Ultraviolet Divergences

In higher-order perturbation theory we encounter Feynman graphs with closed loops, associated with unconstrained momenta.

For every such momentum \( k^\mu \), we have to integrate over all values, i.e.

\[
\int \frac{d^4k}{(2\pi)^4}
\]

E.g. “electron self-energy” in QED:

\[
\mathcal{A}_{fi} = \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{u}(p') \gamma^\mu \frac{-ig_{\mu\nu}}{k^2} \frac{i(q + m)}{q^2 - m^2} \gamma^\nu u(p) \\
\times (-ie)(2\pi)^4 \delta^4(p - q - k)(-ie)(2\pi)^4 \delta^4(q + k - p') \\
= -e^2(2\pi)^4 \delta^4(p - p') \\
\times \bar{u}(p) \int \frac{d^4k}{(2\pi)^4} \gamma^\mu (p' - k + m) \gamma_\mu u(p)
\]

\[
\int_{\infty} \frac{d^4k}{k^2(p - k)^2} \]

is divergent!
We say that $\int d^4 k k^{D-4}$ has superficial degree of divergence $D$

\[
D = 0 \Rightarrow \text{log-divergent} \\
1 \Rightarrow \text{linearly divergent} \\
2 \Rightarrow \text{quadratically divergent}
\]

The actual degree of divergence may be less, e.g. due to cancellations required by gauge invariance. For example, the electron self-energy is actually only log-divergent. Putting an upper cut-off $\Lambda$ on the integral, one finds

\[
\mathcal{A}_{fi} \sim -i(2\pi)^4 \delta^4(p - p') \frac{3\alpha}{2\pi} m \ln \left( \frac{\Lambda}{m} \right) + \ldots
\]

If the theory has only a finite set of (classes of) divergent (i.e. cut-off dependent) diagrams, their contributions can be absorbed into redefinitions of the coupling constant(s) and masses. This is called renormalization.

For example, iteration of the electron self-energy leads to renormalization of the electron mass. Defining $\Sigma = -\frac{3m}{8\pi^2} \ln \frac{\Lambda}{m} + \ldots$ we have
\[ \frac{i(\not{p} + m)}{p^2 - m^2} \equiv \frac{i}{\not{p} - m} \]
\[ \frac{i}{\not{p} - m} ie^2 \sum \frac{i}{\not{p} - m} \]
\[ = \frac{i}{\not{p} - m} \left[ 1 - ie^2 \sum \frac{i}{\not{p} - m} \right]^{-1} = \frac{i}{\not{p} - m + e^2 \sum} \]

- Hence \( m \to m + \delta m \) where

\[ \frac{\delta m}{m} = \frac{3\alpha}{2\pi} \ln \frac{\Lambda}{m} + \ldots \]

The real, observed mass is \( m + \delta m \). The bare mass, i.e. the parameter in the Lagrangian, is not observable, and indeed depends on \( \Lambda \) if we keep the observed mass fixed.
Renormalizability

● How many classes of superficially divergent graphs are there in QED? We have

❖ \( \int d^4k \) for every loop (unconstrained momentum)
❖ \( \frac{i}{k^2-m} \) for every internal fermion line (electron)
❖ \( \frac{-ig^{\mu\nu}}{k^2} \) for every internal boson line (photon)

\[ \Rightarrow D = 4L - F_I - 2B_I \]

where

\[
\begin{align*}
L &= \text{number of unconstrained momenta} \\
F_I &= \text{number of internal fermion lines} \\
B_I &= \text{number of internal boson lines}
\end{align*}
\]

● But if \( V \) is the number of vertices,

\[ L = F_I + B_I - V + 1 \]
If vertex involves $F_V$ fermions and $B_V$ bosons, we have by ‘conservation of ends’:

\[
\sum_V F_V = 2F_I + F_E \\
\sum_V B_V = 2B_I + B_E
\]

where

$F_E = \text{number of external fermion lines}$

$B_E = \text{number of external boson lines}$

In QED, $F_V = 2$, $B_V = 1$

\[
\Rightarrow V = F_I + \frac{1}{2}F_E = 2B_I + B_E \\
\Rightarrow D = 4 - \frac{3}{2}F_E - B_E
\]
Note that $D$ is independent of $L$ and $V$.

Thus there is only a finite number of classes of superficially divergent diagrams in QED, with

$$D = 4 - \frac{3}{2} F_E - B_E \geq 0$$

There are only 5 classes of superficially divergent graphs in QED, of which 3 are actually (log) divergent.
<table>
<thead>
<tr>
<th>$F_E$</th>
<th>$B_E$</th>
<th>$D$</th>
<th>Diagrams</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>photon self-energy: log-divergent</td>
<td>$\Rightarrow$ charge renormalization</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>1</td>
<td>$= 0$ to all orders</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>0</td>
<td>light-by-light scattering</td>
<td>actually convergent</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>electron self-energy: log-divergent</td>
<td>$\Rightarrow$ mass &amp; charge renorm’n</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>vertex correction: log-divergent</td>
<td>$\Rightarrow$ charge renormalization</td>
</tr>
</tbody>
</table>
N.B. In QED, charge renormalization from electron self-energy and vertex correction cancel, so it can be ascribed entirely to photon self-energy (vacuum polarization).
Dimensions of Fields and Couplings

