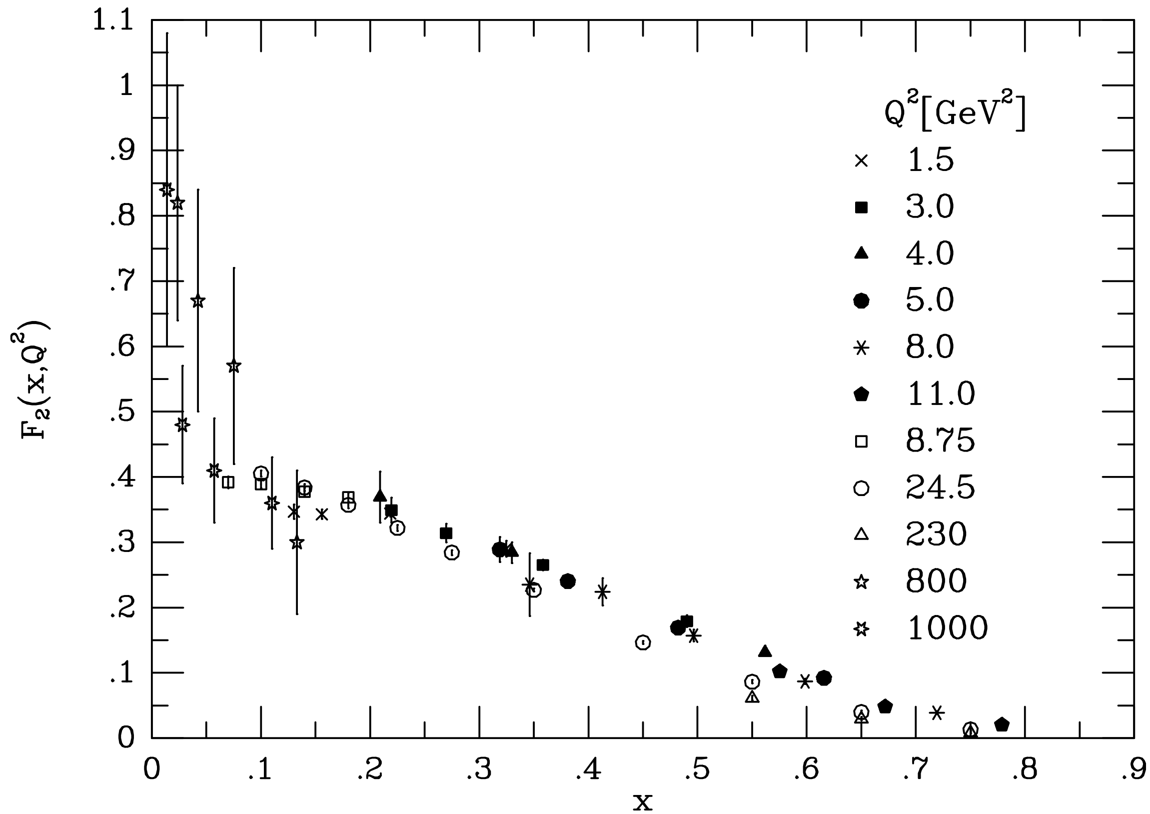
$$\frac{d^2\sigma}{dxdy} = \frac{8\pi\alpha^2 ME}{Q^4}\left[\left(\frac{1+(1-y)^2}{2}\right)2xF_1 \right.$$
$$\left. +(1-y)(F_2 - 2xF_1) - (M/2E)xyF_2\right].$$

- Bjorken limit is Q^2, $p \cdot q \to \infty$ with x fixed. In this limit structure functions obey approximate Bjorken scaling law, i.e. depend only on dimensionless variable x:

$$F_i(x, Q^2) \longrightarrow F_i(x).$$

- Bjorken scaling implies that virtual photon is scattered by almost-free *pointlike constituents* (partons) — otherwise structure functions would depend on ratio Q/Q_0, with $1/Q_0$ a length scale characterizing size of constituents.

- How can partons be bound inside hadrons but still appear almost free at high Q^2?



- Figure shows F_2 structure function for proton target. Although Q^2 varies by two orders of magnitude, in first approximation data lie on universal curve.

- Parton model of DIS is formulated in a frame where target proton is moving very fast — *infinite momentum frame*.
  - Suppose that, in this frame, photon scatters from pointlike quark with fraction ξ of proton's momentum. Since $(\xi p+q)^2 = m_q^2 \ll Q^2$, we must have $\xi = Q^2/2p\cdot q = x$.
  - In terms of Mandelstam variables $\hat{s}, \hat{t}, \hat{u}$, spin-averaged matrix element squared for massless $eq \to eq$ scattering is

$$\overline{\sum}|\mathcal{M}|^2 = 2e_q^2 e^4 \frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$$

    where $\overline{\sum}$ denotes average (sum) over initial (final) colours and spins.
  - In terms of DIS variables, $\hat{t} = -Q^2$, $\hat{u} = \hat{s}(y-1)$ and $\hat{s} = Q^2/y$. Differential cross section is then

$$\frac{d^2\hat{\sigma}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}[1+(1-y)^2]\frac{1}{2}e_q^2\delta(x-\xi).$$

  - From structure function definition (neglecting M)

$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left\{[1+(1-y)^2]F_1 + \frac{(1-y)}{x}(F_2 - 2xF_1)\right\}.$$

  - Hence structure functions for scattering from parton with momentum fraction ξ is

$$\hat{F}_2 = x e_q^2 \delta(x-\xi) = 2x\hat{F}_1 \ .$$

- Suppose probability that quark q carries momentum fraction between ξ and $\xi + d\xi$ is $q(\xi)\, d\xi$. Then

$$\begin{aligned} F_2(x) &= \sum_q \int_0^1 d\xi\ q(\xi)\ x e_q^2 \delta(x-\xi) \\ &= \sum_q e_q^2 x q(x) = 2xF_1(x)\ . \end{aligned}$$

- Relationship $F_2 = 2xF_1$ (Callan-Gross relation) follows from spin-$\frac{1}{2}$ property of quarks ($F_1 = 0$ for spin-0).

- Proton consists of three valence quarks (uud), which carry its electric charge and baryon number, and infinite sea of light $q\bar{q}$ pairs. Probed at scale Q, sea contains all quark flavours with $m_q \ll Q$. Thus at $Q \sim 1$ GeV expect

$$F_2^{em}(x) \simeq \frac{4}{9}x[u(x)+\bar{u}(x)] + \frac{1}{9}x[d(x)+\bar{d}(x)+s(x)+\bar{s}(x)]$$

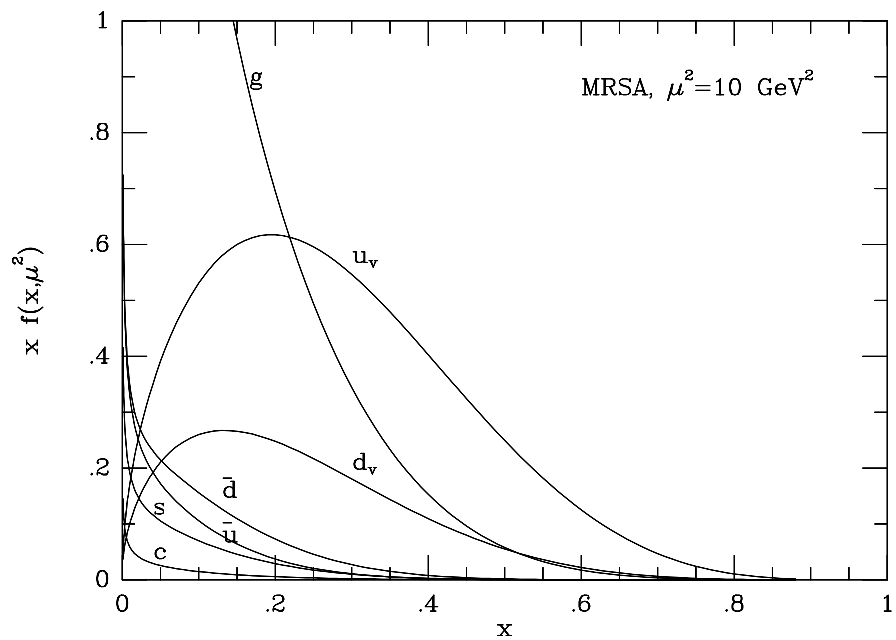
where

$$\begin{aligned} u(x) &= u_V(x) + \bar{u}(x) \\ d(x) &= d_V(x) + \bar{d}(x) \\ s(x) &= \bar{s}(x) \end{aligned}$$

with sum rules

$$\int_0^1 dx\ u_V(x) = 2\ ,\quad \int_0^1 dx\ d_V(x) = 1\ .$$

- Experimentally one finds $\sum_q \int_0^1 dx\ x[q(x) + \bar{q}(x)] \simeq 0.5$.. Thus quarks only carry about 50% of proton's momentum. Rest is carried by *gluons*. Although not directly measured in DIS, gluons participate in other hard scattering processes such as large-p_T jet and prompt photon production.



- Figure shows typical set of parton distributions extracted from fits to DIS data, at $Q^2 = 10$ GeV^2.

QCD Running Coupling

- Consider dimensionless physical observable R which depends on a single large energy scale, $Q \gg m$ where m is any mass. Then we can set $m \to 0$ (assuming this limit exists), and dimensional analysis suggests that R should be independent of Q.
- This is not true in quantum field theory. Calculation of R as a perturbation series in the coupling $\alpha_S = g^2/4\pi$ requires renormalization to remove ultraviolet divergences. This introduces a second mass scale μ — point at which subtractions which remove divergences are performed. Then R depends on the ratio Q/μ and is not constant. The renormalized coupling α_S also depends on μ.
- But μ is arbitrary! Therefore, if we hold bare coupling fixed, R cannot depend on μ. Since R is dimensionless, it can only depend on Q^2/μ^2 and the renormalized coupling α_S. Hence

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_S\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S}\right] R = 0 \; .$$

- Introducing
$$\tau = \ln\left(\frac{Q^2}{\mu^2}\right) , \quad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} ,$$
we have
$$\left[-\frac{\partial}{\partial \tau} + \beta(\alpha_S)\frac{\partial}{\partial \alpha_S}\right] R = 0.$$
This renormalization group equation is solved by defining running coupling $\alpha_S(Q)$:
$$\tau = \int_{\alpha_S}^{\alpha_S(Q)} \frac{dx}{\beta(x)} , \quad \alpha_S(\mu) \equiv \alpha_S .$$
Then
$$\frac{\partial \alpha_S(Q)}{\partial \tau} = \beta(\alpha_S(Q)) , \quad \frac{\partial \alpha_S(Q)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q))}{\beta(\alpha_S)} .$$
and hence $R(Q^2/\mu^2, \alpha_S) = R(1, \alpha_S(Q))$. Thus all scale dependence in R comes from running of $\alpha_S(Q)$.
- We shall see QCD is asymptotically free: $\alpha_S(Q) \longrightarrow 0$ as $Q \longrightarrow \infty$. Thus for large Q we can safely use perturbation theory. Then knowledge of $R(1, \alpha_S)$ to fixed order allows us to predict variation of R with Q.

