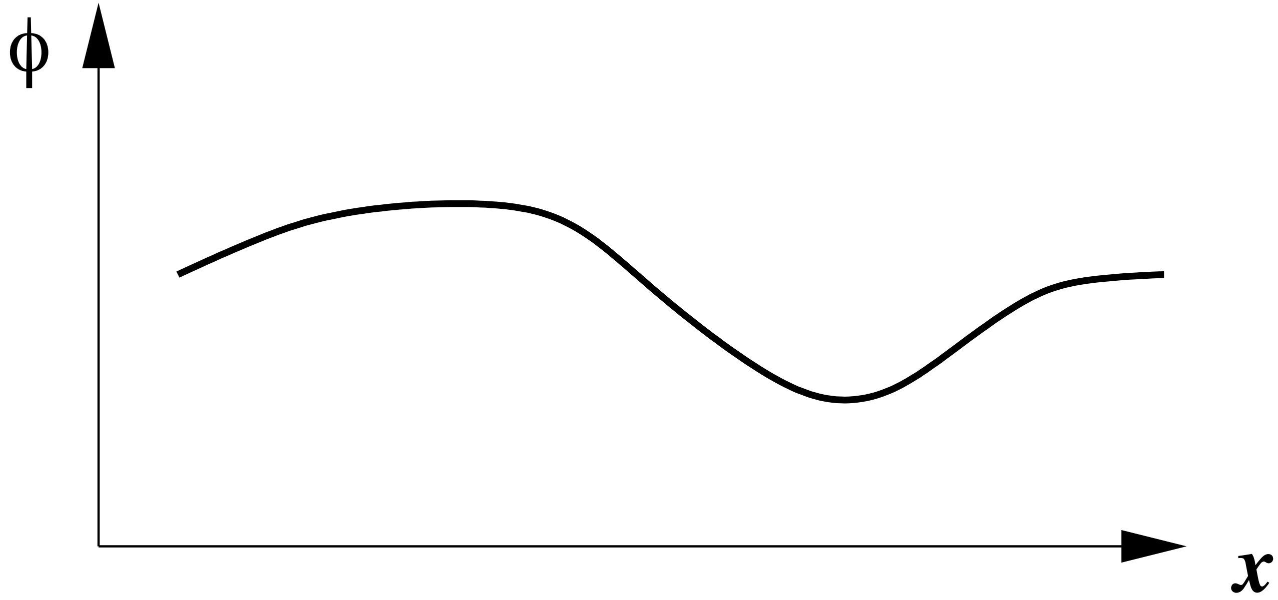
Classical Field Theory



- Consider waves on a string, mass per unit length ρ, tension T, displacement $\phi(x,t)$.

$$\begin{aligned}
\text{K.E.} \qquad T &= \int \tfrac{1}{2}\rho \left(\frac{\partial \phi}{\partial t}\right)^2 dx \\
\text{P.E.} \qquad V &= \int \tfrac{1}{2}T \left(\frac{\partial \phi}{\partial x}\right)^2 dx = \int \tfrac{1}{2}\rho c^2 \left(\frac{\partial \phi}{\partial x}\right)^2 dx
\end{aligned}$$

where wave velocity $c = \sqrt{T/\rho}$.

- Lagrangian $L = T - V = \int \mathcal{L}\, dx$ with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\rho \left[\left(\frac{\partial \phi}{\partial t} \right)^2 - c^2 \left(\frac{\partial \phi}{\partial x} \right)^2 \right]$$

For brevity, write

$$\frac{\partial \phi}{\partial t} = \dot{\phi}\,, \qquad \frac{\partial \phi}{\partial x} = \phi'$$

$$\Rightarrow \qquad \mathcal{L} = \tfrac{1}{2}\rho \left(\dot{\phi}^2 - c^2 \phi'^2 \right)$$

- Equations of motion given by least action $\delta S = 0$ where $S = \int L\, dt$. Now

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi' + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta\dot{\phi} \right) dx\, dt$$

But

$$\int \frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi'\, dx = \int \frac{\partial \mathcal{L}}{\partial \phi'} \frac{\partial}{\partial x} \delta\phi\, dx = \left[\frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi \right]_{-\infty}^{+\infty} - \int \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \phi'} \right) \delta\phi\, dx$$

We can assume the boundary terms vanish. Similarly

$$\int \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta \dot{\phi} \, dt = - \int \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \delta \phi \, dt$$

and so

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \phi'} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \right] \delta \phi \, dx \, dt$$

- This has to vanish for any $\delta\phi(x,t)$, so we obtain the Euler-Lagrange equation of motion for the field $\phi(x,t)$:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \phi'} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = 0$$

- For the string, $\mathcal{L} = \frac{1}{2}\rho \left(\dot{\phi}^2 - c^2 \phi'^2 \right)$. Hence

$$\rho c^2 \frac{\partial \phi'}{\partial x} - \rho \frac{\partial \dot{\phi}}{\partial t} = 0$$

which gives the wave equation

$$c^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = 0$$

- We shall also need the Hamiltonian

$$H = \int \mathcal{H} \, dx$$

Recall that for a single coordinate q we have $L = L(\dot{q}, q)$ and

$$H = p\dot{q} - L$$

where p is the generalized momentum

$$p = \frac{\partial L}{\partial \dot{q}}$$

- Similarly, for a field $\phi(x, t,)$ we define momentum density

$$\pi(x, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

Then

$$\mathcal{H}(\pi, \phi) = \pi\dot{\phi} - \mathcal{L}$$

- For the string $\pi = \rho\dot{\phi}$ (as expected) and

$$\begin{aligned} \mathcal{H} &= \pi\left(\frac{\pi}{\rho}\right) - \tfrac{1}{2}\rho\left(\frac{\pi}{\rho}\right)^2 + \tfrac{1}{2}\rho c^2 \phi'^2 \\ &= \frac{1}{2\rho}\pi^2 + \tfrac{1}{2}\rho c^2\left(\frac{\partial\phi}{\partial x}\right)^2 \end{aligned}$$

Klein-Gordon Field

- We choose the Lagrangian density ($\hbar = c = 1$)

$$\mathcal{L} = \tfrac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \tfrac{1}{2}m^2\phi^2$$

  to obtain the Klein-Gordon equation of motion:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) &= 0 \\ \Rightarrow \qquad -m^2\phi + \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} &= 0 \end{aligned}$$

- The momentum density is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\phi}{\partial t}$$

  and so the Klein-Gordon Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \tfrac{1}{2}m^2\phi^2$$

- In 3 spatial dimensions $\partial\phi/\partial x \rightarrow \nabla\phi$,

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x)}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial y)}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial z)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = 0$$

$$\Rightarrow \qquad -m^2\phi + \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = 0$$

- Covariant notation:

$$\mathcal{L} = \tfrac{1}{2}g_{\mu\nu}\left(\partial^\mu\phi\right)\left(\partial^\nu\phi\right) - \tfrac{1}{2}m^2\phi^2$$

Euler-Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right) = -m^2\phi - \partial^\mu\left(g_{\mu\nu}\partial^\nu\phi\right) = 0$$

i.e.

$$\partial^\mu\partial_\mu\phi + m^2\phi = 0$$

- Note that the Lagrangian density $\mathcal{L}$ and the action $S = \int \mathcal{L}\, d^3\boldsymbol{r}\, dt = \int \mathcal{L}\, d^4x$ are scalars (invariant functions), like ϕ.

