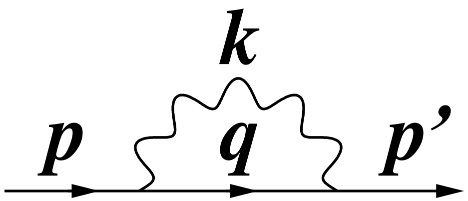
Ultraviolet Divergences

- In higher-order perturbation theory we encounter Feynman graphs with closed loops, associated with unconstrained momenta.
- For every such momentum k^μ, we have to integrate over all values, i.e.

$$\int \frac{d^4k}{(2\pi)^4}$$

E.g. "electron self-energy" in QED: 

$$\begin{aligned}
\mathcal{A}_{fi} &= \int \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\bar{u}(p')\gamma^\mu \frac{-ig_{\mu\nu}}{k^2}\frac{i(\not{q}+m)}{q^2-m^2}\gamma^\nu u(p) \\
&\times \quad (-ie)(2\pi)^4\delta^4(p-q-k)(-ie)(2\pi)^4\delta^4(q+k-p') \\
&= -e^2(2\pi)^4\delta^4(p-p') \\
&\times \quad \bar{u}(p)\int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu(\not{p}-\not{k}+m)\gamma_\mu}{k^2[(p-k)^2-m^2]}u(p)
\end{aligned}$$

- $\int^\infty d^4k \frac{k}{k^2(p-k)^2}$ is divergent!

- We say that $\int d^4k\, k^{D-4}$ has superficial degree of divergence D

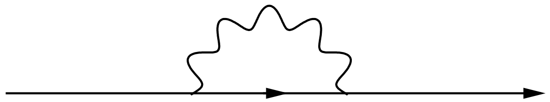
$$\begin{aligned} D \;\; = \;\; & 0 \Rightarrow \text{log-divergent} \\ & 1 \Rightarrow \text{linearly divergent} \\ & 2 \Rightarrow \text{quadratically divergent} \end{aligned}$$

- The actual degree of divergence may be less, e.g. due to cancellations required by gauge invariance. For example, the electron self-energy is actually only log-divergent. Putting an upper cut-off Λ on the integral, one finds

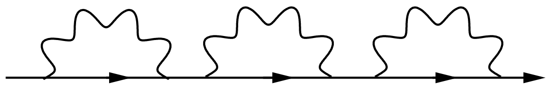
$$\mathcal{A}_{fi} \sim -i(2\pi)^4\, \delta^4(p-p')\frac{3\alpha}{2\pi} m \ln\left(\frac{\Lambda}{m}\right) + \ldots$$

- If the theory has only a finite set of (classes of) divergent (i.e. cut-off dependent) diagrams, their contributions can be absorbed into redefinitions of the coupling constant(s) and masses. This is called renormalization.
- For example, iteration of the electron self-energy leads to renormalization of the electron mass. Defining $\Sigma = -\frac{3m}{8\pi^2}\ln\frac{\Lambda}{m} + \ldots$ we have

$$\frac{i(\not{p}+m)}{p^2-m^2} \equiv \frac{i}{\not{p}-m}$$



$$\frac{i}{\not{p}-m} ie^2\Sigma \frac{i}{\not{p}-m}$$



$$\frac{i}{\not{p}-m} \sum_n \left[ie^2\Sigma \frac{i}{\not{p}-m} \right]^n$$

$$= \frac{i}{\not{p}-m} \left[1 - ie^2\Sigma \frac{i}{\not{p}-m} \right]^{-1} = \frac{i}{\not{p}-m+e^2\Sigma}$$

- Hence $m \to m + \delta m$ where

$$\frac{\delta m}{m} = \frac{3\alpha}{2\pi} \ln \frac{\Lambda}{m} + \ldots$$

The real, observed mass is $m + \delta m$. The bare mass, i.e. the parameter in the Lagrangian, is not observable, and indeed depends on Λ if we keep the observed mass fixed.