QCD Beta Function

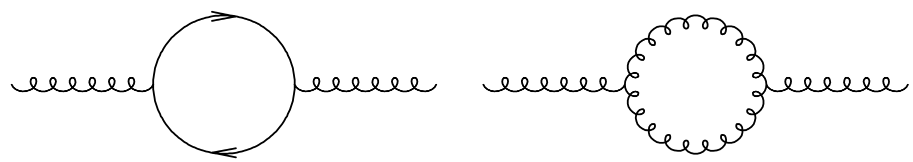
- Running of of the QCD coupling α_S is determined by the β function, which has the expansion

$$\beta(\alpha_S) = -b\alpha_S^2(1 + b'\alpha_S) + \mathcal{O}(\alpha_S^4)$$

$$b = \frac{(11C_A - 2n_f)}{12\pi}, \quad b' = \frac{(17C_A^2 - 5C_A n_f - 3C_F n_f)}{2\pi(11C_A - 2n_f)},$$

where n_f is number of "active" light flavours. Terms up to $\mathcal{O}(\alpha_S^5)$ are known.

- Roughly speaking, quark loop "vacuum polarisation" diagram (a) contributes negative n_f term in b, while gluon loop (b) gives positive C_A contribution, which makes β function negative overall.



a) b)

- QED β function is

$$\beta_{QED}(\alpha) = \frac{1}{3\pi}\alpha^2 + \ldots$$

Thus b coefficients in QED and QCD have opposite signs.

- From previous section,

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = -b\alpha_S^2(Q)\Big[1 + b'\alpha_S(Q)\Big] + \mathcal{O}(\alpha_S^4).$$

Neglecting b' and higher coefficients gives

$$\alpha_S(Q) = \frac{\alpha_S(\mu)}{1 + \alpha_S(\mu) b\tau}\,, \quad \tau = \ln\left(\frac{Q^2}{\mu^2}\right).$$

- As Q becomes large, $\alpha_S(Q)$ decreases to zero: this is asymptotic freedom. Notice that sign of b is crucial. In QED, $b < 0$ and coupling *increases* at large Q.
Including next coefficient b' gives implicit equation for $\alpha_S(Q)$:

$$b\tau = \frac{1}{\alpha_S(Q)} - \frac{1}{\alpha_S(\mu)} + b'\ln\Big(\frac{\alpha_S(Q)}{1 + b'\alpha_S(Q)}\Big) - b'\ln\Big(\frac{\alpha_S(\mu)}{1 + b'\alpha_S(\mu)}\Big)$$

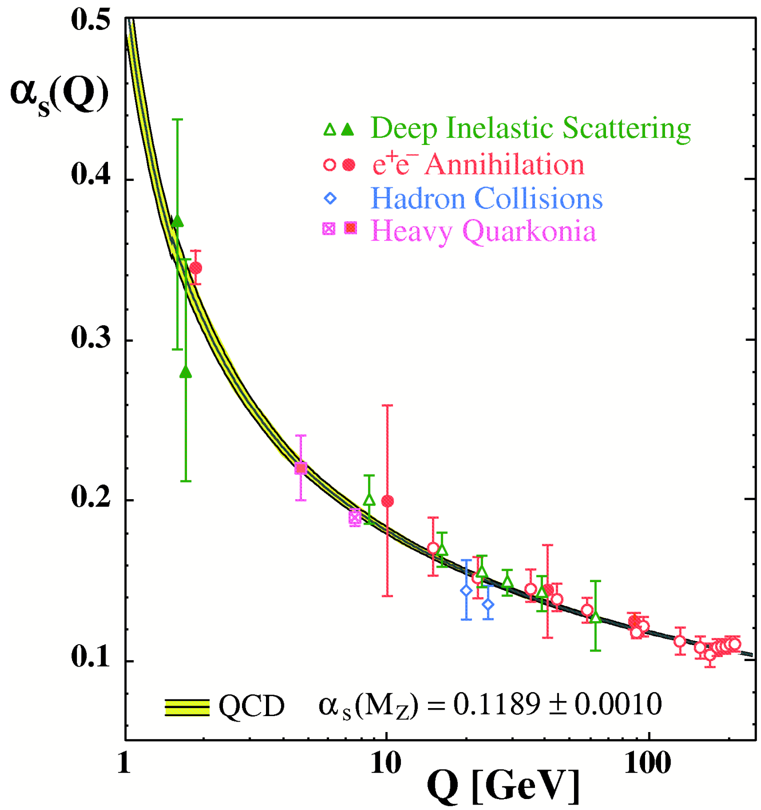
- What type of terms does the solution of the renormalization group equation take into account in the dimensionless physical quantity $R(Q^2/\mu^2, \alpha_S)$?
  Assume that R has perturbative expansion

$$R(1, \alpha_S) = R_1 \alpha_S + R_2 \alpha_S^2 + \mathcal{O}(\alpha_S^3)$$

  RGE solution $R(1, \alpha_S(Q))$ can be re-expressed in terms of $\alpha_S(\mu)$:

$$\begin{aligned} \alpha_S(Q) &= \alpha_S(\mu) - b\tau[\alpha_S(\mu)]^2 + \mathcal{O}(\alpha_S^3) \\ R(1, \alpha_S(Q)) &= R_1 \alpha_S(\mu) + (R_2 - b\tau)\alpha_S(\mu)^2 + \mathcal{O}(\alpha_S^3) \end{aligned}$$

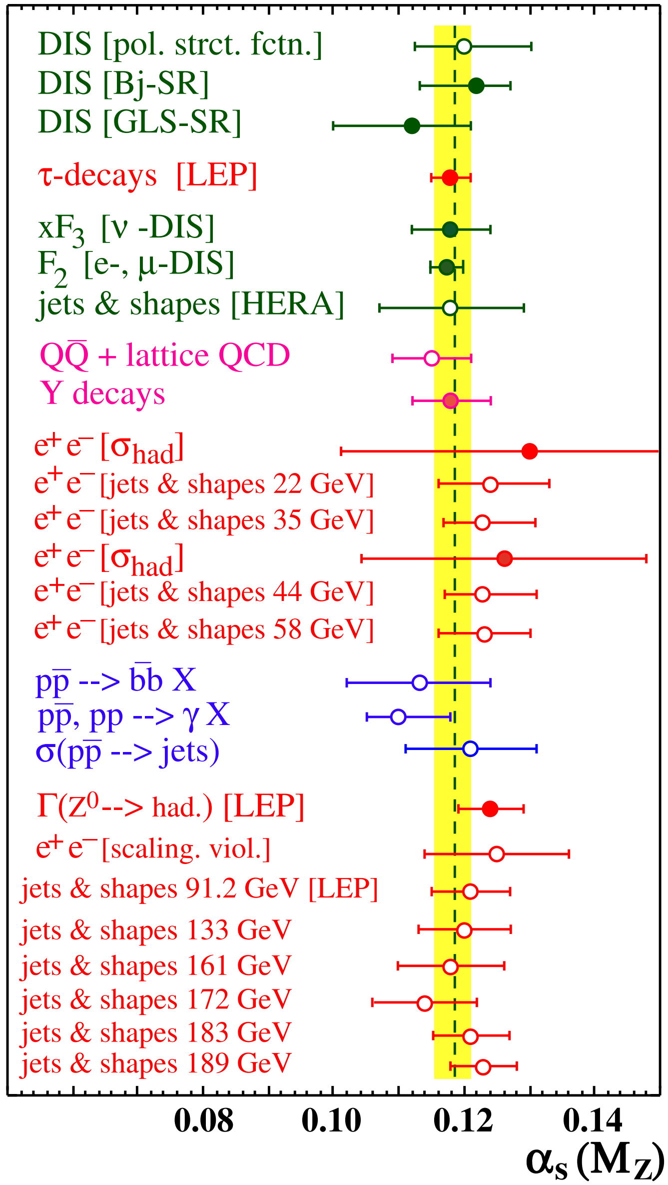
  Thus there are powers of $\tau = \log(Q^2/\mu^2)$ that are automatically resummed by using the running coupling.
- Notice that a *leading order* (LO) evaluation of R (i.e. the coefficient R_1) is not very useful since $\alpha_S(\mu)$ can be given any value by varying the scale μ.
  - We need the *next-to-leading order* (NLO) coefficient $(R_2 - b\tau)$ to gain some control of scale dependence: the μ dependence of τ starts to compensate that of $\alpha_S(\mu)$.



- Current best fit value of α_S at mass of Z is [Bethke, hep-ex/0606035]

$$\alpha_S(M_Z) = 0.1189 \pm 0.0010$$

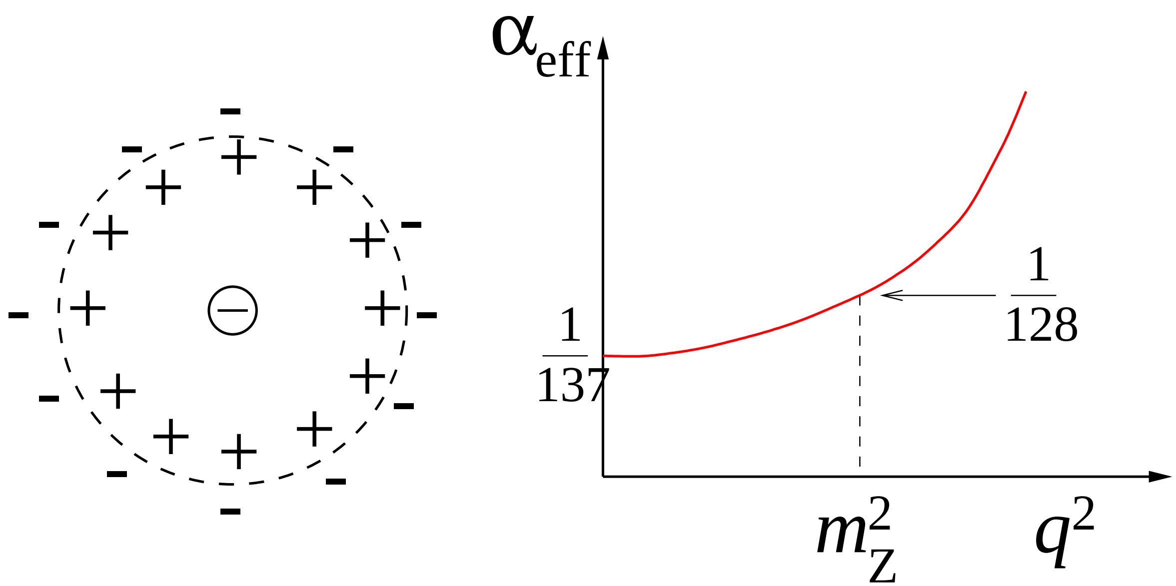
- Uncertainty in α_S propagates directly into QCD cross sections. Thus we expect errors at the percent level (at least) in prediction of cross sections which begin in order α_S.