- On the other hand the momentum density $\pi = \partial\phi/\partial t$ and the Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}\left(\nabla\phi\right)^2 + \tfrac{1}{2}m^2\phi^2$$

are not: time development $\Rightarrow$ frame dependence.

Fourier Analysis

- We can express any real field $\phi(x,t)$ as a Fourier integral:

$$\phi(x,t) = \int dk\, N(k)\left[a(k)e^{ikx-i\omega t} + a^*(k)e^{-ikx+i\omega t}\right]$$

where $N(k)$ is a convenient normalizing factor for the Fourier transform $a(k)$. The frequency $\omega(k)$ is obtained by solving the equation of motion: KG equation $\Rightarrow$ $\omega = +\sqrt{k^2+m^2}$.

- The Hamiltonian

$$H = \int \left(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}\phi'^2 + \tfrac{1}{2}m^2\phi^2\right) dx$$

takes a simpler form in terms of the Fourier amplitudes $a(k)$. We can write e.g.

$$\phi^2 = \int dk\, N(k)\,[\ldots] \int dk'\, N(k')\,[\ldots]$$

and use

$$\int dx\, e^{i(k\pm k')x} = 2\pi\,\delta(k\pm k')$$

to show that

$$\begin{aligned}\int \phi^2\, dx &= 2\pi \int dk\, dk'\, N(k)\, N(k') \big[a(k)a(k')\delta(k+k')e^{-i(\omega+\omega')t} \\ &+ a^*(k)a^*(k')\delta(k+k')e^{i(\omega+\omega')t} + a(k)a^*(k')\delta(k-k')e^{-i(\omega-\omega')t} \\ &+ a^*(k)a(k')\delta(k-k')e^{i(\omega-\omega')t} \big]\end{aligned}$$

- Noting that $\omega(-k) = \omega(k)$ and choosing $N(k)$ such that $N(-k) = N(k)$, this gives

$$\begin{aligned}\int \phi^2\, dx &= 2\pi \int dk\, [N(k)]^2 \big[a(k)a(-k)e^{-2i\omega t} \\ &+ a^*(k)a^*(-k)e^{2i\omega t} + a(k)a^*(k) + a^*(k)a(k) \big]\end{aligned}$$

- Similarly

$$\begin{aligned}\int \phi'^2\, dx &= 2\pi \int dk\, [kN(k)]^2 \big[a(k)a(-k)e^{-2i\omega t} \\ &+ a^*(k)a^*(-k)e^{2i\omega t} + a(k)a^*(k) + a^*(k)a(k) \big]\end{aligned}$$

while

$$\int \dot{\phi}^2 \, dx \quad = \quad 2\pi \int dk \, [\omega(k) N(k)]^2 \left[-a(k)a(-k)e^{-2i\omega t} \right.$$
$$\left. -a^*(k)a^*(-k)e^{+2i\omega t} + a(k)a^*(k) + a^*(k)a(k) \right]$$

and hence, using $k^2 = \omega^2 - m^2$,

$$H = 2\pi \int dk \, [N(k)\omega(k)]^2 [a(k)a^*(k) + a^*(k)a(k)]$$

or, choosing

$$N(k) = \frac{1}{2\pi \cdot 2\omega(k)}$$

$$H = \int dk \, N(k) \, \tfrac{1}{2}\omega(k) \, [a(k)a^*(k) + a^*(k)a(k)]$$

i.e. the integrated density of modes $N(k)$ times the energy per mode $\omega(k)|a(k)|^2$.

$\Rightarrow$ Each normal mode of the system behaves like an independent harmonic oscillator with amplitude $a(k)$.

- In 3 spatial dimensions we write

$$\phi(\boldsymbol{r},t) = \int d^3\boldsymbol{k}\, N(\boldsymbol{k}) \left[a(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} + a^*(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega t}\right]$$

and use

$$\int d^3\boldsymbol{r}\, e^{i(\boldsymbol{k}\pm\boldsymbol{k}')\cdot\boldsymbol{r}} = (2\pi)^3\,\delta^3(\boldsymbol{k}\pm\boldsymbol{k}')$$

Therefore we should choose

$$N(\boldsymbol{k}) = \frac{1}{(2\pi)^3\, 2\omega(\boldsymbol{k})}$$

to obtain an integral with the usual relativistic phase space (density of states) factor:

$$H = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega(\boldsymbol{k})}\, \omega(\boldsymbol{k})\, |a(\boldsymbol{k})|^2$$

Second Quantization

- First quantization was the procedure of replacing classical dynamical variables q and p by quantum operators $\hat{q}$ and $\hat{p}$ such that

$$[\hat{q}, \hat{p}] = i \qquad\qquad (\hbar = 1)$$

- Second quantization is replacing the field variable $\phi(x,t)$ and its conjugate momentum density $\pi(x,t)$ by operators such that

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\,\delta(x - x')$$

N.B. x and x' are not dynamical variables but labels for the field values at different points. Compare (and contrast)

$$[\hat{q}_j, \hat{p}_k] = i\,\delta_{jk} \qquad\qquad (j, k = x, y, z)$$

- The wave function ϕ satisfying the Klein-Gordon equation is replaced by the field operator $\hat{\phi}$, satisfying the same equation.

- The Fourier representation becomes

$$\hat{\phi}(x,t) = \int dk\, N(k)\, \left[\hat{a}(k)e^{ikx-i\omega t} + \hat{a}^\dagger(k)e^{-ikx+i\omega t}\right]$$

i.e. $\hat{\phi}$ is hermitian but the Fourier conjugate operator $\hat{a}$ is not.

- Keeping track of the order of operators, the Hamiltonian operator is

$$\hat{H} = \int dk\, N(k)\, \tfrac{1}{2}\omega(k)\, [\hat{a}(k)\hat{a}^\dagger(k) + \hat{a}^\dagger(k)\hat{a}(k)]$$

- Comparing this with the simple harmonic oscillator,

$$\hat{H}_{\mathrm{SHO}} = \tfrac{1}{2}\omega\, (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$$

we see that $\hat{a}^\dagger(k)$ and $\hat{a}(k)$ must be the ladder operators for the mode of wave number k. They add/remove one quantum of excitation of the mode. These quanta are the particles corresponding to that field:

$$\Rightarrow \qquad \hat{a}^\dagger(k) = \text{creation operator}\ , \qquad \hat{a}(k) = \text{annihilation operator}$$

for Klein-Gordon particles.