Renormalizability

- How many classes of superficially divergent graphs are there in QED? We have
  - $\int d^4k$ for every loop (unconstrained momentum)
  - $\frac{i}{\not{k}-m}$ for every internal fermion line (electron)
  - $\frac{-ig^{\mu\nu}}{k^2}$ for every internal boson line (photon)

$$\Rightarrow \quad D = 4L - F_I - 2B_I \qquad \text{where}$$

$$\begin{aligned} L &= \text{number of unconstrained momenta} \\ F_I &= \text{number of internal fermion lines} \\ B_I &= \text{number of internal boson lines} \end{aligned}$$

- But if V is the number of vertices,

$$L = F_I + B_I - V + 1$$

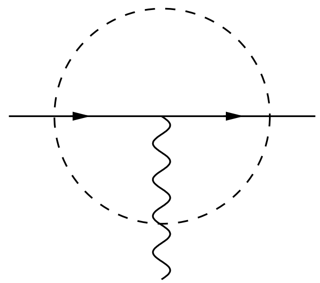
- If vertex involves F_V fermions and B_V bosons, we have by 'conservation of ends':

$$\begin{aligned}\sum_V F_V &= 2F_I + F_E \\ \sum_V B_V &= 2B_I + B_E\end{aligned}$$

where

$$\begin{aligned}F_E &= \text{number of external fermion lines} \\ B_E &= \text{number of external boson lines}\end{aligned}$$

- In QED, $F_V = 2$, $B_V = 1$



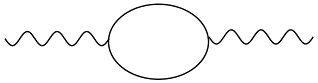
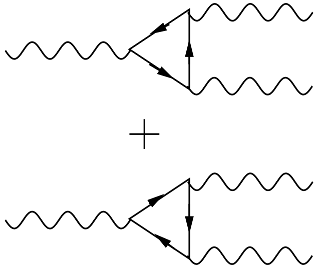
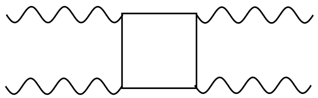
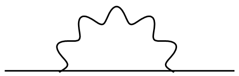
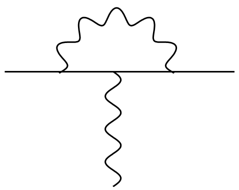
$$\begin{aligned}\Rightarrow \qquad V &= F_I + \tfrac{1}{2}F_E = 2B_I + B_E \\ \Rightarrow \qquad D &= 4 - \tfrac{3}{2}F_E - B_E\end{aligned}$$

Note that D is independent of L and V.

- Thus there is only a finite number of classes of superficially divergent diagrams in QED, with

$$D = 4 - \frac{3}{2} F_E - B_E \geq 0$$

- There are only 5 classes of superficially divergent graphs in QED, of which 3 are actually (log) divergent.

F_E	B_E	D	Diagrams	Remarks
0	2	2		photon self-energy: log-divergent $\Rightarrow$ charge renormalization
0	3	1		$= 0$ to all orders
0	4	0		light-by-light scattering actually convergent
2	0	1		electron self-energy: log-divergent $\Rightarrow$ mass & charge renorm'n
2	1	0		vertex correction: log-divergent $\Rightarrow$ charge renormalization

N.B. In QED, charge renormalization from electron self-energy and vertex correction cancel, so it can be ascribed entirely to photon self-energy (vacuum polarization).

Dimensions of Fields and Couplings

- In natural units we have only mass (equivalently, energy or momentum) dimensions: $x \sim ct \sim \hbar c/E \sim \hbar/mc$.

$$\hbar = c = 1 \Rightarrow [L] = [T] = [E]^{-1} = [M]^{-1}$$

- Hence action S (units $\hbar$) is dimensionless, and

$$S = \int \mathcal{L}\, d^4x \qquad \Rightarrow \qquad [\mathcal{L}] = [x]^{-4} = [M]^4$$

Furthermore $[\partial^\mu] = [p^\mu] = [M]$. From this we can deduce dimensions of fields and couplings:

$$\begin{aligned}
\mathcal{L}_{\mathrm{KG}} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \quad &\Rightarrow \quad [\phi] = [M] \\
\mathcal{L}_{\mathrm{D}} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \quad &\Rightarrow \quad [\psi] = [M]^{3/2} \\
\mathcal{L}_{\mathrm{em}} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad &\Rightarrow \quad [F^{\mu\nu}] = [M]^2 \\
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad &\Rightarrow \quad [A^\mu] = [M]
\end{aligned}$$