- Using the formula for running $\alpha_S(Q)$ to rescale all measurements to $Q = M_Z$ gives excellent agreement.

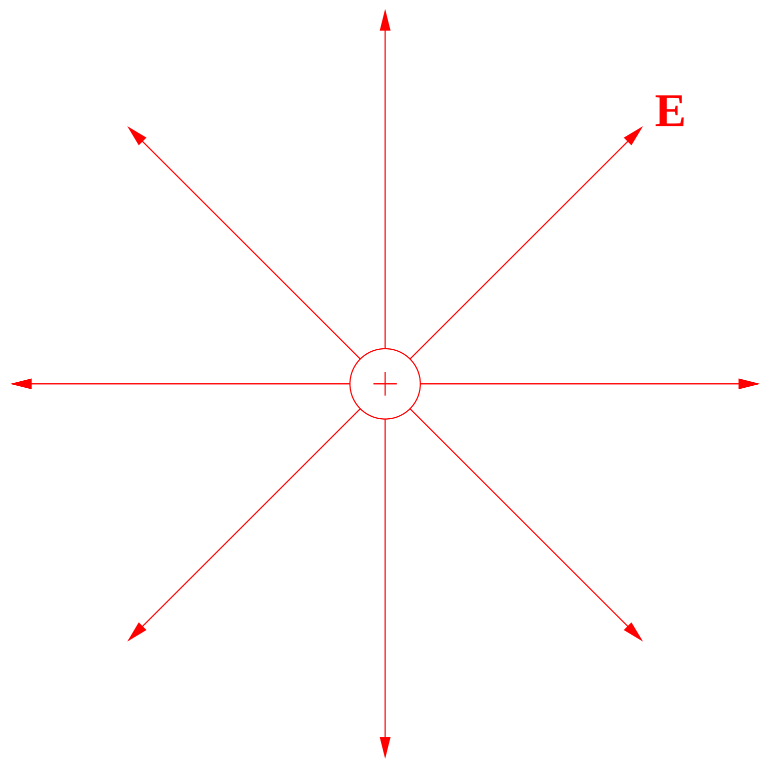
Charge Screening

- In QED the observed electron charge is distance-dependent ($\Rightarrow$ momentum transfer dependent) due to charge screening by the vacuum polarisation:



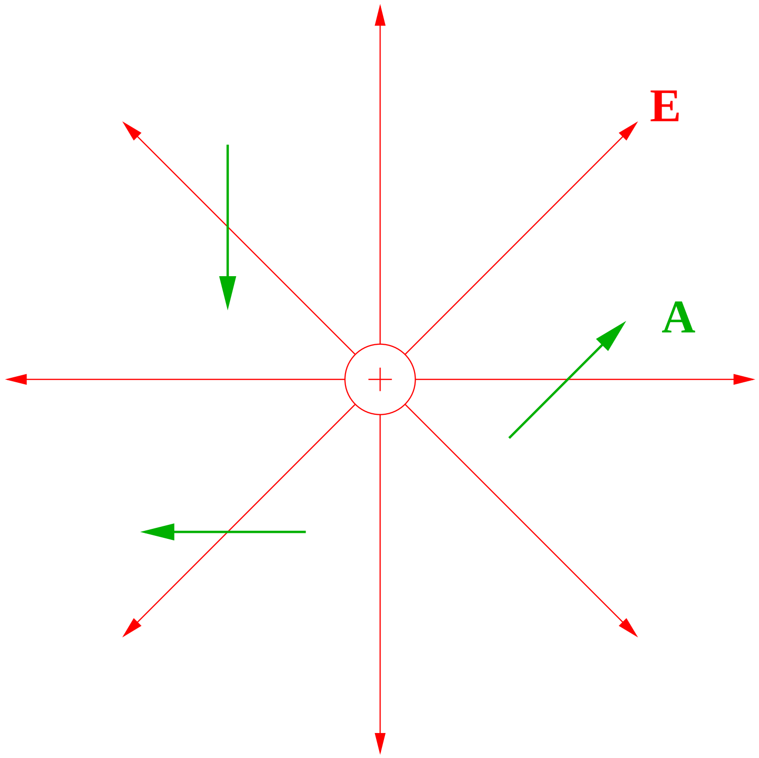
- At short distances (high momentum scales) we see more of the "bare" charge $\Rightarrow$ effective charge (coupling) increases.
- In contrast, the vacuum polarisation of a non-Abelian gauge field gives anti-screening.
  - ❖ Consider for simplicity an SU(2) gauge field: this has 3 "colours". . .

Non-Abelian Vacuum Polarization



$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = g\, \delta^3(\boldsymbol{r}) + g\, (\boldsymbol{A} \cdot \boldsymbol{E} - \boldsymbol{A} \cdot \boldsymbol{E})$$

Non-Abelian Vacuum Polarization



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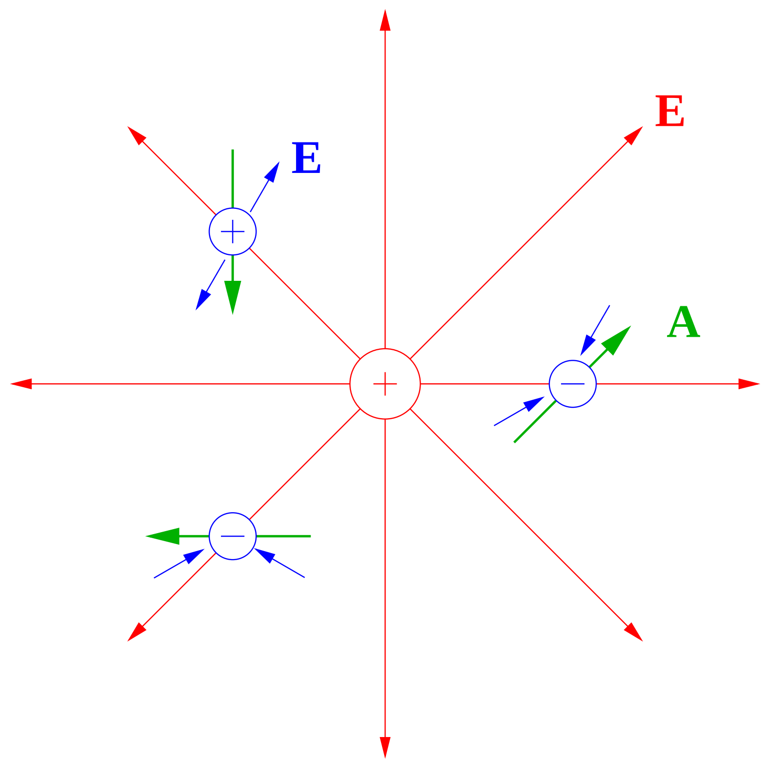

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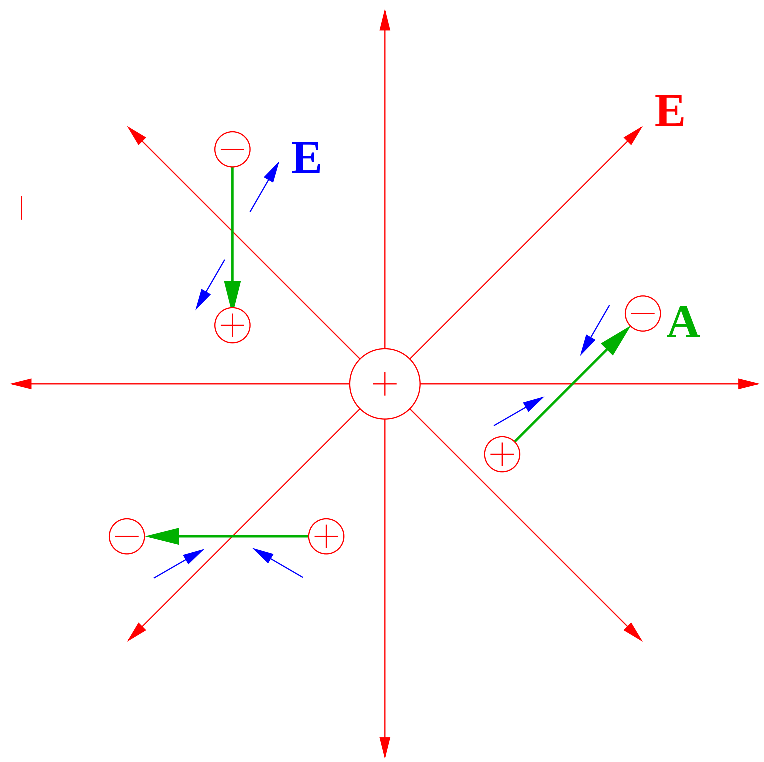


$$\nabla \cdot \boldsymbol{E} = g\,\delta^3(\boldsymbol{r}) + g\,(\boldsymbol{A} \cdot \boldsymbol{E} - \boldsymbol{A} \cdot \boldsymbol{E})$$

History of Asymptotic Freedom

1954 Yang & Mills study vector field theory with non-Abelian gauge invariance.

1965 Vanyashin & Terentyev compute vacuum polarization due to a massive charged vector field. In our notation, they found

$$b = \frac{1}{12\pi}\left(\frac{21}{2} = 11 - \frac{1}{2}\right)$$

The $\frac{1}{2}$ comes from longitudinal polarization states (absent for massless gluons)

❖ They concluded that this result ". . . seems extremely undesirable"

1969 Khriplovich correctly computes the one-loop β-function in SU(2) Yang-Mills theory using the Coulomb ($\nabla \cdot A = 0$) gauge

$$b = \frac{C_A}{12\pi}(12 - 1 = 11)$$

In Coulomb gauge the anti-screening (12) is due to an instantaneous Coulomb interaction

❖ He did not make a connection with strong interactions

1971 't Hooft computes the one-loop β-function for SU(3) gauge theory but does not publish it.