- Ladder operators of simple harmonic oscillator satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1$$

The analogous commutation relation for the creation and annihilation operators is

$$\begin{aligned} N(k)\,[\hat{a}(k), \hat{a}^\dagger(k')] &= \delta(k - k') \\ \Rightarrow \qquad [\hat{a}(k), \hat{a}^\dagger(k')] &= 2\pi \cdot 2\omega(k)\,\delta(k - k') \end{aligned}$$

or in 3 spatial dimensions

$$[\hat{a}(\boldsymbol{k}), \hat{a}^\dagger(\boldsymbol{k}')] = (2\pi)^3 \cdot 2\omega(\boldsymbol{k})\,\delta^3(\boldsymbol{k} - \boldsymbol{k}')$$

On the other hand

$$[\hat{a}(\boldsymbol{k}), \hat{a}(\boldsymbol{k}')] = [\hat{a}^\dagger(\boldsymbol{k}), \hat{a}^\dagger(\boldsymbol{k}')] = 0$$

- The commutators of the creation and annihilation operators correspond to the field commutation relation

$$[\hat{\phi}(\boldsymbol{r},t),\hat{\pi}(\boldsymbol{r}',t)] = \int d^3\boldsymbol{k}\,d^3\boldsymbol{k}'\,N(\boldsymbol{k})\,N(\boldsymbol{k}')[-i\omega(\boldsymbol{k}')]\times$$
$$\left[\hat{a}(\boldsymbol{k})e^{-ik\cdot x}+\hat{a}^\dagger(\boldsymbol{k})e^{ik\cdot x},\hat{a}(\boldsymbol{k}')e^{-ik'\cdot x'}-\hat{a}^\dagger(\boldsymbol{k}')e^{ik'\cdot x'}\right]$$
$$= i\int d^3\boldsymbol{k}\,N(\boldsymbol{k})\omega(\boldsymbol{k})\left[e^{-ik\cdot(x-x')}+e^{ik\cdot(x-x')}\right]$$

where $x^\mu = (t,\boldsymbol{r})$ and $x'^\mu = (t,\boldsymbol{r}')$. Hence

$$[\hat{\phi}(\boldsymbol{r},t),\hat{\pi}(\boldsymbol{r}',t)] = i\,\delta^3(\boldsymbol{r}-\boldsymbol{r}')$$

as expected. On the other hand

$$[\hat{\phi}(\boldsymbol{r},t),\hat{\phi}(\boldsymbol{r}',t)] = [\hat{\pi}(\boldsymbol{r},t),\hat{\pi}(\boldsymbol{r}',t)] = 0$$

- The fact that the field operator has positive- and negative-frequency parts now appears quite natural:
  - positive frequency part $\hat{a}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}$ annihilates particles
  - negative frequency part $\hat{a}^\dagger(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega t}$ creates particles
- $\pm\hbar\omega$ is the energy released/absorbed in the annihilation/creation process.
  N.B. The hermitian field describes particles that are identical to their antiparticles, e.g. π^0 mesons.
- More generally (as we shall see shortly) the negative-frequency part of $\hat{\phi}$ creates antiparticles.

Single Particle States

- If $|0\rangle$ represents the state with no particles present (the vacuum), then $\hat{a}^\dagger(\boldsymbol{k})|0\rangle$ is a state containing a particle with wave vector $\boldsymbol{k}$, i.e. momentum $\hbar\boldsymbol{k}$. More generally, to make a state with wave function $\phi(\boldsymbol{r}, t)$ where

$$\phi(\boldsymbol{r}, t) = \int d^3\boldsymbol{k}\, \tilde{\phi}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$$

  we should operate on the vacuum with the operator

$$\int d^3\boldsymbol{k}\, \tilde{\phi}(\boldsymbol{k}) \hat{a}^\dagger(\boldsymbol{k})$$

- Writing this state as $|\phi\rangle = \int d^3\boldsymbol{k}\, \tilde{\phi}(\boldsymbol{k}) \hat{a}^\dagger(\boldsymbol{k})|0\rangle$ we can find the wave function using the relation

$$\langle 0|\hat{\phi}(\boldsymbol{r}, t)|\phi\rangle = \phi(\boldsymbol{r}, t)$$

- Thus the field operator is an operator for "finding out the wave function" (for single-particle states).

Two Particle States

- Similarly, we can make a state $|\phi_{12}\rangle$ of two particles:

$$|\phi_{12}\rangle = \int d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2\, \tilde{\phi}(\boldsymbol{k}_1, \boldsymbol{k}_2)\hat{a}^\dagger(\boldsymbol{k}_1)\hat{a}^\dagger(\boldsymbol{k}_2)|0\rangle$$

  Since the two particles are identical bosons, this state is symmetric in the labels 1 and 2 even if $\tilde{\phi}$ isn't, because $\hat{a}^\dagger(\boldsymbol{k}_1)$ and $\hat{a}^\dagger(\boldsymbol{k}_2)$ commute:

$$\begin{aligned} |\phi_{12}\rangle &= \int d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2\, \tilde{\phi}(\boldsymbol{k}_1, \boldsymbol{k}_2)\hat{a}^\dagger(\boldsymbol{k}_1)\hat{a}^\dagger(\boldsymbol{k}_2)|0\rangle \\ &= \int d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2\, \tilde{\phi}(\boldsymbol{k}_2, \boldsymbol{k}_1)\hat{a}^\dagger(\boldsymbol{k}_2)\hat{a}^\dagger(\boldsymbol{k}_1)|0\rangle \\ &= |\phi_{21}\rangle \end{aligned}$$

- The 2-particle wave function is

$$\begin{aligned} \phi(\boldsymbol{r}_1, \boldsymbol{r}_2, t) &= \langle 0|\hat{\phi}(\boldsymbol{r}_1, t)\hat{\phi}(\boldsymbol{r}_2, t)|\phi_{12}\rangle \\ &= \phi(\boldsymbol{r}_2, \boldsymbol{r}_1, t) \end{aligned}$$

  Thus quantum field theory is also a good way of dealing with systems of many identical particles.

Number Operator

- Recall that for the simple harmonic oscillator the operator $\hat{a}^\dagger \hat{a}$ tells us the number of quanta of excitation:

$$\hat{a}^\dagger \hat{a}|\phi_n\rangle = n|\phi_n\rangle$$

when $|\phi_n\rangle$ is the nth excited state.

- Similarly, in field theory the operator

$$\hat{\mathcal{N}} = \int dk\, N(k)\, \hat{a}^\dagger(k)\hat{a}(k)$$

counts the number of particles in a state. For example,

$$\hat{\mathcal{N}}|\phi_{12}\rangle = \int dk\, dk_1\, dk_2\, N(k)\tilde{\phi}(k_1, k_2)\hat{a}^\dagger(k)\hat{a}(k)\hat{a}^\dagger(k_1)\hat{a}^\dagger(k_2)|0\rangle$$

where

$$\begin{aligned}
& N(k)\hat{a}^\dagger(k)\hat{a}(k)\hat{a}^\dagger(k_1)\hat{a}^\dagger(k_2) \\
= \quad & N(k)\hat{a}^\dagger(k)\hat{a}^\dagger(k_1)\hat{a}(k)\hat{a}^\dagger(k_2) + \hat{a}^\dagger(k)\hat{a}^\dagger(k_2)\delta(k-k_1) \\
= \quad & N(k)\hat{a}^\dagger(k)\hat{a}^\dagger(k_1)\hat{a}^\dagger(k_2)\hat{a}(k) + \hat{a}^\dagger(k)\hat{a}^\dagger(k_1)\delta(k-k_2) + \hat{a}^\dagger(k)\hat{a}^\dagger(k_2)\delta(k-k_1)
\end{aligned}$$

However, $\hat{a}(k)|0\rangle = 0$ (by definition of the vacuum) and so

$$\hat{\mathcal{N}}|\phi_{12}\rangle = 0 + |\phi_{12}\rangle + |\phi_{12}\rangle \; = \; 2|\phi_{12}\rangle$$

N.B. In general, states do not need to be eigenstates of $\hat{\mathcal{N}}$: they need not contain a definite number of particles.