- In natural units we have only mass (equivalently, energy or momentum) dimensions: $x \sim ct \sim \hbar c/E \sim \hbar/mc$.

  \[ \hbar = c = 1 \Rightarrow [L] = [T] = [E]^{-1} = [M]^{-1} \]

- Hence action $S$ (units $\hbar$) is dimensionless, and

  \[ S = \int \mathcal{L} \, d^4x \quad \Rightarrow \quad [\mathcal{L}] = [x]^{-4} = [M]^4 \]

  Furthermore $[\partial^\mu] = [p^\mu] = [M]$. From this we can deduce dimensions of fields and couplings:

  \[
  \mathcal{L}_{\text{KG}} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \quad \Rightarrow \quad [\phi] = [M] \\
  \mathcal{L}_D = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \quad \Rightarrow \quad [\psi] = [M]^{3/2} \\
  \mathcal{L}_{\text{em}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad \Rightarrow \quad [F^{\mu\nu}] = [M]^2 \\
  F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \Rightarrow \quad [A^\mu] = [M] \]
Higgs self-coupling:
\[ \lambda (\phi^* \phi)^2 \Rightarrow [\lambda] = [M]^0 \]

Gauge couplings:
\[ D^\mu = \partial^\mu + ieA^\mu (+igW^\mu) \Rightarrow [e] = [g] = [M]^0 \]

Fermi coupling:
\[ G_F (\bar{\psi} \gamma^\mu \psi)(\bar{\psi} \gamma_\mu \psi) \Rightarrow [G_F] = [M]^{-2} \]

Yukawa coupling:
\[ g_f \phi \bar{\psi} \psi \Rightarrow [g_f] = [M]^0 \]

Thus in any theory we can associate dimension 4 with any vertex, as follows

\[ 4 = \frac{3}{2} F_V + B_V + P_V + g_V \]

where \( P_V \) = number of momentum factors, \( g_V \) = dimension of coupling.

For example...
Fermi

$4 = \frac{3}{2}(4) + 0 + 0 + (-2)$

3-gauge-boson

$4 = \frac{3}{2}(0) + 3 + 1 + 0$

- Now we can derive superficial degree of divergence in any theory:

$$D = 4L - F_I - 2B_I + \sum V P_V$$

Recall that $L = F_I + B_I - V + 1$ and

$$\sum V F_V = 2F_I + F_E, \quad \sum V B_V = 2B_I + B_E$$

$$D = 4 - 4V + 3F_I + 2B_I + \sum V P_V$$
\[
= 4 - 4V - \frac{3}{2} F_E - B_E + \sum_V \left( \frac{3}{2} F_V + B_V + P_V = 4 - g_V \right)
\]
\[
= 4 - \frac{3}{2} F_E - B_E - \sum_V g_V
\]

● Standard Model couplings are all \textit{dimensionless}, so \(\sum_V g_V = 0\) and the situation is similar to QED:

❖ Finite number of divergent sub-graphs (‘primitive divergences’)
❖ Can absorb cut-off dependence in bare parameters of Lagrangian
❖ Hence theory is \textit{renormalizable}

N.B. Lots of work needed to \textit{prove} this (’t Hooft and Veltman \(\Rightarrow\) Nobel prize).

● Non-standard vertices have \(g_V < 0\), so \(D\) gets larger and larger in higher orders of perturbation theory \(\Rightarrow\) theory becomes \textit{unrenormalizable}. For example

\begin{align*}
\text{6-Higgs coupling:} \\
\lambda_6 (\phi^\dagger \phi)^3 & \Rightarrow [\lambda_6] = [M]^{-2} \\
\text{Fermi coupling:} \\
G_F (\bar{\psi} \gamma^\mu \psi) (\bar{\psi} \gamma_\mu \psi) & \Rightarrow [G_F] = [M]^{-2}
\end{align*}
2-boson Yukawa coupling:

\[ \lambda_f \phi^\dagger \phi \bar{\psi} \psi \Rightarrow [\lambda_f] = [M]^{-1} \]

- Is it surprising that Nature provides only renormalizable interactions? Maybe not, because unrenormalizability \( \Rightarrow \) bad (divergent) high-energy behaviour. E.g. Fermi theory:

\[
\sigma(\nu_e e) \sim G_F^2 \\
[G_F] = [M]^{-2}, \quad [\sigma] = [M]^{-2} \\
\Rightarrow \sigma(\nu_e e) \sim G_F^2 E^2 \rightarrow \infty
\]

- Thus if we suppose there exists a finite theory at very high energies (GUT? SUSY? Strings?), all unrenormalizable interactions will have shrunk to negligible values in going from that high scale to present energies:
$\sigma$ vs. $E$

- Present scale
- GUT scale

- Renormalized
- Unrenormalized