❖ He wrote it on the blackboard at a conference

❖ His supervisor (Veltman) told him it wasn't interesting

❖ 't Hooft & Veltman received the 1999 Nobel Prize for proving the *renormalizability* of QCD (and the whole Standard Model).

1972 Fritzsch & Gell-Mann propose that the strong interaction is an SU(3) gauge theory, later named QCD by Gell-Mann

1973 Gross & Wilczek, and independently Politzer, compute and publish the 1-loop β-function for QCD:

$$b = \frac{1}{12\pi}(11C_A - 2n_f)$$

⇒2004 Nobel Prize (now that 't Hooft has one anyway . . .)

1974 Caswell† and Jones compute the 2-loop β-function for QCD.

1980 Tarasov, Vladimirov & Zharkov compute the 3-loop β-function for QCD.

1997 van Ritbergen, Vermaseren & Larin compute the 4-loop β-function for QCD ($\sim 50,000$ Feynman diagrams):

"... We obtained in this way the following result for the 4-loop beta function in the $\overline{\mathrm{MS}}$-scheme:

$$q^2\frac{\partial a_s}{\partial q^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6)$$

where $a_s = \alpha_S/4\pi$ and . . .

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f \; , \quad \beta_1 = \frac{34}{3}C_A^2 - 4C_F T_F n_f - \frac{20}{3}C_A T_F n_f$$

$$\beta_2 = \frac{2857}{54}C_A^3 + 2C_F^2 T_F n_f - \frac{205}{9}C_F C_A T_F n_f - \frac{1415}{27}C_A^2 T_F n_f + \frac{44}{9}C_F T_F^2 n_f^2 + \frac{158}{27}C_A T_F^2 n_f^2$$

$$\begin{aligned}\beta_3 = {} & C_A^4 \left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right) + C_A^3 T_F n_f \left(-\frac{39143}{81} + \frac{136}{3}\zeta_3\right) \\ & + C_A^2 C_F T_F n_f \left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right) + C_A C_F^2 T_F n_f \left(-\frac{4204}{27} + \frac{352}{9}\zeta_3\right) \\ & + 46 C_F^3 T_F n_f + C_A^2 T_F^2 n_f^2 \left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right) + C_F^2 T_F^2 n_f^2 \left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right) \\ & + C_A C_F T_F^2 n_f^2 \left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right) + \frac{424}{243}C_A T_F^3 n_f^3 + \frac{1232}{243}C_F T_F^3 n_f^3 \\ & + \frac{d_A^{abcd} d_A^{abcd}}{N_A}\left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right) + n_f \frac{d_F^{abcd} d_A^{abcd}}{N_A}\left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right) \\ & + n_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A}\left(-\frac{704}{9} + \frac{512}{3}\zeta_3\right)\end{aligned}$$

Here ζ is the Riemann zeta-function ($\zeta_3 = 1.202\cdots$) and the colour factors for SU(N) are

$$T_F = \frac{1}{2}, \quad C_A = N, \quad C_F = \frac{N^2-1}{2N}, \quad \frac{d_A^{abcd}d_A^{abcd}}{N_A} = \frac{N^2(N^2+36)}{24},$$
$$\frac{d_F^{abcd}d_A^{abcd}}{N_A} = \frac{N(N^2+6)}{48}, \quad \frac{d_F^{abcd}d_F^{abcd}}{N_A} = \frac{N^4-6N^2+18}{96N^2}$$

- Substitution of these colour factors for $N = 3$ yields the following numerical results for QCD:

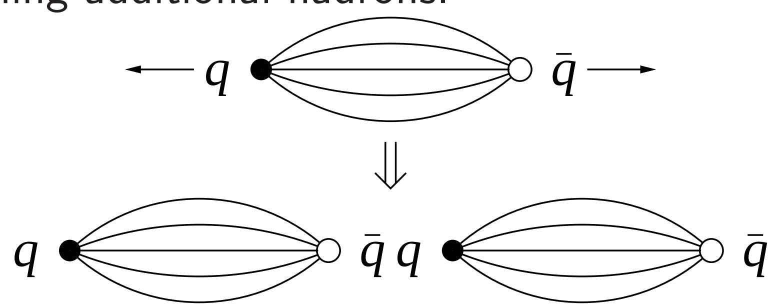
$$\begin{aligned}
\beta_0 &\approx 11 - 0.66667 n_f \\
\beta_1 &\approx 102 - 12.6667 n_f \\
\beta_2 &\approx 1428.50 - 279.611 n_f + 6.01852 n_f^2 \\
\beta_3 &\approx 29243.0 - 6946.30 n_f + 405.089 n_f^2 + 1.49931 n_f^3
\end{aligned}$$

- Expansion parameter $a_s = \alpha_S/4\pi \approx 0.01 \Rightarrow$ good convergence.


$\beta_g = \frac{-g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$
$\frac{-g^5}{(16\pi^2)^2}\left(\frac{34}{3}N_c^2 + \cdots\right)$
$\frac{-g^7}{(16\pi^2)^3}\left(\frac{2857}{54}N_c^3 + \cdots\right)$
$\frac{-g^9}{(16\pi^2)^4}\left(\cdots\right)$

Nonperturbative QCD

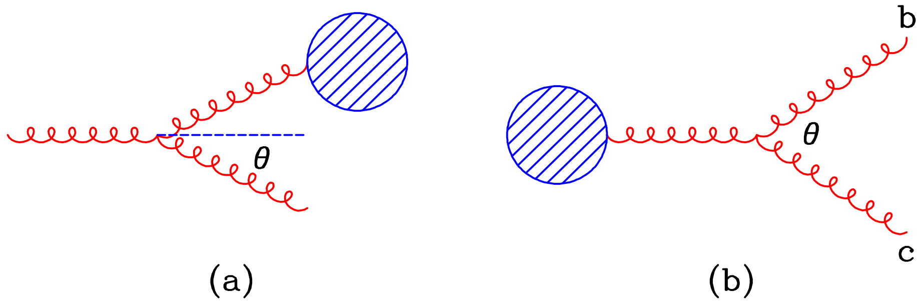
- Corresponding to asymptotic freedom at high momentum scales (short distances), we have infrared slavery: $\alpha_S(Q)$ becomes large at low momenta (long distances). Perturbation theory (PT) not reliable for large α_S, so nonperturbative methods (e.g. lattice) must be used.
- Important low momentum-scale phenomena:
  - Confinement: partons (quarks and gluons) found only in colour-singlet bound states (hadrons), size ~ 1 fm. If we try to separate them, it becomes energetically favourable to create extra partons, forming additional hadrons.



  - Hadronization: partons produced in short-distance interactions reorganize themselves (and multiply) to make observed hadrons.
- Note that confinement is a static (long-distance) property of QCD, treatable by lattice techniques whereas hadronization is a dynamical (long timescale) phenomenon: only models are available at present (see later).

Infrared Divergences

- Even in high-energy, short-distance regime, long-distance aspects of QCD cannot be ignored. Soft or collinear gluon emission gives infrared divergences in PT. Light quarks ($m_q \ll \Lambda$) also lead to divergences in the limit $m_q \longrightarrow 0$ (mass singularities).



❖ Spacelike branching: gluon splitting on incoming line (a)

$$p_b^2 = -E_a E_c (1 - \cos\theta) \leq 0 \ .$$

Propagator factor $1/p_b^2$ diverges as $E_c \longrightarrow 0$ (soft singularity) or $\theta \longrightarrow 0$ (collinear or mass singularity). If a and b are quarks, inverse propagator factor is

$$p_b^2 - m_q^2 = -E_a E_c (1 - v_a \cos\theta) \leq 0 \ ,$$

Hence $E_c \longrightarrow 0$ soft divergence remains; collinear enhancement becomes a divergence as $v_a \longrightarrow 1$, i.e. when quark mass is negligible. If emitted parton c is a quark, vertex factor cancels $E_c \longrightarrow 0$ divergence.

- **Timelike branching**: gluon splitting on outgoing line (b)

$$p_a^2 = E_b E_c (1 - \cos\theta) \geq 0 \; .$$

  Diverges when either emitted gluon is soft (E_b or $E_c \longrightarrow 0$) or when opening angle $\theta \longrightarrow 0$. If b and/or c are quarks, collinear/mass singularity in $m_q \longrightarrow 0$ limit. Again, soft quark divergences cancelled by vertex factor.

- Similar infrared divergences in loop diagrams, associated with soft and/or collinear configurations of virtual partons within region of integration of loop momenta.
- Infrared divergences indicate dependence on long-distance aspects of QCD not correctly described by PT. Divergent (or enhanced) propagators imply propagation of partons over long distances. When distance becomes comparable with hadron size ~ 1 fm, quasi-free partons of perturbative calculation are confined/hadronized non-perturbatively, and apparent divergences disappear.
- Can still use PT to perform calculations, provided we limit ourselves to two classes of observables:
  - Infrared safe quantities, i.e. those insensitive to soft or collinear branching. Infrared divergences in PT calculation either cancel between real and virtual contributions or are removed by kinematic factors. Such quantities are determined primarily by hard, short-distance physics; long-distance effects give power corrections, suppressed by inverse powers of a large momentum scale.
  - Factorizable quantities, i.e. those in which infrared sensitivity can be absorbed into an overall non-perturbative factor, to be determined experimentally.