Electromagnetic Field

- Each component of A^μ (in Lorenz gauge) is quantized like a massless Klein-Gordon field (i.e. with $\omega = |\boldsymbol{k}|$):

$$\hat{A}^\mu(\boldsymbol{r}, t) = \sum_{P=L,R} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega}$$

$$\left[\varepsilon^\mu_P(\boldsymbol{k})\hat{a}_P(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} + \varepsilon^{*\mu}_P(\boldsymbol{k})\hat{a}^\dagger_P(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega t}\right]$$

  where $\hat{a}_P(\boldsymbol{k})$ annihilates a photon with momentum $\hbar\boldsymbol{k}$ and polarization P and $\hat{a}^\dagger_P(\boldsymbol{k})$ creates one.

- $\varepsilon^\mu_{L,R}(\boldsymbol{k})$ are the left/right-handed circular polarization 4-vectors for wave vector $\boldsymbol{k}$ and

$$\left[\hat{a}_P(\boldsymbol{k}), \hat{a}^\dagger_{P'}(\boldsymbol{k}')\right] = (2\pi)^3\, 2\omega(\boldsymbol{k})\, \delta_{PP'}\delta^3(\boldsymbol{k}-\boldsymbol{k}')$$

  N.B. Different polarizations commute.

- The Hamiltonian for the e.m. field is

$$\hat{H} = \tfrac{1}{2}\int d^3\boldsymbol{r}\, (\boldsymbol{E}^2 + \boldsymbol{B}^2)$$

where (in $A^0 = 0$ gauge) from $\hat{\boldsymbol{E}} = -\partial \hat{\boldsymbol{A}}/\partial t$, $\hat{\boldsymbol{B}} = \nabla \times \hat{\boldsymbol{A}}$ we have

$$\hat{\boldsymbol{E}} = \sum_P \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega} i\omega \left[\boldsymbol{\varepsilon}_P \hat{a}_P e^{-ik\cdot x} - \boldsymbol{\varepsilon}^*_P \hat{a}^\dagger_P e^{+ik\cdot x}\right]$$

$$\hat{\boldsymbol{B}} = \sum_P \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega} i\boldsymbol{k} \times \left[\boldsymbol{\varepsilon}_P \hat{a}_P e^{-ik\cdot x} - \boldsymbol{\varepsilon}^*_P \hat{a}^\dagger_P e^{+ik\cdot x}\right]$$

- Using $\boldsymbol{k} \times \boldsymbol{\varepsilon}_L = i\omega \boldsymbol{\varepsilon}_L$, $\boldsymbol{k} \times \boldsymbol{\varepsilon}_R = -i\omega \boldsymbol{\varepsilon}_R$ [recall that for $\boldsymbol{k}$ along the z-axis we have $\varepsilon^\mu_{L,R} = (0, 1, \mp i, 0)/\sqrt{2}$] we find, as expected, that

$$\hat{H} = \sum_P \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega} \tfrac{1}{2}\omega(\boldsymbol{k}) \left[\hat{a}_P(\boldsymbol{k}) a^\dagger_P(\boldsymbol{k}) + \hat{a}^\dagger_P(\boldsymbol{k}) a_P(\boldsymbol{k})\right]$$

Vacuum Energy and Normal Ordering

- Using the above expression for the e.m. Hamiltonian and the commutation relation for the photon annihilation and creation operators, we find that the energy of the vacuum is given by

$$\hat{H}|0\rangle = \int d^3\boldsymbol{k}\,\omega(\boldsymbol{k}) \lim_{k'\to k} \delta^3(\boldsymbol{k}-\boldsymbol{k}')|0\rangle$$

- In a finite volume V, we should interpret

$$\lim_{k'\to k} \delta^3(\boldsymbol{k}-\boldsymbol{k}') = \lim_{k'\to k} \int_V \frac{d^3\boldsymbol{r}}{(2\pi)^3} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = \frac{V}{(2\pi)^3}$$

  The energy density of the vacuum is thus U_0 where $\hat{H}|0\rangle = U_0 V|0\rangle$

$$\Rightarrow\ U_0 = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\omega(\boldsymbol{k}) = \infty \qquad !!$$

- This is due to the zero-point fluctuations of the e.m. field ($\frac{1}{2}\hbar\omega$ per mode). In reality, we don't understand physics at very short distances (very large wave numbers) and presumably the integral gets cut off at some very large $|\boldsymbol{k}| \sim \Lambda$. We shall see that quantities we can measure don't depend much on Λ or the form of the cut-off.

- For most purposes we can throw away the term in $\hat{H}$ that gives rise to the vacuum energy, which corresponds to measuring all energies relative to the vacuum. This means writing

$$\hat{H} = \sum_P \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\, 2\omega} \omega(\boldsymbol{k}) \hat{a}_P^\dagger(\boldsymbol{k}) \hat{a}_P(\boldsymbol{k})$$

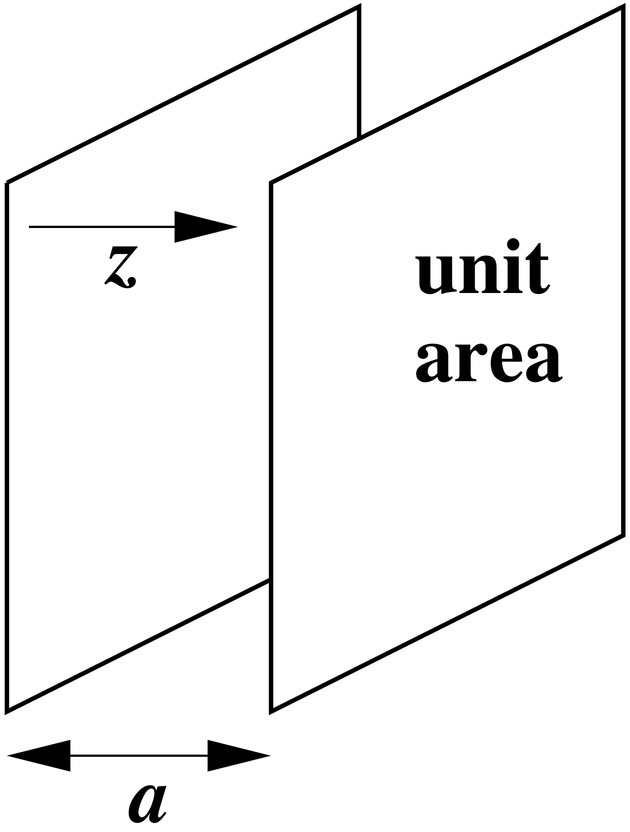
  which is called the normal-ordered form of $\hat{H}$, i.e. a form with the annihilation operator $\hat{a}_P$ on the right.
  N.B. After normal-ordering, $\hat{H}$ is still hermitian.

  Clearly, a normal-ordered operator has the vacuum as an eigenstate with eigenvalue zero.

- However, we should not think that this means vacuum fluctuations are not there. They give rise to real effects, such as spontaneous transitions and ...

The Casimir Effect

- Suppose a region of vacuum is bounded by the plane surfaces of two semi-infinite conductors. Then some (low-frequency) vacuum fluctuations are forbidden by the boundary conditions ($\boldsymbol{E}_{\parallel} = \boldsymbol{B}_{\perp} = 0$). Thus the vacuum energy is reduced, corresponding to an attractive force between the conductors.