Higgs self-coupling:

$$\lambda(\phi^\dagger\phi)^2 \quad \Rightarrow \quad [\lambda] = [M]^0$$

Gauge couplings:

$$D^\mu = \partial^\mu + ieA^\mu(+igW^\mu) \quad \Rightarrow \quad [e] = [g] = [M]^0$$

Fermi coupling:

$$G_F(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi) \quad \Rightarrow \quad [G_F] = [M]^{-2}$$

Yukawa coupling:

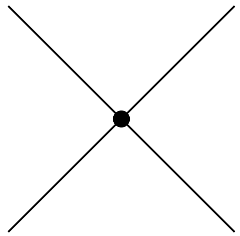
$$g_f\phi\bar\psi\psi \quad \Rightarrow \quad [g_f] = [M]^0$$

- Thus in any theory we can associate dimension 4 with any vertex, as follows

$$4 = \tfrac{3}{2}F_V + B_V + P_V + g_V$$

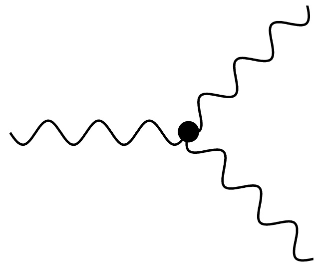
where P_V = number of momentum factors, g_V = dimension of coupling.

For example...



Fermi

$$4 = \tfrac{3}{2}(4) + 0 + 0 + (-2)$$



3-gauge-boson

$$4 = \tfrac{3}{2}(0) + 3 + 1 + 0$$

- Now we can derive superficial degree of divergence in any theory:

$$D = 4L - F_I - 2B_I + \sum_V P_V$$

Recall that $L = F_I + B_I - V + 1$ and

$$\sum_V F_V = 2F_I + F_E\ , \quad \sum_V B_V = 2B_I + B_E$$

$$D \quad = \quad 4 - 4V + 3F_I + 2B_I + \sum_V P_V$$

$$
\begin{aligned}
&= 4 - 4V - \tfrac{3}{2}F_E - B_E + \sum_V (\tfrac{3}{2}F_V + B_V + P_V = 4 - g_V) \\
&= 4 - \tfrac{3}{2}F_E - B_E - \sum_V g_V
\end{aligned}
$$

- Standard Model couplings are all dimensionless, so $\sum_V g_V = 0$ and the situation is similar to QED:
  - Finite number of divergent sub-graphs ('primitive divergences')
  - Can absorb cut-off dependence in bare parameters of Lagrangian
  - Hence theory is renormalizable

  N.B. Lots of work needed to *prove* this ('t Hooft and Veltman $\Rightarrow$ Nobel prize).

- Non-standard vertices have $g_V < 0$, so D gets larger and larger in higher orders of perturbation theory $\Rightarrow$ theory becomes unrenormalizable. For example

6-Higgs coupling:

$$\lambda_6(\phi^\dagger\phi)^3 \quad \Rightarrow \quad [\lambda_6] = [M]^{-2}$$

Fermi coupling:

$$G_F(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi) \quad \Rightarrow \quad [G_F] = [M]^{-2}$$

2-boson Yukawa coupling:

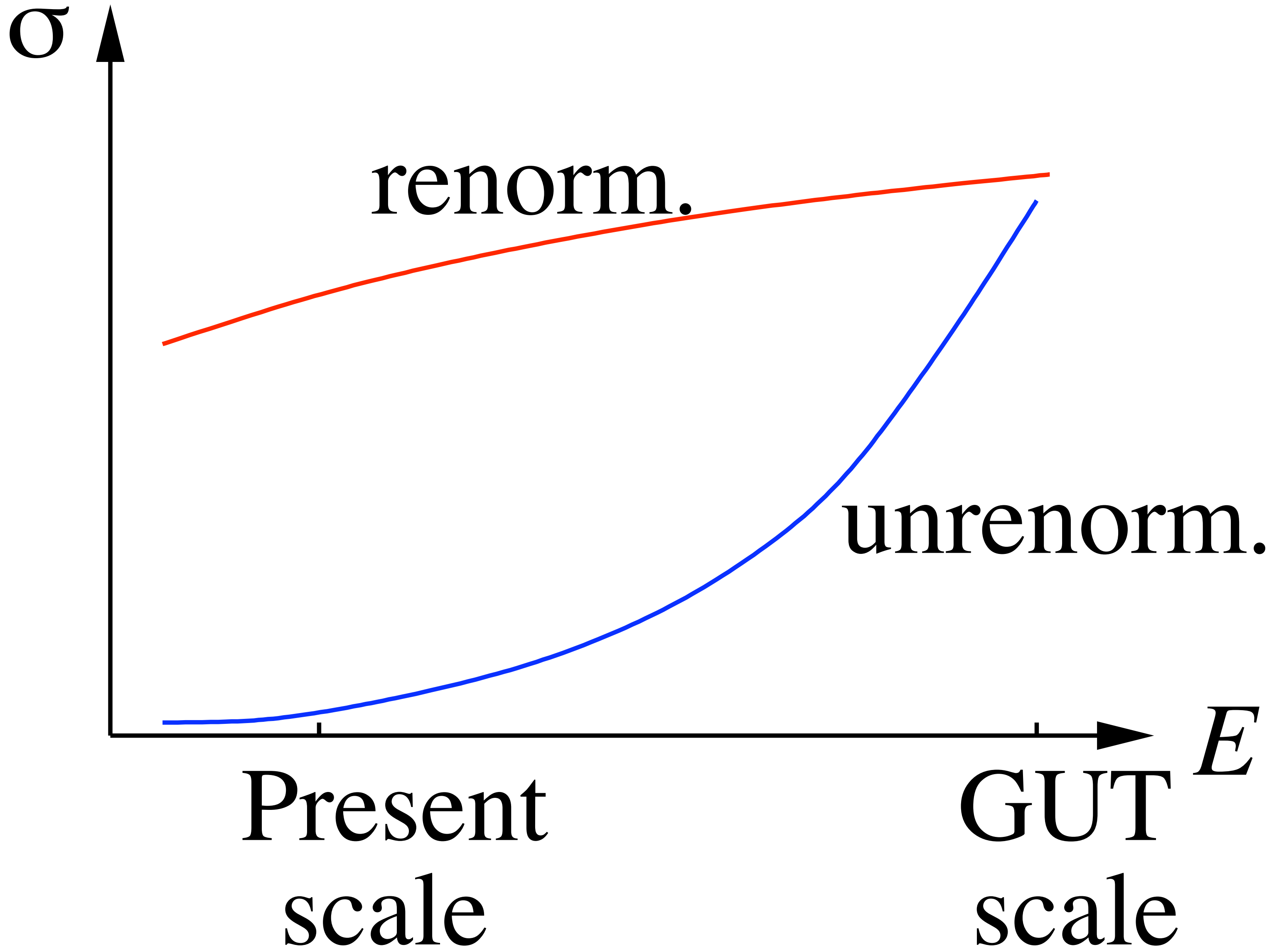
$$\lambda_f \phi^\dagger \phi \bar{\psi} \psi \quad \Rightarrow \quad [\lambda_f] = [M]^{-1}$$

- Is it surprising that Nature provides only renormalizable interactions? Maybe not, because unrenormalizability $\Rightarrow$ bad (divergent) high-energy behaviour.

  E.g. Fermi theory:

$$\begin{aligned} & \sigma(\nu_e e) \sim G_F^2 \\ & [G_F] = [M]^{-2}\,, \quad [\sigma] = [M]^{-2} \\ \Rightarrow \quad & \sigma(\nu_e e) \sim G_F^2 E^2 \to \infty \end{aligned}$$

- Thus if we suppose there exists a finite theory at very high energies (GUT? SUSY? Strings?), all unrenormalizable interactions will have shrunk to negligible values in going from that high scale to present energies:


σ
renorm.
unrenorm.
E
Present
scale
GUT
scale