- Infrared safe quantities:
  - Total cross section for $e^+e^- \longrightarrow$ hadrons: real and virtual divergences cancel. In $4-2\epsilon$ dimensions $[H(\epsilon) = 1 + \mathcal{O}(\epsilon)]$

$$\frac{\sigma_{\rm real}}{\sigma_{\rm Born}} = C_F \frac{\alpha_{\rm S}}{2\pi} H(\epsilon) \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon)\right]$$

$$\frac{\sigma_{\rm virt}}{\sigma_{\rm Born}} = 1 + C_F \frac{\alpha_{\rm S}}{2\pi} H(\epsilon) \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon)\right]$$

$$\frac{\sigma_{\rm tot}}{\sigma_{\rm Born}} = 1 + C_F \frac{3\alpha_{\rm S}}{4\pi} + \mathcal{O}(\alpha_{\rm S}^2) \ .$$

  - Event shapes: singularities cancelled by kinematics. Examples are Thrust and C-parameter:

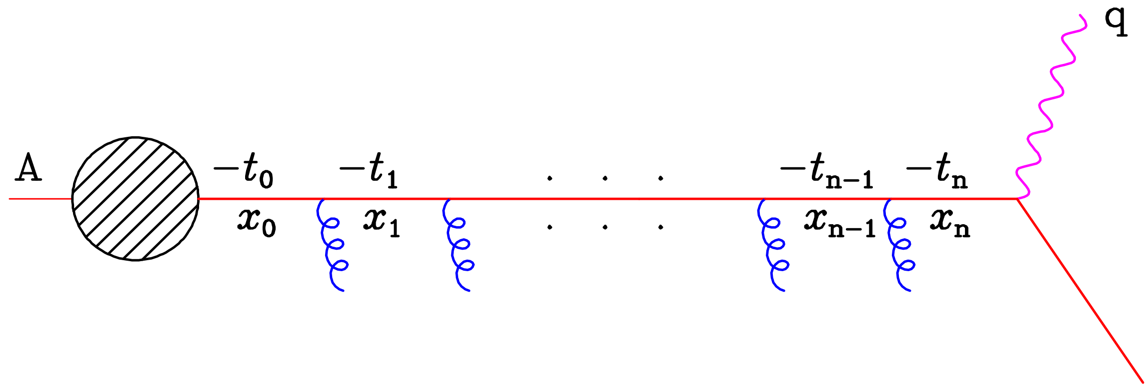
$$T = \max \frac{\sum_i |\boldsymbol{p}_i \cdot \boldsymbol{n}|}{\sum_i |\boldsymbol{p}_i|}$$

$$C = \frac{3}{2} \frac{\sum_{i,j} |\boldsymbol{p}_i|\,|\boldsymbol{p}_j|\,\sin^2\theta_{ij}}{(\sum_i |\boldsymbol{p}_i|)^2}$$

- Factorizable quantities: hadronic structure functions; jet fragmentation functions.

Scaling Violation and DGLAP Equation

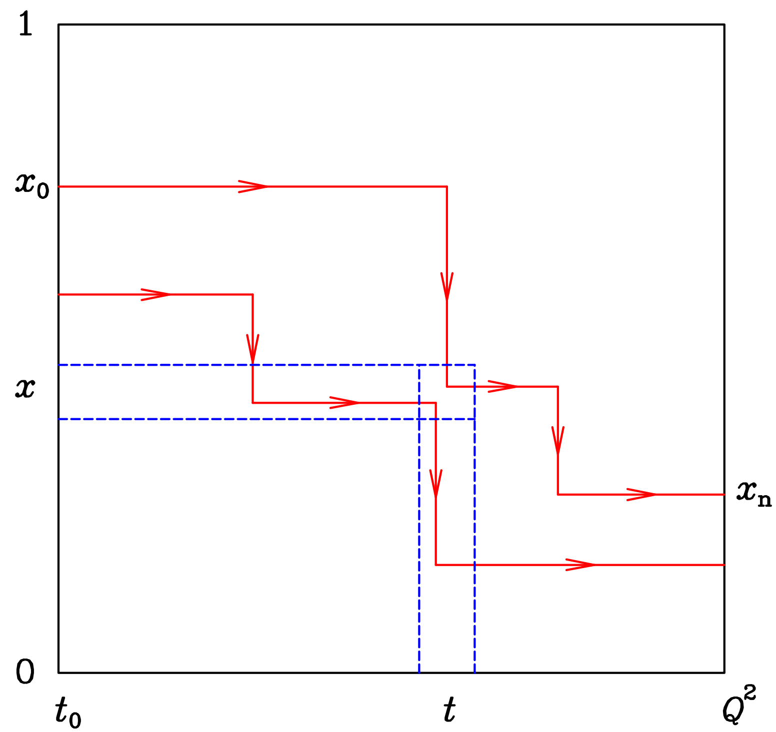
- Bjorken scaling is not exact. This is due to enhancement of higher-order contributions from small-angle parton branching, discussed earlier.



- Incoming quark from target hadron, initially with low virtual mass-squared $-t_0$ and carrying a fraction x_0 of hadron's momentum, moves to more virtual masses and lower momentum fractions by successive small-angle emissions, and is finally struck by photon of virtual mass-squared $q^2 = -Q^2$.

- Cross section will depend on Q^2 and on momentum fraction distribution of partons seen by virtual photon at this scale, $D(x, Q^2)$.

- To derive evolution equation for Q^2-dependence of $D(x, Q^2)$, first introduce pictorial representation of evolution, also useful for Monte Carlo simulation.



- Represent sequence of branchings by path in (t, x)-space. Each branching is a step downwards in x, at a value of t equal to (minus) the virtual mass-squared after the branching.
- At $t = t_0$, paths have distribution of starting points $D(x_0, t_0)$ characteristic of target hadron at that scale. Then distribution $D(x, t)$ of partons at scale t is just the x-distribution of paths at that scale.
- Consider change in the parton distribution $D(x, t)$ when t is increased to $t + \delta t$. This is number of paths arriving in element $(\delta t, \delta x)$ minus number leaving that element, divided by δx.

- Number arriving is branching probability times parton density integrated over all higher momenta $x' = x/z$,

$$\delta D_{\text{in}}(x,t) = \frac{\delta t}{t}\int_x^1 dx'\, dz \frac{\alpha_S}{2\pi}\hat{P}(z) D(x',t)\,\delta(x - zx')$$
$$= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z) D(x/z,t)$$

- For the number leaving element, must integrate over lower momenta $x' = zx$:

$$\delta D_{\text{out}}(x,t) = \frac{\delta t}{t} D(x,t)\int_0^x dx'\, dz \frac{\alpha_S}{2\pi}\hat{P}(z)\,\delta(x' - zx)$$
$$= \frac{\delta t}{t} D(x,t)\int_0^1 dz \frac{\alpha_S}{2\pi}\hat{P}(z)$$

- Change in population of element is

$$\delta D(x,t) = \delta D_{\text{in}} - \delta D_{\text{out}}$$
$$= \frac{\delta t}{t}\int_0^1 dz\,\frac{\alpha_S}{2\pi}\hat{P}(z)\left[\frac{1}{z}D(x/z,t) - D(x,t)\right] .$$

- Introduce plus-prescription with definition

$$\int_0^1 dz\ f(z)\, g(z)_+ \ = \ \int_0^1 dz\ [f(z) - f(1)]\, g(z)\ .$$

Using this we can define regularized splitting function

$$P(z) = \hat{P}(z)_+\ ,$$

and obtain Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation:

$$t\frac{\partial}{\partial t}D(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P(z)D(x/z,t)\ .$$

Beware! Note that

$$\begin{aligned}\int_x^1 dz\, f(z)g(z)_+ \quad &= \quad \int_0^1 dz\, \Theta(z-x)f(z)g(z)_+ \\ &= \quad \int_x^1 dz\, [f(z) - f(1)]g(z) - f(1)\int_0^x dz\, g(z)\end{aligned}$$

- Here $D(x,t)$ represents parton momentum fraction distribution inside incoming hadron probed at scale t. In timelike branching, it represents instead hadron momentum fraction distribution produced by an outgoing parton. Boundary conditions and direction of evolution are different, but evolution equation remains the same.

Quark and Gluon Distributions

- For several different types of partons, must take into account different processes by which parton of type i can enter or leave the element $(\delta t, \delta x)$. This leads to coupled DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}D_i(x,t) = \sum_j \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P_{ij}(z)D_j(x/z,t) \equiv \frac{\alpha_S}{2\pi}P_{ij}\otimes D_j$$

- Quark $(i = q)$ can enter element via either $q \longrightarrow qg$ or $g \longrightarrow q\bar{q}$, but can only leave via $q \longrightarrow qg$. Thus plus-prescription applies only to $q \longrightarrow qg$ part, giving

$$\begin{aligned} P_{qq}(z) &= \hat{P}_{qq}(z)_+ = C_F\left(\frac{1+z^2}{1-z}\right)_+ \\ P_{qg}(z) &= \hat{P}_{qg}(z) = T_R\left[z^2 + (1-z)^2\right] \end{aligned}$$

where $C_F = 4/3$ and $T_R = 1/2$ for colour group SU(3).

- Gluon can arrive either from $g \longrightarrow gg$ (2 contributions) or from $q \longrightarrow qg$ (or $\bar{q} \longrightarrow \bar{q}g$). Thus number arriving is

$$\delta D_{g,\mathrm{in}} = \frac{\delta t}{t}\int_0^1 dz \frac{\alpha_S}{2\pi}\left\{\hat{P}_{gg}(z)\left[\frac{D_g(x/z,t)}{z} + \frac{D_g(x/(1-z),t)}{1-z}\right]\right.$$
$$\left. + \frac{\hat{P}_{qq}(z)}{1-z}\left[D_q\left(\frac{x}{1-z},t\right) + D_{\bar{q}}\left(\frac{x}{1-z},t\right)\right]\right\}$$
$$= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}\left\{2\hat{P}_{gg}(z) D_g\left(\frac{x}{z},t\right) + \hat{P}_{qq}(1-z)\left[D_q\left(\frac{x}{z},t\right) + D_{\bar{q}}\left(\frac{x}{z},t\right)\right]\right\}$$

- Gluon can leave by splitting into either gg or $q\bar{q}$, so that

$$\delta D_{g,\mathrm{out}} = \frac{\delta t}{t} D_g(x,t)\int_0^1 dz \frac{\alpha_S}{2\pi}\left[\hat{P}_{gg}(z) + n_f \hat{P}_{qg}(z)\, dz\right] .$$

- After some manipulation we find ($C_A = 3$, n_f light flavours)

$$P_{gg}(z) = 2C_A\left[\left(\frac{z}{1-z} + \tfrac{1}{2}z(1-z)\right)_+ + \frac{1-z}{z} + \tfrac{1}{2}z(1-z)\right]$$
$$-\frac{2}{3} n_f T_R\, \delta(1-z) ,$$

$$P_{gq}(z) \quad = \quad P_{g\bar{q}}(z) = \hat{P}_{qq}(1-z) = C_F \frac{1+(1-z)^2}{z} \; .$$

- Using definition of the plus-prescription, can check that

$$\begin{aligned}
\left(\frac{z}{1-z} + \tfrac{1}{2} z(1-z)\right)_+ &= \frac{z}{(1-z)_+} + \tfrac{1}{2} z(1-z) + \frac{11}{12}\delta(1-z) \\
\left(\frac{1+z^2}{1-z}\right)_+ &= \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \; ,
\end{aligned}$$

so P_{qq} and P_{gg} can be written in more common forms

$$\begin{aligned}
P_{qq}(z) &= C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right] \\
P_{gg}(z) &= 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] \\
&\quad + \frac{1}{6}(11 C_A - 4 n_f T_R)\, \delta(1-z) \; .
\end{aligned}$$

Solution by Moments

- Given $D_i(x,t)$ at some scale $t = t_0$, factorized structure of DGLAP equation means we can compute its form at any other scale.
- One strategy for doing this is to take moments (Mellin transforms) with respect to x:

$$\tilde{D}_i(N,t) = \int_0^1 dx\; x^{N-1}\; D_i(x,t)\ .$$

Inverse Mellin transform is

$$D_i(x,t) = \frac{1}{2\pi i}\int_C dN\; x^{-N}\; \tilde{D}_i(N,t)\ ,$$

where contour C is parallel to imaginary axis to right of all singularities of integrand.