- Since the vacuum energy density is proportional to $\hbar c$, by dimensional analysis this Casimir force per unit area must be

$$F_C \propto \frac{\hbar c}{a^4}$$

- To find the constant of proportionality, note that $k_z = n\pi/a$ where $n = 0, 1, 2, \ldots$, so

$$\omega = c|\boldsymbol{k}| = c\sqrt{\boldsymbol{k}_\parallel^2 + (n\pi/a)^2}$$

- For $n = 1, 2, 3\ldots$ there are 2 polarizations; for $n = 0$ only one polarization is allowed, since $\boldsymbol{E}_\parallel = 0$. Hence the energy per unit area is

$$E = \tfrac{1}{2}\hbar c \int \frac{d^2\boldsymbol{k}_\parallel}{(2\pi)^2}\left[|\boldsymbol{k}_\parallel| + 2\sum_{n=1}^{\infty}\sqrt{\boldsymbol{k}_\parallel^2 + (n\pi/a)^2}\right]$$

- This is assumed to be cut off by new physics at wave numbers $|\boldsymbol{k}| > \Lambda$. Thus

$$E = \frac{\hbar c}{2\pi}\left[\tfrac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n)\right]$$

where

$$F(n) = \int k_{\parallel}\, dk_{\parallel} \sqrt{\boldsymbol{k}_{\parallel}^2 + (n\pi/a)^2}\, f\left(\sqrt{\boldsymbol{k}_{\parallel}^2 + (n\pi/a)^2}\right)$$

with $f(k) = 1$ for $k \ll \Lambda$ and $f(k) = 0$ for $k \gg \Lambda$.

- Changing variable from $k_{\parallel}$ to $k = \sqrt{\boldsymbol{k}_{\parallel}^2 + (n\pi/a)^2}$,

$$F(n) = \int_{n\pi/a}^{\infty} k^2\, dk\, f(k)$$

Removing the boundary conditions would allow n to be a continuous variable:

$$E_0 = \frac{\hbar c}{2\pi} \int_0^{\infty} dn\, F(n)$$

Hence the change in the vacuum energy per unit area is

$$\delta E = E - E_0 = \frac{\hbar c}{2\pi}\left[\tfrac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^{\infty} dn\, F(n)\right]$$

- The Euler-McLaurin formula tells us that

$$\int_0^{\infty} dn\, F(n) = \tfrac{1}{2}F(0) + \sum_{n=1}^{\infty}\left[F(n) + \frac{1}{(2n)!}B_{2n}F^{(2n-1)}(0)\right]$$

where B_{2n} are Bernoulli numbers:

$$B_2 = \frac{1}{6}\,, \qquad B_4 = -\frac{1}{30}\,, \ldots$$

Hence

$$\delta E = \frac{\hbar c}{2\pi}\left[-\frac{1}{12}F'(0) + \frac{1}{720}F'''(0) + \cdots\right]$$

But

$$F(n) = \int_{n\pi/a}^{\infty} k^2\, dk\, f(k) \qquad \Rightarrow \qquad F'(n) = -\frac{\pi}{a}\left(\frac{n\pi}{a}\right)^2 f\left(\frac{n\pi}{a}\right)$$

and so $F'(0) = 0$, $F'''(0) = -2(\pi/a)^3$ and

$$\delta E = -\frac{\pi^2}{720}\frac{\hbar c}{a^3}$$

- Thus the Casimir force is

$$F_C = \frac{d}{da}\delta E = \frac{\pi^2}{240}\frac{\hbar c}{a^4}$$

This is very small,

$$F_C = \frac{1.3 \times 10^{-27}}{a^4} \text{ Pa m}^4$$

but it has been measured (Sparnaay, 1957).

Complex Fields

- Suppose $\hat{\phi}$ is the second-quantized version of a complex field, i.e. $\hat{\phi}^\dagger \neq \hat{\phi}$. We can always decompose it into

$$\hat{\phi} = \frac{1}{\sqrt{2}}\left(\hat{\phi}_1 + i\hat{\phi}_2\right)$$

where $\hat{\phi}_1$ and $\hat{\phi}_2$ are hermitian. Then

$$\hat{\phi}(x,t) = \int dk\, N(k)\left[\hat{a}(k)e^{ikx-i\omega t} + \hat{b}^\dagger(k)e^{-ikx+i\omega t}\right]$$

where

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{a}_1 + i\hat{a}_2)\ , \qquad \hat{b}^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + i\hat{a}_2^\dagger) \neq \hat{a}^\dagger$$

- From the canonical commutation relations

$$\left[\hat{\phi}_j(x,t), \hat{\pi}_l(x',t)\right] = i\delta_{jl}\delta(x-x') \qquad (j,l = 1,2)$$

we can deduce

$$N(k)\left[\hat{a}_j(k), \hat{a}_l^\dagger(k')\right] = \delta_{jl}\,\delta(k-k')$$

and hence that

$$N(k)[\hat{a}(k), \hat{a}^\dagger(k')] = N(k)[\hat{b}(k), \hat{b}^\dagger(k')] = \delta(k-k') \ ,$$
$$[\hat{a}(k), \hat{b}^\dagger(k')] = [\hat{a}(k), \hat{b}(k')] = [\hat{a}^\dagger(k), \hat{b}^\dagger(k')] = 0 \ .$$

Hence $\hat{a}^\dagger$ and $\hat{b}^\dagger$ are creation operators for different particles.

- The Lagrangian density

$$\mathcal{L} = \mathcal{L}[\hat{\phi}_1] + \mathcal{L}[\hat{\phi}_2]$$

can be written as

$$\mathcal{L} = \frac{\partial \hat{\phi}^\dagger}{\partial t}\frac{\partial \hat{\phi}}{\partial t} - \frac{\partial \hat{\phi}^\dagger}{\partial x}\frac{\partial \hat{\phi}}{\partial x} - m^2 \hat{\phi}^\dagger \hat{\phi}$$

The canonical momentum density is thus

$$\hat{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{\hat{\phi}}} = \frac{\partial \hat{\phi}^\dagger}{\partial t}$$

and the Hamiltonian density is

$$\hat{\mathcal{H}} = \hat{\pi}\dot{\hat{\phi}} + \hat{\pi}^\dagger \dot{\hat{\phi}}^\dagger - \hat{\mathcal{L}} = \hat{\pi}^\dagger \hat{\pi} + \frac{\partial \hat{\phi}^\dagger}{\partial x}\frac{\partial \hat{\phi}}{\partial x} + m^2 \hat{\phi}^\dagger \hat{\phi}$$

- Using the Fourier expansion of $\hat{\phi}$ and integrating over all space, we find

$$\hat{H} = \int dx\, \hat{\mathcal{H}} = \tfrac{1}{2} \int dk\, N(k)\, \omega(k) \times$$
$$[\hat{a}(k)\hat{a}^\dagger(k) + \hat{a}^\dagger(k)\hat{a}(k) + \hat{b}(k)\hat{b}^\dagger(k) + \hat{b}^\dagger(k)\hat{b}(k)]$$

or after normal ordering

$$\hat{H} = \int dk\, N(k)\, \omega(k)\, [\hat{a}^\dagger(k)\hat{a}(k) + \hat{b}^\dagger(k)\hat{b}(k)]$$

Thus both $\hat{a}^\dagger$ and $\hat{b}^\dagger$ create particles with positive energy $\hbar\omega(k)$.