- After Mellin transformation, convolution in DGLAP equation becomes simply a product:

$$t\frac{\partial}{\partial t}\tilde{D}_i(N,t) = \sum_j \gamma_{ij}(N,\alpha_S)\tilde{D}_j(N,t)$$

where moments of splitting functions give PT expansion of anomalous dimensions γ_{ij}:

$$\begin{aligned}\gamma_{ij}(N,\alpha_S) &= \sum_{n=0}^{\infty}\gamma_{ij}^{(n)}(N)\left(\frac{\alpha_S}{2\pi}\right)^{n+1}\\ \gamma_{ij}^{(0)}(N) &= \tilde{P}_{ij}(N) = \int_0^1 dz\; z^{N-1}\; P_{ij}(z)\end{aligned}$$

- From above expressions for $P_{ij}(z)$ we find

$$
\begin{aligned}
\gamma_{qq}^{(0)}(N) &= C_F\left[-\frac{1}{2}+\frac{1}{N(N+1)}-2\sum_{k=2}^{N}\frac{1}{k}\right] \\
\gamma_{qg}^{(0)}(N) &= T_R\left[\frac{(2+N+N^2)}{N(N+1)(N+2)}\right] \\
\gamma_{gq}^{(0)}(N) &= C_F\left[\frac{(2+N+N^2)}{N(N^2-1)}\right] \\
\gamma_{gg}^{(0)}(N) &= 2C_A\left[-\frac{1}{12}+\frac{1}{N(N-1)}+\frac{1}{(N+1)(N+2)}\right. \\
&\qquad \left.-\sum_{k=2}^{N}\frac{1}{k}\right]-\frac{2}{3}n_f T_R\ .
\end{aligned}
$$

- Consider combination of parton distributions which is flavour non-singlet, e.g. $D_V = D_{q_i} - D_{\bar{q}_i}$ or $D_{q_i} - D_{q_j}$. Then mixing with the flavour-singlet gluons drops out and solution for fixed α_S is

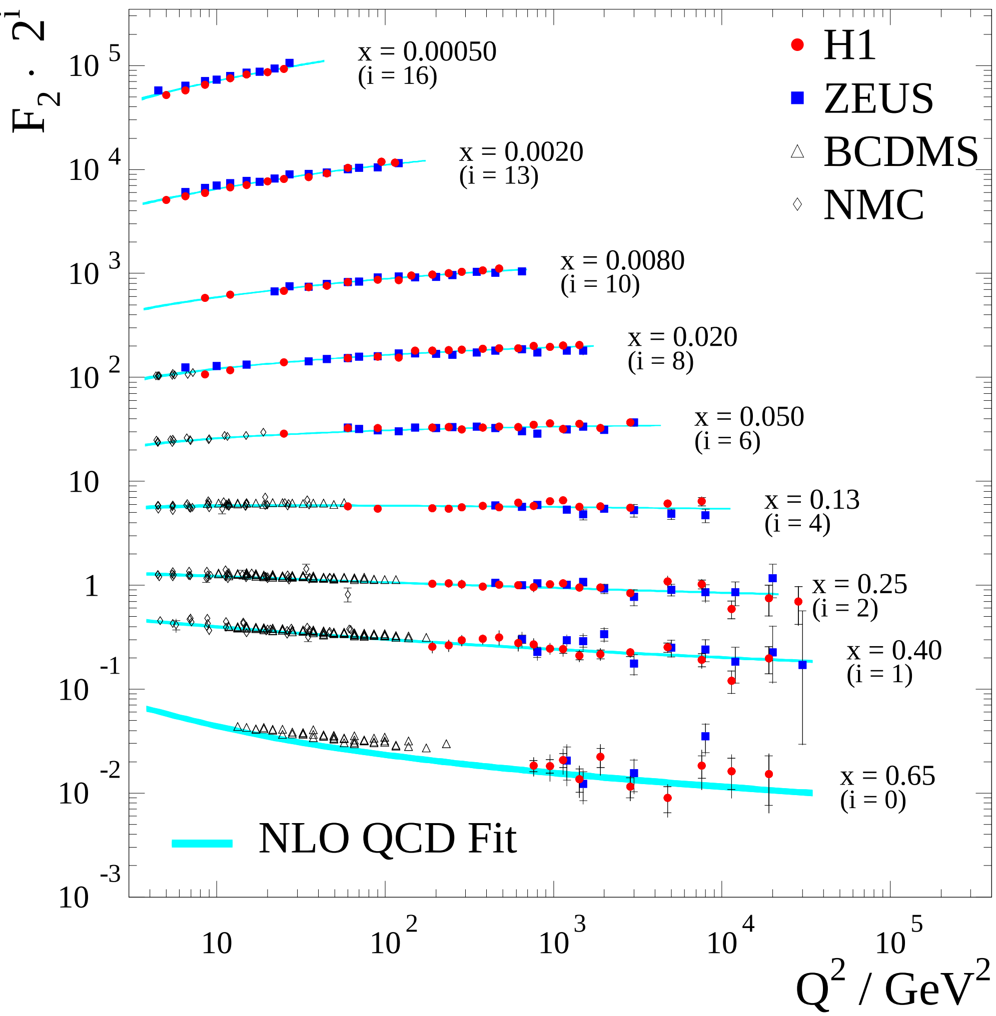
$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{t}{t_0}\right)^{\gamma_{qq}(N,\alpha_S)} \quad ,$$

- We see that dimensionless function D_V, instead of being scale-independent function of x as expected from dimensional analysis, has scaling violation: its moments vary like powers of scale t (hence the name anomalous dimensions).
- For running coupling $\alpha_S(t)$, scaling violation is power-behaved in $\ln t$ rather than t. Using leading-order formula $\alpha_S(t) = 1/b\ln(t/\Lambda^2)$, we find

$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{\alpha_S(t_0)}{\alpha_S(t)}\right)^{d_{qq}(N)}$$

  where $d_{qq}(N) = \gamma_{qq}^{(0)}(N)/2\pi b$.
- Now $d_{qq}(1) = 0$ and $d_{qq}(N) < 0$ for $N \geq 2$. Thus as t increases $\tilde{D}_V(N,t)$ is constant for $N = 1$ (*valence sum rule*) and decreases at larger N.

- Since larger-N moments emphasise larger x, this means that $D_V(x,t)$ *decreases* at large x and *increases* at small x. Physically, this is due to increase in the phase space for gluon emission by quarks as t increases, leading to loss of momentum. This is clearly visible in data:



- For flavour-singlet combination, define $\Sigma = \sum_i \left(D_{q_i} + D_{\bar{q}_i}\right)$. Then we obtain

$$t\frac{\partial \Sigma}{\partial t} = \frac{\alpha_S(t)}{2\pi}\left[P_{qq}\otimes\Sigma + 2n_f P_{qg}\otimes D_g\right]$$
$$t\frac{\partial D_g}{\partial t} = \frac{\alpha_S(t)}{2\pi}\left[P_{gq}\otimes\Sigma + P_{gg}\otimes D_g\right] \; .$$

- Thus flavour-singlet quark distribution Σ mixes with gluon distribution D_g: evolution equation for moments has matrix form

$$t\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\Sigma} \\ \tilde{D}_g \end{pmatrix} = \begin{pmatrix} \gamma_{qq} & 2n_f\gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix}\begin{pmatrix} \tilde{\Sigma} \\ \tilde{D}_g \end{pmatrix}$$

- Singlet anomalous dimension matrix has two real eigenvalues $\gamma_\pm$ given by

$$\gamma_\pm = \tfrac{1}{2}[\gamma_{gg} + \gamma_{qq} \pm \sqrt{(\gamma_{gg}-\gamma_{qq})^2 + 8n_f\gamma_{gq}\gamma_{qg}}] \; .$$

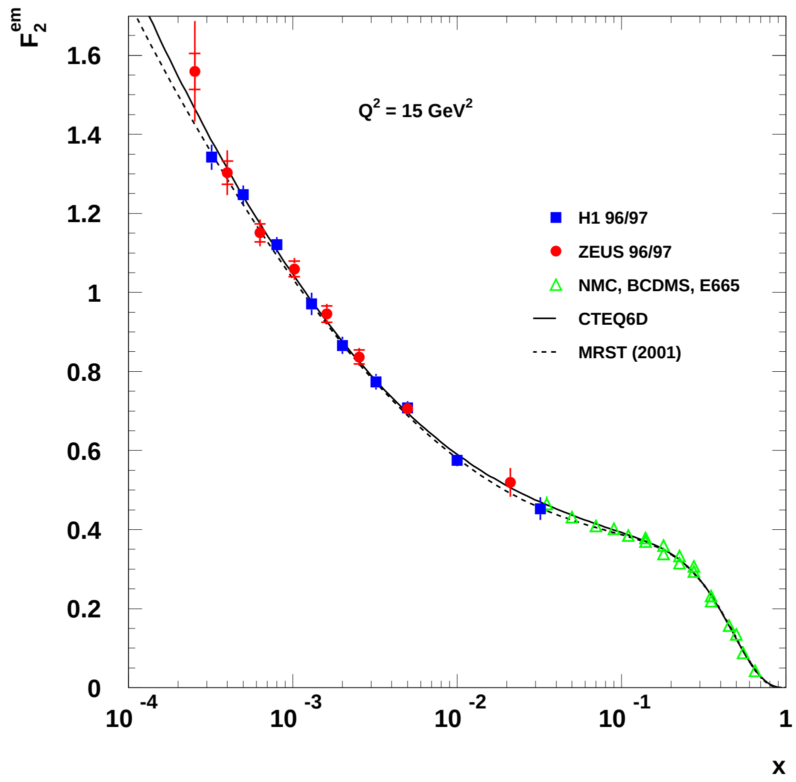
- Expressing $\tilde{\Sigma}$ and $\tilde{D}_g$ as linear combinations of eigenvectors $\tilde{\Sigma}_+$ and $\tilde{\Sigma}_-$, we find they evolve as superpositions of terms of above form with $\gamma_\pm$ in place of γ_{qq}.

Small x

- At small x, corresponding to $N \longrightarrow 1$,

$$\gamma_+ \longrightarrow \gamma_{gg} \longrightarrow \infty \ , \qquad \gamma_- \longrightarrow \gamma_{qq} \longrightarrow 0 \ ,$$

Therefore we expect structure functions to grow rapidly at small x, which is as observed:



- Higher-order corrections also become large in this region:

$$\begin{aligned}\gamma_{qq}^{(1)}(N) &\rightarrow \frac{40C_F n_f T_R}{9(N-1)}\\ \gamma_{qg}^{(1)}(N) &\rightarrow \frac{40C_A T_R}{9(N-1)}\\ \gamma_{gq}^{(1)}(N) &\rightarrow \frac{9C_F C_A - 40C_F n_f T_R}{9(N-1)}\\ \gamma_{gg}^{(1)}(N) &\rightarrow \frac{(12C_F - 46C_A) n_f T_R}{9(N-1)}\,.\end{aligned}$$

- Thus we find

$$\begin{aligned}\gamma_+ &\rightarrow \frac{2C_A}{N-1}\frac{\alpha_S}{2\pi}\left[1 + \frac{(26C_F - 23C_A)n_f}{18C_A}\frac{\alpha_S}{2\pi} + \ldots\right]\\ &= \frac{2C_A}{N-1}\frac{\alpha_S}{2\pi}\left[1 - 0.64 n_f\frac{\alpha_S}{2\pi} + \ldots\right]\end{aligned}$$

where neglected terms are either non-singular at $N=1$ or higher-order in α_S. Thus NLO correction is relatively small.

- In general one finds (BFKL) that for $N \to 1$

$$\gamma_+ \to \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{\gamma^{(n,m)}}{(N-1)^m} \left(\frac{\alpha_S}{2\pi}\right)^n$$

  - Each inverse power of $(N-1)$ corresponds to a $\log x$ enhancement at small x.
  - However, it happens that $\gamma^{(2,2)}$ and $\gamma^{(3,3)}$ are zero.
  - This is the main reason why substantial deviations from NLO QCD are not yet seen in DIS at small x.

Parton Showers

- DGLAP equations are convenient for evolution of parton distributions. To study structure of final states, a slightly different form is useful. Consider again simplified treatment with only one type of parton branching. Introduce the Sudakov form factor:

$$\Delta(t) \equiv \exp\left[-\int_{t_0}^{t}\frac{dt'}{t'}\int dz\,\frac{\alpha_S}{2\pi}\hat{P}(z)\right] ,$$

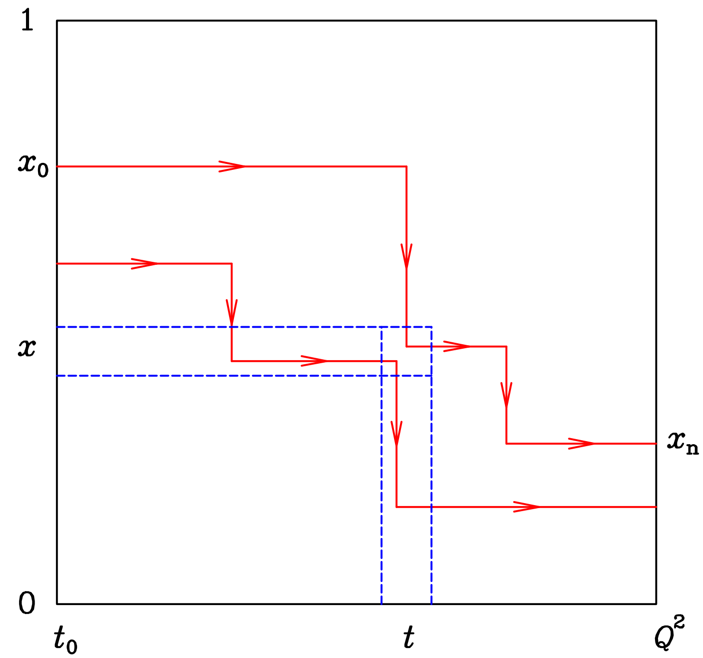
Then

$$\begin{aligned} t\frac{\partial}{\partial t}D(x,t) &= \int\frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)D(x/z,t) + \frac{D(x,t)}{\Delta(t)}t\frac{\partial}{\partial t}\Delta(t) , \\ t\frac{\partial}{\partial t}\left(\frac{D}{\Delta}\right) &= \frac{1}{\Delta}\int\frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)D(x/z,t) . \end{aligned}$$

- This is similar to DGLAP, except D is replaced by D/Δ and regularized splitting function P replaced by unregularized $\hat{P}$. Integrating,

$$\begin{aligned} D(x,t) &= \Delta(t)D(x,t_0) \\ &\quad + \int_{t_0}^{t}\frac{dt'}{t'}\frac{\Delta(t)}{\Delta(t')}\int\frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)D(x/z,t') . \end{aligned}$$

- This has simple interpretation. First term is contribution from paths that do not branch between scales t_0 and t. Thus Sudakov form factor $\Delta(t)$ is probability of evolving from t_0 to t without branching. Second term is contribution from paths which have their last branching at scale t'. Factor of $\Delta(t)/\Delta(t')$ is probability of evolving from t' to t without branching.



- Generalization to several species of partons straightforward. Species i has Sudakov form factor

$$\Delta_i(t) \equiv \exp\left[-\sum_j \int_{t_0}^{t} \frac{dt'}{t'} \int dz\, \frac{\alpha_S}{2\pi} \hat{P}_{ji}(z)\right] ,$$

which is probability of it evolving from t_0 to t without branching. Then

$$t\frac{\partial}{\partial t}\left(\frac{D_i}{\Delta_i}\right) = \frac{1}{\Delta_i}\sum_j \int \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}_{ij}(z) D_j(x/z, t) .$$

Infrared Cutoff

- In DGLAP equation, infrared singularities of splitting functions at $z = 1$ are regularized by plus-prescription. However, in above form we must introduce an explicit infrared cutoff, $z < 1 - \epsilon(t)$. Branchings with z above this range are unresolvable: emitted parton is too soft to detect. Sudakov form factor with this cutoff is probability of evolving from t_0 to t without any resolvable branching.
- Sudakov form factor sums enhanced virtual (parton loop) as well as real (parton emission) contributions. No-branching probability is the sum of virtual and unresolvable real contributions: both are divergent but their sum is finite.
- Infrared cutoff $\epsilon(t)$ depends on what we classify as resolvable emission. For timelike branching, natural resolution limit is given by cutoff on parton virtual mass-squared, $t > t_0$. When parton energies are much larger than virtual masses, transverse momentum in $a \to bc$ is

$$p_T^2 = z(1-z)p_a^2 - (1-z)p_b^2 - zp_c^2 > 0 \ .$$

Hence for $p_a^2 = t$ and $p_b^2,\ p_c^2 > t_0$ we require

$$z(1-z) > t_0/t \ ,$$

that is,

$$z,\ 1-z > \epsilon(t) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 4t_0/t} \simeq t_0/t \ .$$

- Quark Sudakov form factor is then

$$\Delta_q(t) \simeq \exp\left[-\int_{2t_0}^{t}\frac{dt'}{t'}\int_{t_0/t'}^{1-t_0/t'} dz\frac{\alpha_S}{2\pi}\hat{P}_{qq}(z)\right] \; .$$

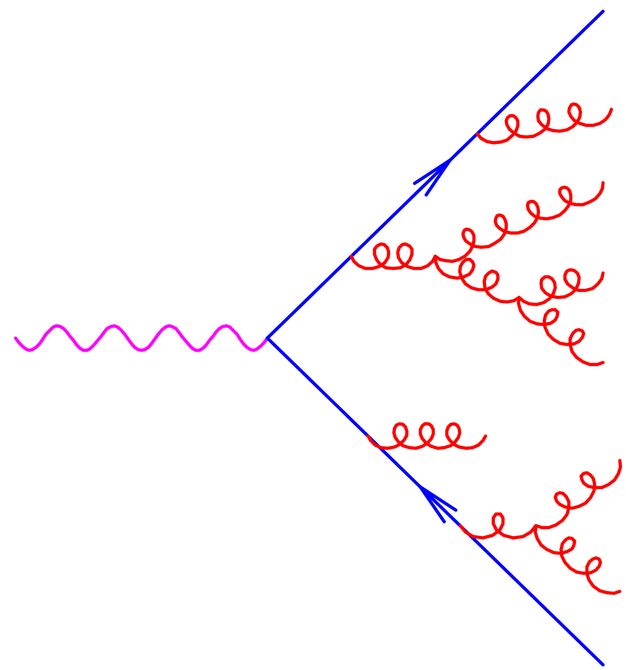
- Careful treatment of running coupling suggests its argument should be $p_T^2 \sim z(1-z)t'$. Then at large t

$$\Delta_q(t) \sim \left(\frac{\alpha_S(t)}{\alpha_S(t_0)}\right)^{p\ln t} \; ,$$

  ($p =$ a constant), which tends to zero faster than any negative power of t.

- Infrared cutoff discussed here follows from kinematics. We shall see later that QCD dynamics effectively reduces phase space for parton branching, leading to a more restrictive effective cutoff.

- Each emitted (timelike) parton can itself branch. In that case t evolves downwards towards cutoff value t_0, rather than upwards towards hard process scale Q^2. Due to successive branching, a parton cascade or shower develops. Each outgoing line is source of new cascade, until all outgoing lines have stopped branching. At this stage, which depends on cutoff scale t_0, outgoing partons have to be converted into hadrons via a hadronization model.



- Figure shows (schematically) a typical parton shower in $Z^0 \longrightarrow$ hadrons: for a resolution scale $t_0 \sim 1$ GeV^2, about 7 gluons are emitted.

Polarization Effects

- Correlation between plane of polarization of initial gluon and plane of branching (angle ϕ) in $g \to gg$:

$$\hat{P}_{gg}(z,\phi) = 2C_A \left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z) + z(1-z)\cos 2\phi\right] \, .$$

  Hence branching in plane of gluon polarization preferred.