Symmetries and Conservation Laws

- We want to find a current and a density that satisfy the continuity equation for the complex Klein-Gordon field. We use an important general result called Noether's theorem (Emmy Noether, 1918), which tells us that there is a conserved current associated with every continuous symmetry of the Lagrangian, i.e. with symmetry under a transformation of the form

$$\phi \to \phi + \delta\phi$$

where $\delta\phi$ is infinitesimal. Symmetry means

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi'}\delta\phi' + \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\delta\dot{\phi} = 0$$

where

$$\begin{aligned} \delta\phi' &= \delta\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\delta\phi \\ \delta\dot{\phi} &= \delta\left(\frac{\partial\phi}{\partial t}\right) = \frac{\partial}{\partial t}\delta\phi \end{aligned}$$

(easily generalized to 3 spatial dimensions).

- The Euler-Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0$$

then implies that

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right)\delta\phi + \frac{\partial \mathcal{L}}{\partial \phi'}\frac{\partial}{\partial x}(\delta\phi) \\
&+ \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)\delta\phi + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\frac{\partial}{\partial t}(\delta\phi) = 0 \\
\Rightarrow \quad & \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\delta\phi\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\delta\phi\right) = 0
\end{aligned}$$

- Comparing with the conservation/continuity equation (in 1 dimension)

$$\frac{\partial}{\partial x}(J_x) + \frac{\partial \rho}{\partial t} = 0$$

shows that the conserved density and current are (proportional to)

$$\rho = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\delta\phi\,, \qquad J_x = \frac{\partial \mathcal{L}}{\partial \phi'}\delta\phi$$

- In 3 spatial dimensions

$$J_x = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}\delta\phi\ , \quad J_y = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial y)}\delta\phi\ , \ldots$$

and hence in covariant notation

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi$$

- If the Lagrangian involves several fields $\phi_1, \phi_2, \ldots$, the symmetry may involve changing them all: invariance w.r.t. $\phi_j \to \phi_j + \delta\phi_j \Rightarrow$ conserved Noether current

$$J^\mu = \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j$$

- After second quantization, the same procedure (being careful about the order of operators) can be used to define conserved current and density operators.

Phase (Gauge) Invariance

- The Klein-Gordon Lagrangian density

$$\mathcal{L} = \frac{\partial \hat{\phi}^\dagger}{\partial t}\frac{\partial \hat{\phi}}{\partial t} - \frac{\partial \hat{\phi}^\dagger}{\partial x}\frac{\partial \hat{\phi}}{\partial x} - m^2 \hat{\phi}^\dagger \hat{\phi}$$

or in 3 spatial dimensions

$$\begin{aligned} \mathcal{L} &= \frac{\partial \hat{\phi}^\dagger}{\partial t}\frac{\partial \hat{\phi}}{\partial t} - \nabla \hat{\phi}^\dagger \cdot \nabla \hat{\phi} - m^2 \hat{\phi}^\dagger \hat{\phi} \\ &= \partial_\mu \hat{\phi}^\dagger \partial^\mu \hat{\phi} - m^2 \hat{\phi}^\dagger \hat{\phi} \end{aligned}$$

is invariant under a global phase change in $\hat{\phi}$:

$$\begin{aligned} \hat{\phi} &\rightarrow e^{-i\varepsilon}\hat{\phi} \simeq \hat{\phi} - i\varepsilon\hat{\phi} \\ \hat{\phi}^\dagger &\rightarrow e^{+i\varepsilon}\hat{\phi}^\dagger \simeq \hat{\phi}^\dagger + i\varepsilon\hat{\phi}^\dagger \end{aligned}$$

i.e. $\delta\hat{\phi} \propto -i\hat{\phi}$, $\delta\hat{\phi}^\dagger \propto +i\hat{\phi}^\dagger$.

- The corresponding conserved Noether current is just that derived earlier:

$$\hat{j}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \hat{\phi})}\delta\hat{\phi} + \delta\hat{\phi}^\dagger \frac{\partial \mathcal{L}}{\partial(\partial_\mu \hat{\phi}^\dagger)} = -i(\partial^\mu \hat{\phi}^\dagger)\hat{\phi} + i\hat{\phi}^\dagger(\partial^\mu \hat{\phi})$$

- We can define an associated conserved charge, which is the integral of $\hat{\rho}$ over all space:

$$\hat{Q} = \int \hat{\rho}\, d^3\boldsymbol{r}$$

$$\frac{d\hat{Q}}{dt} = \int \frac{\partial \hat{\rho}}{\partial t} d^3\boldsymbol{r} = -\int \nabla \cdot \hat{\boldsymbol{J}}\, d^3\boldsymbol{r} = -\int_{\infty\ \text{sphere}} \hat{\boldsymbol{J}} \cdot d\boldsymbol{S} = 0$$

- In this case

$$\hat{Q} = -i \int \left(\frac{\partial \hat{\phi}^\dagger}{\partial t} \hat{\phi} - \hat{\phi}^\dagger \frac{\partial \hat{\phi}}{\partial t} \right) d^3\boldsymbol{r}$$

Inserting the Fourier decomposition of the field,

$$\hat{\phi} = \int d^3\boldsymbol{k}\, N(\boldsymbol{k}) \left[\hat{a}(\boldsymbol{k}) e^{-ik\cdot x} + \hat{b}^\dagger(\boldsymbol{k}) e^{ik\cdot x} \right]$$

where $N(\boldsymbol{k}) = [(2\pi)^3\, 2\omega(\boldsymbol{k})]^{-1}$, we find

$$\hat{Q} = \int d^3\boldsymbol{k}\, N(\boldsymbol{k}) \left[\hat{a}^\dagger(\boldsymbol{k}) \hat{a}(\boldsymbol{k}) - \hat{b}^\dagger(\boldsymbol{k}) \hat{b}(\boldsymbol{k}) \right]$$

- Comparing with the energy

$$\hat{H} = \int d^3\boldsymbol{k}\, N(\boldsymbol{k})\omega(\boldsymbol{k}) \left[\hat{a}^\dagger(\boldsymbol{k})\hat{a}(\boldsymbol{k}) + \hat{b}^\dagger(\boldsymbol{k})\hat{b}(\boldsymbol{k})\right]$$

we see that the particles created by $\hat{a}^\dagger$ and $\hat{b}^\dagger$ have opposite charge.

- Thus we see that in quantum field theory all the 'problems' with the negative-energy solutions of the Klein-Gordon equation are resolved, as follows
  - The object ϕ that satisifies the KG equation is in fact the field operator $\hat{\phi}$.
  - The Fourier decomposition of ϕ has a positive frequency part that annihilates a particle (with energy $\hbar\omega$ and charge $+1$) AND a negative frequency part that creates an antiparticle (with energy $\hbar\omega$ and charge -1).
  - Similarly, $\hat{\phi}^\dagger$ creates a particle or annihilates an antiparticle.