- For $g \to q\bar{q}$:

$$\hat{P}_{qg}(z,\phi) = T_R \left[z^2 + (1-z)^2 - 2z(1-z)\cos 2\phi\right]$$

  i.e. strong preference for splitting perpendicular to polarization.

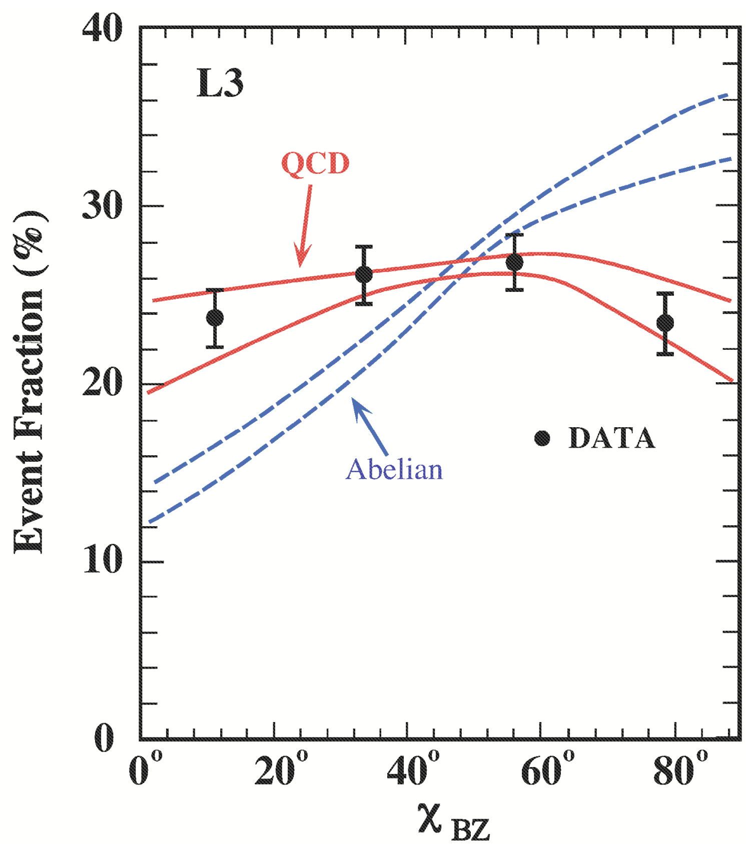
- Branching $q \to qg$:
  - Helicity conservation ensures that quark does not change helicity in branching.
  - Gluon polarized in plane of branching preferred:

$$\hat{P}_{qq}(z,\phi) = C_F \left[\frac{1+z^2}{1-z} + \frac{2z}{1-z}\cos 2\phi\right] \, .$$

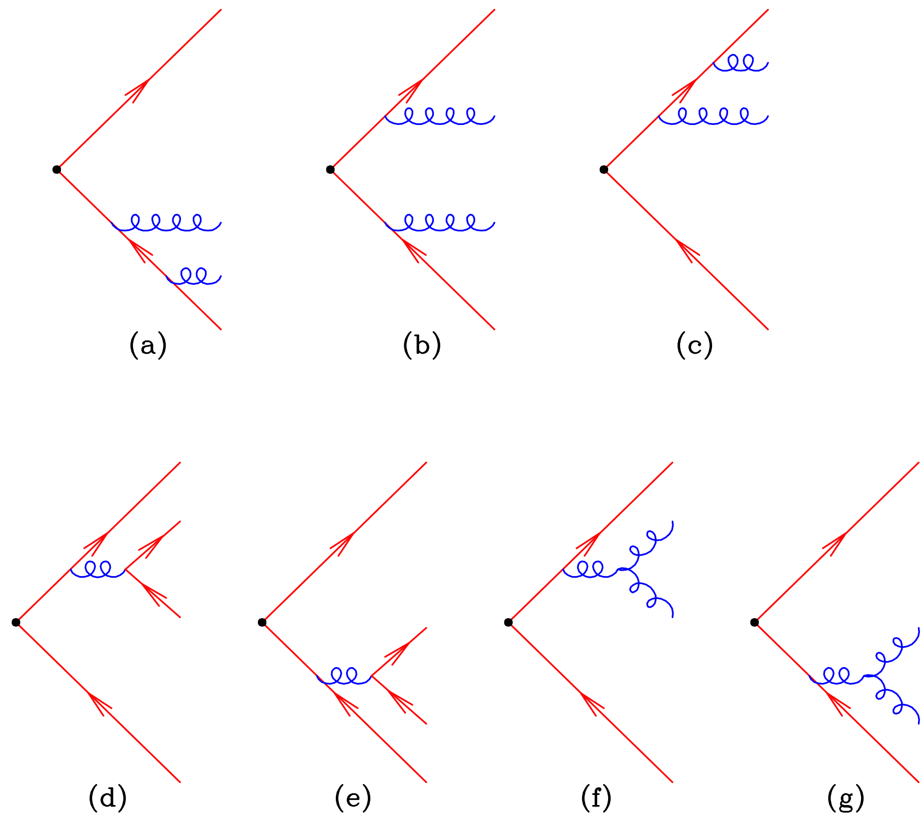
Four-Jet Angular Distribution

- Angular correlations are illustrated by the angular distribution in $e^+e^- \longrightarrow 4$ jets. Bengtsson-Zerwas angle χ_{BZ} is angle between the planes of two lowest and two highest energy jets:

$$\cos\chi_{BZ} = \frac{(\boldsymbol{p}_1 \times \boldsymbol{p}_2) \cdot (\boldsymbol{p}_3 \times \boldsymbol{p}_4)}{|\boldsymbol{p}_1 \times \boldsymbol{p}_2|\ |\boldsymbol{p}_3 \times \boldsymbol{p}_4|}\ .$$

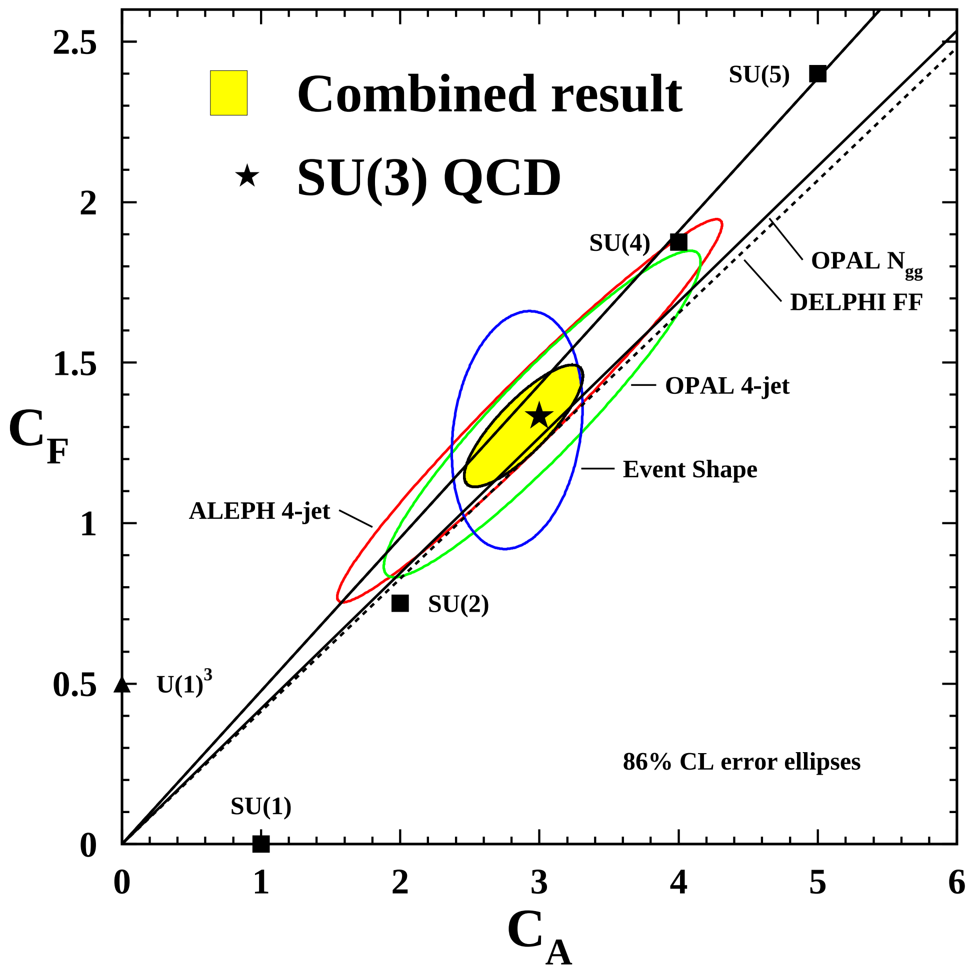


- Lowest-order diagrams for 4-jet production shown below. Two hardest jets tend to follow directions of primary $q\bar{q}$.



- "Double bremsstrahlung" diagrams give negligible correlations.
- $g \longrightarrow q\bar{q}$ give strong anti-correlation ("Abelian" curve), because gluon tends to be polarized in plane of primary jets and prefers to split perpendicular to polarization.
- $g \longrightarrow gg$ occurs more often parallel to polarization. Although its correlation is much weaker than in $g \longrightarrow q\bar{q}$, $g \longrightarrow gg$ is dominant in QCD due to larger colour factor and soft gluon enhancements.
- Thus B-Z angular distribution is flatter than in an Abelian theory.

- Combining with fits to e^+e^- event shape distributions allows determination of the colour factors C_A and C_F.



Summary of Lecture 1

- Deep inelastic lepton scattering (DIS) reveals parton structure of hadrons.
  - Pointlike constituents $\Rightarrow$ Bjorken scaling.
  - Sum rules reveal properties of partons.
  - Gluons inferred from missing momentum.
- QCD charge anti-screening$\Rightarrow$ asymptotic freedom
  - Infrared safe quantities perturbatively computable.
  - Factorization $\Rightarrow$ violation of Bjorken scaling also computable.
  - Leading contribution due to multiple small-angle parton branching.
- Parton distributions evolve according to DGLAP equations.
  - These involve convolutions $\Rightarrow$ solve by taking moments (x^{N-1})
  - Divergences as $N \to 1$ lead to rapid increase in parton distributions at small x.
- Emitted partons can also branch, leading to parton showers.
  - Showers determine broad structure of final state.
  - Sudakov form factor gives probability of evolution without resolvable branching.
  - Can follow parton showers until evolution scale becomes too low for perturbation theory $\Rightarrow$ infrared cutoff. Then we need hadronization model.
  - Gluon polarization leads to angular correlations in showers.