The Dirac Field

- The Lagrangian density that gives the Dirac equation of motion,

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

is

$$\mathcal{L}_{\mathrm{D}} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

- As in the Klein-Gordon case, we should treat ψ and $\bar{\psi} = \psi^\dagger \gamma^0$ as independent fields. The Euler-Lagrange equation for $\bar{\psi}$ gives the Dirac equation for ψ:

$$\frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial \bar{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial(\partial_\mu \bar{\psi})} \right) = i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

while that for ψ gives the Dirac equation for $\bar{\psi}$:

$$\frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial \psi} = \partial_\mu \left(\frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial(\partial_\mu \psi)} \right) = -m\bar{\psi} - i\partial_\mu \left(\bar{\psi}\gamma^\mu \right) = 0$$

Check: $-i\partial_\mu \psi^\dagger \gamma^{\mu\dagger} - m\psi^\dagger = 0$, $\bar{\psi} = \psi^\dagger \gamma^0$, $\gamma^{\mu\dagger}\gamma^0 = \gamma^0 \gamma^\mu \Rightarrow -i\partial_\mu \bar{\psi}\gamma^\mu = m\bar{\psi}$.

- The generalized momentum densities are

$$\pi = \frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0 = i\psi^\dagger$$
$$\bar{\pi} = \frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial \dot{\bar{\psi}}} = 0$$

- Hence the Hamiltonian density is

$$\mathcal{H}_{\mathrm{D}} = \pi\dot{\psi} - \mathcal{L}_{\mathrm{D}} = \bar{\psi} i \gamma^0 \frac{\partial \psi}{\partial t} - \mathcal{L}_{\mathrm{D}}$$
$$= -\bar{\psi} i \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \psi + m\bar{\psi}\psi$$

and the total energy is given by

$$H = \int d^3\boldsymbol{r}\, \mathcal{H}_{\mathrm{D}} = \int d^3\boldsymbol{r}\, \bar{\psi}(-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m)\psi$$
$$= \int d^3\boldsymbol{r}\, \psi^\dagger(-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m)\psi$$

which is indeed the expectation value of the original Dirac Hamiltonian.

- Now we second quantize by expressing the field operator $\hat{\psi}$ as a Fourier integral over plane-wave spinors with operator coefficients:

$$\hat{\psi}(\boldsymbol{r},t) = \int d^3\boldsymbol{k}\, N(\boldsymbol{k}) \sum_s [\hat{c}_s(\boldsymbol{k}) u_s(\boldsymbol{k}) e^{-ik\cdot x} + \hat{d}_s^\dagger(\boldsymbol{k}) v_s(\boldsymbol{k}) e^{+ik\cdot x}]$$

where $s = 1, 2$ label spin up/down, i.e. $u_1 = u^\uparrow$ free particle spinor, etc.

- We expect that the operator $\hat{c}_s(\boldsymbol{k})$ annihilates a particle (e.g. an electron) of momentum $\hbar\boldsymbol{k}$, spin orientation s. while $\hat{d}_s^\dagger(\boldsymbol{k})$ creates an antiparticle (positron) of the same momentum and spin. But the Hamiltonian operator is

$$\hat{H} = \int d^3\boldsymbol{r}\, \hat{\psi}^\dagger(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m)\hat{\psi}$$

and using the Dirac equation

$$\begin{aligned}(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m)u_s &= Eu_s = \omega u_s \\ (-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m)v_s &= -Ev_s = -\omega v_s\end{aligned}$$

we find that

$$\hat{H} = \int d^3\boldsymbol{k}\, N(\boldsymbol{k})\omega(\boldsymbol{k}) \sum_s \left[\hat{c}_s^\dagger(\boldsymbol{k})\hat{c}_s(\boldsymbol{k}) - \hat{d}_s(\boldsymbol{k})\hat{d}_s^\dagger(\boldsymbol{k})\right]$$

- Thus there appears to be a problem: unlike the Klein-Gordon case, we seem to get a negative contribution to the energy from the antiparticles!
- The Dirac equation also has phase (gauge) symmetry:

$$\begin{aligned} \hat{\psi} &\rightarrow e^{-i\varepsilon}\hat{\psi} \simeq \hat{\psi} - i\varepsilon\hat{\psi} \\ \hat{\psi}^\dagger &\rightarrow e^{+i\varepsilon}\hat{\psi}^\dagger \simeq \hat{\psi}^\dagger + i\varepsilon\hat{\psi}^\dagger \end{aligned}$$

with corresponding Noether current

$$\begin{aligned} \hat{j}^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\hat{\psi})}\delta\hat{\psi} + \delta\hat{\psi}^\dagger\frac{\partial\mathcal{L}}{\partial(\partial_\mu\hat{\psi}^\dagger)} \\ &= \hat{\bar{\psi}}\, i\gamma^\mu(-i\hat{\psi}) = \hat{\bar{\psi}}\gamma^\mu\hat{\psi} \end{aligned}$$

and conserved charge

$$\hat{Q} = \int d^3\boldsymbol{r}\,\hat{J}^0 = \int d^3\boldsymbol{r}\,\hat{\psi}^\dagger\hat{\psi} = \int d^3\boldsymbol{k}\,N(\boldsymbol{k})\sum_s\left[\hat{c}_s^\dagger(\boldsymbol{k})\hat{c}_s(\boldsymbol{k}) + \hat{d}_s(\boldsymbol{k})\hat{d}_s^\dagger(\boldsymbol{k})\right]$$

which also seems to have the wrong sign for the antiparticle contribution.

- However, we actually need the normal-ordered operators, which involve $\hat{d}_s^\dagger\hat{d}_s$, not $\hat{d}_s\hat{d}_s^\dagger$. Hence if

$$d_s^\dagger\hat{d}_s = -\hat{d}_s\hat{d}_s^\dagger + \text{const.}$$

then all signs are correct.

- Thus we are forced to give the creation and annihilation operators for spin one-half particles anticommutation relations:

$$\left\{\hat{c}_s(\boldsymbol{k}), \hat{c}_{s'}^\dagger(\boldsymbol{k}')\right\} \equiv \hat{c}_s(\boldsymbol{k})\hat{c}_{s'}^\dagger(\boldsymbol{k}') + \hat{c}_{s'}^\dagger(\boldsymbol{k}')\hat{c}_s(\boldsymbol{k})$$

$$= (2\pi)^3\,2\omega(\boldsymbol{k})\,\delta_{ss'}\,\delta^3(\boldsymbol{k}-\boldsymbol{k}') = \left\{\hat{d}_s(\boldsymbol{k}), \hat{d}_{s'}^\dagger(\boldsymbol{k}')\right\}$$

while

$$\{\hat{c}_s(\boldsymbol{k}), \hat{c}_{s'}(\boldsymbol{k}')\} = \{\hat{d}_s(\boldsymbol{k}), \hat{d}_{s'}(\boldsymbol{k}')\} = 0$$

- This means that two-particle states are antisymmetric:

$$
\begin{aligned}
|\phi_{12}\rangle &= \int d^3\boldsymbol{k}_1\, d^3\boldsymbol{k}_2 \sum_{s_1,s_2} \tilde{\phi}_{s_1 s_2}(\boldsymbol{k}_1,\boldsymbol{k}_2)\hat{c}^\dagger_{s_1}(\boldsymbol{k}_1)\hat{c}^\dagger_{s_2}(\boldsymbol{k}_2)|0\rangle \\
&= -\int d^3\boldsymbol{k}_1\, d^3\boldsymbol{k}_2 \sum_{s_1,s_2} \tilde{\phi}_{s_1 s_2}(\boldsymbol{k}_1,\boldsymbol{k}_2)\hat{c}^\dagger_{s_2}(\boldsymbol{k}_2)\hat{c}^\dagger_{s_1}(\boldsymbol{k}_1)|0\rangle \\
&= -\int d^3\boldsymbol{k}_1\, d^3\boldsymbol{k}_2 \sum_{s_1,s_2} \tilde{\phi}_{s_2 s_1}(\boldsymbol{k}_2,\boldsymbol{k}_1)\hat{c}^\dagger_{s_1}(\boldsymbol{k}_1)\hat{c}^\dagger_{s_2}(\boldsymbol{k}_2)|0\rangle \\
&= -|\phi_{21}\rangle
\end{aligned}
$$

- Thus spin one-half particles must be fermions. This is the spin-statistics theorem.

Interacting Fields

- We introduce e.m. interactions into the Dirac Lagrangian by the usual minimal substitution, $\partial^\mu \to \partial^\mu + ieA^\mu$:

$$\begin{aligned}\mathcal{L}_\mathrm{D} &= \bar{\psi} i\gamma^\mu \left(\partial_\mu + ieA_\mu\right)\psi - m\bar{\psi}\psi \\ &= \mathcal{L}_0 - eA_\mu \bar{\psi}\gamma^\mu\psi\end{aligned}$$

  where $\mathcal{L}_0$ is the free-particle Lagrangian density.

- Notice that the canonical momentum $\pi = \partial\mathcal{L}_\mathrm{D}/\partial\dot{\psi}$ is unchanged, and so the Hamiltonial density is

$$\mathcal{H}_\mathrm{D} = \pi\dot{\psi} - \mathcal{L}_\mathrm{D} = \mathcal{H}_0 + \mathcal{H}_\mathrm{I}$$

  where the interaction Hamiltonian density is

$$\mathcal{H}_\mathrm{I} = eA_\mu \bar{\psi}\gamma^\mu\psi$$

- In the second-quantized theory $\mathcal{H}_\mathrm{I}$ becomes an operator,

$$\hat{\mathcal{H}}_\mathrm{I} = e\hat{A}_\mu \hat{\bar{\psi}}\gamma^\mu\hat{\psi}$$

where all the field operators are capable of creating or annihilating particles:

$$\begin{aligned}
\hat{A}_\mu &= \int d^3\boldsymbol{k}\, N(\boldsymbol{k}) \sum_P \left[\hat{a}_P(\boldsymbol{k})\varepsilon_{P\mu} e^{-ik\cdot x} + \hat{a}_P^\dagger(\boldsymbol{k})\varepsilon_{P\mu}^* e^{+ik\cdot x}\right] \\
\hat{\bar{\psi}} &= \int d^3\boldsymbol{p}'\, N(\boldsymbol{p}') \sum_{s'} \left[\hat{c}_{s'}^\dagger(\boldsymbol{p}')\bar{u}_{s'}(\boldsymbol{p}') e^{+ip'\cdot x} + \hat{d}_{s'}(\boldsymbol{p}')\bar{v}_{s'}(\boldsymbol{p}') e^{-ip'\cdot x}\right] \\
\hat{\psi} &= \int d^3\boldsymbol{p}\, N(\boldsymbol{p}) \sum_s \left[\hat{c}_s(\boldsymbol{p})u_s(\boldsymbol{p}) e^{-ip\cdot x} + \hat{d}_s^\dagger(\boldsymbol{p})v_s(\boldsymbol{p}) e^{+ip\cdot x}\right]
\end{aligned}$$

- In first-order perturbation theory, the transition matrix element is

$$\mathcal{A}_{fi} = -i \int d^4x\, \langle f|\mathcal{H}_\mathrm{I}|i\rangle$$

- Suppose for example that the initial state $|i\rangle$ contains an electron of momentum p_i, spin orientation s_i, and a photon of momentum q, polarization R:

$$|i\rangle = \hat{c}_{s_i}^\dagger(\boldsymbol{p}_i)\hat{a}_R^\dagger(\boldsymbol{q})|0\rangle$$

- Using the (anti)commutation relations for the $\hat{c}$'s and $\hat{a}$'s, the positive-frequency parts of $\hat{A}$ and $\hat{\psi}$ give

$$\hat{A}_\mu^{(+)}\hat{\psi}^{(+)}|i\rangle = \varepsilon_{R\mu}(\boldsymbol{q})u_{s_i}(\boldsymbol{p}_i)e^{-i(p_i+q)\cdot x}|0\rangle$$

- Similarly, if $|f\rangle$ contains an electron of momentum $\boldsymbol{p}_f$, spin s_f,

$$\begin{aligned} |f\rangle &= \hat{c}^\dagger_{s_f}(\boldsymbol{p}_f)|0\rangle \\ \langle f| &= \langle 0|\hat{c}_{s_f}(\boldsymbol{p}_f) \end{aligned}$$

and hence the negative-frequency part of $\hat{\bar{\psi}}$ gives

$$\langle f|\hat{\bar{\psi}}^{(-)} = \bar{u}_{s_f}(\boldsymbol{p}_f)e^{+ip_f\cdot x}\langle 0|$$

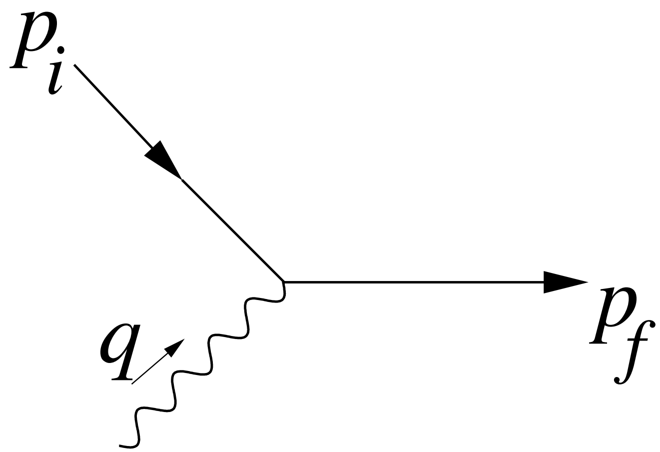
- Putting everything together, we have

$$\langle f|\hat{A}_\mu^{(+)}\hat{\bar{\psi}}^{(-)}\gamma^\mu\hat{\psi}^{(+)}|i\rangle = \varepsilon_{R\mu}\bar{u}_{s_f}(\boldsymbol{p}_f)\gamma^\mu u_{s_i}(\boldsymbol{p}_i)e^{i(p_f-p_i-q)\cdot x}$$

which gives

$$\mathcal{A}_{fi} = -ie(2\pi)^4\, \delta^4(p_f - p_i - q)\, \varepsilon_{R\mu}\bar{u}_{s_f}(\boldsymbol{p}_f)\gamma^\mu u_{s_i}(\boldsymbol{p}_i)$$

corresponding to the Feynman rule for the vertex



- Similarly, if the initial and final fermions are positrons, then the term

$$\langle f|\hat{A}_\mu^{(+)}\hat{\bar{\psi}}^{(+)}\gamma^\mu\hat{\psi}^{(-)}|i\rangle$$

gives the expected result

$$\mathcal{A}_{fi} = -ie(2\pi)^4\, \delta^4(p_f - p_i - q)\, \varepsilon_{R\mu}\bar{v}_{s_i}(\boldsymbol{p}_i)\gamma^\mu v_{s_f}(\boldsymbol{p}_f)$$